Sequential Robust Auctions of Used Cars
MATH80624A
Term Paper

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1 Motivation

We study the problem of used car auctioning, more specifically, how sellers should sequence the offerings in a sequential auction to maximize revenues. Our study is based on the paper by Ma et al. (2021), which presents a robust linear program to determine the optimal sequencing and maximum revenue that can be generated from auctioning used cars while accounting for uncertainty in the selling price. The authors study four uncertainty sets: box, ellipsoidal, polyhedron, and box-polyhedron.

Upon reading the paper, we noted some shortcomings in the reformulations presented by the authors. Surprised, this therefore motivated us to first investigate further the reformulations for the four uncertainty sets presented.

Furthermore, although Ma et al. (2021) present an interesting application of robust optimization, we feel it could be extended. In particular, the uncertainty in the prices could be derived from a data-driven procedure (that necessarily includes more information on what affects the price variation) in the definition of the robust linear program, as well as the uncertainty set.

Finally, we noted Ma et al. (2021) do not provide any explanation on the trade-off between risk and return. This motivates us to use a chance constraint approach for our reformulation, to derive potential managerial insights from our model.

Research Questions. This study aims to answer the following research questions:

RQ1. How are our reformulations comparing with the ones presented in the paper?

RQ2. How can we extend the problem formulation by developing a data-driven robust optimization framework that accounts for uncertainty in the selling prices of the cars in each sequences, given which cars have been shown previously?

The rest of the study is organized as follows. The next section summarizes the studies guiding this project. The following section touches the data used by the authors of the paper in question. The nominal model and robust counterpart sections then present mathematical models proposed by the authors and new reformulations proposed by us. Following these sections, an extension of the current study will be presented and numerical results for both research questions will be given. Finally, conclusion section will summarize the study and will explain future research directions.

2 Relevant Literature

The problem of multi-item sequential auction is well studied as a deterministic or stochastic problem. A short review of such studies can be found in Ozturk and Karabatı (2017). However, applications of robust optimization to the problem are rather scant.

Broadly speaking, auctions can be studied from the bidder's perspective, or the seller's perspective. Some, such as Ozturk and Karabatı (2017) and Bandi and Bertsimas (2014) approach the problem from the bidder's perspective. Interestingly, Bandi and Bertsimas (2014) propose a robust optimization approach to the problem, where the uncertainty comes from the bidders' valuations. Conversely, Ma et al. (2021) approach the problem from the seller's perspective, as they are interested in deriving optimal sequencing decisions (for the seller).

On the optimal sequencing, specific to applications of used cars auctioning, Elmaghraby (2007) reports an interesting dichotomy between theory and practice. While theory suggests presenting high-values cars first, and low-valued cars last (high-low sequencing), it was observed that practitioners adopt the opposite strategy, that is, to show low-values cars first, and high-valued cars last (low-high sequencing). Interestingly, Ozturk and Karabatı (2017) find that such strategies are not optimal (in a limited information setting), and that the optimal strategy depends on the bidder's behaviour (myopic, strategic, etc.). Similarly, Ma et al. (2021) report simple strategies such as high-low or low-high are not optimal.

In terms of data-driven approaches, Rooderkerk and Van Heerde (2016) present an interesting application of a data-driven robust optimization approach in which a linear regression is used to define the uncertainty set. We use a similar intuition in the development of our approach.

3 Data

The authors share the data they use in their paper. Their study uses two data sets, a small one consisting of 10 cars and 5 sequences, and a larger one, consisting of 100 cars and 10 sequences. We focus on the smaller data set. The data is provided in Table 5 (appendix A).

4 Nominal Model

Throughout Sections 4 and 5, we follow the notation as presented in Ma et al. (2021). The notation is repeated in Table 1.

Table 1: Key notation used in the paper in Ma et al. (2021)

Sets	
I	cars ($\|I\|$ = n)
T	sequence ($\|T\|$ = m)
Parameters	
v_i	value estimated by seller for car i
a_i	price when the ith car is new
c_i	storage cost of car i
Variables	
Decision variable	
$x_{it} \in \{0,1\}$	whether to sell the ith car in the sequence t
Uncertain variable	
P_{it}	selling price of the ith car in the sequence t

According to this, cars will be denoted by the set I, and T will denote the set of sequences. In this problem, the seller has m sequences of auction and n different second-hand cars to be sold where the number of cars n is larger than the number of sequences m.

We have three parameters which will take different values for each car. v_i is the seller's buying price of car i, a_i is the price of the i^{th} car when it is new, and if the seller decide not to sell a car, s/he will incur holding cost c_i. It is also assumed that there are enough buyers so that each car, if shown, will be bought by someone.

The mathematical model will be used to decide whether to sell the i^{th} car in the sequence t, and the decision variable will be denoted as x_{it}. The defined problem is modeled as follows by the author of the paper.

$$
\begin{aligned}
\max_{x} \quad & \sum_{i=1}^{n}\sum_{t=1}^{m}(P_{it}-v_i)x_{it}-\sum_{i=1}^{n}(c_i(1-\sum_{t=1}^{m}x_{it})) && && \text{(1a)}\\
s.t. \quad & \sum_{t=1}^{m}P_{it}x_{it}\le a_i && \forall i\in I && \text{(1b)}\\
& \sum_{i=1}^{n}x_{it}\ge 1 && \forall t\in T && \text{(1c)}\\
& 0\le \sum_{t=1}^{m}x_{it}\le 1 && \forall i\in I && \text{(1d)}\\
& x_{it}\in\{0,1\} && \forall i\in I, t\in T && \text{(1e)}
\end{aligned}
$$

The aim is to maximize the total profit obtained. The first part of the objective function will give the revenue obtained from selling the used cars. With the second part, authors account the holding cost of cars if the seller decides not to sell a car in any sequence.

The first constraint ensures that the selling price of a used car is not more than the price when that car was new, a_i. Constraint (1c) is used for saying that there should at least one car in each sequence. Please note that, the constraint was written as strict inequality constraint in the paper, but we decided to continue with the correct representation. Lastly, Constraint (1d) ensures each car can only be sold at most in one sequence.

The above mathematical model is rewritten in epigraph form and it will be used in the following sections of this study.

$$
\begin{aligned}
\max_{x,w} \quad & w && && \text{(2a)}\\
s.t. \quad & w\le \sum_{i=1}^{n}\sum_{t=1}^{m}(P_{it}-v_i)x_{it}-\sum_{i=1}^{n}(c_i(1-\sum_{t=1}^{m}x_{it})) && && \text{(2b)}\\
& \sum_{t=1}^{m}P_{it}x_{it}\le a_i && \forall i\in I && \text{(2c)}\\
& \sum_{i=1}^{n}x_{it}\ge 1 && \forall t\in T && \text{(2d)}\\
& 0\le \sum_{t=1}^{m}x_{it}\le 1 && \forall i\in I && \text{(2e)}\\
& x_{it}\in\{0,1\} && \forall i\in I, t\in T && \text{(2f)}
\end{aligned}
$$

5 Robust Counterparts

As mentioned in Section 1, the price consists of two parts; determinate price and uncertainty price. The determinate price p_{it} is the average price of the similar used cars, and the uncertainty price includes price fluctuation $\tilde{p_{it}}$ and uncertainty factor $\xi_{it} \in U$ where U is the uncertainty set. Therefore, the robust counterpart of the above given mathematical model will be obtain by replacing the P_{it} with its new definition $P_{it} = p_{it} + \tilde{p}_{it}\xi_{it}$. It is provided below where S is used to simplify the first constraint, $S = \sum_{i=1}^{n}\sum_{t=1}^{m}(p_{it} - v_i)x_{it} - \sum_{i=1}^{n}(c_i(1 - \sum_{t=1}^{m} x_{it}))$.

$$
\begin{aligned}
\max_{x,w} \quad & w && && \text{(3a)}\\
s.t. \quad & w \leq S + \sum_{i=1}^{n}\sum_{t=1}^{m}(\tilde{p}_{it}\xi_{it})x_{it} && \forall \xi \in U && \text{(3b)}\\
& \sum_{t=1}^{m}(p_{it} + \tilde{p}_{it}\xi_{it})x_{it} \leq a_i && \forall i \in I, \forall \xi \in U && \text{(3c)}\\
& \sum_{i=1}^{n} x_{it} \geq 1 && \forall t \in T && \text{(3d)}\\
& 0 \leq \sum_{t=1}^{m} x_{it} \leq 1 && \forall i \in I && \text{(3e)}\\
& x_{it} \in \{0, 1\} && \forall i \in I, t \in T && \text{(3f)}
\end{aligned}
$$

5.1 Reformulations presented in the paper

We disagree with the robust reformulations presented in Ma et al. (2021). To support our claim, in the following subsections, we present our reformulations in addition to the limitations that we observed in the reformulations provided in the paper. Numerical comparison of the reformulations is provided in Section 7.1.

For the rest of this section, we follow the steps presented in Ben-Tal and Nemirovski (2000) to reformulate the problems given the different uncertainty sets.

5.1.1 Box Uncertainty Set

In the paper, the box uncertainty set is defined as: $Z^{Box} := \{\xi \in U \ : ||\xi||_\infty \leq \Theta\} = \{\xi : |\xi_{it}| \leq \Theta\}$. We first note that this definition is ambiguous given the ∞-norm on a matrix is not equivalent to $|\xi_{it}|$.

We disagree with the assumption the authors made to obtain the robust constraints to constraints (3b) and (3c). First, the authors wrongly assume that the minimum ξ can take over the interval $[-\Theta, \Theta]$ is zero for constraint (3b). Second, the assumption that the maximum (minimum) value ξ can take over the interval $[-\Theta, \Theta]$ is Θ $(-\Theta)$ is inadequate given the data used. As presented in Table 5, some of the variances $\tilde{p}_{it}$ are negative. The assumption used by the authors, in this case, would not always provide the maximum (minimum) for a given car. On the other hand, if the variance had been defined as a non-negative scalar, and the perturbation as being within the interval $[-1, 1]$, then the assumption used by the authors would have been valid.

We now present our robust reformulation for the Box uncertainty set.

Proposition 1 *Let the Box uncertainty set be defined as*

$$Z^{Box} := \{\xi \in U \ : ||\xi||_\infty \leq \Theta\} = \{\xi : |\xi_{it}| \leq \Theta\}$$

then, the robust reformulation of the proposed mathematical model is

$$\begin{aligned}
\max_{x,w,\lambda^+,\lambda^-,\gamma^+,\gamma^-} \quad & w && && \text{(4a)}\\
s.t. \quad & \Theta \sum_{i=1}^{n}\sum_{t=1}^{m}(\lambda_{it}^+ + \lambda_{it}^-) \geq w - S && && \text{(4b)}\\
& \lambda_{it}^+ - \lambda_{it}^- = \tilde{p}_{it}x_{it} && \forall i \in I, \forall t \in T && \text{(4c)}\\
& \Theta \sum_{t=1}^{m}(\gamma_{it}^+ + \gamma_{it}^-) \leq a_i - \sum_{t=1}^{m} p_{it}x_{it} && \forall i \in I && \text{(4d)}\\
& \gamma_{it}^+ - \gamma_{it}^- = \tilde{p}_{it}x_{it} && \forall i \in I, \forall t \in T && \text{(4e)}\\
& \sum_{i=1}^{n} x_{it} \geq 1 && \forall t \in T && \text{(4f)}\\
& 0 \leq \sum_{t=1}^{m} x_{it} \leq 1 && \forall i \in I && \text{(4g)}\\
& x_{it} \in \{0,1\} && \forall i \in I, \forall t \in T && \text{(4h)}\\
& \lambda_{it}^+, \lambda_{it}^- \leq 0 && \forall i \in I, \forall t \in T && \text{(4i)}\\
& \gamma_{it}^+, \gamma_{it}^- \geq 0 && \forall i \in I, \forall t \in T && \text{(4j)}
\end{aligned}$$

Proof. See Appendix B ■

5.1.2 Ellipsoidal Uncertainty Set

Ellipsoidal uncertainty set is defined as follows: $Z^{Ellipsoidal} := \{\xi \in U | ||\xi||_2 \leq \Omega\} = \{\xi \in U | \sqrt{(\sum_t \xi_{it}^2)} \leq \Omega\}$. Again, we note that this definition is ambiguous given the the 2-norm on a matrix is not equivalent to $\sqrt{(\sum_t \xi_{it}^2)}$. We understand it as taking the 2-norm over the nx1 vector of the row sums.

We disagree with their reformulation for constraint (3b) due to the same reason discussed in the box uncertainty set. We now present our robust reformulation for the Ellipsoidal uncertainty set.

Proposition 2 *Let the Ellipsoidal uncertainty set be defined as*

$$Z^{Ellipsoidal} := \{\xi \in U | \sqrt{(\sum_t \xi_{it}^2)} \leq \Omega\}$$

then, the robust reformulation of the mathematical model is

$$\begin{aligned}
\max_{x,w} \quad & w && && \text{(5a)}\\
s.t. \quad & \Omega|| \sum_{i=1}^{n}\sum_{t=1}^{m} \tilde{p}_{it}x_{it}||_2 \geq w - S && && \text{(5b)}\\
& \Omega|| \sum_{t=1}^{m} \tilde{p}_{it}x_{it}||_2 \leq a_i - \sum_{t=1}^{m} p_{it}x_{it} && \forall i \in I && \text{(5c)}\\
& \sum_{i=1}^{n} x_{it} \geq 1 && \forall t \in T && \text{(5d)}\\
& 0 \leq \sum_{t=1}^{m} x_{it} \leq 1 && \forall i \in I && \text{(5e)}\\
& x_{it} \in \{0,1\} && \forall i \in I, \forall t \in T && \text{(5f)}
\end{aligned}$$

Proof. See Appendix B ■

5.1.3 Polyhedron Uncertainty Set

Since the authors continue using the same misconception about the minimum and maximum of uncertain variable ξ, we will directly provide our own reformulations for the polyhedron and box-polyhedron uncertainty sets without going into details about their reformulations.

Proposition 3 *Let the Polyhedron uncertainty set be defined as*

$$Z^{Polyhedron} = \{\xi \in U : ||\xi||_1 \leq \Gamma_i\} = \{\xi : \sum_t |\xi_{it}| \leq \Gamma_i\}$$

then, the robust reformulation of the mathematical model is

$$\begin{aligned}
\max_{x,w,\Lambda,\gamma,\lambda^+,\lambda^-,\pi^+,\pi^-} \quad & w && \text{(6a)}\\
s.t. \quad & \Gamma\gamma \leq S - w && \text{(6b)}
\end{aligned}$$

$$-\gamma_i + \lambda^+_{it} = \tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{6c}$$

$$-\gamma_i + \lambda^-_{it} = -\tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{6d}$$

$$\Gamma_i \Lambda_i \leq a_i - \sum_{t=1}^{m} p_{it} x_{it} \qquad \forall i \in I \tag{6e}$$

$$\Lambda_i - \pi^+_{it} = \tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{6f}$$

$$\Lambda_i - \pi^-_{it} = -\tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{6g}$$

$$\sum_{i=1}^{n} x_{it} \geq 1 \qquad \forall t \in T \tag{6h}$$

$$0 \leq \sum_{t=1}^{m} x_{it} \leq 1 \qquad \forall i \in I \tag{6i}$$

$$x_{it} \in \{0, 1\} \qquad \forall i \in I, \forall t \in T \tag{6j}$$

$$\lambda^+_{it}, \lambda^-_{it}, \pi^+_{it}, \pi^-_{it} \geq 0 \qquad \forall i \in I, \forall t \in T \tag{6k}$$

$$\gamma_i, \Lambda_i \geq 0 \qquad \forall i \in I \tag{6l}$$

Proof. See Appendix B ■

5.1.4 Box-Polyhedron (Budgeted) Uncertainty Set

Proposition 4 *Let the Box-Polyhedron uncertainty set be defined as*

$$Z^{Box-Polyhedron} = \{\xi \in U : ||\xi||_\infty \leq \Theta, ||\xi||_\infty \leq \Gamma\} = \{\xi : |\xi_{it}| \leq \Theta, \sum_t |\xi_{it}| \leq \Gamma_i\}$$

then, the robust reformulation of the mathematical model is

$$\max_{\substack{x,w,\gamma,\Lambda,\lambda^+,\lambda^-,\\ \mu^+,\mu^-,\pi^+,\pi^-,\eta^+,\eta^-}} \quad w \tag{7a}$$

$$s.t. \quad \sum_{i=1}^{n} \Gamma_i \gamma_i + \theta \sum_{i=1}^{n} \sum_{t=1}^{m} (\mu^+_{it} + \mu^-_{it}) \leq S - w \tag{7b}$$

$$-\gamma_i - \mu^+_{it} + \mu^-_{it} + \lambda^+_{it} = \tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{7c}$$

$$-\gamma_i + \mu^+_{it} - \mu^-_{it} + \lambda^-_{it} = -\tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{7d}$$

$$\Gamma_i \Lambda_i + \theta \sum_{t=1}^{m} (\eta^+_{it} + \eta^-_{it}) \leq a_i - \sum_{t=1}^{m} p_{it} x_{it} \qquad \forall i \in I \tag{7e}$$

$$\Lambda_i + \eta^+_{it} - \eta^-_{it} - \pi^+_{it} = \tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{7f}$$

$$\Lambda_i - \eta^+_{it} + \eta^-_{it} - \pi^-_{it} = -\tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \tag{7g}$$

$$\sum_{i=1}^{n} x_{it} \geq 1 \qquad \forall t \in T \tag{7h}$$

$$0 \leq \sum_{t=1}^{m} x_{it} \leq 1 \qquad \forall i \in I \qquad (7i)$$

$$x_{it} \in \{0, 1\} \qquad \forall i \in I, \forall t \in T \qquad (7j)$$

$$\lambda_{it}^{+}, \lambda_{it}^{-}, \mu_{it}^{+}, \mu_{it}^{-}, \pi_{it}^{+}, \pi_{it}^{-}, \eta_{it}^{+}, \eta_{it}^{-} \geq 0 \qquad \forall i \in I, \forall t \in T \qquad (7k)$$

$$\gamma_i, \Lambda_i \geq 0 \qquad \forall i \in I \qquad (7l)$$

Proof. See Appendix B ∎

6 Sequential Robust Auctions (SRA)

This section presents our extension of the problem of sequential auctions as studied by Ma et al. (2021). Our goal is to determine the optimal sequencing of cars. However, we slightly modify the setting for our application from that presented in Ma et al. (2021). The main assumptions for the definition of the Sequential Robust Auction (SRA) model we are studying in this sections are listed below.

1. The main difference between our model (SRA) and Ma et al. (2021) model (SAN) is that we consider one car per sequence. Therefore, the sequence in which the car is shown also corresponds to its ordering. Note that in the text that follows, sequence and position are used interchangeably.

2. Similar to Ma et al. (2021), we assume that there are enough buyers such that each car shown is sold.

3. We drop the holding cost term included in the SAN formulation as it is not relevant to our setting. In addition, results from Ma et al. (2021) shown that it was never optimal for the seller to hold on to a car.

4. The uncertainty in the selling price of the cars now comes from which cars are shown before.

Table 2 below presents the updated notation that is used for our SRA model.

Table 2: Key Notation for the SRA Model

Sets	
I	cars ($|I| = $ n)
T	sequence ($|T| = $ n)
Parameters	
v_i	value estimated by seller for car i
Variables	
Decision variable	
$x_{it} \in \{0, 1\}$	whether to sell the ith car in the sequence t
r_i	selling price of car i
$w_{ik} \in \{0, 1\}$	whether the kth car is sold before the ith car
$y_{ikt} \in \{0, 1\}$	whether the kth car is sold before the ith car in the tth sequence
Uncertain variable	
p_{it}	selling price of the ith car in the tth sequence
p_i^0	nominal price of the ith car
δ_{ik}	price variation of the ith car if the kth car is sold before

The Sequential Robust Auction (SRA) nominal model is presented below. Note it takes the form of a classical assignment problem.

$$\max_{x,r} \quad \sum_{i=1}^{n} (r_i - v_i) \tag{8a}$$

$$s.t. \quad r_i \leq \sum_{t=1}^{n} p_{it} x_{it} \qquad \forall i \in I \tag{8b}$$

$$\sum_{t=1}^{n} x_{it} = 1 \qquad \forall i \in I \tag{8c}$$

$$\sum_{i=1}^{n} x_{it} = 1 \qquad \forall t \in T \tag{8d}$$

$$x_{it} \in \{0, 1\} \qquad \forall i \in I, \forall t \in T \tag{8e}$$

The first constraint corresponds to the epigraph form of the selling price for car i, and the two last constraints ensure that there can be only one car assigned to each sequence. Assume now that to explicitly account the sequencing of the cars in our model, the selling price of the i^{th} car in the t^{th} sequence, p_{it}, is defined as

$$p_{it} = p_i^0 + \sum_{k=1}^{n} \delta_{ik} w_{ik} \tag{9}$$

Substituting the previous equation in 8, and introducing variable y to linearize the product of x and w, the SRA nominal model becomes

$$
\begin{aligned}
\max_{x,r,w,y} \quad & \sum_{i=1}^{n} (r_i - v_i) && && \text{(10a)}\\
s.t. \quad & r_i \leq \sum_{t=1}^{n} \left(p_i^0 x_{it} + \sum_{k=1}^{n} \delta_{ik} y_{ikt} \right) && \forall i \in I && \text{(10b)}\\
& \sum_{t=1}^{n} x_{it} = 1 && \forall i \in I && \text{(10c)}\\
& \sum_{i=1}^{n} x_{it} = 1 && \forall t \in T && \text{(10d)}\\
& \sum_{k=1}^{n} w_{ik} \leq \sum_{t=1}^{n} t x_{it} - 1 && \forall i \in I && \text{(10e)}\\
& w_{ik} + w_{ki} \leq 1 && \forall i \in I,\ \forall k \in I && \text{(10f)}\\
& w_{ik} = 0 && \forall i = k \in I && \text{(10g)}\\
& y_{ikt} \leq w_{ik} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(10h)}\\
& y_{ikt} \leq x_{it} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(10i)}\\
& y_{ikt} \geq w_{ik} + x_{it} - 1 && \forall i \in I, \forall k \in I, \forall t \in T && \text{(10j)}\\
& x_{it} \in \{0, 1\} && \forall i \in I, \forall t \in T && \text{(10k)}\\
& w_{ik} \in \{0, 1\} && \forall i \in I, \forall k \in I && \text{(10l)}\\
& y_{ikt} \in \{0, 1\} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(10m)}
\end{aligned}
$$

The first three constraints are unchanged from 8. The fourth constraint ensures there cannot be more cars shown before than the position the car is in, the fifth constraint is used to ensure that only one of the variables can take the value 1 since both cars cannot be shown before each other at the same time, and the sixth constraint ensures the diagonal of w is always zero. Note that it is also assumed the diagonal of δ is always 0. Finally, the last three constraints are introduced to linearize the product of two binary variables, namely, x and w.

6.1 The Data

We were unable to obtain a real-life data set of used car sequential auctions. All of the data sets found online on auctions were data sets for single-item auctions, such as sales on the online auction platform eBay, and thus unusable in the context of our study.

Consequently, we generate synthetic data for our numerical study. We start from the data used in Ma et al. (2021) for the larger data set (see Table 6). The synthetic data is generated as follows: for each car, for each sequence, we draw a sample of 25 observations from a normal distribution centered at the value given in Table 6, and of scale parameter varying between 1000 and 10,000 (varies for each observation). For each observation, we also randomly generate an indicator vector to specify which cars were shown before. Note that the choice of 25 observations is arbitrary, but was chosen to be rather small to induce small sample bias in the draws to be used in the study. The full generation procedure can be viewed from the Google Colab file used for the study.

Using the data generated, for each car, we regress the vectors of indicator values on the prices to obtain estimates of p_i^0 and δ_{ik}. Such a linear regression can be written as

$$p_t = \beta_0 + \sum_{k=1}^{n-1} \beta_k \mathbb{1}_{\{\textbf{car } k \textbf{ shown before car } i\}} \tag{11}$$

In the regression above, the estimates for β_0 correspond to the estimate for p_i^0, while the set of estimates for β_k correspond to the estimates for the set of δ_{ik}. Furthermore, from the regression, we get the standard error for each coefficient estimate. Recall that one of the underlying assumptions of linear regression is that the coefficients of the regression are normally distributed about their true mean (β_k) with true variance (σ_k).

6.2 Robust Reformulation

Let us now consider the model that accounts for uncertainty in the selling prices of the cars in each sequences, given which cars have been shown previously. In particular, we are interested in re-expressing the first constraint in (10) by the chance constraint

$$\mathbb{P}((\delta^i)^T \mathbf{w}^i \geq r_i - p_i^0) \geq 1 - \varepsilon \tag{12}$$

where δ^i is the vector of price variations for car i. As presented in the previous section, δ^i follows a multivariate normal distribution centered around mean $\bar{\delta}^i$ and scale parameter $\tilde{\delta}^i$. In the equation above, the parameter ε characterizes the level of risk we are willing to include in the model, and corresponds to the maximum probability of the constraint being not violated. Furthermore, in our formulation we assume the nominal price of car i (p_i^0) to be deterministic, although it is an estimate from the linear regression. The price of car i in sequence t can therefore be expressed as

$$p_{it} = p_i^0 + \sum_{k=1}^{n} \left(\bar{\delta}_{ik} w_{ik} + \tilde{\delta}_{ik} z w_{ik} \right) \tag{13}$$

where $z \sim N(0, 1)$.

The uncertain model can then be written as

$$
\begin{aligned}
\max_{x,r,w,y} \quad & \sum_{i=1}^{n} (r_i - v_i) && && \text{(14a)}\\
s.t. \quad & r_i \le \sum_{t=1}^{n} \left(p_i^0 x_{it} + \sum_{k=1}^{n} \left(\bar{\delta}_{ik} y_{ikt} + \tilde{\delta}_{ik} z y_{ikt} \right) \right) && \forall i \in I, \forall z \in \mathcal{U} && \text{(14b)}\\
& \sum_{t=1}^{n} x_{it} = 1 && \forall i \in I && \text{(14c)}\\
& \sum_{i=1}^{n} x_{it} = 1 && \forall t \in T && \text{(14d)}\\
& \sum_{k=1}^{n} w_{ik} \le \sum_{t=1}^{n} t x_{it} - 1 && \forall i \in I && \text{(14e)}\\
& w_{ik} + w_{ki} \le 1 && \forall i \in I, \ \forall k \in I && \text{(14f)}\\
& w_{ik} = 0 && \forall i = k \in I && \text{(14g)}\\
& y_{ikt} \le w_{ik} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(14h)}\\
& y_{ikt} \le x_{it} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(14i)}\\
& y_{ikt} \ge w_{ik} + x_{it} - 1 && \forall i \in I, \forall k \in I, \forall t \in T && \text{(14j)}\\
& x_{it} \in \{0, 1\} && \forall i \in I, \forall t \in T && \text{(14k)}\\
& w_{ik} \in \{0, 1\} && \forall i \in I, \forall k \in I && \text{(14l)}\\
& y_{ikt} \in \{0, 1\} && \forall i \in I, \forall k \in I, \forall t \in T && \text{(14m)}
\end{aligned}
$$

where we define the uncertainty set $\mathcal{U}$ as

$$
\mathcal{U} := \{ z \in \mathbb{R}^n \mid ||z||_2 \le \gamma \}
$$

We now present the robust reformulation to our nominal problem.

Proposition 5 *Given the uncertainty set $\mathcal{U}$ as defined above, the SRA model is*

$$
\begin{aligned}
\max_{x,r,w,y} \quad & \sum_{i=1}^{n} (r_i - v_i) && && \text{(15a)}\\
s.t. \quad & r_i \le \sum_{t=1}^{n} \left(p_i^0 x_{it} + \sum_{k=1}^{n} \bar{\delta}_{ik} y_{ikt} - \Phi^{-1}(1-\varepsilon) ||\mathbf{y}^i \tilde{\delta}^i||_2 \right) && \forall i \in I && \text{(15b)}\\
& \sum_{t=1}^{n} x_{it} = 1 && \forall i \in I && \text{(15c)}\\
& \sum_{i=1}^{n} x_{it} = 1 && \forall t \in T && \text{(15d)}
\end{aligned}
$$

$$\sum_{k=1}^{n} w_{ik} \leq \sum_{t=1}^{n} t x_{it} - 1 \qquad \forall i \in I \quad (15e)$$

$$w_{ik} + w_{ki} \leq 1 \qquad \forall i, k \in I \quad (15f)$$

$$w_{ik} = 0 \qquad \forall i = k \in I \quad (15g)$$

$$y_{ikt} \leq w_{ik} \qquad \forall i, k \in I, \forall t \in T \quad (15h)$$

$$y_{ikt} \leq x_{it} \qquad \forall i, k \in I, \forall t \in T \quad (15i)$$

$$y_{ikt} \geq w_{ik} + x_{it} - 1 \qquad \forall i, k \in I, \forall t \in T \quad (15j)$$

$$x_{it} \in \{0, 1\} \qquad \forall i \in I, \forall t \in T \quad (15k)$$

$$w_{ik} \in \{0, 1\} \qquad \forall i, k \in I \quad (15l)$$

$$y_{ikt} \in \{0, 1\} \qquad \forall i, k \in I, \forall t \in T \quad (15m)$$

Proof. We only need to robustify constraint (14b). First rewrite the constraint as

$$-\sum_{t=1}^{n} \sum_{k=1}^{n} \tilde{\delta}_{ik} z y_{ikt} \leq \sum_{t=1}^{n} p_i^0 x_{it} + \sum_{t=1}^{n} \sum_{k=1}^{n} \bar{\delta}_{ik} y_{ikt} - r_i$$

We are therefore interested in the following robustness linear program on uncertain variable z

$$\Psi := \max_{z \in \mathcal{U}} -\sum_{t=1}^{n} \sum_{k=1}^{n} \tilde{\delta}_{ik} z y_{ikt}$$

Using Cauchy-Schwarz inequality, we have that this is equivalent to

$$\Psi := \gamma ||\mathbf{y}^i \tilde{\delta}^i||_2$$

where $\mathbf{y}^i \in \mathbb{R}^{nxn}$ and $\tilde{\delta}^i \in \mathbb{R}^n$ represent the matrix of variable y and vector of variable $\tilde{\delta}$ associated to car i.

Because the uncertain variable z follows a standard normal distribution, $\gamma = \Phi^{-1}(1-\varepsilon)$, where Φ^{-1} corresponds to the inverse cumulative distribution function of the standard normal distribution. ■

Finally, we note that because decision variable x always corresponds to a vector for which all elements are 0 except for a 1 in the tth position, we know that the product of x and w will always return the null vector, except for sequence t. Furthermore, because decision variable y is a linear variable that represents the product of x and w, we know it is either equal to 0, or w in the tth sequence. We take full advantage of this observation to simplify our model.

7 Results

In order to answer our research questions, we use the RSOME package. Subsection 7.1 will show the comparison of the proposed robust models with our reformulations, and the numerical study for the extension proposed by us will be given in Subsection 7.2.

7.1 Comparative study of reformulations

To compare our reformulations with the reformulations proposed by the authors, we will use this Google Colab file. We solved the problem for 11 different parameter values. Change in the optimal objective function value depending on the parameter can be seen in below Figure 1. In this figure, dashed lines show the profit obtained from the reformulations in the paper, whereas full lines represent the optimal objective function value of our reformulations. As it can bee seen from Figure 1, we obtained different profit from those reported in the paper. After analyzing the results, it is obvious that the authors ignore the properties of their parameters while reformulating the robust mathematical model.

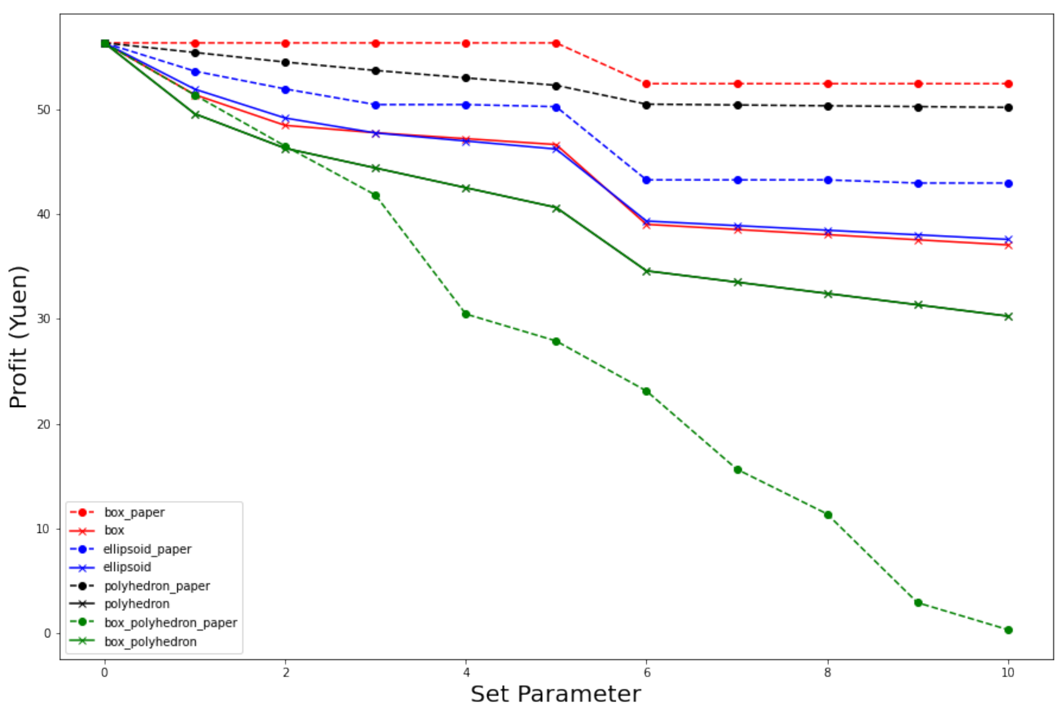

Figure 1: Profit as a function of uncertainty set parameters

7.2 Sequential Robust Auctions

We use the RSOME package in Google Colab to solve the SRA Model. Because of the nonlinearity of decision variable y in the euclidean norm of (15), we instead solve for the model presented in (14) directly, defining the uncertainty set, in RSOME. Details of the model implementation can be found in the Google Colab file used for the study.

We solve the SRA model for 11 different values of risk parameter ε. Results of the optimal profit achieved as a function of the level of risk are presented in Figure 2.

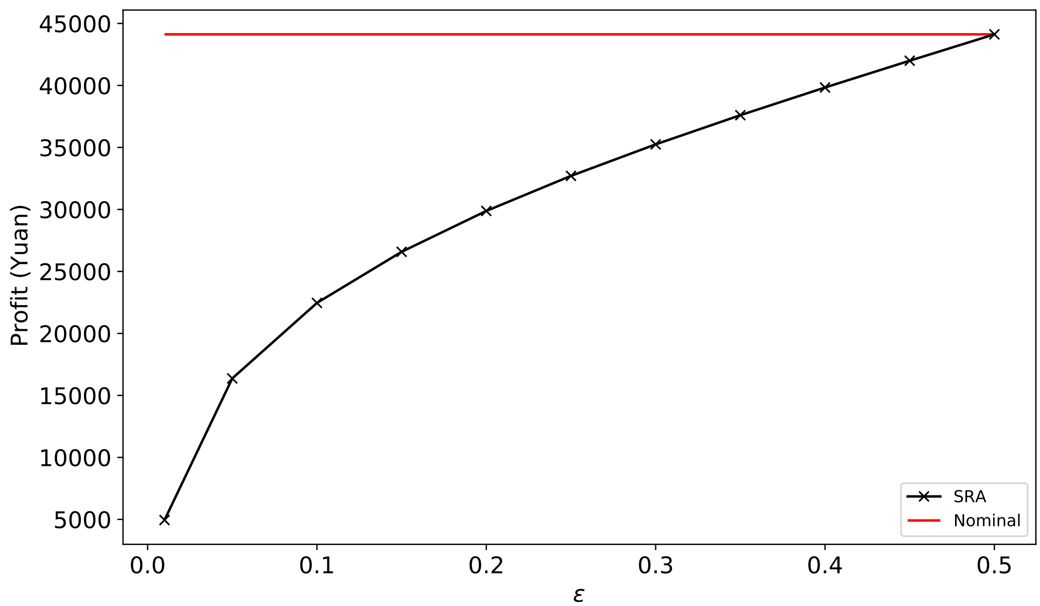

Figure 2: Profit as a function of ε (SRA Model)

We see from Figure 2 that profit rapidly increases as ε varies from 0.01 to around 0.2, and then varies almost linearly to $\varepsilon = 0.5$. $\varepsilon = 0.5$ corresponds to the nominal model, and we see in fact that the profit obtained from the SRA model for $\varepsilon = 0.5$ corresponds to the profit obtained from the nominal model.

Figure 2 provides interesting insights in the relationship between risk and returns. As ε tends to 0, the SRA will return more conservative results, as it considers an increasing range of possible scenarios, of which some can be potentially very pessimistic. Conversely, as ε tends to 0.5, the SRA model will tend to the nominal (i.e. expected values) model.

Table 3 presents the optimal sequencing obtained for each level of ε. We see a change in the optimal sequencing between values of ε of 0.1 and 0.15, but remains otherwise unchanged before and after. From 0.1 to 0.15, we see all cars move down one place in the sequencing, and car 8 moves from first to last position. This obesrvation can be explained by the fact that as ε increases, a more restricted set of scenarios is considered in the model. In our case, this change between values of ε of 0.1 and 0.15 was significant enough to change the prices of each car, and thus the ordering.

Table 3: Optimal Sequencing of Cars (SRA Model)

ε	Sequencing	Profit (Yuan)
0.01	6, 4, 3, 7, 10, 5, 9, 1, 2, 8	4,945.94
0.05	6, 4, 3, 7, 10, 5, 9, 1, 2, 8	16,368.94
0.1	6, 4, 3, 7, 10, 5, 9, 1, 2, 8	22,458.50
0.15	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	26,581.67
0.2	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	29,878.59
0.25	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	32,707.06
0.3	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	35,247.12
0.35	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	37,600.85
0.4	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	39,834.32
0.45	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	41,995.23
0.5	5, 3, 2, 6, 9, 4, 8, 10, 1, 7	44,121.87

It is also interesting to look at the resulting sequencing of prices. Recall the dichotomy between theory and practice in multi-item sequential auctions (high-valued cars first vs. high-valued cars last) reported by Elmaghraby (2007). Table 4 presents the two optimal sequencing obtained from the SRA model with the corresponding nominal price (p_i^0) of each car.

Table 4: Optimal Car Sequencing with Nominal Price (SRA Model)

Sequencing 1 ($\varepsilon = 0.01$ to 0.1)	6	4	3	7	10	5	9	1	2	8
Price (Yuen)	28,548.53	24,011.44	28,169.39	21,834.79	25,828.40	21,901.05	23,965.17	19,249.80	26,998.82	28,408.60
Sequencing 2 ($\varepsilon = 0.15$ to 0.5)	5	3	2	6	9	4	8	10	1	7
Price (Yuen)	24,011.44	28,169.39	21,834.79	25,828.40	21,901.05	23,965.17	19,249.80	26,998.82	28,408.60	28,548.53

Results do not seem to indicate any of the strategies (high-low or low-high) to be optimal in our case. This observation provides some evidence that there might be more at play than simply ordering the cars by value in the optimal sequencing of cars for multi-item auctions. Indeed, the SRA corresponds to a data-driven robust approach that includes all underlying dynamics (buyer behaviour, cross-product effects, etc.) that would have affected the auctioning and prices of cars in the past (given our data was not synthetic), thus providing evidence towards disproving the simplistic high-low or low-high policies.

8 Conclusion & Future Work

This study was concerned with two main research questions. First, after noticing shortcomings in the reformulations presented by Ma et al. (2021), we reformulated the problems and compared the results with those presented by the authors. We found differences in all reformulations. This leads us to believe Ma et al. (2021) developed erroneous reformulations. Second, we were interested in extending the study by Ma et al. (2021), by accounting for uncertainty in the selling prices of the cars in each sequences, given which cars had been shown previously. We presented how, given some data about selling prices and which cars have been shown previously, we can obtain a nominal price and estimates of prices variations for each cars, and how it can be used to define a robust optimization linear program. Additionally, we presented a tractable finite dimensional reformulation of the problem. By using a chance constraint approach, we were able to present insights into the risk and return trade-off performance of our model. Finally, we also provided evidence that the simple high-low or low-high policies, presented in the literature and used in practice respectively, are not optimal in our case, and that the SRA model developed in this study may be more suitable to derive optimal policies in sequential multi-item auctions.

Using synthetic data allowed us to prove our approach works, and can provide an optimal sequencing while explicitly accounting for the influence previous cars on price variation. However, one limitation of this study is the lack of a real data set to derive managerial insights from the model.

One possible extension of this work would be the use of a more sophisticated estimation method (e.g. Deep Learning) to extract value from a data set, to be used to define the uncertainty set. Indeed, a method that could allow for the inclusion of car features could prove powerful.

References

Bandi, C. and Bertsimas, D. (2014). Optimal design for multi-item auctions: A robust optimization approach. *Mathematics of Operations Research*, 39(4):1012–1038.

Ben-Tal, A. and Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical programming*, 88(3):411–424.

Elmaghraby, W. (2007). Auctions within e-sourcing events. *Production and Operations Management*, 16(4):409–422.

Ma, G., Zheng, J., Wei, J., Wang, S., and Han, Y. (2021). Robust optimization strategies for seller based on uncertainty sets in context of sequential auction. *Applied Mathematics and Computation*, 390:125650.

Ozturk, O. C. and Karabatı, S. (2017). A decision support framework for evaluating revenue performance in sequential purchase contexts. *European Journal of Operational Research*, 263(3):922–934.

Rooderkerk, R. P. and Van Heerde, H. J. (2016). Robust optimization of the 0-1 knapsack problem: Balancing risk and return in assortment optimization. *European Journal of Operational Research*, 250(3):842–854.

Appendix A. Data

Table 5: $p_{it}/\tilde{p}_{it}$ for the first 10 cars for five sequences

$p_{it}/\tilde{p}_{it}$ Unit: ten thousand yuan					
car	sequence 1	sequence 2	sequence 3	sequence 4	sequence 5
1	20.5/0.0571	23.1/−0.6687	23.6/0.204	22.8/−0.4741	21.5/0.3082
2	20.4/0.3784	22.5/0.4963	25.5/−0.0989	22.6/−0.8324	24.4/−0.5420
3	27.6/0.8267	26.1/−0.6952	26.9/0.6516	33.7/0.0767	30.8/0.9923
4	22.3/−0.8436	24.5/−0.1146	24.7/−0.7867	23.5/0.9238	22.5/−0.9907
5	28.5/0.5498	33.4/0.6346	29.5/0.7374	27.1/−0.8311	28.5/−0.2004
6	26.9−0.4803	25.8/0.6001	26.3/−0.1372	23.6/0.8213	27.3/−0.6363
7	29.9/−0.4724	27.5/−0.7089	19.2/−0.7279	28.7/0.7386	27.9/0.1594
8	29.2/0.0997	30.9/−0.7101	27.3/0.7061	28.2/0.2441	29.5/−0.2981
9	20.8/0.0265	24.0/−0.1964	23.7/−0.8481	24.6/−0.5202	22.1/−0.7534
10	19.2/−0.6322	18.7/−0.5201	19.3/0.1655	20.1/−0.9007	19.2/0.8054
Note: $\tilde{p}_{it}$ is a random number in (-1,1)					

Table 6: p_{it} for the first 10 cars for 10 sequences

p_{it} Unit: ten thousand yuan										
car	seq1	seq2	seq3	seq4	seq5	seq6	seq7	seq8	seq9	seq10
1	20.5	23.1	23.6	22.8	21.5	20.8	23.7	20.2	21.8	23.3
2	20.4	22.5	25.5	22.6	24.4	21.2	21.4	23.6	23.1	23.0
3	27.6	26.1	26.9	33.7	30.8	24.2	25.3	24.7	26.3	24.2
4	22.3	24.5	24.7	23.5	22.5	24.3	23.6	23.4	23.7	24.7
5	28.5	33.4	29.5	27.1	28.5	28.7	30.2	32.9	31.8	34.1
6	26.9	25.8	26.3	23.6	27.3	22.4	22.5	24.4	26.8	22.2
7	29.9	27.5	19.2	28.7	27.9	24.2	24.8	24.9	25.9	23.7
8	29.2	30.9	27.3	28.2	29.5	26.4	26.9	27.8	29.1	26.1
9	20.8	24.0	23.7	24.6	22.1	20.4	20.1	19.9	23.3	20.2
10	19.2	18.7	19.3	20.1	19.2	15.6	20.4	17.3	21.3	22.7

Appendix B. Proofs of Robust Reformulations - SAN Model

Proof for Box uncertainty set.

We only have to robustify two constraints, namely constraints (3b) and (3c). We start with (3b). Throughout this proof, let

$$S = \sum_{i=1}^{n}\sum_{t=1}^{m}(p_{it} - v_i)x_{it} - \sum_{i=1}^{n}(c_i(1 - \sum_{t=1}^{m} x_{it}))$$

Constraint (3b). We rewrite the constraint as

$$\sum_{i=1}^{n}\sum_{t=1}^{m}\tilde{p}_{it}x_{it}\xi_{it} \geq w - S, \quad \forall \xi \in Z^{Box}$$

The robustness linear problem on the uncertain variable ξ for constraint (3b) can be written as given in below on the left with its dual provided on the right

$$\begin{aligned}
\min_{\xi} \quad & \sum_{i=1}^{n}\sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \\
s.t. \quad & \xi \leq \Theta 1 \\
& \xi \geq -\Theta 1 \\
& \xi \in \mathbb{R}^m
\end{aligned}
\qquad\qquad
\begin{aligned}
\max_{\lambda^+,\lambda^-} \quad & \Theta \sum_{i=1}^{n}\sum_{t=1}^{m} (\lambda_{it}^+ + \lambda_{it}^-) \\
s.t. \quad & \lambda_{it}^+ - \lambda_{it}^- = \tilde{p}_{it} x_{it} \qquad \forall i \in I, \forall t \in T \\
& \lambda^+, \lambda^- \leq 0
\end{aligned}$$

Thus, constraint (3b) is replaced by the set of robust constraints

$$\begin{aligned}
& \Theta \sum_{i=1}^{n}\sum_{t=1}^{m} (\lambda_{it}^+ + \lambda_{it}^-) \geq w - S \\
& \lambda_{it}^+ - \lambda_{it}^- = \tilde{p}_{it} x_{it} && \forall i \in I, \forall t \in T \\
& \lambda^+, \lambda^- \leq 0
\end{aligned}$$

Constraint (3c). We rewrite the constraint as

$$\sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \leq a_i - \sum_{t=1}^{m} p_{it} x_{it}, \quad \forall i \in I, \forall \xi \in Z^{box}$$

The robustness linear problem on the uncertain variable ξ for constraint (3c) can be written as given in below on the left with its dual provided on the right

$$\begin{aligned}
\max_{\xi} \quad & \sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \\
s.t. \quad & \xi \leq \Theta 1 \\
& \xi \geq -\Theta 1 \\
& \xi \in \mathbb{R}^m
\end{aligned}
\qquad\qquad
\begin{aligned}
\min_{\gamma^+,\gamma^-} \quad & \Theta \sum_{t=1}^{m} (\gamma_{it}^+ + \gamma_{it}^-) \\
s.t. \quad & \gamma_{it}^+ - \gamma_{it}^- = \tilde{p}_{it} x_{it} \quad \forall i \in I, \forall t \in T \\
& \gamma^+, \gamma^- \geq 0
\end{aligned}$$

Thus, constraint (3c) is replaced by the set of robust constraints

$$\begin{aligned}
& \Theta \sum_{t=1}^{m} (\gamma_{it}^+ + \gamma_{it}^-) \leq a_i - \sum_{t=1}^{m} p_{it} x_{it} && \forall i \in I \\
& \gamma_{it}^+ - \gamma_{it}^- = \tilde{p}_{it} x_{it} && \forall i \in I, \forall t \in T \\
& \gamma^+, \gamma^- \geq 0
\end{aligned}$$

Proof for Ellipsoidal set.

As for the proof of Proposition 1, let

$$S = \sum_i \sum_t (p_{it} - v_i) x_{it} - \sum_i (c_i(1 - \sum_t x_{it}))$$

Let

$$\hat{\xi} := \sum_t \xi_{it} \in \mathbb{R}^n$$

and rewrite the uncertainty set as

$$Z^{Ellipsoidal} := \{\xi \in U | \hat{\xi}^T \Sigma^{-1} \hat{\xi} \leq \Omega^2\} \tag{16}$$

where Σ is assumed positive semi-definite. In our case, $\Sigma = I$, the identity matrix, such that this condition is met.

Note that, we agree with the reformulation of the constraint (3c) given in the paper. Therefore, in this part, we will only reformulate the constraint (3b).

Constraint (3b). The robustness linear problem on the uncertainty variable ξ for constraint (3b) can be written as

$$\Psi := \min_{\xi : \hat{\xi}^T \Sigma^{-1} \hat{\xi} \leq \Omega^2} \sum_i \sum_t \tilde{p}_{it} x_{it} \xi_{it} \iff \Psi := - \max_{\xi : \xi^T \Sigma^{-1} \xi \leq \Omega^2} - \sum_i \sum_t \tilde{p}_{it} x_{it} \xi_{it}$$

Using Cauchy-Schwarz inequality, $a^T b \leq ||a||_2 ||b||_2$, we can say that the optimal solution to the robustness linear problem is thus

$$\Psi = \Omega || \sum_i \sum_t \tilde{p}_{it} x_{it} ||_2$$

Therefore, the constraint (3b) becomes

$$\Omega || \sum_i \sum_t \tilde{p}_{it} x_{it} ||_2 \geq w - S$$

Proof for Polyhedron set.

We will start with rewriting the uncertainty set;

$$Z^{Polyhedron} = \{\xi | \exists \Delta_{it}^{+} \geq 0, \exists \Delta_{it}^{-} \geq 0, \xi_{it} = \Delta_{it}^{+} - \Delta_{it}^{-}, \sum_{t=1}^{m} (\Delta_{it}^{+} + \Delta_{it}^{-}) \leq \Gamma_i\}$$

Then, we will robustify the constraints (3b) and (3c) in order.

Constraint (3b). We rewrite the constraint as

$$\sum_{i=1}^{n} \sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \geq w - S, \quad \forall \xi \in Z^{Polyhedron}$$

The robustness linear problem on the uncertain variable ξ for constraint (3b) can be written as given in below on the left with its dual provided on the right

$$\begin{aligned}
\min_{\Delta_{it}^{+}, \Delta_{it}^{-}} \quad & \sum_{i=1}^{n} \sum_{t=1}^{m} \tilde{p}_{it} x_{it} (\Delta_{it}^{+} - \Delta_{it}^{-}) & \\
s.t. \quad & -\sum_{t=1}^{m} (\Delta_{it}^{+} + \Delta_{it}^{-}) \geq -\Gamma_i & \forall i \in I \\
& \Delta_{it}^{+} \geq 0 & \forall i \in I, \forall t \in T \\
& \Delta_{it}^{-} \geq 0 & \forall i \in I, \forall t \in T
\end{aligned}$$

$$\begin{aligned}
\max_{\gamma_i, \lambda_{it}^{+}, \lambda_{it}^{-}} \quad & \sum_{i=1}^{n} -\Gamma_i \gamma_i & \\
s.t. \quad & -\gamma_i + \lambda_{it}^{+} = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& -\gamma_i + \lambda_{it}^{-} = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \lambda_{it}^{+}, \lambda_{it}^{-} \geq 0 & \forall i \in I, \forall t \in T \\
& \gamma_i \geq 0 & \forall i \in I
\end{aligned}$$

Therefore, constraint (3b) is replaced by the set of robust constraints

$$\begin{aligned}
& \Gamma \gamma \leq S - w & \\
& -\gamma_i + \lambda_{it}^{+} = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& -\gamma_i + \lambda_{it}^{-} = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \lambda_{it}^{+}, \lambda_{it}^{-} \geq 0 & \forall i \in I, \forall t \in T \\
& \gamma_i \geq 0 & \forall i \in I
\end{aligned}$$

Constraint (3c). We rewrite the constraint as

$$\sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \leq a_i - \sum_{t=1}^{m} p_{it} x_{it}, \quad \forall i \in I, \forall \xi \in Z^{Polyhedron}$$

The robustness linear problem on the uncertain variable ξ for constraint (3c) can be written as given in below on the left with its dual provided on the right

$$
\begin{aligned}
\max_{\Delta^+_{it},\Delta^-_{it}} \quad & \sum_{t=1}^{m} \tilde{p}_{it} x_{it} (\Delta^+_{it} - \Delta^-_{it}) \\
s.t. \quad & \sum_{t=1}^{m} (\Delta^+_{it} + \Delta^-_{it}) \le \Gamma_i & \forall i \in I \\
& -\Delta^+_{it} \le 0 & \forall i \in I, \forall t \in T \\
& -\Delta^-_{it} \le 0 & \forall i \in I, \forall t \in T
\end{aligned}
\qquad
\begin{aligned}
\min_{\Lambda_i,\pi^+_{it},\pi^-_{it}} \quad & \Gamma_i \Lambda_i \\
s.t. \quad & \Lambda_i - \pi^+_{it} = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \Lambda_i - \pi^-_{it} = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \pi^+_{it}, \pi^-_{it} \ge 0 & \forall i \in I, \forall t \in T \\
& \Lambda_i \ge 0 & \forall i \in I
\end{aligned}
$$

Therefore, constraint (3c) is replaced by the set of robust constraints

$$
\begin{aligned}
& \Gamma_i \Lambda_i \le a_i - \sum_{t=1}^{m} p_{it} x_{it} & \forall i \in I \\
& \Lambda_i - \pi^+_{it} = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \Lambda_i - \pi^-_{it} = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \pi^+_{it}, \pi^-_{it} \ge 0 & \forall i \in I, \forall t \in T \\
& \Lambda_i \ge 0 & \forall i \in I
\end{aligned}
$$

Proof for Box-Polyhedron set.

Constraint (3b). We rewrite the constraint as

$$
\sum_{i=1}^{n} \sum_{t=1}^{m} \tilde{p}_{it} x_{it} \xi_{it} \ge w - S, \qquad \forall \xi \in Z^{Box-Polyhedron}
$$

The robustness linear problem on the uncertain variable ξ for constraint (3b) can be written as given in below

$$
\begin{aligned}
\min_{\Delta^+_{it},\Delta^-_{it}} \quad & \sum_{i=1}^{n} \sum_{t=1}^{m} \tilde{p}_{it} x_{it} (\Delta^+_{it} - \Delta^-_{it}) \\
s.t. \quad & -\sum_{t=1}^{m} (\Delta^+_{it} + \Delta^-_{it}) \ge -\Gamma_i & \forall i \in I \\
& -\Delta^+_{it} + \Delta^-_{it} \ge -\Theta & \forall i \in I, \forall t \in T \\
& \Delta^+_{it} - \Delta^-_{it} \ge -\Theta & \forall i \in I, \forall t \in T \\
& \Delta^+_{it} \ge 0 & \forall i \in I, \forall t \in T \\
& \Delta^-_{it} \ge 0 & \forall i \in I, \forall t \in T
\end{aligned}
$$

The dual of the above problem will be

$$
\begin{aligned}
\max_{\gamma_i,\lambda_{it}^+,\lambda_{it}^-,\mu_{it}^+,\mu_{it}^-} \quad & \sum_{i=1}^{n} -\Gamma_i\gamma_i - \theta\sum_{i=1}^{n}\sum_{t=1}^{m}(\mu_{it}^+ + \mu_{it}^-) & \\
s.t. \quad & -\gamma_i - \mu_{it}^+ + \mu_{it}^- + \lambda_{it}^+ = \tilde{p}_{it}x_{it} & \forall i \in I, \forall t \in T \\
& -\gamma_i + \mu_{it}^+ - \mu_{it}^- + \lambda_{it}^- = -\tilde{p}_{it}x_{it} & \forall i \in I, \forall t \in T \\
& \lambda_{it}^+, \lambda_{it}^-, \mu_{it}^+, \mu_{it}^- \geq 0 & \forall i \in I, \forall t \in T \\
& \gamma_i \geq 0 & \forall i \in I
\end{aligned}
$$

Therefore, constraint (3b) is replaced by the set of robust constraints

$$
\begin{aligned}
& \sum_{i=1}^{n}\Gamma_i\gamma_i + \theta\sum_{i=1}^{n}\sum_{t=1}^{m}(\mu_{it}^+ + \mu_{it}^-) \leq S - w & \\
& -\gamma_i - \mu_{it}^+ + \mu_{it}^- + \lambda_{it}^+ = \tilde{p}_{it}x_{it} & \forall i \in I, \forall t \in T \\
& -\gamma_i + \mu_{it}^+ - \mu_{it}^- + \lambda_{it}^- = -\tilde{p}_{it}x_{it} & \forall i \in I, \forall t \in T \\
& \lambda_{it}^+, \lambda_{it}^-, \mu_{it}^+, \mu_{it}^- \geq 0 & \forall i \in I, \forall t \in T \\
& \gamma_i \geq 0 & \forall i \in I
\end{aligned}
$$

Constraint (3c). We rewrite the constraint as

$$
\sum_{t=1}^{m}\tilde{p}_{it}x_{it}\xi_{it} \leq a_i - \sum_{t=1}^{m}p_{it}x_{it}, \quad \forall i \in I, \forall \xi \in Z^{Box-Polyhedron}
$$

The robustness linear problem on the uncertain variable ξ for constraint (3c) can be written as given in below

$$
\begin{aligned}
\max_{\Delta_{it}^+,\Delta_{it}^-} \quad & \sum_{t=1}^{m}\tilde{p}_{it}x_{it}(\Delta_{it}^+ - \Delta_{it}^-) & \\
s.t. \quad & \sum_{t=1}^{m}(\Delta_{it}^+ + \Delta_{it}^-) \leq \Gamma_i & \forall i \in I \\
& \Delta_{it}^+ - \Delta_{it}^- \leq \Theta & \forall i \in I, \forall t \in T \\
& -\Delta_{it}^+ + \Delta_{it}^- \leq \Theta & \forall i \in I, \forall t \in T \\
& -\Delta_{it}^+ \leq 0 & \forall i \in I, \forall t \in T \\
& -\Delta_{it}^- \leq 0 & \forall i \in I, \forall t \in T
\end{aligned}
$$

The dual of the corresponding problem will be

$$
\begin{aligned}
\min_{\Lambda_i, \pi_{it}^+, \pi_{it}^-, \eta_{it}^+, \eta_{it}^-} \quad & \sum_{i=1}^{n} \Gamma_i \Lambda_i + \theta \sum_{i=1}^{n} \sum_{t=1}^{m} (\eta_{it}^+ + \eta_{it}^-) & \\
s.t. \quad & \Lambda_i + \eta_{it}^+ - \eta_{it}^- - \pi_{it}^+ = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \Lambda_i - \eta_{it}^+ + \eta_{it}^- - \pi_{it}^- = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \pi_{it}^+, \pi_{it}^-, \eta_{it}^+, \eta_{it}^- \geq 0 & \forall i \in I, \forall t \in T \\
& \Lambda_i \geq 0 & \forall i \in I
\end{aligned}
$$

Thus, constraint (3c) is replaced by the set of robust constraints

$$
\begin{aligned}
& \Gamma_i \Lambda_i + \theta \sum_{t=1}^{m} (\eta_{it}^+ + \eta_{it}^-) \leq a_i - \sum_{t=1}^{m} p_{it} x_{it} & \forall i \in I \\
& \Lambda_i + \eta_{it}^+ - \eta_{it}^- - \pi_{it}^+ = \tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \Lambda_i - \eta_{it}^+ + \eta_{it}^- - \pi_{it}^- = -\tilde{p}_{it} x_{it} & \forall i \in I, \forall t \in T \\
& \pi_{it}^+, \pi_{it}^-, \eta_{it}^+, \eta_{it}^- \geq 0 & \forall i \in I, \forall t \in T \\
& \Lambda_i \geq 0 & \forall i \in I
\end{aligned}
$$