Quatitative Risk Management Using Robust Optimization
MATH80624A
Assignment 1

Jean-Sébastien Matte   Sena Onen Oz

Due date: February 12, 2021

Exercises Chapter 2

Our solution file for exercises from Chapter 2 can be accessed by clicking here.

2.1) Convex Hull Set

Raw Formulation

We are interested in the convex hull uncertainty set

$$\begin{aligned} \max_x \quad & c^T x && (1)\\ \text{subject to} \quad & (a+z)^T x \le b\,,\ \forall z \in \mathcal{Z} && (2)\\ & 0 \le x \le 1\ , && (3) \end{aligned}$$

where

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^n \,\middle|\, \exists \theta \in \mathbb{R}^K,\ z = \sum_{i=1}^{K} \theta_i \bar{z}_i,\ \theta \ge 0,\ \sum_{i=1}^{K} \theta_i = 1 \right\}$$

After solving the problem, our result is

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0200s
The objective of Raw Robust Counterpart is 3.2742
```

Reduced Formulation

We reformulate the above problem into a reduced tractable linear reformulation as follows:

First rewrite the uncertainty set as

$$\mathcal{Z}_1 := \left\{ z = \sum_{i=1}^{K} \theta_i \bar{z}_i \in \mathbb{R}^n \,\middle|\, \theta \ge 0, \sum_{i=1}^{K} \theta_i = 1 \right\}$$

which we rewrite in the familiar form of $Wz \leq v$ giving

$$\mathcal{Z}_1 := \left\{ \sum_{i=1}^{K} \theta_i \bar{z}_i \in \mathbb{R}^n \middle| \begin{bmatrix} -I \in \mathbb{R}^{KxK} \\ 1^T \\ -1^T \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

where 0 is a K-vector of zeros, and 1^T is an K-vector of ones. Note that we replaced the equality constraint

$$\sum_{i=1}^{K} \theta_i = 1$$

by

$$\sum_{i=1}^{K} \theta_i \leq 1, \sum_{i=1}^{K} \theta_i \geq 1 \tag{4}$$

We can then rewrite the first constraint of the main program such that

$$z^T x \leq b - a^T x \Leftrightarrow \left(\sum_{i=1}^{K} \theta_i \bar{z}_i \right)^T x \leq b - a^T x$$

Our robustness linear problem on the uncertain variable z can be written as

$$\max_{\theta} \quad \left(\sum_{i=1}^{K} \theta_i \bar{z}_i \right)^T x \tag{5}$$

$$\text{s. t.} \quad \begin{bmatrix} -I \\ 1^T \\ -1^T \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{6}$$

with dual problem

$$\min_{\lambda} \quad \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} \tag{7}$$

$$\text{s. t.} \quad -\lambda_i + \lambda_{K+1} - \lambda_{K+2} = \bar{z}_i^T x, \forall i = 1, ..., K \tag{8}$$

$$\lambda \geq 0 \tag{9}$$

where $\lambda \in \mathbb{R}^{K+2}$

Thus, our reduced reformulation can be written as

$$\max_{x,\lambda} \quad c^T x \tag{10}$$

$$\text{s. t.} \quad \lambda_{K+1} - \lambda_{K+2} \leq b - a^T x \tag{11}$$

$$-\lambda_i + \lambda_{K+1} - \lambda_{K+2} = \bar{z}_i^T x, \forall i = 1, ..., K \tag{12}$$

$$\lambda \geq 0 \tag{13}$$

$$0 \leq x \leq 1 \tag{14}$$

After reformulation, we obtain:

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0199s
The objective of Reduced Robust Counterpart is 3.2742
```

2.2) Budgeted Uncertainty Set

Raw Formulation

We are interested in the budgeted uncertainty set

$$
\begin{aligned}
\max_{x} \quad & c^T x && (15)\\
\text{subject to} \quad & (a+z)^T x \le b\,, \forall z \in \mathcal{Z} && (16)\\
& 0 \le x \le 1\,, && (17)
\end{aligned}
$$

where

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^m \,\middle|\, -1 \le z \le 1,\ \sum_i |z_i| \le \Gamma \right\}$$

After solving, we obtain:

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0357s
The objective of Raw Robust Counterpart is 0.1314
```

Reduced Formulation

We reformulate the above problem into a reduced tractable linear reformulation as follows:

First rewrite the uncertainty set as given in below to be able to use the equivalence idea that we discussed in class

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^m \,\middle|\, \exists \Delta^+ \ge 0, \exists \Delta^- \ge 0, z = \Delta^+ - \Delta^-, \Delta^+ + \Delta^- \le 1,\ \sum_i (\Delta_i^+ + \Delta_i^-) \le \Gamma \right\}$$

$$\mathcal{Z} := \left\{ \begin{bmatrix} \Delta^+ \in \mathbb{R}^m \\ \Delta^- \in \mathbb{R}^m \end{bmatrix} \,\middle|\, \begin{bmatrix} -I & 0 \\ 0 & -I \\ I & I \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta^+ \\ \Delta^- \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 1 \\ \Gamma \end{bmatrix} \right\}$$

We can then rewrite the first constraint of the main program such that

$$z^T x \le b - a^T x \Leftrightarrow \left(\Delta^+ - \Delta^-\right)^T x \le b - a^T x$$

Our robustness linear problem on the uncertain variable z can be written as

$$\max_{\Delta^+,\Delta^-} \quad \begin{bmatrix} \Delta^+ \\ \Delta^- \end{bmatrix}^T \begin{bmatrix} x \\ -x \end{bmatrix} \tag{18}$$

$$\text{s. t.} \quad \begin{bmatrix} -I & 0 \\ 0 & -I \\ I & I \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta^+ \\ \Delta^- \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ \Gamma \end{bmatrix} \tag{19}$$

with dual problem

$$\min_{\alpha^+,\alpha^-,\gamma,\lambda} \quad \begin{bmatrix} 0 & 0 & 1 & \Gamma \end{bmatrix} \begin{bmatrix} \alpha^+ \\ \alpha^- \\ \gamma \\ \lambda \end{bmatrix} \tag{20}$$

$$\text{s. t.} \quad \begin{bmatrix} -I & 0 & I & 1 \\ 0 & -I & I & 1 \end{bmatrix} \begin{bmatrix} \alpha^+ \\ \alpha^- \\ \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} \alpha^+ & \alpha^- & \gamma & \lambda \end{bmatrix}^T \geq 0 \tag{22}$$

where $\alpha^+, \alpha^-, \gamma \in \mathbb{R}^m$.

Thus, our reduced reformulation can be written as

$$\max_{x,\gamma,\lambda} \quad c^T x \tag{23}$$

$$\text{s. t.} \quad \sum_{i=1}^{m} \gamma_i + \lambda\Gamma \leq b - a^T x \tag{24}$$

$$x_i \geq -\gamma_i - \lambda, \forall i = 1, ..., m \tag{25}$$

$$x_i \leq \gamma_i + \lambda, \forall i = 1, ..., m \tag{26}$$

$$0 \leq x \leq 1 \tag{27}$$

$$\gamma, \lambda \geq 0 \tag{28}$$

After reformulation we obtain:

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0593s
The objective of Reduced Robust Counterpart is 0.1314
```

2.3) CVaR Uncertainty Set

Raw Formulation

We are interested in the CVaR uncertainty set

$$\max_{x} \; c^T x \tag{29}$$
$$\text{subject to} \; (a+z)^T x \leq b \,, \forall z \in \mathcal{Z} \tag{30}$$
$$0 \leq x \leq 1 \,, \tag{31}$$

where

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^n \,\middle|\, \exists \theta \in \mathbb{R}^K ,\, z = \sum_{i=1}^{K} \theta_i \bar{z}_i ,\, \theta \geq 0 ,\, \sum_{i=1}^{K} \theta_i = 1 ,\, \theta \leq \frac{1}{K\alpha} \right\}$$

Our solution is

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0329s
The objective of Raw Robust Counterpart is 3.4787
```

Reduced Formulation

We reformulate the above problem into a reduced tractable linear reformulation as follows:

First rewrite the uncertainty set as

$$\mathcal{Z}_1 := \left\{ z = \sum_{i=1}^{K} \theta_i \bar{z}_i \in \mathbb{R}^n \,\middle|\, \theta \geq 0, \sum_{i=1}^{K} \theta_i = 1, \theta \leq \frac{1}{K\alpha} \right\}$$

which we rewrite in the familiar form of $Wz \leq v$ giving

$$\mathcal{Z}_1 := \left\{ \sum_{i=1}^{K} \theta_i \bar{z}_i \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} -I \in \mathbb{R}^{KxK} \\ 1^T \\ -1^T \\ I \in \mathbb{R}^{KxK} \end{bmatrix} \big[\theta\big] \leq \begin{bmatrix} 0 \\ 1 \\ -1 \\ (\frac{1}{K\alpha})1 \end{bmatrix} \right\}$$

where 0 is a K-vector of zeros, 1^T is an K-vector of ones, and $(\frac{1}{K\alpha})1$ is also a K-vector with value $\frac{1}{K\alpha}$ for each of its entries. Note that we replaced the equality constraint

$$\sum_{i=1}^{K} \theta_i = 1$$

by

$$\sum_{i=1}^{K} \theta_i \leq 1, \sum_{i=1}^{K} \theta_i \geq 1$$

We can then rewrite the first constraint of the main program such that

$$z^T x \le b - a^T x \Leftrightarrow \left(\sum_{i=1}^{K} \theta_i \bar{z}_i\right)^T x \le b - a^T x$$

Our robustness linear problem on the uncertain variable z can be written as

$$\max_{\theta} \quad \left(\sum_{i=1}^{K} \theta_i \bar{z}_i\right)^T x \tag{32}$$

$$\text{s. t.} \quad \begin{bmatrix} -I \\ 1^T \\ -1^T \\ I \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \le \begin{bmatrix} 0 \\ 1 \\ -1 \\ (\frac{1}{K\alpha})1 \end{bmatrix} \tag{33}$$

with dual problem

$$\min_{\lambda} \quad \begin{bmatrix} 0 & 1 & -1 & (\frac{1}{K\alpha})1 \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} \tag{34}$$

$$\text{s. t.} \quad -\lambda_i + \lambda_{K+1} - \lambda_{K+2} + \lambda_{K+2+i} = \bar{z}_i^T x, \forall i = 1, ..., K \tag{35}$$

$$\lambda \ge 0 \tag{36}$$

where $\lambda \in \mathbb{R}^{2K+2}$

Thus, our reduced reformulation can be written as

$$\max_{x,\lambda} \quad c^T x \tag{37}$$

$$\text{s. t.} \quad \lambda_{K+1} - \lambda_{K+2} + \frac{1}{K\alpha} \sum_{i=1}^{K} \lambda_{K+2+i} \le b - a^T x \tag{38}$$

$$-\lambda_i + \lambda_{K+1} - \lambda_{K+2} + \lambda_{K+2+i} = \bar{z}_i^T x, \forall i = 1, ..., K \tag{39}$$

$$\lambda \ge 0 \tag{40}$$

$$0 \le x \le 1 \tag{41}$$

After reformulation, we obtain:

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0314s
The objective of Reduced Robust Counterpart is 3.4787
```

Exercises Chapter 3

Our solution file for exercises from Chapter 3 can be accessed by clicking here.

3.1) Calibration of uncertainty sets using data

Calibrating the boxed ellipsoidal uncertainty set

```
Calibrating the ellipsoid set: gamma=1.457444
```

Calibrating the budgeted uncertainty set

```
Calibrating the budgeted set: Gamma=3.606350
```

Calibrating the CVaR uncertainty set

In order to find the α, we used the below mathematical model as suggested in question.

$$\min_{\theta,\lambda} \quad \lambda \tag{42}$$

$$\text{subject to} \quad z = \sum_{i=1}^{K} \theta_i \bar{z}_i \tag{43}$$

$$0 \leq \theta \leq \lambda/K \tag{44}$$

$$\sum_{i=1}^{K} \theta_i = 1 \ , \tag{45}$$

where $\alpha = 1/\lambda$. We have

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0113s
The lambda is 1.0000
The alpha is 1.0000
```

3.2) Calibration of uncertainty sets using distribution hypothesis

Calibrating the budgeted uncertainty set

Following the Corollary 3.8, we used the given formula $\Gamma := \sqrt{2mln(1/\epsilon)}$. As a result, we have

```
Calibrating the budgeted set: Gamma=7.740455
```

Calibrating the ellipsoid set

For this part of the homework, we followed the Theorem 3.5 and directly used the given formula $\gamma := \sqrt{2ln(1/\epsilon)}$. We have

```
Calibrating the ellipsoid set: gamma=2.447747
```

3.3) Evaluation of performance

Boxed ellipsoidal set from Exercise 3.1)

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0104s
Estimated VaR is 0.2074
VaR from 2000 to 2009 is 0.0712
VaR from 2010 to 2014 is 0.0866
VaR with extreme distribution is 0.2396
```

Budgeted uncertainty set from Exercise 3.1)

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0092s
Estimated VaR is 0.1671
VaR from 2000 to 2009 is 0.0641
VaR from 2010 to 2014 is 0.0776
VaR with extreme distribution is 0.2964
```

CVaR uncertainty set from Exercise 3.1)

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0231s
Estimated VaR is -0.0431
VaR from 2000 to 2009 is 0.1681
VaR from 2010 to 2014 is 0.1483
VaR with extreme distribution is 0.3243
```

CVaR uncertainty set with =0.05

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0239s
Estimated VaR is 0.0556
VaR from 2000 to 2009 is 0.0497
VaR from 2010 to 2014 is 0.0661
VaR with extreme distribution is 0.3223
```

Budgeted uncertainty set for ambiguous chance constraint found in 3.2)

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0091s
Estimated VaR is 0.3243
VaR from 2000 to 2009 is 0.1681
VaR from 2010 to 2014 is 0.1483
```

```
VaR with extreme distribution is 0.3243
```

Boxed ellipsoidal uncertainty set for ambiguous chance constraint constraint found in 3.2)

```
Being solved by Mosek...
Solution status: optimal
Running time: 0.0100s
Estimated VaR is 0.3243
VaR from 2000 to 2009 is 0.1681
VaR from 2010 to 2014 is 0.1483
VaR with extreme distribution is 0.3243
```

Discussion of Results

Budgeted Boxed Ellipsoidal Sets. As proposed in Corollary 3.3 of the notes, we observe for those two sets that models using an ambiguous chance constraint formulation perform better than the models using uncertainty sets.

We plotted the variation of the four results as a function of Γ and γ for the budgeted and boxed ellipsoidal sets respectively, and obtained increasing stepping functions for both, which further aligns with Corollary 3.3, as we observe Γ and γ increase in value from the calibration from the uncertainty set to the ambiguous chance constraint. This also aligns with an example seen in class (example 3.1.1), where it was shown that as the size parameter increases, the proportion of scenario covered also increases. Therefore, we can conclude that an ambiguous change constraint would include more scenario in its formulation, and lead to better performances.

Finally, we note that we obtain the same results for the budgeted and boxed ellipsoidal sets using an ambiguous chance constraint formulation as both size parameters locate the models on the same plateau, meaning that both models include the same scenario, leading to the same results.

CVaR Set. Results obtained for the CVaR show that performance for in- and out-of-sample VaR is greater than the bound estimated, for both models. In addition, we observe that the values for the estimated bound and the VaR with extreme distribution are somewhat similar, but that the in- and out-of-sample performance with $\alpha = 1$ is significantly better than with $\alpha = 0.05$.

We note that the value for α that we get when calibrating the CVaR uncertainty set in 3.1 actually provides us with the largest possible value for α, and thus the tightest upper bound for θ. In that case, $z = 0$, and x is equal to 0 for all entries except for that corresponding to the stock with the highest return. In fact, the estimated VaR obtained with $\alpha = 1$ is exactly equal to the maximal mean return. From this observation, we conclude that the tighter the bounds around θ, the more conservative you are, and thus you are not mixing between stocks, but rather betting everything on the best (historically) performing stock. This is also in line with the derivation of the CVaR set, where $1/K$ is similar to a draw from the uniform distribution, and $1/\alpha$ is similar to $1/\epsilon$.

We plotted variation of the four results as a function of α, and noticed that the variation was mostly constant. As such, we conclude that the CVaR set has less variance in its results, irrespective of the calibration of the size parameter.