VaR and Out-of-Sample Evaluation

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Contents

Part I

The value at risk (VaR) is defined as the α quantile of the profit of an investment. In this lab we will conduct simulation to investigate the role of the this important parameter in risk management.

We use the example discussed in class as illustration. Consider a \$10,000 investment in *Amazon* for 1 month. Assume that

$$
\begin{aligned}
r &= \text{cc monthly return on Amazon} \\
r &\sim N(0.05, 0.01),\ \mu_r = 0.05,\ \sigma_r = 0.10.
\end{aligned}
$$

Define the profit of the investment as

$$L_1 = W_1 - W_0 = W_0 \cdot (e^r - 1) = 10,000(e^r - 1)$$

where W_0 and W_1 denote the initial value and future value of the investment. Then the VaR_α of the investment is

$$\text{VaR}_\alpha = 10,000(e^{q_\alpha^r} - 1)$$

where q_α^r is the α quantile of the cc return r.

```r
# Declare global variables
W0 <- 10000 # Initial value of the investment
N_Exp <- 500 # Number of simulations
```

Problem 1

Calculate $\text{VaR}_{0.1}$. Find the probability that L_1 is less than $\text{VaR}_{0.1}$ using 500 replicated experiments.

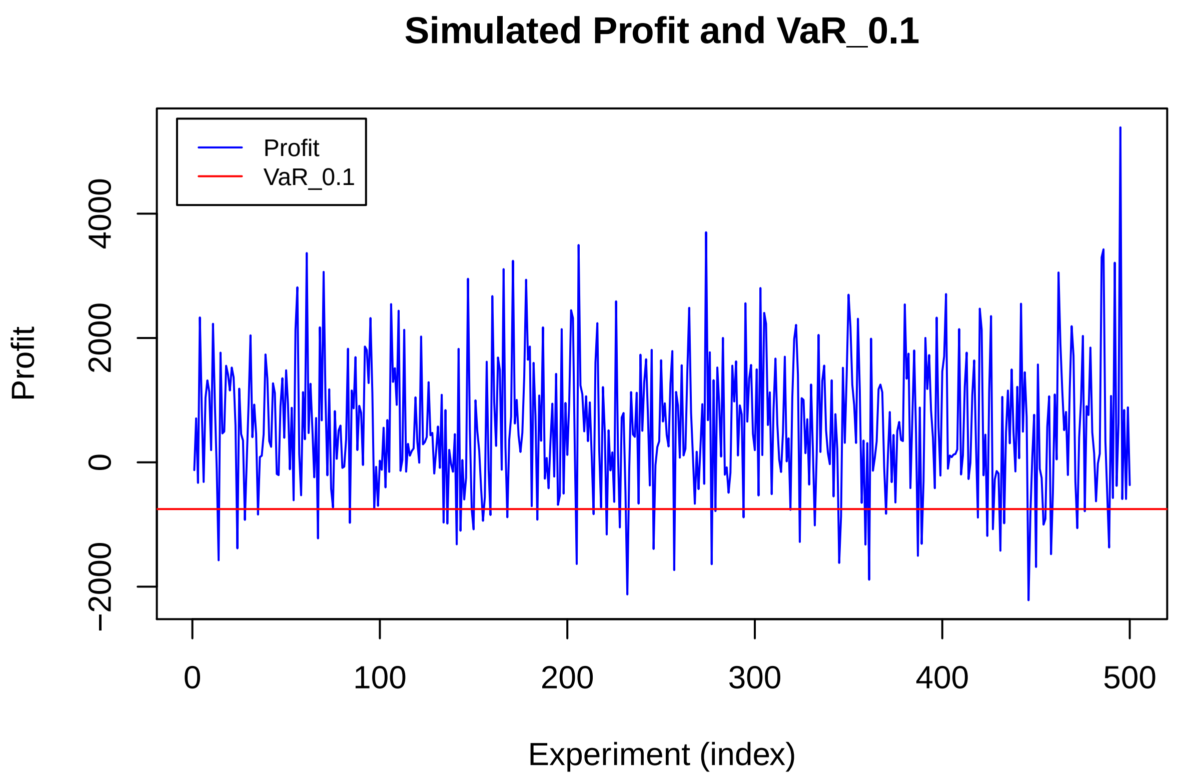
```
set.seed(1)
# Simulate monthly cc returns
r <- rnorm(N_Exp, mean = 0.05, sd = 0.1)

# Simulate profits
L1 <- W0 * (exp(r) - 1)

# Paramatric VaR / Paramatric Estimation
VaR_01 <- W0 * (exp(qnorm(0.1, 0.05, 0.1)) - 1)

# Empirical Probability
Out_v1 <- as.numeric(L1 < VaR_01)
mean(Out_v1)
```

```
## [1] 0.1
```

Problem 2

Plot the simulated profits (as a function of their index in the replicated experiments). Plot the line of $\text{VaR}_{0.1}$ in the same picture. Explain what you find in the picture.

```
index <- 1:N_Exp

plot(index,
     L1,
     type = "l",
     col = "blue",
     xlab = "Experiment (index)",
     ylab = "Profit",
     main = "Simulated Profit and VaR_0.1")

abline(h = VaR_01, col = "red")

legend("topleft",
       legend = c("Profit", "VaR_0.1"),
       lty = c(1,1),
       col = c("blue", "red"),
       cex = 0.75,
       inset =  0.02)
```




```r
Out_v2 <- as.numeric(L1 < VaR_01)
sum(Out_v2)
```

```
## [1] 50
```

```r
mean(Out_v2)
```

```
## [1] 0.1
```

Problem 3

Calculate $W_0 + \text{VaR}_{0.1}$. Find the probability that W_1 is less than $W_0 + \text{VaR}_{0.1}$ using 500 replicated experiments.

```r
W1 <- W0 + L1
W0_VaR <- W0 + VaR_01
Out_v3 <- as.numeric(W1 < W0_VaR)
mean(Out_v3)
```

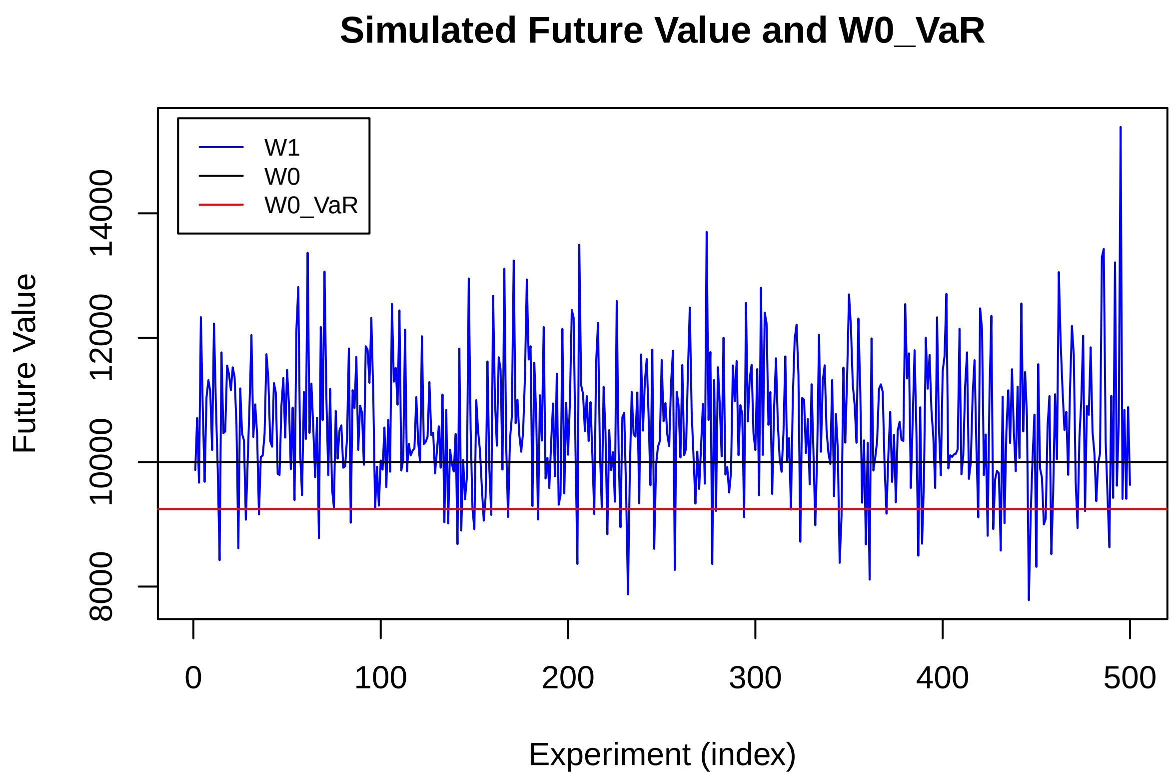
```
## [1] 0.1
```

Problem 4

Plot the simulated future values (as a function of their index in the replicated experiments). Plot the lines of W_0 and $W_0 + \text{VaR}_{0.1}$ in the same picture. Explain what you find in the picture.

```r
plot(index,
     W1,
     type ="l",
     col = "blue",
     xlab = "Experiment (index)",
     ylab = "Future Value",
```

```r
     main = "Simulated Future Value and W0_VaR")

abline(h = W0, col = "black")

abline(h = W0_VaR, col = "red")

legend("topleft",
       legend = c("W1", "W0", "W0_VaR"),
       lty = c(1,1,1),
       col = c("blue", "black", "red"),
       cex = 0.75,
       inset =  0.02)
```



Comment: 1. There are more W1 values above the black line (W0) then below the black line, meaning the expected future value should be larger than the initial value. 2. The number of W1 values below the red line (W0_VaR) are about the same as the number of L1 values below the red line (VaR_0.1) in problem 3, which is consistent with our expectation.

Part II

Next we consider estimation of the mean, variance and VaR_α using the *simulated data*. For VaR_α, we consider $\alpha = 0.1$ and both the parametric and nonparametric estimators discussed in class.

We first generate 500 cc returns $\{r_1, r_2, \ldots, r_{500}\}$ from $N(\mu, \sigma^2)$ where $\mu = 0.05$ and $\sigma^2 = 0.01$. These 500 simulated values are treated as cc returns of 500 trading dates. In this simulation study, for any date t ($t > 300$), we use 300 historic data before t to estimate μ, σ^2 and $\mathrm{VaR}_{0.1}$. For example, at time 301 we use $\{r_1, r_2, \ldots, r_{300}\}$ for estimation, and at time 302 we use $\{r_2, r_3, \ldots, r_{301}\}$ for estimation and etc. Therefore, we will get 200 estimators of μ, σ^2 and $\mathrm{VaR}_{0.1}$ respectively.

```r
S <- 500 # Total number of cc returns generated
E <- 300 # Length of the "window" used for estimation
N_Est <- S - E # We will have 200 estimations (500 - 300)
```

Problem 5

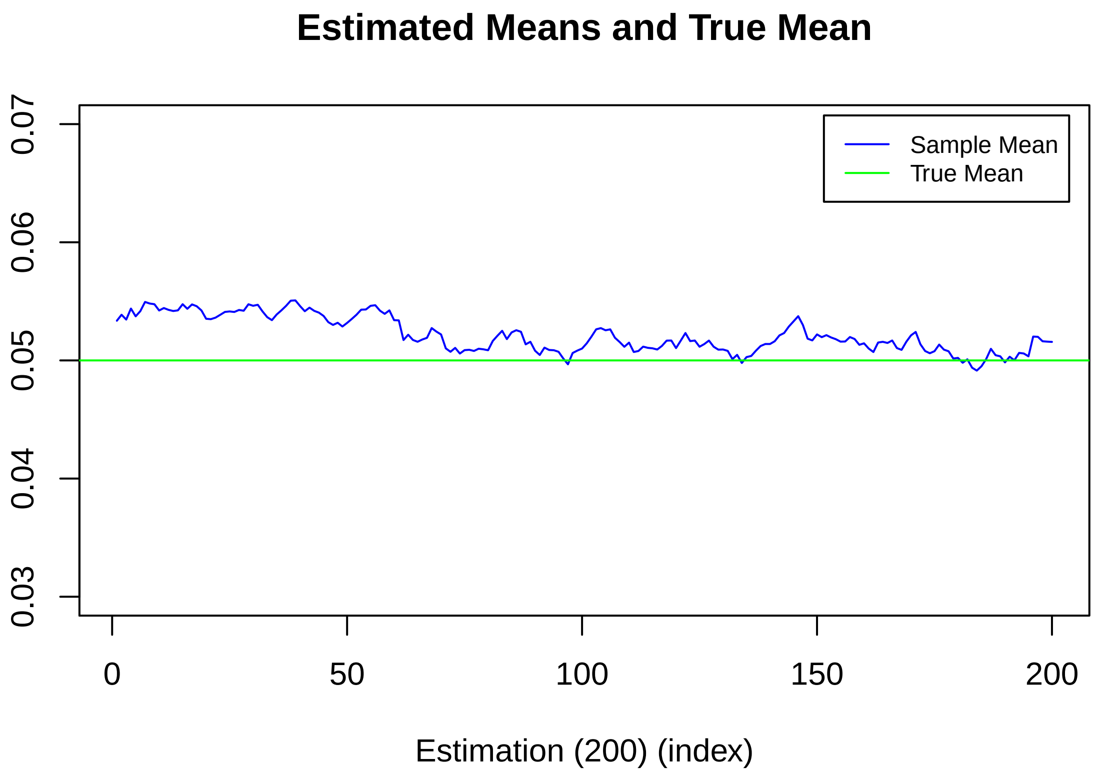
Plot the (200) estimated mean and the true mean. Explain what you find in the picture.

```r
set.seed(1)
r <- rnorm(S, mean = 0.05, sd = 0.1)

# Create empty vectors to store mean, variance, W1, L1, parametric VaR, and non-parametric VaR
Out_m <- rep(0, N_Est)
Out_v <- rep(0, N_Est)
Out_W1 <- rep(0, N_Est)
Out_L1 <- rep(0, N_Est)
Out_VaR1 <- rep(0, N_Est)
Out_VaR2 <- rep(0, N_Est)

for (s in seq_len(N_Est)) {
  s1 <- s + E - 1 # The last date we will use to estimate, e.g. 300 for the 1st iteration
  Data <- r[s:s1]

  Out_m[s] <- mean(Data)
  Out_v[s] <- mean(Data^2) - (mean(Data))^2 * E/(E-1) # Sample variance

  Out_L1[s] <- W0 * (exp(r[s + E]) - 1)
  Out_W1[s] <- W0 + Out_L1[s]

  # Two methods to calculate VaR
  # Method 1: parametric
  q1 <- qnorm(0.1, Out_m[s], Out_v[s]^(0.5))
  Out_VaR1[s] <- W0 * (exp(q1) - 1)

  # Method 2: non-parametric (no assumption about the structure of the distribution)
  q2 <- quantile(Data, probs = 0.1, type = 1)
  Out_VaR2[s] <- W0 * (exp(q2) - 1)
}
```

Plot the estimated means.

```r
plot(1:N_Est,
     Out_m,
     type = "l",
     col = "blue",
     ylim = range(c(0.03, 0.07)),
     xlab = "Estimation (200) (index)",
     ylab = "",
     main = "Estimated Means and True Mean")

par(new = TRUE)

abline(h = 0.05, col = "green")

legend(x = "topright",
```

```
       legend = c("Sample Mean", "True Mean"),
       lty = c(1, 1),
       col = c("blue", "green"),
       cex = 0.75,
       inset = 0.02)
```



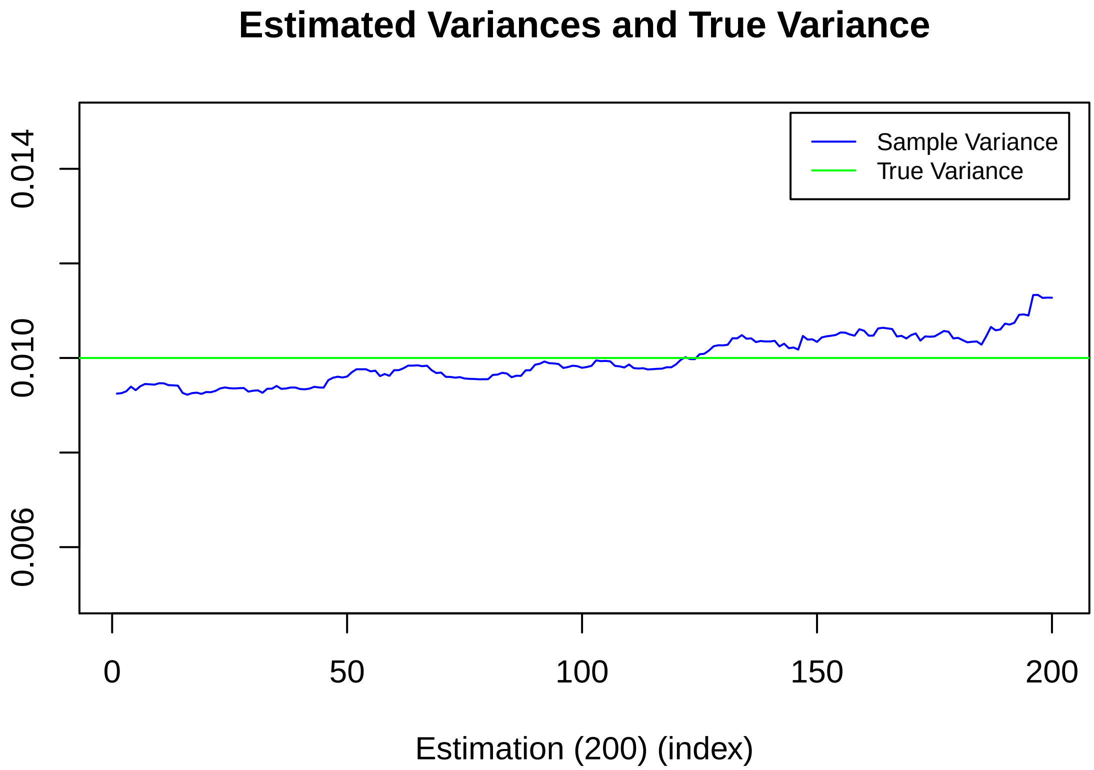
Comment: The estimated (sample) means are relatively close to the true mean. For the most of the instance, the sample mean is slightly higher than the true mean.

Problem 6

Plot the (200) estimated variance and the true variance. Explain what you find in the picture.

```
plot(1:N_Est,
     Out_v,
     type = "l",
     col = "blue",
     ylim = range(c(0.005, 0.015)),
     xlab = "Estimation (200) (index)",
     ylab = "",
     main = "Estimated Variances and True Variance")

par(new = TRUE)

abline(h = 0.01, col = "green")

legend(x = "topright",
       legend = c("Sample Variance", "True Variance"),
       lty = c(1, 1),
       col = c("blue", "green"),
```

```r
       cex = 0.75,
       inset = 0.02)
```



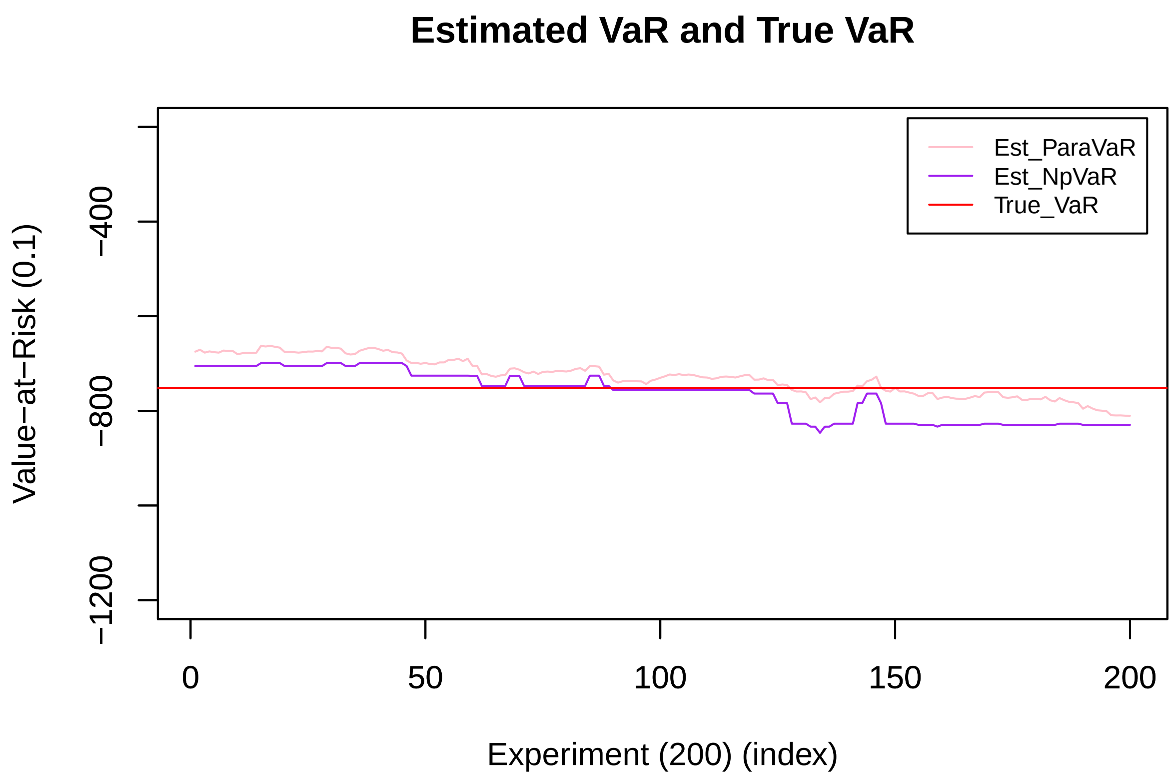
Comment: The estimated (sample) variances are relatively close to the true variance. However, there is a stochastic trend on the graph - sample variance increases as we move the window from 1-300 to 200-500.

Problem 7

Plot the true $\text{VaR}_{0.1}$ and the (200) estimated parametric and nonparametric VaRs. Explain what you find in the picture.

```r
# Plot the estimated parametric VaRs
plot(1:N_Est,
     Out_VaR1,
     type = "l",
     col = "pink",
     xlab = "Experiment (200) (index)",
     ylab = "Value-at-Risk (0.1)",
     xlim = range(c(1, N_Est)),
     ylim = range(c(-1200, -200)),
     main = "Estimated VaR and True VaR")

par(new = TRUE)

# Plot the estimated non-parametric VaRs
plot(1:N_Est,
     Out_VaR2,
     type = "l",
     col = "purple",
     xlab = "",
```

```r
     ylab = "",
     xlim = range(c(1, N_Est)),
     ylim = range(c(-1200, -200)))

# Plot the true VaR_0.1
q_0.1 <- qnorm(0.1, 0.05, 0.1)
VaR_0.1 <- W0 * (exp(q_0.1) - 1)
abline(h = VaR_0.1, col = "red")

# Legend
legend(x = "topright",
       legend = c("Est_ParaVaR", "Est_NpVaR", "True_VaR"),
       lty = c(1, 1, 1),
       col = c("pink", "purple", "red"),
       cex = 0.75,
       inset = 0.02)
```



Comment: Both of the estimated VaRs are close to the true value. The parametric estimation is usually more accurate.

Problem 8

Suppose at each trading date, we invest $W_0 = \$10,000$ at the beginning of the date and close the position at the end of the date. Calculate the profits of the investment at each trading date. Also calculate the future values of the investment at the end of the trading date.

```r
# We have already done this in question 5.
head(Out_L1)
```

```
## [1] 1495.4633 -532.5909 2803.5033  117.0476 2403.7596 2228.9533
```

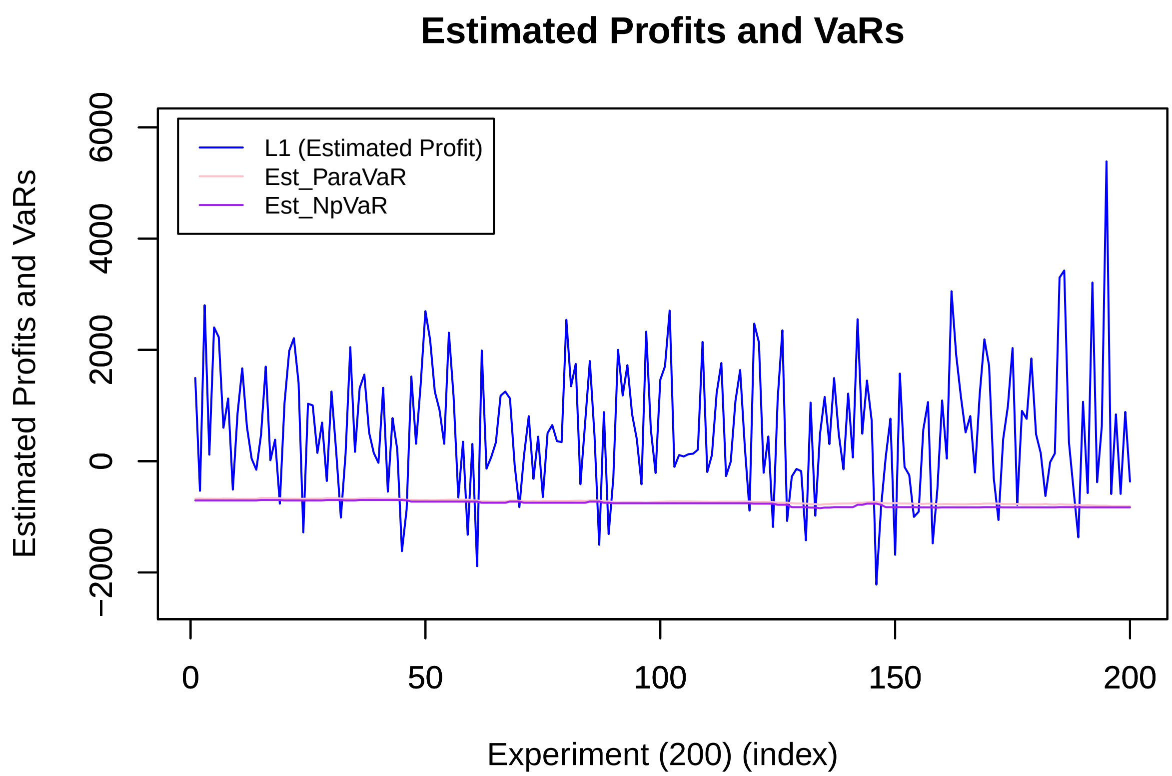
```r
head(Out_W1)
```

```
## [1] 11495.463  9467.409 12803.503 10117.048 12403.760 12228.953
```

Problem 9

Plot the profits from the trading date 301 to the trading date 500. Plot the parametric and nonparametric estimators of the $\mathrm{VaR}_{0.1}$ in the same picture. Calculate the relative frequencies that the profits cross the estimators of the $\mathrm{VaR}_{0.1}$.

```r
# Plot the profits
plot(1:N_Est,
     Out_L1,
     type = "l",
     col = "blue",
     xlab = "Experiment (200) (index)",
     ylab = "Estimated Profits and VaRs",
     xlim = range(c(1, N_Est)),
     ylim = range(c(-2500, 6000)),
     main = "Estimated Profits and VaRs")

par(new = TRUE)

# Plot the parametric estimates of the VaR_0.1
plot(1:N_Est,
     Out_VaR1,
     type = "l",
     col = "pink",
     xlab = "",
     ylab = "",
     xlim = range(c(1, N_Est)),
     ylim = range(c(-2500, 6000)))

par(new = TRUE)

# Plot the non-parametric estimates of the VaR_0.1
plot(1:N_Est,
     Out_VaR2,
     type = "l",
     col = "purple",
     xlab = "",
     ylab = "",
     xlim = range(c(1, N_Est)),
     ylim = range(c(-2500, 6000)))

legend(x = "topleft",
       legend = c("L1 (Estimated Profit)", "Est_ParaVaR", "Est_NpVaR"),
       lty = c(1, 1, 1),
       col = c("blue", "pink", "purple"),
       cex = 0.75,
       inset = 0.02)
```




```
a1 <- as.numeric(Out_L1 < Out_VaR1)
a2 <- as.numeric(Out_L1 < Out_VaR2)
mean(a1)
```

```
## [1] 0.12
```

```
mean(a2)
```

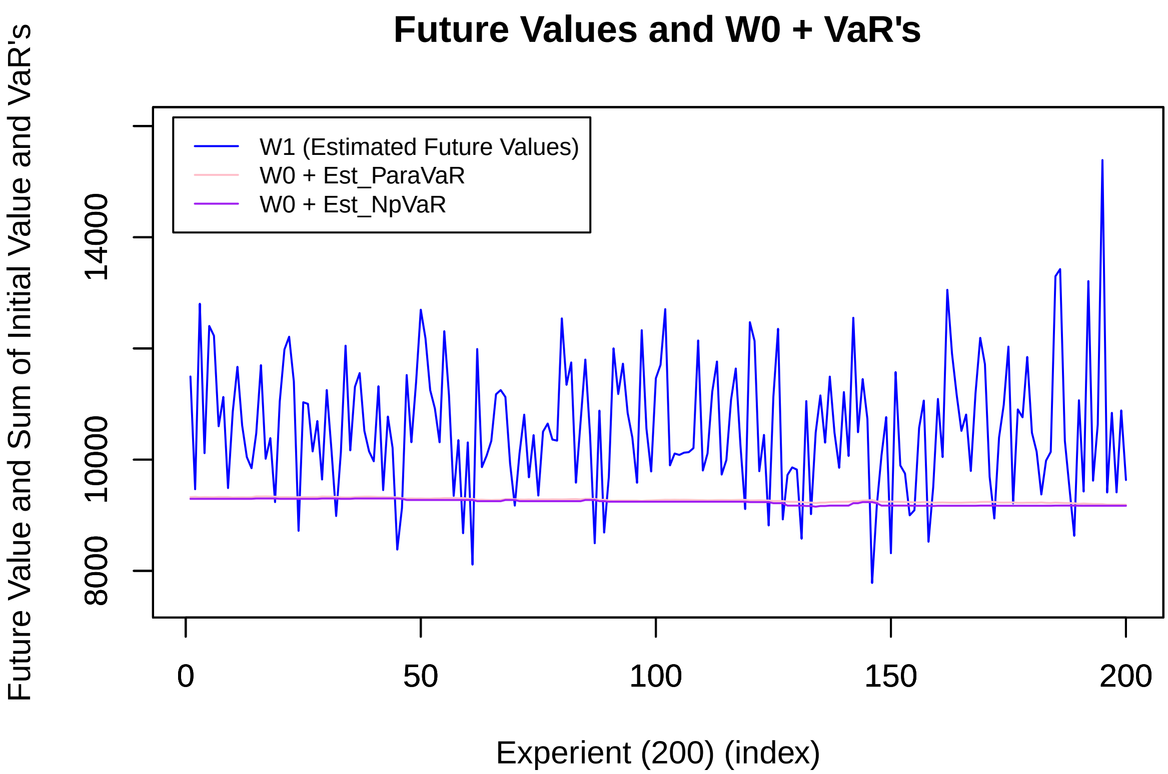
```
## [1] 0.115
```

Problem 10

Plot the future values of the investment from the trading date 301 to the trading date 500. Calculate the sum of $\$10,000 + \mathrm{VaR}_{0.1}$ for $\mathrm{VaR}_{0.1}$ being the parametric and nonparametric estimators respectively. Plot the future values, the two summations ($$10,000 + $VaR\_\{0.1\}$ for $\mathrm{VaR}_{0.1}$ being the parametric and nonparametric estimators) in the same picture. Explain what you find in the picture.

```
QW1 <- W0 + Out_VaR1
QW2 <- W0 + Out_VaR2

plot(1:N_Est,
     Out_W1,
     type = "l",
     col = "blue",
     xlab = "Experient (200) (index)",
     ylab = "Future Value and Sum of Initial Value and VaR's",
     xlim = range(c(1, N_Est)),
     ylim = range(c(7500, 16000)),
     main = "Future Values and W0 + VaR's")

par(new = TRUE)
```

```r
plot(1:N_Est,
     QW1,
     type = "l",
     col = "pink",
     xlab = "",
     ylab = "",
     xlim = range(c(1, N_Est)),
     ylim = range(c(7500, 16000)))

par(new = TRUE)

plot(1:N_Est,
     QW2,
     type = "l",
     col = "purple",
     xlab = "",
     ylab = "",
     xlim = range(c(1, N_Est)),
     ylim = range(c(7500, 16000)))

legend(x = "topleft",
       legend = c("W1 (Estimated Future Values)", "W0 + Est_ParaVaR", "W0 + Est_NpVaR"),
       lty = c(1, 1, 1),
       col = c("blue", "pink", "purple"),
       cex = 0.75,
       inset = 0.02)
```



Comment: This graph is almost the same as the plot we made in problem 9. Except the mean value (+W0), everything else is the same.