ECON662D1_ass3_JSM_code

August 29, 2022

[1]:
```
# ECON662D1 Assignment 3 - Jean-Sébastien Matte 260913682
```

[2]:
```
# import modules
import numpy as np
import pandas as pd
import scipy as sp
import statsmodels.api as sm
import matplotlib
matplotlib.use("pgf")
matplotlib.rcParams.update({
    "pgf.texsystem": "pdflatex",
    'font.family': 'serif',
    'text.usetex': True,
    'pgf.rcfonts': False,
})

np.random.seed(123)
```

[3]:
```
# load data
in_dir = '/Users/jsmatte/github/ECON662D1/Assignment3'
in_file = str(in_dir + '/e662.as3.20.dat.txt')
df = pd.read_csv(in_file, sep = '\s+', header = None)
print(df)
```

```
           0          1          2          3
0  -0.429193   91.52508  -7.879150  -22.82027
1  -1.719862   58.32108 -10.377350  -46.89894
2  -5.007316   71.19264   8.240685  -83.72890
3  -0.287603   85.37536  -7.939428  -98.93214
4  -5.385808   84.63815  28.309510  -80.68243
..       …          …          …          …
95 -4.539189   26.54987 -40.947840 -146.96500
96 -9.798897   69.94029  75.729620 -126.41690
97 -2.504187   89.50351 -44.335000  -76.11952
98 -2.118638  100.67600  48.674720  -16.33207
99 -0.782366  133.57730   4.789128  -10.57532

[100 rows x 4 columns]
```

[4]:
```
df.columns = ['y', 'x1', 'x2', 'x3']
print(df)
```

```
           y         x1         x2         x3
0  -0.429193   91.52508  -7.879150  -22.82027
1  -1.719862   58.32108 -10.377350  -46.89894
2  -5.007316   71.19264   8.240685  -83.72890
3  -0.287603   85.37536  -7.939428  -98.93214
4  -5.385808   84.63815  28.309510  -80.68243
..       …         …         …         …
95 -4.539189   26.54987 -40.947840 -146.96500
96 -9.798897   69.94029  75.729620 -126.41690
97 -2.504187   89.50351 -44.335000  -76.11952
98 -2.118638  100.67600  48.674720  -16.33207
99 -0.782366  133.57730   4.789128  -10.57532

[100 rows x 4 columns]
```

[5]:
```
# create matrix X of the regressors (unrestricted model)
X = np.zeros((len(df),4))
for i in range(X.shape[0]):
    X[i] = [1, df.x1.values[i], df.x2.values[i], df.x3.values[i]]
print(X.shape)

# create vector y of observations
y = np.array([[j] for j in df.y.values])
print(y.shape)
```

```
(100, 4)
(100, 1)
```

[6]:
```
# Create the different OLS models of different sample sizes, and collect data
n = 17
H1 = []
H2 = []
for i in range(n):
    print('MODEL_' + str(i))

    # unrestricted model & results
    globals()['model_' + str(i)] = sm.OLS(y[0:(20+5*i), :], X[0:(20+5*i), :]).
 ↪fit()
    temp_reg = globals()['model_' + str(i)]
    print(temp_reg.summary())
    print('')

    temp_resid = temp_reg.resid
    temp_fitval = temp_reg.fittedvalues
```

```
   # Hypothesis 1: H0: B1 = 0
   print('Hopothesis 1: H0: B1 = 0')
   temp_P_H1 = []

   # restricted model & results
   X_rest1 = X[0:(20+5*i), [0, 2, 3]]
   globals()['model_' + str(i) + '_rest1'] = sm.OLS(y[0:(20+5*i), :], X_rest1).
↪fit()
   temp_reg_rest1 = globals()['model_' + str(i) + '_rest1']

   temp_resid_rest1 = temp_reg_rest1.resid
   temp_fitval_rest1 = temp_reg_rest1.fittedvalues


   # p-value based on Student t-test
   print('P-value based on Student t-test')
   print(temp_reg.t_test('x1 = 0'))
   temp_P_H1.append(temp_reg.pvalues[1])
   print('')


   # P-value from rescaled restricted residuals bootstrap
   temp_nobs = temp_reg.nobs

   # rescale residuals
   rescale_factor = np.sqrt(temp_nobs / (temp_nobs - 3))
   rescaled_temp_resid_rest1 = rescale_factor * temp_resid_rest1

   boot_t = []
   for b in range(999):
       # draw from rescaled restricted distrubances
       draws_rest1 = np.random.choice(rescaled_temp_resid_rest1, size =␣
↪len(rescaled_temp_resid_rest1))

       # update observations using rescaled, restricted, resampled residuals
       temp_y_star_rest1 = temp_fitval_rest1 + draws_rest1
       temp_y_star_rest1 = np.array([[j] for j in temp_y_star_rest1])

       # Regress y* on the original covariates
       temp_reg_star = sm.OLS(temp_y_star_rest1, X[0:(20+5*i), :]).fit()

       # Bootstrap statistic
       boot_t.append(temp_reg_star.tvalues[1])

   boot_cnt_right = 0
```

```
    boot_cnt_left = 0
    for bt in boot_t:
        if bt <= temp_reg.tvalues[1]:
            boot_cnt_right += 1
        elif bt > temp_reg.tvalues[1]:
            boot_cnt_left += 1
    boot_p = 2 * min(boot_cnt_right / 999, boot_cnt_left / 999)
    temp_P_H1.append(boot_p)
    print('P-value from rescaled restricted residuals bootstrap')
    print(boot_p)
    print('')


    # P-value from wild bootstrap
    # define the pmf of s* as proposed by Mammen
    xk = [-1, 1]  # -> the rv_discrete class is only defined for xk values that␣
↪are integers, thus we define the
                  #    pmf as such. after random draws, we reassign the correct␣
↪values to the s* vector and carry on
                  #    with our computation
    pk = [((np.sqrt(5)+1) / (2*np.sqrt(5))), ((np.sqrt(5)-1) / (2*np.sqrt(5)))]
    s_pmf = sp.stats.rv_discrete(name = 's_pmf', values = (xk, pk))

    boot_t_wild = []
    for b_wild in range(999):
        # s_star
        s_star_raw = s_pmf.rvs(size = len(temp_resid_rest1))

        s_star = []
        for s in s_star_raw:
            if s == -1:
                s_star.append(-(np.sqrt(5) - 1) / 2)
            elif s == 1:
                s_star.append((np.sqrt(5) + 1) / 2)

        # update observations using wild disturbances
        resid_star_rest1 = []
        for resid_idx in range(len(temp_resid_rest1)):
            resid_star_rest1.append(s_star[resid_idx] *␣
↪temp_resid_rest1[resid_idx])

        temp_y_star_wild_rest1 = temp_fitval_rest1 + resid_star_rest1
        temp_y_star_wild_rest1 = np.array([[q] for q in temp_y_star_wild_rest1])

        # Regress y* on the original covariates
        temp_reg_wild = sm.OLS(temp_y_star_wild_rest1, X[0:(20+5*i), :]).fit()
```

```
        # Wild Bootstrap statistic
        boot_t_wild.append(temp_reg_wild.tvalues[1])

    boot_wild_cnt_right = 0
    boot_wild_cnt_left = 0
    for bt_wild in boot_t_wild:
        if bt_wild <= temp_reg.tvalues[1]:
            boot_wild_cnt_right += 1
        elif bt_wild > temp_reg.tvalues[1]:
            boot_wild_cnt_left += 1
    boot_p_wild = 2 * min(boot_wild_cnt_right / 999, boot_wild_cnt_left / 999)
    temp_P_H1.append(boot_p_wild)

    print('P-value from wild bootstrap')
    print(boot_p_wild)
    print('')

    H1.append(temp_P_H1)



    # Hypothesis 2: H0: B1 = 0.1
    print('Hopothesis 2: H0: B1 = 0.1')
    temp_P_H2 = []

    # restricted model & results
    temp_X2 = X[0:(20+5*i), 1]
    temp_X2 = np.array([[x] for x in temp_X2])
    y_rest2 = y[0:(20+5*i), :] - 0.1 * temp_X2
    X_rest2 = X_rest1
    globals()['model_' + str(i) + '_rest2'] = sm.OLS(y_rest2, X_rest2).fit()
    temp_reg_rest2 = globals()['model_' + str(i) + '_rest2']

    temp_resid_rest2 = temp_reg_rest2.resid
    temp_fitval_rest2 = temp_reg_rest2.fittedvalues


    # P-value based on Student t-test
    print('P-value based on Student t-test')
    print(temp_reg.t_test('x1 = 0.1'))
    temp_P_H2.append(temp_reg.t_test('x1 = 0.1').pvalue)
    print('')


    # P-value from rescaled restricted residuals bootstrap
    # rescale residuals
    rescaled_temp_resid_rest2 = rescale_factor * temp_resid_rest2
```

```
    boot_t2 = []
    for b2 in range(999):
        # draw from rescaled restricted distrubances
        draws_rest2 = np.random.choice(rescaled_temp_resid_rest2, size =␣
↪len(rescaled_temp_resid_rest2))

        # update observations using rescaled, restricted, resampled residuals
        temp_y_star_rest2 = temp_fitval_rest2 + draws_rest2
        temp_y_star_rest2 = np.array([[j] for j in temp_y_star_rest2])

        # Regress y* on the original covariates
        temp_reg_star2 = sm.OLS(temp_y_star_rest2, X[0:(20+5*i), :]).fit()

        # Bootstrap statistic
        boot_t2.append(temp_reg_star2.t_test('x1 = 0.1').tvalue[0][0])

    boot_cnt_right2 = 0
    boot_cnt_left2 = 0
    for bt2 in boot_t2:
        if bt2 <= temp_reg.tvalues[1]:
            boot_cnt_right2 += 1
        elif bt2 > temp_reg.tvalues[1]:
            boot_cnt_left2 += 1
    boot_p2 = 2 * min(boot_cnt_right2 / 999, boot_cnt_left2 / 999)
    temp_P_H2.append(boot_p2)
    print('P-value from rescaled restricted residuals bootstrap')
    print(boot_p2)
    print('')


    # P-value from wild bootstrap
    boot_t_wild2 = []
    for b_wild2 in range(999):
        # s_star
        s_star_raw2 = s_pmf.rvs(size = len(temp_resid_rest2))

        s_star2 = []
        for s2 in s_star_raw2:
            if s2 == -1:
                s_star2.append(-(np.sqrt(5) - 1) / 2)
            elif s2 == 1:
                s_star2.append((np.sqrt(5) + 1) / 2)

        # update observations using wild disturbances
        resid_star_rest2 = []
        for resid_idx2 in range(len(temp_resid_rest2)):
```

```
            resid_star_rest2.append(s_star2[resid_idx2] *␣
↪temp_resid_rest2[resid_idx2])

        temp_y_star_wild_rest2 = temp_fitval_rest2 + resid_star_rest2
        temp_y_star_wild_rest2 = np.array([[q] for q in temp_y_star_wild_rest2])

        # Regress y* on the original covariates
        temp_reg_wild2 = sm.OLS(temp_y_star_wild_rest2, X[0:(20+5*i), :]).fit()

        # Wild Bootstrap statistic
        boot_t_wild2.append(temp_reg_wild2.t_test('x1 = 0.1').tvalue[0][0])

    boot_wild_cnt_right2 = 0
    boot_wild_cnt_left2 = 0
    for bt_wild2 in boot_t_wild2:
        if bt_wild2 <= temp_reg.tvalues[1]:
            boot_wild_cnt_right2 += 1
        elif bt_wild2 > temp_reg.tvalues[1]:
            boot_wild_cnt_left2 += 1
    boot_p_wild2 = 2 * min(boot_wild_cnt_right2 / 999, boot_wild_cnt_left2 /␣
↪999)
    temp_P_H2.append(boot_p_wild2)

    print('P-value from wild bootstrap')
    print(boot_p_wild2)
    print('')

    H2.append(temp_P_H2)

    print('-----------------------------------------------------------')
    print('')
```

```
MODEL_0
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.544
Model:                            OLS   Adj. R-squared:                  0.458
Method:                 Least Squares   F-statistic:                     6.351
Date:                Tue, 01 Dec 2020   Prob (F-statistic):            0.00485
Time:                        14:30:45   Log-Likelihood:                -34.074
No. Observations:                  20   AIC:                             76.15
Df Residuals:                      16   BIC:                             80.13
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
```

```
const         -0.6174      1.558     -0.396      0.697      -3.920       2.685
x1             0.0056      0.015      0.366      0.719      -0.027       0.038
x2            -0.0424      0.012     -3.534      0.003      -0.068      -0.017
x3             0.0332      0.013      2.478      0.025       0.005       0.062
==============================================================================
Omnibus:                        3.991   Durbin-Watson:                   2.331
Prob(Omnibus):                  0.136   Jarque-Bera (JB):                2.583
Skew:                           0.878   Prob(JB):                        0.275
Kurtosis:                       3.138   Cond. No.                         479.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0056      0.015      0.366      0.719      -0.027       0.038
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.7267267267267268

P-value from wild bootstrap
0.6486486486486487

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0056      0.015     -6.202      0.000      -0.027       0.038
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_1
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.648
Model:                            OLS   Adj. R-squared:                  0.597
Method:                 Least Squares   F-statistic:                     12.86
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           5.48e-05
Time:                        14:30:51   Log-Likelihood:                -42.146
No. Observations:                  25   AIC:                             92.29
Df Residuals:                      21   BIC:                             97.17
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0281      1.275      0.022      0.983      -2.623       2.679
x1             0.0075      0.013      0.571      0.574      -0.020       0.035
x2            -0.0402      0.011     -3.807      0.001      -0.062      -0.018
x3             0.0473      0.009      5.354      0.000       0.029       0.066
==============================================================================
Omnibus:                        8.026   Durbin-Watson:                   2.311
Prob(Omnibus):                  0.018   Jarque-Bera (JB):                6.397
Skew:                           1.210   Prob(JB):                       0.0408
Kurtosis:                       3.533   Cond. No.                         456.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0075      0.013      0.571      0.574      -0.020       0.035
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.5945945945945946

P-value from wild bootstrap
0.47047047047047047

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
```

```
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0075      0.013     -7.078      0.000      -0.020       0.035
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

----------------------------------------------------

MODEL_2
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.601
Model:                            OLS   Adj. R-squared:                  0.555
Method:                 Least Squares   F-statistic:                     13.05
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           2.16e-05
Time:                        14:30:57   Log-Likelihood:                -52.025
No. Observations:                  30   AIC:                             112.1
Df Residuals:                      26   BIC:                             117.7
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.4020      1.276     -0.315      0.755      -3.024       2.220
x1             0.0116      0.013      0.896      0.378      -0.015       0.038
x2            -0.0379      0.010     -3.686      0.001      -0.059      -0.017
x3             0.0447      0.009      5.177      0.000       0.027       0.062
==============================================================================
Omnibus:                        6.526   Durbin-Watson:                   2.496
Prob(Omnibus):                  0.038   Jarque-Bera (JB):                5.529
Skew:                           1.048   Prob(JB):                       0.0630
Kurtosis:                       3.162   Cond. No.                         475.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
```

```
------------------------------------------------------------------------------
c0             0.0116      0.013      0.896      0.378      -0.015       0.038
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.36236236236236236

P-value from wild bootstrap
0.25825825825825827

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0116      0.013     -6.798      0.000      -0.015       0.038
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_3
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.603
Model:                            OLS   Adj. R-squared:                  0.565
Method:                 Least Squares   F-statistic:                     15.71
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           2.15e-06
Time:                        14:31:03   Log-Likelihood:                -59.019
No. Observations:                  35   AIC:                             126.0
Df Residuals:                      31   BIC:                             132.3
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.1940      1.166     -0.166      0.869      -2.571       2.183
x1             0.0082      0.012      0.711      0.482      -0.015       0.032
x2            -0.0394      0.009     -4.296      0.000      -0.058      -0.021
x3             0.0455      0.008      5.657      0.000       0.029       0.062
==============================================================================
Omnibus:                       11.785   Durbin-Watson:                   2.394
```

```
Prob(Omnibus):                  0.003   Jarque-Bera (JB):                11.410
Skew:                           1.332   Prob(JB):                      0.00333
Kurtosis:                       3.854   Cond. No.                        503.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0082      0.012      0.711      0.482      -0.015       0.032
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.4964964964964965

P-value from wild bootstrap
0.3303303303303303

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0082      0.012     -7.942      0.000      -0.015       0.032
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_4
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.607
Model:                            OLS   Adj. R-squared:                  0.574
Method:                 Least Squares   F-statistic:                     18.54
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           1.93e-07
```

```
Time:                        14:31:09   Log-Likelihood:                -66.268
No. Observations:                40   AIC:                             140.5
Df Residuals:                    36   BIC:                             147.3
Df Model:                         3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.2863      1.084     -0.264      0.793      -2.484       1.911
x1             0.0080      0.010      0.776      0.443      -0.013       0.029
x2            -0.0381      0.009     -4.341      0.000      -0.056      -0.020
x3             0.0445      0.007      6.067      0.000       0.030       0.059
==============================================================================
Omnibus:                       12.970   Durbin-Watson:                   2.441
Prob(Omnibus):                  0.002   Jarque-Bera (JB):               13.203
Skew:                           1.318   Prob(JB):                      0.00136
Kurtosis:                       3.989   Cond. No.                         515.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0080      0.010      0.776      0.443      -0.013       0.029
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.44644644644644643

P-value from wild bootstrap
0.3383383383383383

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0080      0.010     -8.925      0.000      -0.013       0.029
==============================================================================

P-value from rescaled restricted residuals bootstrap
```

0.0

P-value from wild bootstrap
0.0

MODEL_5

OLS Regression Results

Dep. Variable:	y	R-squared:	0.631
Model:	OLS	Adj. R-squared:	0.604
Method:	Least Squares	F-statistic:	23.41
Date:	Tue, 01 Dec 2020	Prob (F-statistic):	5.44e-09
Time:	14:31:15	Log-Likelihood:	-72.622
No. Observations:	45	AIC:	153.2
Df Residuals:	41	BIC:	160.5
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	-0.0868	0.942	-0.092	0.927	-1.989	1.815
x1	0.0059	0.009	0.652	0.518	-0.012	0.024
x2	-0.0380	0.008	-4.808	0.000	-0.054	-0.022
x3	0.0447	0.007	6.838	0.000	0.032	0.058

Omnibus:	14.700	Durbin-Watson:	2.429
Prob(Omnibus):	0.001	Jarque-Bera (JB):	15.915
Skew:	1.327	Prob(JB):	0.000350
Kurtosis:	4.200	Cond. No.	489.

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test

Test for Constraints

	coef	std err	t	P>|t|	[0.025	0.975]
c0	0.0059	0.009	0.652	0.518	-0.012	0.024

P-value from rescaled restricted residuals bootstrap
0.5205205205205206

```
P-value from wild bootstrap
0.47847847847847846

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0059      0.009    -10.302      0.000      -0.012       0.024
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

----------------------------------------------------

MODEL_6
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.705
Model:                            OLS   Adj. R-squared:                  0.686
Method:                 Least Squares   F-statistic:                     36.70
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           2.90e-12
Time:                        14:31:20   Log-Likelihood:                -79.054
No. Observations:                  50   AIC:                             166.1
Df Residuals:                      46   BIC:                             173.8
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0614      0.879      0.070      0.945      -1.708       1.831
x1             0.0060      0.008      0.705      0.484      -0.011       0.023
x2            -0.0397      0.007     -5.365      0.000      -0.055      -0.025
x3             0.0479      0.006      8.614      0.000       0.037       0.059
==============================================================================
Omnibus:                       19.469   Durbin-Watson:                   2.409
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               24.528
Skew:                           1.485   Prob(JB):                     4.72e-06
Kurtosis:                       4.718   Cond. No.                         511.
==============================================================================

Notes:
```

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test

Test for Constraints

	coef	std err	t	P>|t|	[0.025	0.975]
c0	0.0060	0.008	0.705	0.484	-0.011	0.023

P-value from rescaled restricted residuals bootstrap
0.4844844844844845

P-value from wild bootstrap
0.42042042042042044

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test

Test for Constraints

	coef	std err	t	P>|t|	[0.025	0.975]
c0	0.0060	0.008	-11.118	0.000	-0.011	0.023

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

MODEL_7

OLS Regression Results

Dep. Variable:	y	R-squared:	0.712
Model:	OLS	Adj. R-squared:	0.695
Method:	Least Squares	F-statistic:	42.07
Date:	Tue, 01 Dec 2020	Prob (F-statistic):	7.89e-14
Time:	14:31:26	Log-Likelihood:	-84.668
No. Observations:	55	AIC:	177.3
Df Residuals:	51	BIC:	185.4
Df Model:	3		
Covariance Type:	nonrobust		

```
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0656      0.832      0.079      0.937      -1.604       1.735
x1             0.0054      0.008      0.683      0.498      -0.011       0.021
x2            -0.0396      0.007     -6.022      0.000      -0.053      -0.026
x3             0.0479      0.005      9.117      0.000       0.037       0.058
==============================================================================
Omnibus:                       24.346   Durbin-Watson:                   2.396
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               35.789
Skew:                           1.617   Prob(JB):                     1.69e-08
Kurtosis:                       5.270   Cond. No.                         530.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0054      0.008      0.683      0.498      -0.011       0.021
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.5225225225225225

P-value from wild bootstrap
0.45245245245245247

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0054      0.008    -11.844      0.000      -0.011       0.021
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------
```

MODEL_8

OLS Regression Results

Dep. Variable:	y	R-squared:	0.695
Model:	OLS	Adj. R-squared:	0.678
Method:	Least Squares	F-statistic:	42.46
Date:	Tue, 01 Dec 2020	Prob (F-statistic):	1.91e-14
Time:	14:31:32	Log-Likelihood:	-93.707
No. Observations:	60	AIC:	195.4
Df Residuals:	56	BIC:	203.8
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.0668	0.818	0.082	0.935	-1.573	1.706
x1	0.0048	0.008	0.619	0.539	-0.011	0.020
x2	-0.0377	0.007	-5.750	0.000	-0.051	-0.025
x3	0.0475	0.005	9.080	0.000	0.037	0.058

Omnibus:	22.942	Durbin-Watson:	2.402
Prob(Omnibus):	0.000	Jarque-Bera (JB):	31.724
Skew:	1.519	Prob(JB):	1.29e-07
Kurtosis:	4.860	Cond. No.	537.

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test

Test for Constraints

	coef	std err	t	P>|t|	[0.025	0.975]
c0	0.0048	0.008	0.619	0.539	-0.011	0.020

P-value from rescaled restricted residuals bootstrap
0.4984984984984985

P-value from wild bootstrap
0.4624624624624625

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test

Test for Constraints

	coef	std err	t	P>|t|	[0.025	0.975]
c0	0.0048	0.008	-12.341	0.000	-0.011	0.020

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

MODEL_9

OLS Regression Results

Dep. Variable:	y	R-squared:	0.701
Model:	OLS	Adj. R-squared:	0.687
Method:	Least Squares	F-statistic:	47.78
Date:	Tue, 01 Dec 2020	Prob (F-statistic):	5.15e-16
Time:	14:31:38	Log-Likelihood:	-99.692
No. Observations:	65	AIC:	207.4
Df Residuals:	61	BIC:	216.1
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	-0.1765	0.742	-0.238	0.813	-1.661	1.308
x1	0.0069	0.007	0.995	0.323	-0.007	0.021
x2	-0.0382	0.006	-6.221	0.000	-0.051	-0.026
x3	0.0469	0.005	9.331	0.000	0.037	0.057

Omnibus:	26.144	Durbin-Watson:	2.358
Prob(Omnibus):	0.000	Jarque-Bera (JB):	39.406
Skew:	1.578	Prob(JB):	2.77e-09
Kurtosis:	5.142	Cond. No.	516.

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test

Test for Constraints

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0069      0.007      0.995      0.323      -0.007       0.021
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.36236236236236236

P-value from wild bootstrap
0.28428428428428426

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0069      0.007    -13.376      0.000      -0.007       0.021
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_10
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.720
Model:                            OLS   Adj. R-squared:                  0.707
Method:                 Least Squares   F-statistic:                     56.43
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           3.38e-18
Time:                        14:31:44   Log-Likelihood:                -106.59
No. Observations:                  70   AIC:                             221.2
Df Residuals:                      66   BIC:                             230.2
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.1127      0.708     -0.159      0.874      -1.526       1.300
x1             0.0065      0.007      0.977      0.332      -0.007       0.020
x2            -0.0394      0.006     -6.719      0.000      -0.051      -0.028
x3             0.0473      0.005     10.051      0.000       0.038       0.057
```

```
==============================================================================
Omnibus:                       27.727   Durbin-Watson:                   2.389
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               42.925
Skew:                           1.588   Prob(JB):                     4.78e-10
Kurtosis:                       5.152   Cond. No.                         519.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0065      0.007      0.977      0.332      -0.007       0.020
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.2902902902902903

P-value from wild bootstrap
0.3023023023023023

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0065      0.007    -14.031      0.000      -0.007       0.020
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_11
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.738
Model:                            OLS   Adj. R-squared:                  0.727
```

```
Method:                 Least Squares   F-statistic:                     66.74
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           1.27e-20
Time:                        14:31:52   Log-Likelihood:                -112.36
No. Observations:                  75   AIC:                             232.7
Df Residuals:                      71   BIC:                             242.0
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0612      0.684     -0.089      0.929      -1.426       1.303
x1             0.0058      0.006      0.911      0.366      -0.007       0.018
x2            -0.0394      0.006     -7.033      0.000      -0.051      -0.028
x3             0.0476      0.004     10.747      0.000       0.039       0.056
==============================================================================
Omnibus:                       31.414   Durbin-Watson:                   2.392
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               53.210
Skew:                           1.654   Prob(JB):                     2.79e-12
Kurtosis:                       5.466   Cond. No.                         532.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0058      0.006      0.911      0.366      -0.007       0.018
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.34434434434434436

P-value from wild bootstrap
0.3383383383383383

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0058      0.006    -14.826      0.000      -0.007       0.018
==============================================================================
```

```
P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

-----------------------------------------------------

MODEL_12
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.734
Model:                            OLS   Adj. R-squared:                  0.724
Method:                 Least Squares   F-statistic:                     70.06
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           7.95e-22
Time:                        14:32:00   Log-Likelihood:                -119.82
No. Observations:                  80   AIC:                             247.6
Df Residuals:                      76   BIC:                             257.2
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0231      0.658     -0.035      0.972      -1.333       1.287
x1             0.0056      0.006      0.939      0.351      -0.006       0.018
x2            -0.0409      0.005     -7.592      0.000      -0.052      -0.030
x3             0.0478      0.004     10.949      0.000       0.039       0.057
==============================================================================
Omnibus:                       31.431   Durbin-Watson:                   2.430
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               52.029
Skew:                           1.611   Prob(JB):                     5.03e-12
Kurtosis:                       5.286   Cond. No.                         536.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0056      0.006      0.939      0.351      -0.006       0.018
==============================================================================
```

```
P-value from rescaled restricted residuals bootstrap
0.36436436436436437

P-value from wild bootstrap
0.2982982982982983

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0056      0.006    -15.794      0.000      -0.006       0.018
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_13
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.725
Model:                            OLS   Adj. R-squared:                  0.714
Method:                 Least Squares   F-statistic:                     71.05
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           1.29e-22
Time:                        14:32:08   Log-Likelihood:                -127.58
No. Observations:                  85   AIC:                             263.2
Df Residuals:                      81   BIC:                             272.9
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.2322      0.649     -0.358      0.722      -1.524       1.059
x1             0.0074      0.006      1.240      0.218      -0.004       0.019
x2            -0.0417      0.005     -7.800      0.000      -0.052      -0.031
x3             0.0462      0.004     10.913      0.000       0.038       0.055
==============================================================================
Omnibus:                       30.066   Durbin-Watson:                   2.332
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               46.956
Skew:                           1.533   Prob(JB):                     6.36e-11
Kurtosis:                       4.964   Cond. No.                         547.
==============================================================================
```

```
Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0074      0.006      1.240      0.218      -0.004       0.019
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.23423423423423423

P-value from wild bootstrap
0.1881881881881882

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0074      0.006    -15.633      0.000      -0.004       0.019
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_14
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.736
Model:                            OLS   Adj. R-squared:                  0.726
Method:                 Least Squares   F-statistic:                     79.78
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           9.13e-25
Time:                        14:32:16   Log-Likelihood:                -133.14
No. Observations:                  90   AIC:                             274.3
Df Residuals:                      86   BIC:                             284.3
Df Model:                           3
```

```
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.1376      0.617     -0.223      0.824      -1.365       1.090
x1             0.0060      0.006      1.068      0.288      -0.005       0.017
x2            -0.0418      0.005     -8.265      0.000      -0.052      -0.032
x3             0.0463      0.004     11.605      0.000       0.038       0.054
==============================================================================
Omnibus:                       34.542   Durbin-Watson:                   2.375
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               59.314
Skew:                           1.620   Prob(JB):                     1.32e-13
Kurtosis:                       5.306   Cond. No.                         552.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0060      0.006      1.068      0.288      -0.005       0.017
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.24824824824824826

P-value from wild bootstrap
0.24624624624624625

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0060      0.006    -16.847      0.000      -0.005       0.017
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0
```

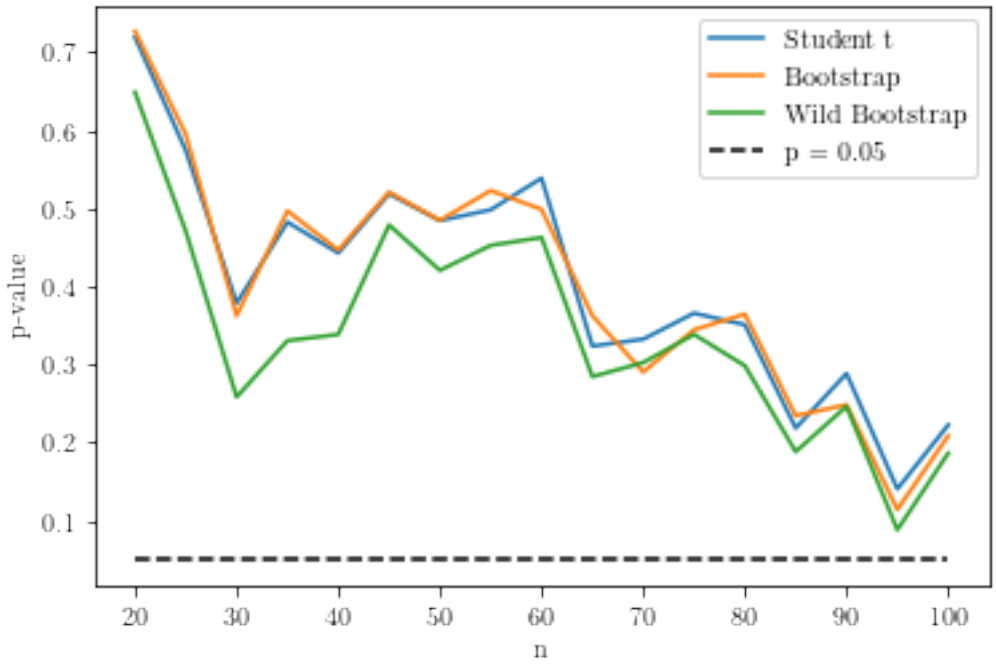
```
-----------------------------------------------------

MODEL_15
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.732
Model:                            OLS   Adj. R-squared:                  0.723
Method:                 Least Squares   F-statistic:                     82.92
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           6.13e-26
Time:                        14:32:23   Log-Likelihood:                -139.57
No. Observations:                  95   AIC:                             287.1
Df Residuals:                      91   BIC:                             297.4
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.3144      0.580     -0.542      0.589      -1.466       0.837
x1             0.0078      0.005      1.487      0.140      -0.003       0.018
x2            -0.0422      0.005     -8.529      0.000      -0.052      -0.032
x3             0.0457      0.004     11.767      0.000       0.038       0.053
==============================================================================
Omnibus:                       34.800   Durbin-Watson:                   2.361
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               59.065
Skew:                           1.584   Prob(JB):                     1.49e-13
Kurtosis:                       5.212   Cond. No.                         535.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0078      0.005      1.487      0.140      -0.003       0.018
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.11411411411411411

P-value from wild bootstrap
0.08808808808808809
```

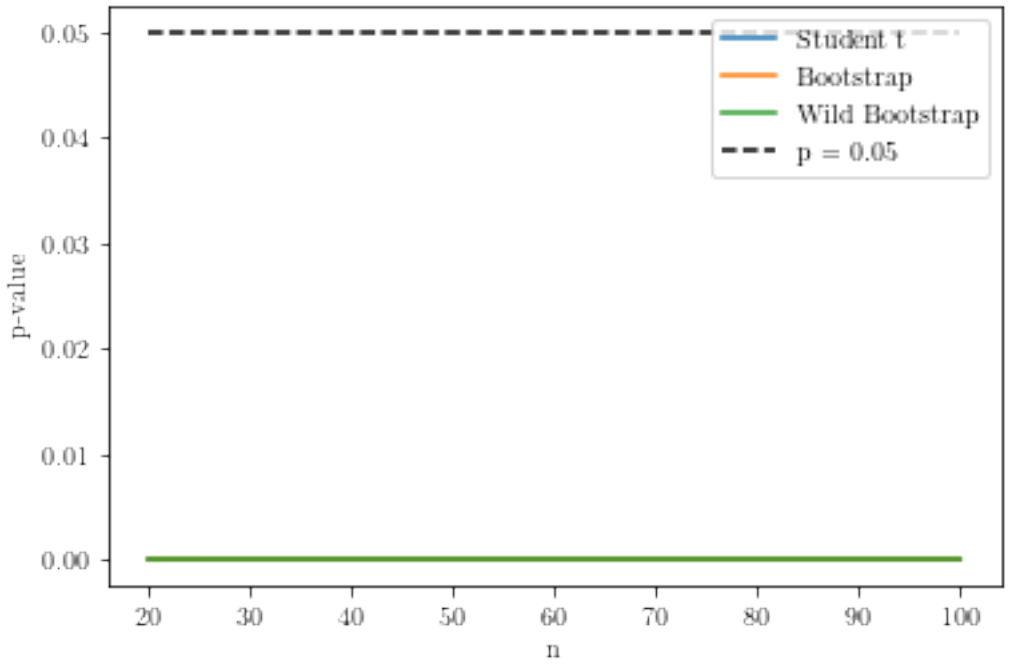
```
Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                            Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0078      0.005    -17.522      0.000      -0.003       0.018
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------

MODEL_16
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.760
Model:                            OLS   Adj. R-squared:                  0.752
Method:                 Least Squares   F-statistic:                     101.3
Date:                Tue, 01 Dec 2020   Prob (F-statistic):           1.23e-29
Time:                        14:32:30   Log-Likelihood:                -146.04
No. Observations:                 100   AIC:                             300.1
Df Residuals:                      96   BIC:                             310.5
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.1631      0.567     -0.288      0.774      -1.288       0.962
x1             0.0063      0.005      1.228      0.222      -0.004       0.016
x2            -0.0425      0.004     -9.609      0.000      -0.051      -0.034
x3             0.0465      0.004     12.692      0.000       0.039       0.054
==============================================================================
Omnibus:                       37.704   Durbin-Watson:                   2.361
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               67.499
Skew:                           1.617   Prob(JB):                     2.20e-15
Kurtosis:                       5.396   Cond. No.                         545.
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.

Hopothesis 1: H0: B1 = 0
```

```
P-value based on Student t-test
                             Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0063      0.005      1.228      0.222      -0.004       0.016
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.2082082082082082

P-value from wild bootstrap
0.18618618618618618

Hopothesis 2: H0: B1 = 0.1
P-value based on Student t-test
                             Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0             0.0063      0.005    -18.404      0.000      -0.004       0.016
==============================================================================

P-value from rescaled restricted residuals bootstrap
0.0

P-value from wild bootstrap
0.0

------------------------------------------------------
```

[7]:
```python
# Plot p-value results

H11 = []
H12 = []
H13 = []
for H in H1:
    H11.append(H[0])
    H12.append(H[1])
    H13.append(H[2])

H21 = []
H22 = []
H23 = []
for HH in H2:
    H21.append(HH[0])
```

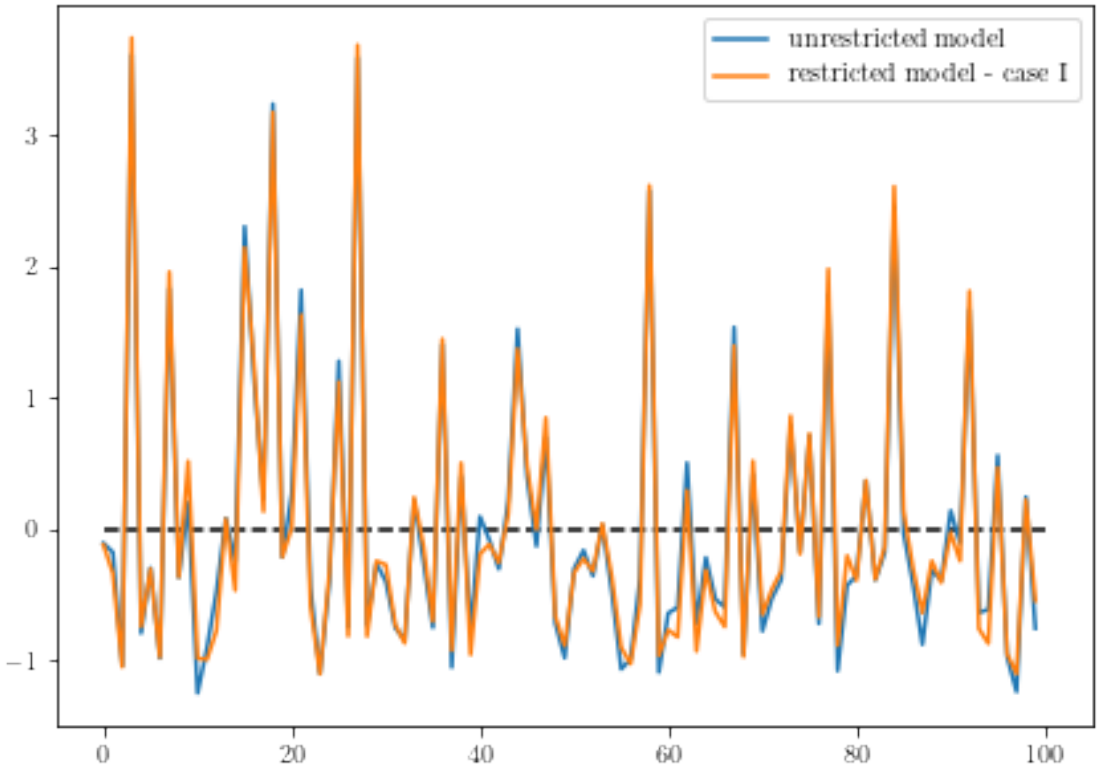
```
    H22.append(HH[1])
    H23.append(HH[2])

nobs = np.arange(20, 105, 5)

import matplotlib.pyplot as plt
%matplotlib inline
plt.plot(nobs, H11, label = 'Student t')
plt.plot(nobs, H12, label = 'Bootstrap')
plt.plot(nobs, H13, label = 'Wild Bootstrap')
plt.hlines(0.05, 20, 100, linestyle = 'dashed', label = 'p = 0.05')
plt.xlabel('n')
plt.ylabel('p-value')
plt.legend(loc = 'upper right')
plt.savefig('Varp.pgf')
plt.show()

import matplotlib.pyplot as plt
%matplotlib inline
plt.plot(nobs, H21, label = 'Student t')
plt.plot(nobs, H22, label = 'Bootstrap')
plt.plot(nobs, H23, label = 'Wild Bootstrap')
plt.hlines(0.05, 20, 100, linestyle = 'dashed', label = 'p = 0.05')
plt.xlabel('n')
plt.ylabel('p-value')
plt.legend(loc = 'upper right')
plt.savefig('Varp2.pgf')
plt.show()
```





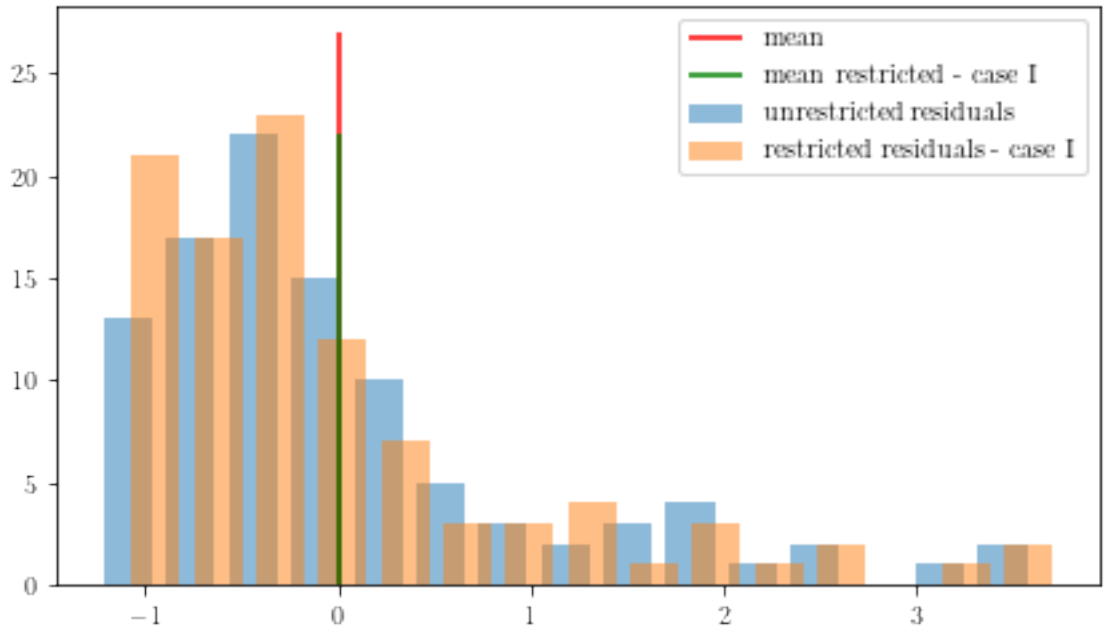
[8]:
```
resid_16 = model_16.resid
resid_rest1_16 = model_16_rest1.resid
```

```
x = np.arange(len(resid_16))

import matplotlib.pyplot as plt
%matplotlib inline
plt.figure(figsize=(7, 5))
plt.plot(resid_16, label = 'unrestricted model')
plt.plot(resid_rest1_16, label = 'restricted model - case I')
plt.hlines(0, 0, len(resid_16), color = 'k', linestyle = 'dashed')
plt.legend(loc = 'upper right')
plt.savefig('resid.pgf', dpi = 400)
plt.show()
```



[9]:
```
resid_unique, resid_counts = np.unique(resid_16, return_counts = True)
resid_rest1_unique, resid_rest1_counts = np.unique(resid_rest1_16,␣
 ↪return_counts = True)
nbins = 15
counts, bin_edges = np.histogram(resid_16, nbins)
ticks = []
for i_e in range(1, len(bin_edges)):
    ticks.append(((bin_edges[i_e] - bin_edges[i_e - 1]) / 2) + bin_edges[i_e -␣
 ↪1])
```

```
counts_rest1, bin_edges_rest1 = np.histogram(resid_rest1_16, nbins)
ticks_rest1 = []
for i_e_rest1 in range(1, len(bin_edges_rest1)):
    ticks_rest1.append(((bin_edges_rest1[i_e_rest1] - bin_edges_rest1[i_e_rest1␣
 ↪- 1]) / 2) + bin_edges_rest1[i_e_rest1 - 1])

import matplotlib.pyplot as plt
%matplotlib inline
plt.figure(figsize=(7, 4))
plt.bar(ticks, counts, width = 0.25, alpha = 0.5, label = 'unrestricted␣
 ↪residuals')
plt.bar(ticks_rest1, counts_rest1, width = 0.25, alpha = 0.5, label =␣
 ↪'restricted residuals - case I')
plt.vlines(np.mean(resid_16), 0, max(counts)+5, colors = 'r', label = 'mean')
plt.vlines(np.mean(resid_rest1_16), 0, max(counts), colors = 'g', label = 'mean␣
 ↪restricted - case I')
plt.legend(loc = 'upper right')
plt.savefig('resid_dist.pgf', dpi = 100)
plt.show()
```



[10]:
```
# Results

print(H11)
print('')
```

```
print(H12)
print('')
print(H13)
print('')
print('--------')
print(H21)
print('')
print(H22)
print('')
print(H23)
print('')
print('--------')
```

```
[0.7191648727895876, 0.5739307657161, 0.3783781052307009, 0.4822052779834689,
0.44263342877850886, 0.5183102694510442, 0.48410435202344426,
0.4979882646508451, 0.5385403316897768, 0.3234446951074834, 0.33229488641697214,
0.36556678284711674, 0.3507738743613309, 0.21839316036920003,
0.2883284369379734, 0.14044919427501004, 0.22242857874816382]

[0.7267267267267268, 0.5945945945945946, 0.36236236236236236,
0.4964964964964965, 0.44644644644644643, 0.5205205205205206, 0.4844844844844845,
0.5225225225225225, 0.4984984984984985, 0.36236236236236236, 0.2902902902902903,
0.34434434434434436, 0.36436436436436437, 0.23423423423423423,
0.24824824824824826, 0.11411411411411411, 0.2082082082082082]

[0.6486486486486487, 0.47047047047047047, 0.25825825825825827,
0.3303303303303303, 0.3383383383383383, 0.47847847847847846,
0.42042042042042044, 0.45245245245245247, 0.4624624624624625,
0.28428428428428426, 0.3023023023023023, 0.3383383383383383, 0.2982982982982983,
0.1881881881881882, 0.24624624624624625, 0.08808808808808809,
0.18618618618618618]

--------
[array(1.26771034e-05), array(5.53146685e-07), array(3.25099313e-07),
array(5.75686023e-09), array(1.1819935e-10), array(6.07786539e-13),
array(1.25945357e-14), array(2.95333443e-16), array(1.31067221e-17),
array(8.51666214e-20), array(1.75990338e-21), array(1.97131039e-23),
array(1.02937534e-25), array(3.52550924e-26), array(5.61418023e-29),
array(6.55491244e-31), array(3.00207401e-33)]

[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,
0.0]

[0.9409409409409409, 0.918918918918919, 0.8288288288288288, 0.7547547547547547,
0.7107107107107107, 0.7847847847847848, 0.7207207207207207, 0.7207207207207207,
0.6926926926926927, 0.5465465465465466, 0.6086086086086087, 0.5625625625625625,
0.6446446446446447, 0.46646646646646645, 0.5645645645645646,
0.37037037037037035, 0.4444444444444444]
```

[ ]: