Quatitative Risk Management Using Robust Optimization
MATH80624A
Assignment 2

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Due date: March 12, 2021

Exercises Chapter 4

Our solution file for exercises from Chapter 4 can be accessed by clicking here.

4.1) Implementing vertex enumeration

The mathematical model that we use is given in below:

$$\max_{x,\{y^k\}_{k=1}^{K'},s} -\sum_{i=1}^{n} c_i x_i + s \tag{1}$$

$$\text{s.t.} \sum_{i=1}^{n} y_{ij}^k \leq \bar{D}_j + \hat{D}_j z_j^k \qquad \forall j \in \{1,\cdots,m\}, \forall k \in \{1,\cdots,K'\} \tag{2}$$

$$\sum_{j=1}^{m} y_{ij}^k \leq P_i x_i \qquad \forall i \in \{1,\cdots,n\}, \forall k \in \{1,\cdots,K'\} \tag{3}$$

$$s \leq \sum_{i=1}^{n}\sum_{j=1}^{m} (r_{ij} - d_{ij}) y_{ij}^k \quad \forall k \in \{1,\cdots,K'\} \tag{4}$$

$$y_{ij}^k \geq 0 \qquad \forall i \in \{1,\cdots,n\}, \forall j \in \{1,\cdots,m\}, \forall k \in \{1,\cdots,K'\} \tag{5}$$

$$x \in \{0,1\}^n, \tag{6}$$

$\Gamma = 1$

```
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.0415s
when Gamma= 1 the objective is 76.57 and the
          optimal facility locations are [1. 1. 1. 1.]
```

$\Gamma = m$

```
Being solved by Mosek...
Solution status: integer_optimal
```

```
Running time: 9.5839s
when Gamma= 12 the objective is 28.51 and the optimal facility locations are
            [-1.66533454e-17  1.00000000e+00  0.00000000e+00  1.00000000e+00]
```

4.2) RC = multi-stage ARC under $\Gamma = m$

We summarize our thought process below:

1. Since $\Gamma = m$, the uncertainty set will be;

$$\mathcal{Z}(m) := \left\{ z \in \mathbb{R}^m \;\middle|\; -1 \leq z \leq 1,\ \sum_i |z_i| \leq m \right\}$$

   where the $\sum_i |z_i| \leq m$ will be redundant. Hence, we can say that z_j's may take any value between 1 and -1 without any other restriction.

2. We checked the conditions of Theorem 4.3; our y's are bounded (condition 2) by the second inequality of the original problem, coefficients in the objective function do not depend on z, each constraint only includes z_j(condition 4), and condition 1 is also satisfied.

3. If $\sum_{i=1}^n y_{ij}$ satisfies the first constraint with the worst-case of the right hand side, it will satisfy for all other realizations. Since we obtain the worst-case when $z_j = -1$, the right hand side will be $\bar{D}_j - \hat{D}_j$.

$$\max_{x,y} -\sum_{i=1}^n c_i x_i + \sum_{i=1}^n \sum_{j=1}^m (r_{ij} - d_{ij}) y_{ij} \tag{7}$$

$$\text{s.t.} \sum_{i=1}^n y_{ij} \leq \bar{D}_j - \hat{D}_j \qquad \forall j \in \{1, \cdots, m\}, \tag{8}$$

$$\sum_{j=1}^m y_{ij} \leq P_i x_i \qquad \forall i \in \{1, \cdots, n\}, \tag{9}$$

$$y_{ij} \geq 0 \qquad \forall i \in \{1, \cdots, n\}, \forall j \in \{1, \cdots, m\}, \tag{10}$$

$$x \in \{0,1\}^n, \tag{11}$$

That means, Z can be represented as the Cartesian product of independent sets and one can look at each Z_j independently and optimize sub-problems separately.

4.3) Implementing column-and-constraint generation

We reformulate the problem as a master and a slave problem to implement column&constraint generation.

The master problem is:

$$\max_{x,\{y^k\}_{k=1}^{K'}} -\sum_{i=1}^{n} c_i x_i + s \tag{12}$$

$$\text{s.t.} \sum_{i=1}^{n} y_{ij}^k \leq \bar{D}_j + \hat{D}_j z_j^k \qquad \forall j \in \{1, \cdots, m\}, \forall k \in \{1, \cdots, K'\} \tag{13}$$

$$\sum_{j=1}^{m} y_{ij}^k \leq P_i x_i \qquad \forall i \in \{1, \cdots, n\}, \forall k \in \{1, \cdots, K'\} \tag{14}$$

$$s \leq \sum_{i=1}^{n} \sum_{j=1}^{m} (r_{ij} - d_{ij}) y_{ij}^k \quad \forall k \in \{1, \cdots, K'\} \tag{15}$$

$$y_{ij}^k \geq 0 \qquad \forall i \in \{1, \cdots, n\}, \forall j \in \{1, \cdots, m\}, \forall k \in \{1, \cdots, K'\} \tag{16}$$

$$x \in \{0, 1\}^n, \tag{17}$$

where z^k are a subset of $K' < K$ vertices.

The slave problem is:

$$
\begin{aligned}
h(x,z) := \min_{x,y,\lambda,\gamma,\theta,u,v,w,z} \quad & -\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{m}(r_{ij}-d_{ij})y_{ij} && && (18)\\
\text{s.t.} \quad & \sum_{i=1}^{n} y_{ij} \leq \bar{D}_j + \hat{D}_j z_j && \forall j \in \{1,\cdots,m\} && (19)\\
& \sum_{j=1}^{m} y_{ij} \leq P_i x_i && \forall i \in \{1,\cdots,n\} && (20)\\
& -y_{ij} \leq 0 && \forall i \in \{1,\cdots,n\}, \forall j \in \{1,\cdots,m\} && (21)\\
& x \in \{0,1\}^n && && (22)\\
& \lambda_j + \gamma_i - \theta_{ij} = r_{ij} - d_{ij} && \forall i \in \{1,\cdots,n\}, \forall j \in \{1,\cdots,m\} && (23)\\
& 0 \leq \lambda_j \leq M u_j && \forall j \in \{1,\cdots,m\} && (24)\\
& 0 \leq \gamma_i \leq M v_i && \forall i \in \{1,\cdots,n\} && (25)\\
& 0 \leq \theta_{ij} \leq M w_{ij} && \forall i \in \{1,\cdots,n\}, \forall j \in \{1,\cdots,m\} && (26)\\
& \bar{D}_j + \hat{D}_j z_j - \sum_{i=1}^{n} y_{ij} \leq M(1-u_j) && \forall j \in \{1,\cdots,m\} && (27)\\
& P_i x_i - \sum_{j=1}^{m} y_{ij} \leq M(1-v_i) && \forall i \in \{1,\cdots,n\} && (28)\\
& y_{ij} \leq M(1-w_{ij}) && \forall i \in \{1,\cdots,n\}, \forall j \in \{1,\cdots,m\} && (29)\\
& |z| \leq 1 && && (30)\\
& \sum_{i} |z_i| \leq \Gamma && && (31)
\end{aligned}
$$

We follow the steps presented in section 4.6.2 of the notes to find the optimal solution. We use as an initial $\hat{x}$ the vector [0, 0, 0, 0], as it in $\mathcal{X}$, and allows us to satisfy the relatively complete recourse condition. We get the optimal value and solution to the problem after 4 iterations.

```
iteration  4
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.0206s
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.0126s
when Gamma= 4 the objective is 45.05 and the optimal facility locations
are [1. 1. 1. 1.]
```

Note that, as a comparison, we ran the vertex enumeration for $\Gamma = 4$, and got the same optimal value and solution, but required significantly more computing time.

Exercises Chapter 5

Our solution file for exercises from Chapter 5 can be accessed by clicking here.

5.1) Implementing Static RC

Using a static decision rule implies that $y(z) = y$. We incorporate this in the model to get

$$
\begin{aligned}
\max_{x,y} \; & -\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{m}(r_{ij} - d_{ij}) y_{ij} && && (32)\\
\text{s.t.} \; & \sum_{i=1}^{n} y_{ij} \le \bar{D}_j + \hat{D}_j z_j && \forall j \in \{1, \cdots, m\}, \forall z \in \mathcal{Z}(\Gamma) && (33)\\
& \sum_{j=1}^{m} y_{ij} \le P_i x_i && \forall i \in \{1, \cdots, n\}, && (34)\\
& y_{ij} \ge 0 && \forall i \in \{1, \cdots, n\}, \forall j \in \{1, \cdots, m\}, && (35)\\
& x \in \{0, 1\}^n, && && (36)
\end{aligned}
$$

where

$$
\mathcal{Z}(\Gamma) := \left\{ z \in \mathbb{R}^m \,\middle|\, -1 \le z \le 1, \; \sum_i |z_i| \le \Gamma \right\}
$$

```
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.0362s
The objective is 28.510000000000005 and the optimal faciliy location
        is [0. 1. 0. 1.]
```

5.2) Implementing AARC

Using an affine decision rule implies that $y(z) = y + Y(z)$. We incorporate this in the model to get

$$
\begin{aligned}
\max_{x,y,Y} \min_{z} \; & -\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{m}(r_{ij} - d_{ij})(y_{ij} + Y_{ij}(z)) && && (37)\\
\text{s.t.} \; & \sum_{i=1}^{n}(y_{ij} + Y_{ij}(z)) \le \bar{D}_j + \hat{D}_j z_j && \forall j \in \{1, \cdots, m\}, \forall\, z \in \mathcal{Z}(\Gamma) && (38)\\
& \sum_{j=1}^{m}(y_{ij} + Y_{ij}(z)) \le P_i x_i && \forall i \in \{1, \cdots, n\}, \forall z \in \mathcal{Z}(\Gamma) && (39)\\
& (y_{ij} + Y_{ij}(z)) \ge 0 && \forall i \in \{1, \cdots, n\}, \forall j \in \{1, \cdots, m\}, \forall z \in \mathcal{Z}(\Gamma) && (40)\\
& x \in \{0, 1\}^n, && && (41)
\end{aligned}
$$

where

$$\mathcal{Z}(\Gamma) := \left\{ z \in \mathbb{R}^m \,\middle|\, -1 \le z \le 1,\ \sum_i |z_i| \le \Gamma \right\}$$

```
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.9579s
The objective is 76.57000000000002 and the optimal faciliy location
        is [1. 1. 1. 1.]
```

5.3) Implementing Lifted AARC

Using an lifted affine decision rule implies that $y(z) = y + Y^+(z) + Y^-(z)$. We incorporate this in the model to get

$$\max_{x,y,Y} \min_z \quad -\sum_{i=1}^n c_i x_i + \sum_{i=1}^n \sum_{j=1}^m (r_{ij} - d_{ij})(y_{ij} + Y_{ij}^+(z) + Y_{ij}^-(z)) \tag{42}$$

$$\text{s.t.} \sum_{i=1}^n (y_{ij} + Y_{ij}^+(z) + Y_{ij}^-(z)) \le \bar{D}_j + \hat{D}_j z_j \quad \forall\, j \in \{1, \cdots, m\}, \forall (z, z^+, z^-) \in \mathcal{Z}'(\Gamma) \tag{43}$$

$$\sum_{j=1}^m (y_{ij} + Y_{ij}^+(z) + Y_{ij}^-(z)) \le P_i x_i \qquad \forall i \in \{1, \cdots, n\}, \forall (z^+, z^-) \in \mathcal{Z}'(\Gamma) \tag{44}$$

$$(y_{ij} + Y_{ij}^+(z) + Y_{ij}^-(z)) \ge 0 \qquad \forall i \in \{1, \cdots, n\}, \forall j \in \{1, \cdots, m\}, \forall (z^+, z^-) \in \mathcal{Z}'(\Gamma) \tag{45}$$

$$x \in \{0,1\}^n, \tag{46}$$

where

$$\mathcal{Z}'(\Gamma) := \left\{ (z, z^+, z^-) \in \mathbb{R}^m \text{x} \mathbb{R}^m \text{x} \mathbb{R}^m \,\middle|\, z = z^+ - z^-,\ z^+ \ge 0,\ z^- \ge 0,\ z^+ + z^- \le 1,\ \sum_i (z_i^+ + z_i^-) \le \Gamma \right\}$$

```
Being solved by Mosek...
Solution status: integer_optimal
Running time: 0.9579s
The objective is 76.57000000000002 and the optimal faciliy location
        is [1. 1. 1. 1.]
```

5.4) Comparison of Approximate Worst-case Bounds

```
When Gamma=0 the objectives of the four models are
[89.05 89.05 89.05 89.05]
    while the worst-case profits of the policies are
    [89.05 89.05 89.05 89.05].
When Gamma=1 the objectives of the four models are
```

```
[28.51 76.57 76.57 76.57]
    while the worst-case profits of the policies are
    [69.14 76.57 76.57 76.57].
When Gamma=4 the objectives of the four models are
[28.51 44.31 45.05 45.05]
    while the worst-case profits of the policies are
    [43.28 44.31 45.05 45.05].
When Gamma=11 the objectives of the four models are
[28.51 28.51 28.51 28.51]
    while the worst-case profits of the policies are
    [28.51 28.51 28.51 28.51].
```

Denote ψ^*_{SRC}, ψ^*_{AARC}, ψ^*_{LAARC}, ψ^*_{CC} as the optimal values obtained from the SRC, AARC, LAARC and Constraint & Column generation models respectively, then, we should observe

$$\psi^*_{SRC} \leq \psi^*_{AARC} \leq \psi^*_{LAARC} \leq \psi^*_{CC}$$

which is indeed what we observe from our results.

When $\Gamma = 0$, then all the z must be zero, which is equivalent to only looking at the average demand, which is identical to the nominal problem. This explains why all models return the same optimal value. On the other hand, when $\Gamma =$ m-1, the problem will be almost identical to the one described in 4.2), where we would have the worst case of $z = -1$ for m-1 locations. This is the absolute worst case we could have, and explains the lowest bound. Results for $\Gamma = 1$ and $\Gamma = 4$ should fall between those two extremes, with a decrease in performance as Γ increases. This is what we observe in our results.

We observe that some bounds and true worst-case values differ. For example, the worst-case value from the SRC for $\Gamma = 1$ is higher than its bound. This shows that although some of the bounds calculated are low, the true value of the model obtained may not be as poor.

Finally, we observe that as the sophistication in the approximate method increases, the smaller (or nonexistent) the gap between the worst-case bound and true value, which would be expected. Finally, we note that the worst-case bound and true value for Constraint & Column generation have to be equal as it is an exact method. This is what we observe in our results.