Given adjacency matrix $Q_{n\times n}$ (square, nonnegative) with rowsums $\boldsymbol{L} = [L_1, L_2, L_3, \ldots, L_n]$.
Define S to be the augmentation of Q to include the super-node:

$$S = \left[\begin{array}{c|ccccc} 0 & 1 & 1 & 1 & \ldots & 1 \\ \hline \alpha_1 & q_{11} & q_{12} & q_{13} & & q_{1n} \\ \alpha_2 & q_{21} & q_{22} & q_{23} & & q_{2n} \\ \alpha_3 & q_{31} & q_{32} & q_{33} & & q_{3n} \\ \ldots & & & & & \\ \alpha_n & q_{n1} & q_{n2} & q_{n3} & & q_{nn} \end{array}\right] \tag{1}$$

with $\boldsymbol{\beta} = [\beta_1, \beta_2, \beta_3, \ldots, \beta_n] = \mathbf{1}$ and $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n]$ and Q elements q_{ij}.

Let $R = [r_0, r_1, r_2, r_3, \ldots, r_n]$ represent the Patent Rank eigenvector (sums to one) based[1] on the row-normalization of S:

$$\boldsymbol{R} = \hat{S}^T \boldsymbol{R} \qquad \text{where} \qquad \hat{S} = \text{diag}(\boldsymbol{d})^{-1}\boldsymbol{S} \qquad \text{and} \qquad \boldsymbol{d} = \boldsymbol{S}\boldsymbol{e}. \tag{2}$$

Let $R' = [r'_1, r'_2, r'_3 \ldots, r'_n]$ represent the PageRank eigenvector with[2] damping factor $(1-\alpha)$ based on the row-normalization of Q:

$$\boldsymbol{R'} = \hat{Q}^T \boldsymbol{R'} \qquad \text{where} \qquad \hat{Q} = \text{diag}(\boldsymbol{d})^{-1}\boldsymbol{Q} \qquad \text{and} \qquad \boldsymbol{d} = \boldsymbol{Q}\boldsymbol{e}. \tag{3}$$

Given that the matrix Q contains no dangling nodes ($\boldsymbol{L}$ is strictly positive), if we assign augmentation $\boldsymbol{\alpha}$ elements as follows:

$$\alpha_j = \frac{\alpha L_j}{1-\alpha}, \tag{4}$$

Properties of Equation 4:

- **a:**
$$\alpha_j = \alpha(L_j + \alpha_j)$$
- **b:**
$$L_j + \alpha_j = \frac{L_j}{1-\alpha}$$
- **c:**
$$\alpha = \frac{\alpha_j}{L_j + \alpha_j}$$

then

$$R' = [(1+\alpha)r_1, (1+\alpha)r_2, (1+\alpha)r_3 \ldots, (1+\alpha)r_n]. \tag{5}$$

[1]We denote the vector $\boldsymbol{e}$ as a unitary vector of length n with all elements equal to one. A matrix that is row-stochastic and row-normalized is scaled so the sum of every row is equal to one (that is, $\sum_j s_{i,j} = 1$). The scaling factors $d_i = \sum_j h_{i,j}$ define the vector $\boldsymbol{d} = (d_i)$, which satisfies the equation $\boldsymbol{d} = \boldsymbol{S}\boldsymbol{e}$. We denote the diagonal matrix $\boldsymbol{D}_{n\times n}$ as diag($\boldsymbol{d}$) with diagonal entries composed of vector $\boldsymbol{d} = (d_i)$ for $i = 1, 2, \ldots, n$. $\hat{S}^T$ is the transpose of matrix $\hat{S}$.

[2]For example, $\alpha = 0.15$ corresponds to a damping factor of (0.85)

Proof. From Equation 2, the super-node element r_0 is defined based on multiplication as:

$$\begin{aligned} r_0 &= \sum_{j=0}^{n} \frac{\alpha_j}{L_j + \alpha_j} r_j \\ &= \sum_{j=0}^{n} \alpha r_j \text{ (Substitute Equation 4 c:)} \\ &= \alpha \sum_{j=0}^{n} r_j \\ &= \alpha(1 - r_0) \text{ (eigenvector sums to one)} \end{aligned} \tag{6}$$

therefore,

$$r_0 = \frac{\alpha}{1+\alpha} \tag{7}$$

and we note r_0, the value of the super node, is a function of the damping factor.

We also note that we can construct a single element from R:

$$\boldsymbol{R} = \left[\hat{S}^T\right] \boldsymbol{R} \tag{8}$$

as follows:

$$\begin{aligned} r_i &= \frac{r_0}{n} + \sum_{j=1}^{n} \frac{q_{ji}}{L_j + \alpha_j} r_j \\ &= \frac{r_0}{n} + \sum_{j=1}^{n} \frac{(1-\alpha) q_{ji}}{L_j} r_j \text{ (Substitute Equation 4 b:)} \\ &= \frac{r_0}{n} + (1-\alpha) \sum_{j=1}^{n} \frac{q_{ji}}{L_j} r_j \\ &= \frac{\alpha}{n(1+\alpha)} + (1-\alpha) \sum_{j=1}^{n} \frac{q_{ji}}{L_j} r_j \text{ (Substitute Equation 7)} \end{aligned} \tag{9}$$

Next, we define the PageRank vector from Equation 3

$$\boldsymbol{R'} = \left[\hat{Q}^T (1-\alpha) + \frac{\alpha}{n} \boldsymbol{e} \boldsymbol{e}^T\right] \boldsymbol{R'} \tag{10}$$

Therefore, does $(1+\alpha) r_i \stackrel{?}{=} r'_i$. Consider a single element from Equation 10:

$$\begin{aligned} r'_i &= \sum_{j=1}^{n} \left[(1-\alpha) \frac{q_{ji}}{L_j} + \frac{\alpha}{n}\right] r'_j \\ &= (1-\alpha) \sum_{j=1}^{n} \frac{q_{ji}}{L_j} r'_j + \frac{\alpha}{n} \sum_{j=1}^{n} r'_j \\ &= (1-\alpha) \sum_{j=1}^{n} \frac{q_{ji}}{L_j} r'_j + \frac{\alpha}{n} \text{ (sums to one)} \end{aligned} \tag{11}$$

Substituting Equations 9 and 11:

$$
\begin{aligned}
(1+\alpha)r_i &\stackrel{?}{=} r'_i \\
(1+\alpha)\left[\frac{\alpha}{n(1+\alpha)} + (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}r_j\right] &\stackrel{?}{=} (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}r'_j + \frac{\alpha}{n} \\
\frac{\alpha}{n} + (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}(1+\alpha)r_j &\stackrel{?}{=} (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}r'_j + \frac{\alpha}{n} \\
\frac{\alpha}{n} + (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}(1+\alpha)r_j &\stackrel{?}{=} \frac{\alpha}{n} + (1-\alpha)\sum_{j=1}^{n}\frac{q_{ji}}{L_j}r'_j
\end{aligned}
\tag{12}
$$

□