OPTIONS ON THE MINIMUM OR THE MAXIMUM OF TWO RISKY ASSETS

Analysis and Applications

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This paper provides analytical formulas for European put and call options on the minimum or the maximum of two risky assets. The properties of these formulas are discussed in detail. Options on the minimum or the maximum of two risky assets are useful to price a wide variety of contingent claims of interest to financial economists. Applications discussed in this paper include the valuation of foreign currency debt, option-bonds, compensation plans, risk-sharing contracts, secured debt and growth opportunities involving mutually exclusive investments.

1. Introduction

This paper provides formulas for (European) put and call options on the maximum or the minimum of two risky assets and discusses the properties of these options. This analysis provides an important tool because a wide variety of contingent claims of interest to financial economists have a payoff function which includes the payoff function of a put or a call option on the minimum or the maximum of two risky assets.

Options on the minimum of two risky assets enter the payoff function of some traded assets in a straightforward way. An important example of traded assets whose value depends on the value of an option on the minimum of two risky assets is given by option-bonds. Option-bonds are financial instruments which are primarily sold on the Euro-bond market.[1] On the Euro-bond market, when the issuer of an option-bond makes a

[1]For information on the Euro-bond market, see Park (1974) and Feiger and Jacquillat (1981). Fisher (1979) provides good information on the Euro-dollar bond market. Many Euro-bonds are listed on the Luxembourg, London or Singapore stock exchanges. *Euromoney* provides a description of new offerings in each of its monthly issues.

payment to the bearer, the bearer has a choice of two or more currencies in which the payment is to be made.[2] The exchange rate among these currencies is written in the indenture of the bond. For example, a discount option-bond could let the bearer choose at maturity between U.S. dollars and British pounds at a predetermined exchange rate of two dollars per pound. If the firm is solvent at maturity, the bearer chooses to be paid in pounds only if the price of a pound in dollars is larger than two dollars. However, at maturity, the firm may be able to pay the amount due in dollars but not the amount due in pounds.[3] In this case the bondholders get the whole firm in payment. In this example, the bearers of the option-bond hold a straight dollar bond plus an option on the minimum of the value of the firm and the dollar value of the amount to be paid in pounds. The exercise price of this option is equal to the amount to be paid in dollars.

Other examples of applications of options on the minimum or the maximum of two risky assets are given in the paper. It is shown that compensation plans, risk-sharing contracts, collateralized loans and secured debt, indexed wages and some growth opportunities are contingent claims whose payoff function includes the payoff function of options on the minimum or the maximum of two risky assets.

The plan of the paper is as follows. In section 2, an analytical solution for the pricing of a European call option on the minimum of two risky assets is obtained. The price of a European call option on the maximum of two risky assets can be obtained by using the price of an option on the minimum of two risky assets and the price of two conventional call options. This relationship between options on the minimum of two risky assets and options on the maximum of two risky assets is discussed in section 3. Other properties of options on the minimum or the maximum of two risky assets are also discussed in section 3. Some applications of options on the minimum or the maximum of two risky assets to the study of problems in the theory of corporate finance are discussed in section 4. Section 5 provides a summary of the results of the paper.

2. The pricing of a call option on the minimum of two risky assets

In this section, a European call option[4] on the minimum of two risky assets is priced. If V and H are the prices of two risky assets, the European

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[2] A growing number of option-bonds involve the choice of a commodity numeraire. E.g., the July 1981 issue of *Euromoney* describes a bond which offers a barrel of oil as a numeraire.

[3] Feiger and Jacquillat (1979) discuss the pricing of *default-free* option-bonds using an option-theoretic approach. They do not present an analytical formula and are, therefore, unable to shed much light on the properties of those bonds. Furthermore, in general, option-bonds are issued by firms and involve a default-risk.

[4] A review of early developments in the option pricing literature is given by Smith (1976). The seminal contributions to that literature are Black and Scholes (1973) and Merton (1973).

call to be priced has a payoff at maturity equal to $\max\{\min(V, H) - F, 0\}$, where F is the exercise price. The following assumptions are made:[5]

(A.1) Markets are 'frictionless', in the sense that there are no transactions costs, no taxes, no restrictions on short-sales and no difference between the borrowing rate and the lending rate. Trading takes place continuously.

(A.2) Prices V and H satisfy, respectively, the following stochastic differential equations:[6]

$$dV/V = \mu_V \, dt + \sigma_V \, dZ_V,\qquad (1)$$

$$dH/H = \mu_H \, dt + \sigma_H \, dZ_H.\qquad (2)$$

It is assumed that σ_V^2 and σ_H^2, which are, respectively, the instantaneous variances of the rates of return of assets V and H, are constant. μ_V and μ_H, i.e., the instantaneous expected rate of return of, respectively, assets V and H, can change through time, but must be such that (1) and (2) have solutions. dZ_V and dZ_H are standard Wiener processes whose coefficient of correlation is written ρ_{VH}. ρ_{VH} is assumed to be constant through time.

(A.3) The instantaneous rate of interest R is constant through time.

Let $M(V, H, F, T-t)$ be the price of a European call option on $\min(V, H)$ with maturity at date T and exercise price F. To find M, it is sufficient to find the value of a self-financing portfolio whose value at date T is equal to the value of the option at date T.[7] If such a portfolio can be found, its value at date t must be equal to the value of the option at date t, $\forall t \leq T$, to prevent the possibility of arbitrage profits. Let τ be equal to $T-t$, i.e., the time to maturity of the option, and P be equal to the value of the self-financing portfolio. With the assumptions stated in (A.1)–(A.3), it is natural to conjecture that P is a function of V, H and τ only, i.e., $P = P(V, H, \tau)$. Using Ito's Formula, the dynamics for P can be obtained.

$$dP = P_V \, dV + P_H \, dH - P_\tau \, dt$$

$$+ \tfrac{1}{2}\{P_{VV} V^2 \sigma_V^2 + P_{HH} H^2 \sigma_H^2 + 2P_{VH} VH \rho_{VH} \sigma_V \sigma_H\} \, dt.\qquad (3)$$

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[5]Some of these assumptions could be relaxed. In particular, it would be possible to let the interest rate change over time in the same way as in Merton (1973). The added complexity would not add significant insights to the present paper.

[6]See Merton (1971) for an introduction to those equations.

[7]See Harrison and Kreps (1979) for a rigorous discussion of self-financing portfolios. A self-financing portfolio is a portfolio which does not yield or does not require any cash payments until its maturity. Dynamics for the value of a self-financing portfolio are given in Merton (1971).

If the portfolio is self-financing and if it consists of investments in V, H and the safe asset, its dynamics can also be written

$$dP = x(dV/V)P + y(dH/H)P + (1 - x - y)RP\,dt, \tag{4}$$

where x is the fraction of the portfolio invested in asset V and y is the fraction of the portfolio invested in asset H. x and y are functions of V, H and τ. Setting (4) equal to (3), it follows that x and y must satisfy at each point in time

$$P_V V = xP, \tag{5}$$

$$P_H H = yP. \tag{6}$$

Using (4), (5) and (6), the stochastic terms in (4) can be eliminated to yield (after dividing by dt)

$$-P_\tau = RP - RP_V V - RP_H H$$

$$-\tfrac{1}{2}\{P_{VV}V^2\sigma_V^2 + P_{HH}H^2\sigma_H^2 + 2P_{HV}VH\rho_{VH}\sigma_V\sigma_H\}. \tag{7}$$

The portfolio whose value is P is self-financing if it satisfies the partial differential equation given by (7). For P to be equal to the value of the option, the self-financing portfolio must satisfy the boundary conditions specified in the contract which defines the option,

$$P(V, H, 0) = \max\{\min(V, H) - F, 0\}, \tag{8}$$

$$P(0, H, \tau) = 0, \tag{9}$$

$$P(V, 0, \tau) = 0. \tag{10}$$

To obtain a solution to (7) which satisfies (8), (9) and (10), note that (7) does not depend on the instantaneous expected rate of return of V and H. This means that the value of the option does not depend on the attitude of investors towards risk. It is therefore possible to assume that investors are risk-neutral without thereby changing the value of the option, provided that the dynamics of asset prices are adjusted so that all assets have the same expected instantaneous rate of return, i.e., R, as they would in a risk-neutral world. The value of the call option today is, consequently, equal to its expected value at maturity discounted at the interest rate R, where the expectation is taken with respect to risk-neutral dynamics for assets V and H. Using this approach to find a solution to (7) which satisfies (8), (9) and (10), it

follows that the value of the option is given by[8]

$$M = HN_2(\gamma_1 + \sigma_H\sqrt{\tau}, (\ln(V/H) - \tfrac{1}{2}\sigma^2\sqrt{\tau})/\sigma\sqrt{\tau}, (\rho_{VH}\sigma_V - \sigma_H)/\sigma)$$

$$+ VN_2(\gamma_2 + \sigma_V\sqrt{\tau}, (\ln(H/V) - \tfrac{1}{2}\sigma^2\sqrt{\tau})/\sigma\sqrt{\tau}, (\rho_{VH}\sigma_H - \sigma_V)/\sigma)$$

$$- Fe^{-R\tau}N_2(\gamma_1, \gamma_2, \rho_{VH}), \tag{11}$$

where $N_2(\alpha, \beta, \theta)$ is the bivariate cumulative standard normal distribution with upper limits of integration α and β, and coefficient of correlation θ, and

$$\gamma_1 = (\ln(H/F) + (R - \tfrac{1}{2}\sigma_H^2)\tau)/\sigma_H\sqrt{\tau},$$

$$\gamma_2 = (\ln(V/F) + (R - \tfrac{1}{2}\sigma_V^2)\tau)/\sigma_V\sqrt{\tau},$$

$$\sigma^2 = \sigma_V^2 + \sigma_H^2 - 2\rho_{VH}\sigma_V\sigma_H.$$

It can be verified by substitution that (11) satisfies the partial differential equation given by (7) and its boundary conditions. Appendix 1 sketches the derivation of (11).

If the exercise price of the option on the minimum of two risky assets is equal to zero, the formula for the price of the option takes a simpler form than the formula given by eq. (11). Let $E(V, H, 1, \tau)$ be the price of an option to exchange one unit of asset H for one unit of asset V at maturity. Margrabe (1978) prices such an option. Using his results, it follows that

$$M(V, H, 0, \tau) = V - E(V, H, 1, \tau) = V - VN(d_1) + HN(d_2), \tag{11'}$$

where

$$d_1 = (\ln(V/H) + \tfrac{1}{2}\sigma^2\tau)/\sigma\sqrt{\tau},$$

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

To prove that eq. (11') is correct, remember that the payoff of an option to exchange one unit of asset H for one unit of asset V is equal to max $\{V - H, 0\}$. Let W be the value of a portfolio which consists of holding one unit of asset V and writing one option to exchange one unit of asset H for one unit of asset V at maturity $T = t + \tau$. The value of portfolio W at maturity if V is greater than H is $V - (V - H) = H$. The value of a call option on min(V, H) with an exercise price of zero and maturity at date $T = t + \tau$ is also equal to

[8]The method used here is presented in Cox and Ross (1976) and in Harrison and Kreps (1979).

H at maturity if V is greater than H at that time. If V is smaller than H at maturity, the portfolio W is worth V at that date and $M(V, H, 0, 0) = V$. This completes the proof.

3. Properties of call options on the minimum of two risky assets

This section presents comparative static results for call options on the minimum of two risky assets and shows that call options on the minimum of two risky assets can be used to price other options. The comparative static results can be extended in a natural way to the other options priced in this section. The following compact notation is adopted for the price of a call option on the minimum of two risky assets V and H, with exercise price F, and time to maturity τ:

$$M(V, H, F, \tau) = HN_2(\alpha_1, \alpha_2, \rho_c) + VN_2(\beta_1, \beta_2, \rho_C)$$
$$- F e^{-R\tau} N_2(\gamma_1, \gamma_2, \rho_{VH}).$$

3.1. Some parity relationships

From the price of a call option on the minimum of two risky assets and the price of assets V, H and the interest rate R, it is possible to obtain the price of a call option on the maximum of two risky assets, the price of a put option on the minimum of two risky assets and the price of a put option on the maximum of two risky assets.

3.1.1. Pricing a call option on the maximum of two risky assets

Let $MX(V, H, F, \tau)$ be the price of a European call option whose pay-off at maturity is given by

$$\max\{\max(V, H) - F, 0\}.$$

The value of this option is

$$MX(V, H, F, \tau) = C(V, F, \tau) + C(H, F, \tau) - M(V, H, F, \tau), \tag{12}$$

where $C(A, F, \tau)$ is a European call option on asset A with exercise price F and time to expiration τ. To verify this result, note that if V, $V \geq F$, is the maximum of V and H at maturity, the option on the maximum of two risky assets pays $V - F$. A portfolio which consists of holding a call option on V and a call option on H, and writing a call option on the minimum of V and H (with exercise price F for each of the three call options) pays $V - F$ also, as $M(V, H, F, 0)$ is equal to $C(H, F, 0)$. A similar reasoning holds if H is the maximum of V and H.

3.1.2. Pricing a put option on the minimum of two risky assets

If $PM(V, H, F, \tau)$ is the price of a European put option on the minimum of V and H with exercise price F and time to maturity τ, then

$$PM(V, H, F, \tau) = e^{-R\tau}F - M(V, H, 0, \tau) + M(V, H, F, \tau). \tag{13}$$

To verify this result consider the following two investments:

Portfolio A. Purchase one put option on the minimum of V and H with exercise price F.

Portfolio B. Purchase one discount bond which pays F at maturity. Write one option on the minimum of V and H with an exercise price of zero.
Purchase one option on the minimum of V and H with exercise price F.

At maturity, if $\min(V, H) \geqq F$, Portfolio A pays zero whereas Portfolio B pays $F - \min(V, H) + \min(V, H) - F = 0$. If $\min(V, H) = V < F$, then Portfolio A pays $F - V$, whereas Portfolio B pays $F - V + 0 = F - V$. If $\min(V, H) = H < F$, then Portfolio A pays $F - H$, whereas Portfolio B pays $F - H + 0 = F - H$. It follows that in all states of the world, Portfolio A pays the same as Portfolio B at maturity and therefore must have the same value as Portfolio B at time $T - \tau$. This completes the proof that a European put option on the minimum of two risky assets is indeed priced as in eq. (13).

3.1.3. Pricing a put option on the maximum of two risky assets

If $PX(V, H, F, \tau)$ is a European put option on the maximum of assets V and H with exercise price F and time to maturity τ, then $PX(V, H, F, \tau)$ is given by

$$PX(V, H, F, \tau) = e^{-R\tau}F - MX(V, H, 0, \tau) + MX(V, H, F, \tau). \tag{14}$$

The proof of this result is similar to the proof for the pricing of a put option on the minimum of two risky assets.

3.2. A simple dominance result

It is possible to find upper and lower bounds for the option on the minimum of two risky assets which are easy to compute. Let $E(B, A, 1, \tau)$ be an option to exchange one unit of asset A for one unit of asset B at maturity

of the option. The following result holds:

$$\min\{C(V,F,\tau), C(H,F,\tau)\} \geqq M(V,H,F,\tau)$$

$$\geqq \max\{C(V,F,\tau) - E(V,H,1,\tau)$$

$$C(H,F,\tau) - E(H,V,1,\tau), 0\}.$$

Clearly, there is no state of the world in which a European call option on the minimum of two risky assets has a payoff which exceeds the payoff of a call option on any of the two risky assets. If $CE(H,V,1,F,\tau)$ is a call option to exchange one unit of asset V for one unit of asset H if and only if $V \geqq F$, then a portfolio which consists of holding one call option $C(H,F,\tau)$ and writing one call option $CE(H,V,1,F,\tau)$ has payoffs which match exactly the payoffs of $E(H,V,F,\tau)$. Similarly, a portfolio which consists of holding one call option $C(V,F,\tau)$ and writing one call option $CE(H,V,1,F,\tau)$ has payoffs which match exactly the payoffs of $M(V,H,F,\tau)$. However, it must be true that

$$E(H,V,1,\tau) \geqq CE(H,V,1,F,\tau),$$

$$E(H,V,1,\tau) \geqq CE(H,V,1,F,\tau).$$

This completes the proof of the dominance result.

Notice that the Black and Scholes (1973) formula can be used to compute $C(V,F,\tau)$ and $C(H,F,\tau)$. Margrabe (1978) has shown that $E(H,V,1,\tau)$ is given by $HC(V/H,1,\tau)$, with $R=0$. The instantaneous variance of the rate of return on V/H is $\sigma_V^2 + \sigma_H^2 - 2\rho_{VH}\sigma_V\sigma_H$. Note finally that if $F=0$, then $C(V,F,\tau)=V$, $CE(H,V,1,F,\tau) = E(H,V,1,\tau)$ and $M(V,H,F,\tau) = H - E(H,V,1,\tau)$.

3.3. Comparative statics of the price of the call option on the minimum of two risky assets with respect to V, H, F and R

To simplify the analysis, it is useful to define two artificial assets. Let $A(V,H,\tau)$ be the value at time t of receiving the larger of H or V at time $T = t + \tau$. Furthermore, let $a(V,H,\tau)$ be the value at time t of receiving the smaller of H or V at time $T = t + \tau$. It follows that $A(H,V,\tau) = H + E(V,H,1,\tau)$ and $a(H,V,\tau) = H - E(V,H,1,\tau)$. A call option on the minimum of two risky assets with an exercise price F is a call option on asset $a(H,V,\tau)$ with the same exercise price, i.e., $M(V,H,F,\tau) = C(a,F,\tau)$. [Note also that $MX(V,H,F,\tau) = C(A,F,\tau)$.] Since A and a can be treated as assets, it is possible to use

Merton's (1973) distribution-free results to obtain

$$\partial M/\partial H = C_a(\partial a/\partial H) > 0, \tag{15}$$

$$\partial M/\partial V = C_a(\partial a/\partial V) > 0, \tag{16}$$

$$\partial M/\partial F = C_F < 0. \tag{17}$$

Merton (1973) shows that $C_a > 0$ and $C_F < 0$. $\partial a/\partial H$ is positive because an asset which pays $\min(V, \lambda H)$ at maturity T, with $\lambda > 1$, must be worth more than an asset which pays $\min(V, H)$ at the same maturity, provided that the probability that $\min(V, H) = H < V$ is positive. Eq. (11') shows that the value of $a(V, H, \tau)$ does not depend on the interest rate R. This implies that Merton's (1973) result that the value of a call option is an increasing function of the rate of interest applies here in a straightforward way. It also implies that the value of a call option on the minimum of two risky assets is an increasing function of the rate of interest. Appendix 2 verifies these results for the formula developed in this paper. These results also hold for an option on the maximum of two risky assets.

3.4. Comparative statics of the price of the call option on the minimum of two risky assets with respect to σ_V^2, σ_H^2, ρ_{VH} and τ

3.4.1. Effect of a change in σ_V^2 and σ_H^2

Black and Scholes (1973) show that an increase in the instantaneous variance of the risky asset always increases the price of the option. In the case of an option on the minimum of two risky assets, it is not true, however, that an increase in the instantaneous variance of one of the risky assets always increases the price of the option. The sign of the partial derivative of the option price with respect to the variance of either one of the risky assets is ambiguous.

The ambiguity in the sign of this partial derivative arises from the fact that an increase in the instantaneous variance of a risky asset's return can either increase or decrease the expected payoff of the option, assuming that the expectation is taken with respect to risk-neutral price dynamics. The following argument supports this point. Suppose that for large (small) values of H, the probability that V is larger (smaller) than H at maturity is very small. An increase in the instantaneous variance of the return of asset H puts more weight in the tails of the distribution of H at maturity. If the expected value of H at maturity is higher than F, an increase in the instantaneous variance of the return of H simultaneously increases the probability that H is smaller than F at maturity and increases the probability of large values of H. The increase in the probability that H is smaller than F at maturity

decreases the expected payoff of the option. The increase in the probability of H taking a large value at maturity has a negligible effect on the expected payoff of the option, because it is assumed that the probability that a large value of H is equal to $\min(V, H)$ is very small and that the distribution of V is left unchanged. It follows that an increase in the instantaneous variance of the return of asset H decreases the value of the option. An increase in the instantaneous variance of the return of asset V simultaneously increases the probability of a large payoff for the option at maturity and increases the probability that V is smaller than F at maturity. The increase in the probability of low values of V does not affect the expected payoff of the option very much as the probability that V is smaller than H, for low values of H, is very small by assumption. It follows that an increase in the instantaneous variance of the return of asset V increases the value of the option.

An example may be useful to understand the above discussion. Suppose that the option has an exercise price equal to zero. It has been shown earlier that in this case:

$$M(V, H, 0, \tau) = H - E(H, V, 1, \tau). \tag{18}$$

It immediately follows that

$$\frac{\partial M(V, H, 0, \tau)}{\partial \sigma_V^2} = -H\, e^{-R\tau} N_1'\left(\frac{\ln(V/H) + (R - \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) * \frac{\sqrt{\tau}}{\sigma}\left(1 - \rho_{VH}\frac{\sigma_V}{\sigma_H}\right),$$

$$\tag{19}$$

where $\sigma^2 = (\sigma_V^2 + \sigma_H^2 - 2\rho_{VH}\sigma_V\sigma_H)$ and $N_1'(Q)$ is the derivative of the cumulative standard normal distribution $N_1(Q)$ with respect to Q. The partial derivative given by (19) is negative when $\sigma_H > \rho_{VH}\sigma_V$. If (a) the instantaneous variance of H is large, (b) the instantaneous variance of V is small, and (c) H and V are positively correlated, an increase in σ_V^2 can make the payoffs of V 'closer' to those of H than they were before in most states of the world and consequently decrease the expected payoff of exchanging one unit of asset V for one unit of asset H.

Merton (1973) shows that an increase in the Rothschild–Stiglitz measure of risk of asset a implies an increase in the value of a call option on asset a, irrespectively of the distribution of the return on asset a. It follows from this result that an increase in the Rothschild–Stiglitz measure of risk of asset $a(V, H, \tau)$ increases the value of the option on the minimum of two risky assets. Similarly, an increase in the Rothschild–Stiglitz measure of risk of the artificial asset $A(V, H, \tau)$ increases the value of the option on the maximum of two risky assets.

3.4.2. Effect of a change in ρ_{VH}

The partial derivative of the price of the option with respect to the coefficient of correlation between the two risky assets is given by

$$\partial C/\partial \rho_{VH} = H N_1\left(\frac{\alpha_1 - \rho_c \alpha_2}{\sqrt{1-\rho_c^2}}\right)\frac{e^{-\frac{1}{2}\alpha_2^2}}{\sqrt{2\pi}}\left[\frac{\sigma_V \sigma_H \sqrt{\tau}}{\sigma} + \frac{\sigma_V \sigma_H}{\sigma^2}\alpha_2\right]$$

$$+ V N_1\left(\frac{\beta_1 - \rho_c \beta_2}{\sqrt{1-\rho_c^2}}\right)\frac{e^{-\frac{1}{2}\beta_2^2}}{\sqrt{2\pi}}\left[\frac{\sigma_V \sigma_H \sqrt{\tau}}{\sigma} + \frac{\sigma_V \sigma_H}{\sigma^2}\beta_2\right]. \qquad (20)$$

It is proved in the appendix 3 that (20) is positive everywhere for $-1 \leqq \rho_{VH} \leqq 1$. This means that the value of the option is the highest if the two risky assets have a correlation coefficient of one.[9] Intuitively, as ρ_{VH} increases, the probability that the payoff of V will be 'close' to the payoff of H increases. This implies that V is more likely to be high when H is high. It follows that the probability of asset H being above F when asset V is above F increases. Consequently, the expected payoff of the option is higher.

Contrary to the option studied by Black and Scholes (1973), it is possible for an option on the minimum of two risky assets to be worthless when the two risky assets have a positive value. Indeed, if $\gamma_1 + \gamma_2 \leqq 0$, the option is worthless for $\rho_{VH} = -1$.[10] To understand this, suppose that $R = \frac{1}{2}\sigma_H^2 = \frac{1}{2}\sigma_V^2$, with $\gamma_1 < 0$ and $\gamma_2 < 0$. It is obviously true that the sum of γ_1 and γ_2 is negative with these assumptions. These assumptions also imply that $H < F$ and $V < F$. Because $\rho_{VH} = -1$, if $V > F$ at maturity, it must be true that $H < F$. It follows that since the option is worthless, it is never possible for both H and V to be larger than the exercise price F.

3.4.3. Effect of a change in τ

Merton (1973) shows that the value of a call option on an asset which pays no dividends is an increasing function of time to maturity, irrespectively of the distribution of the return on the optioned asset. The price of the artificial asset $a(V, H, \tau)$ is a decreasing function of the time to maturity, as it is equal to H minus V times the price of a European call option. It follows therefore that

$$\partial M/\partial \tau = C_a(\partial a/\partial \tau) + C_\tau \gtrless 0.$$

To understand why the value of asset $a(V, H, \tau)$ is a decreasing function of time to maturity, it is useful to notice that $a(V, H, \tau) < \min(V, H)$. This is true

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[9]If $\sigma_V = \sigma_H$, (17) may be indefinite. A dominance argument or taking the appropriate limits verifies the claim in the text.

[10]Abramowitz and Stegun (1970) show that $N_2(a, b, -1) = 0$ for $a + b \leqq 0$.

because it is always possible to buy the asset whose price is the lowest at time $T-\tau$, hold it until time T, and sell it at time T for a price which is at least as high as the price of asset $a(V,H,\tau)$ at maturity of this asset. At time $\tau=0$, however, it must be true that $a(V,H,0)=\min(V,H)$. It follows that the price of asset $a(V,H,\tau)$ must be a decreasing function of time to maturity. The partial derivative of the price of the call option on the minimum of two risky assets given by the formula developed in this paper is lengthy and is not reproduced here. Its sign is ambiguous.

Whereas the price of asset $a(V,H,\tau)$ is a decreasing function of time to maturity, the price of asset $A(V,H,\tau)$ is an increasing function of time to maturity. [Notice that the price of asset $A(V,H,\tau)$, i.e., of the asset which pays the maximum of V and H at maturity, is equal to H plus V times the price of a European call option.] Because the price of asset $A(V,H,\tau)$ is an increasing function of time to maturity, it follows that the value of an option on the maximum of two risky assets is unambiguously an increasing function of time to maturity, irrespectively of the distribution of the returns of the two risky assets.

One important implication of this discussion is that the price of a European call option on the minimum of two risky assets is not in general the same as the price of an American call option on the minimum of two risky assets. The formula for the price of a call option on the minimum of two risky assets developed here is correct, in general, only for a European call option on the minimum of two risky assets.

3.5. A comparison with other option pricing formulas

It has already been shown that the formula for the pricing of a call option on the minimum of two risky assets reduces to a formula which involves only the univariate normal distribution in some special cases. In particular, an option on the minimum of two risky assets whose exercise price is equal to zero has a price given by $M(V,H,0,\tau)=V-E(V,H,1,\tau)$, where $E(V,H,1,\tau)$ is the price of an option to exchange one unit of asset H for one unit of asset V at date $T=t+\tau$ and is given in eq. (11').

In general, an option on the minimum of two risky assets is an option on $M(V,H,0,\tau)$ with an exercise price equal to F. Geske (1979) has priced an option on an option. The optioned asset in the case of Geske's formula is a call option on the value of the firm with maturity at date $T^*\geq T$, where T is the maturity of the option he prices. The problem Geske addresses involves pricing the promise of a payment at date T of an option on an asset whose price is V, with maturity at date T^*, $T^*\geq T$, when the promise is contingent on V being larger than some number $\bar{V}$ at date T. The problem addressed in this paper involves pricing the promise of a payment at date T of an option on an asset V with exercise price F and maturity at date T, when the promise

is contingent on V being smaller than some variable H at date T. An extension of the present analysis would be to price the promise of a payment at date T of an option on an asset V with exercise price F and maturity at date T^*, when the promise is contingent on V being smaller than some variable H at date T. This extension would lead to a formula which would reduce either to the formula developed by Geske or to the formula developed here in some special cases.

4. Applications

In this section, the results obtained earlier are used to price a variety of contingent claims. The section is divided into two parts. The first part analyses the pricing of bonds involving one or more foreign currencies. These bonds could be used for empirical tests of the results of this paper. The second part of this section shows that the formulas developed earlier can be usefully applied to a wide range of problems in the theory of corporate finance.

4.1. Pricing bonds whose value depends on exchange rates

The most interesting question about the bonds priced here is how their value depends on the stochastic properties of the risky assets which affect their payoff at maturity. The fairly restrictive assumptions made in this section make it possible to focus the discussion on this question. It is assumed that the term structure of interest rates is flat, that the instantaneous rate of return on the safe asset is constant, and that international markets are frictionless.[11,12] Furthermore, only discount bonds are priced.

4.1.1. Foreign currency bonds

For simplicity, define a foreign currency bond as a bond denominated in a different currency from the currency in which the common stock of the firm is traded. A foreign currency bond is the easiest option-bond to price and therefore is useful as an introduction to a discussion of the pricing of more

[11]It is important to notice that 'fixed but adjustable' exchange rates do not have a continuous sample path. It follows that the discussion of this section is more relevant for a regime of flexible exchange rates than for the exchange rate regime which prevailed before 1973.

[12]Schwartz (1982) prices bonds which have a payoff similar to the payoff of the bonds priced here, except that the value of the bonds depends on the price of commodities instead of the price of foreign currencies. The formula obtained in this paper could be used to price the bonds studied by Schwartz (1982), but one should take into account the fact that commodities are not assets, whereas default-free foreign currency bonds are assets. Ingersoll (1982) also points out that stocks of commodities have a convenience yield and that the assumptions required to use option-pricing techniques may be too strong to provide a realistic characterization of commodity markets.

complex bonds. Let V be the value of the firm in terms of the domestic currency, i.e., the currency in which the common stock is traded, and let F^* be the face value of the bond in foreign currency. Define x to be the domestic price of one unit of foreign currency. The payoff in domestic currency of the bond at maturity is

$$\min (V, xF^*),$$

where xF^* is stochastic. The value at date t of a discount bond which pays xF^* at date T is equal to $x(t)e^{-R^*(T-t)}F^*$, where $x(t)$ is the exchange rate at date t and R^* is the instantaneous rate of return of the safe asset in foreign currency. Let $H(t) = x(t)e^{-R^*\tau}F^*$ and $B^*(V/x, F^*, \tau)$ be the value of the foreign currency bond in foreign currency. It can be verified that

$$B^*(V/x, F^*, \tau) = (1/x)M(V, H, 0, \tau). \tag{21}$$

In section 3, $M(V, H, 0, \tau)$ is evaluated. It is stated that $M(V, H, 0, \tau)$ can be obtained by computing the value of a conventional European call option, with V/H as the stock price, 1 as the exercise price, $R = 0$, and an instantaneous variance given by $\sigma^2 = \sigma_H^2 + \sigma_V^2 - 2\rho_{VH}\sigma_V\sigma_H$.[13]

$M(V, H, 0, \tau)$ is a decreasing function of σ^2. Therefore a bond whose face value is denominated in foreign currency is a decreasing function of the standard deviation of the instantaneous percentage rate of change of the value of the firm and of the value of the exchange rate (holding constant the covariance between the instantaneous percentage rate of change of the value of firm and of the exchange rate). Furthermore, the value of the bond is an increasing function of the covariance between the instantaneous percentage rate of change of the value of the firm and of the exchange rate.

H can be shown to depend on the forward exchange rate. Let $X(t, T)$ be the forward exchange rate at date t on a contract maturing at date T. The interest rate parity theorem[14] states that

$$X(t, T) = x(t)e^{(R-R^*)\tau}. \tag{22}$$

Note that (22) just means that the price today of a dollar to be delivered at date T must be the same whether that dollar is invested at home in a default-free bond or abroad in a default-free bond with the proceeds at maturity of the foreign investment sold on the forward exchange market. H

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[13]Fischer (1978) prices indexed-bonds using the formula for the valuation of an option to exchange one asset for another. Note that one can also price $xB^*(V/x, F^*, \tau)$ by using the fact that $B^*(V/x, F^*, \tau) = (1/x)V - C(V/x, F^*, \tau)$.

[14]See Officer and Willet (1970). Frenkel and Levich (1977), among others, show that interest rate parity holds well on Euro-markets.

can now be rewritten as

$$H = X(t, T)e^{-R\tau}F^*. \tag{23}$$

It follows that, *ceteris paribus*, the higher the forward exchange rate of the foreign currency, the higher the value of the bond. This also means that, *ceteris paribus*, the higher the expected future spot exchange rate, the higher the value of the bond.[15] To understand this result, it is useful to think about the case in which changes in the exchange rate are not stochastic. In this case, the bond is equivalent to a bond in domestic currency with an exercise price equal to $X(t, T)F^*$, where $X(t, T)F^*$ is not stochastic. Any increase in $X(t, T)$ means an increase in the face value of the bond in domestic currency. Introducing uncertainty does not change that fact.

4.1.2. Default-free option-bonds

Feiger and Jacquillat (1979) discuss the pricing of default-free option-bonds. The analytical formulas developed in this paper make it possible to price default-free option-bonds which let the bearer choose among payments in three different currencies. Let $B(x_A F_A^*, x_B F_B^*, F, \tau)$ be the price of a discount bond which at maturity pays, at the choice of the bearer, either F units of domestic currency, F_A^* units of currency of country A or F_B^* units of currency of country B. x_A (x_B) is the current price of one unit of currency of country A (B). The payoffs of the option-bond are equal to the payoffs of a portfolio that consists of holding one safe domestic discount bond, one option on $x_A F_A^*$ with exercise price F, one option on $x_B F_B^*$ with exercise price F, and of writing an option on the minimum of $x_A F_A^*$ and $x_B F_B^*$ with the same exercise price, F. Let H be the value today of an asset which will be worth $x_A F_A^*$ at maturity and let V be the value today of an asset which will be worth $x_B F_B^*$ at maturity. If R_A^* and R_B^* are respectively the instantaneous rates of return of the safe nominal asset in country A and B, then

$$H = x_A e^{-R_A^* \tau} F_A^*, \tag{24}$$

$$V = x_B e^{-R_B^* \tau} F_B^*. \tag{25}$$

With this notation,

$$B(x_A F_A^*, x_B F_B^*, F, \tau) = MX(V, H, F, \tau) + e^{-R\tau}F, \tag{26}$$

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[15]Stulz (1981) discusses the relationship between the forward exchange rate and the future spot exchange rate.

where $MX(V,H,F,\tau)$ is the price of an option on the maximum of V and H with exercise price F and time to maturity τ given in section 3. The default-free option-bond is a decreasing function of the coefficient of correlation between the two exchange rates. This is due to the fact that a call option on the maximum of two risky assets is the least valuable when those risky assets have a correlation coefficient of one. An option on the maximum of two risky assets will always pay at least as much as an option on the minimum of two risky assets, but the only case in which the options are the same is when $V=H$, $\sigma_V=\sigma_H$, $\rho_{VH}=1$, i.e., when the two risky assets are perfect substitutes.

4.1.3. Option-bonds issued by firms

Foreign currency bonds are the risky assets which are the most popular among the risky assets involving foreign currencies priced here. However, currency option-bonds issued by firms are the most interesting. For instance, very little — if anything — is known about why such bonds exist. While an analysis of why such bonds exist is beyond the purpose of this paper, knowing how these bonds are priced could be helpful for such an analysis.

Define $B(V,xF^*,F,\tau)$ to be the price of an option-bond issued by a firm whose value is V, which promises to pay at maturity either F^* units of the foreign currency or F units of the domestic currency, at the choice of the bearer of the bond. To reproduce the payoff of such a bond, one can form a portfolio which consists of holding one safe discount bond with face value F, one option on the minimum of V and H, where $H=xe^{-R^*\tau}F^*$, with exercise price F, and of writing a put on V with exercise price F. Let $P(V,F,\tau)$ be the price of a European put option on V with exercise price F. The price of the bond is given by

$$B(V,F,xF^*,\tau)=F\,e^{-R\tau}-P(V,F,\tau)+M(V,H,F,\tau). \qquad (27)$$

At maturity, if the value of the firm is smaller than F, the currency option included in the bond is irrelevant, since the payment to the bondholder cannot be more than the value of the firm. If, at maturity, $V>xF^*>F$, the option on the minimum of V and H pays xF^*-F, whereas it pays zero if $V>F>xF^*$. The total payoff of the portfolio on the right-hand side of (27) or of the option-bond is, in those cases, either xF^* or F. Finally, if $xF^*>V>F$, the bondholder receives V and the holder of the portfolio receives $F+\{\min(V,H)-F\}=V$.

The value of the option-bond is an increasing function of V, H, F and ρ_{VH}. This means that, *ceteris paribus*, an investor is always better off buying an option-bond of a firm whose value is positively as opposed to negatively correlated with the exchange rate when the bonds have identical prices. The fact that the value of the option-bond is an increasing function of H means

that, *ceteris paribus*, the higher the forward exchange rate with respect to the spot exchange rate the more valuable is the option-bond.

4.2. Some other applications in the theory of corporate finance

This section illustrates a number of other applications of the previous analysis to problems in corporate finance. The goal of this section is to emphasize the usefulness of the results developed earlier rather than to provide a complete analysis of these problems.

4.2.1. Incentive and/or risk-sharing contracts

Many contracts entered into by corporations and individuals involve the promise of a payment at date T of a fixed fee and of a variable amount which depends on the realization of some random variable Z at date T. A special case of such contracts occurs when the total promised payment at date T is equal to $F + \max\{\lambda Z - F, 0\}$. If Z is equal to the value of the firm, the payoff of the contract, at date T, written $Q(V, Z, F, 0)$, is equal to the payoff at that date of a discount bond issued by the firm with face value F and maturity T and of λ times a call option on the value of the firm with exercise price $(1/\lambda)F$ and maturity T. However, in general, such contracts have a payoff at maturity which depends on a random variable which is not perfectly correlated with the value of the firm. If the firm has no debt except for the promised payment considered here and pays no dividends until date T, the creditor of the firm receives at maturity either the promised payment or the value of the firm, V. It follows that if V and Z are jointly lognormally distributed and correspond to the prices of traded assets,[16] the formula for the pricing of an option on the minimum of two risky assets can be used to find $Q(V, \lambda Z, F, \tau)$, i.e., the value of the contract at date $t = T - \tau$,

$$Q(V, \lambda Z, F, \tau) = F e^{-R\tau} - P(V, F, \tau) + M(V, \lambda Z, F, \tau), \tag{28}$$

where $P(V, F, \tau)$ is the price of a European put option on V with exercise price F and maturity T.

Examples of contracts in which Z corresponds to the price of a traded asset can be given. In a number of countries, there exist default-free index-bonds, i.e., bonds whose real return is not affected by unanticipated changes in the price-level. (Israel is such a country.) In such a country, a wage

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[16]The assumption that Z corresponds to the price of a traded asset is made to simplify the discussion, so that it can be focused on the usefulness of the formula for the pricing of an option on the minimum of two risky assets. It is possible to price assets in which Z does not correspond to the price of a traded asset. Cox, Ingersoll and Ross (1978) develop a methodology which can be used to price such assets and Dothan and Williams (1980) show how such a methodology can be used for capital budgeting purposes.

contract with a cost-of-living adjustment clause (COLA) can be valued using the formula for an option on the minimum of two risky assets. Let P be the price-level. COLAs generally imply that the nominal wage, F, is adjusted by an amount proportional to $\max\{P - \bar{P}, 0\}$, where $\bar{P}$ is some price-level chosen at the time the worker enters into the contract. If the value of the firm is lower than the promised payment, the worker receives the value of the firm. If $W(V, P, F, 0)$ is the payment made at date T and if this payment is the last and only payment the worker receives, it follows that

$$W(V, P, F, 0) = F - \max\{F - V, 0\} + \max\{\min(V, \delta FP) - F, 0\}, \quad (29)$$

where δ is a coefficient determined in the wage contract and V is the value of the firm. As, by assumption, there exist traded assets whose value at date T is equal, respectively, to V and δFP, $W(V, P, F, \tau)$ can be computed using eq. (29) with H equal to the current value of a discount bond which pays δFP at date T.[17]

An interesting implication of this example is that the value of a COLA is an increasing function of the coefficient of correlation between the dynamics of the value of the firm and the dynamics of the price-level. Furthermore, an increase in the variance of the rate of inflation could decrease the value of a COLA. In general, one would expect an increase in the variance of the rate of growth of the firm to decrease the value of the wage contract.

A COLA is a risk-sharing arrangement between an employer and an employee. There exists a wide variety of risk-sharing arrangements whose payoff at maturity takes the same form as the payoff at maturity of a wage contract with a COLA. For instance, many contracts for the delivery of some commodity at some future date involve a risk-sharing arrangement similar to a COLA, except that the price-level P is replaced by the spot price of the commodity to be delivered. Many contracts whereby a firm produces a commodity to order for a buyer involve the payment at some date T^*, of a fixed amount, and of a variable amount which depends on the costs of some inputs, if these costs exceed some given number specified in the contract.[18] It follows from this discussion that potentially a wide variety of risk-sharing contracts could be priced by using the formula for an option on the minimum of two risky assets.

Compensation plans for managers often have a payoff at maturity similar to the payoff at maturity of a wage contract with a COLA.[19] As an example,

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[17]It is also necessary that H and V are jointly lognormally distributed, that the firm pays no dividends until date T and has no debt of higher or equal priority.

[18]See Rodriguez and Carter (1979, p. 501) for an example of a contract where P is replaced by the spot price of a commodity. For examples of industrial contracts which have a payment which depends on the cost of inputs if it exceeds some number, see Cummins (1977).

[19]See Smith and Watts (1981) for a description and analysis of compensation plans. See also Miller and Scholes (1980).

some compensation plans include a variable payment which depends on a comparison of the performance of the firm for which the managers work and the performance of some competing firm (or the performance of the industry). For simplicity, it is assumed here that the performance of the firms is measured by their value at maturity of the contract. Let V be the value of the managers' firm and S be the value of the competing firm. Both firms pay no dividends until date T. The firm whose value is S has no debt. The only debt of the firm whose value is V is the promised payment to its managers, which is assumed to be equal to

$$F + \min \{\max \{\gamma V - \delta S, 0\}, \max \{\gamma V - F, 0\}\},$$

where γ and δ are positive constants, $\gamma < 1$, and F is the fixed component of the managers' compensation at date T. No variable payment is made if the firm does poorly, i.e., if $\gamma V < F$. It can be verified that the value of the compensation plan at maturity, i.e., $L(V, S, F, 0)$, is equal to

$$L(V, S, F, 0) = F - P(V, F, 0) + C(\gamma V, F, 0) - M(\gamma V, \delta S, F, 0). \tag{30}$$

If V and S are prices of traded assets and are jointly lognormally distributed, $L(V, S, F, \tau)$ is given by the following equation:

$$L(V, S, F, \tau) = F e^{-R\tau} - P(V, F, \tau) + C(\gamma V, F, \tau) - M(\gamma V, \delta S, F, \tau). \tag{27}$$

No clear statement can be made about the effect of a change in the variance of the rate of change of the value of the firm on the value of the compensation plan. In other words, for different values of the parameters of the plan, it might well happen that an increase in the variance of the rate of change of the value of the firm has effects of opposite signs on the value of the compensation plan. However, a compensation plan like the one discussed here induces managers to choose projects which reduce the coefficient of correlation between the rates of change of V and S.

4.2.2. The valuation of investment opportunities

The formula for a call option on the maximum of two risky assets can be used to value complex projects in which the firm chooses among various streams of cash-flows at a future date. Myers (1977) shows that an investment opportunity to which a firm has exclusive access can be regarded as a call option where the exercise price is the future outlay required to undertake the investment. In many cases, when the future outlay is made, the firm can choose among two mutually exclusive risky streams of cash-flows.

Such an investment opportunity can be valued using the formula for an option on the minimum of two risky assets.

As an example, suppose that an all-equity firm has the opportunity to buy a parcel of land at date t which it can use for two mutually exclusive purposes at some future date T. The firm can use the parcel of land to build either a residential property or an office-building. The cost of each building, to be paid at date T, is known and equal to F. For simplicity, assume that V (H) is the price of an asset which pays no dividends and that V (H) is equal to the value of the residential property (office building) at date T.[20] It is assumed that V and H are jointly lognormally distributed. If an outlay is made at date T, the firm cannot change its use of the parcel of land ever again. If no outlay is made, the firm cannot use the parcel of land after date T. Construction takes a negligible amount of time. Taxes and transaction costs are neglected. With these assumptions, the firm buys the parcel of land if its price is no greater than $MX(V, H, F, \tau)$, i.e., the price of a call option on the maximum of V and H, with exercise price F and time to maturity τ. The approach used here offers three interesting insights: (1) The lower the coefficient of correlation between the value of the alternative uses of the parcel of land, the more valuable the parcel of land is to the firm. (2) The parcel of land is always more valuable if it has two possible uses than if it has only one of these two possible uses, as $MX(V, H, F, \tau) \geq C(V, F, \tau)$ and $MX(V, H, F, \tau) \geq C(H, F, \tau)$. (3) The loss suffered by the firm if a zoning restriction is approved (after it has bought the land) which forbids it from building an office-building is equal to $MX(V, H, F, \tau') - C(V, F, \tau')$, where $\tau' = T - t'$ and $t' < T$ is the time at which the zoning restriction is approved.

4.2.3. The valuation of secured debt

The formula for a call option on the minimum of two risky assets makes it possible to obtain some new insights into the problem of valuing secured debt.[21] Let $D(V, S, F, \tau)$ be the value of a discount bond with maturity at date T and facevalue F, secured by a collateral which pays no dividends and whose price is S. The value of the firm's assets, which include the collateral, is equal to W. There exist claims which mature at date T and have priority over the claim of the holders of the secured bond over the assets of the firm whose value is given by $W - S$. These claims have a facevalue equal to G. Define $W - G$ to be equal to V. The firm pays no dividends until date T and has no

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[20]The assumption that the assets whose price is, respectively, V and H, do not pay dividends could be relaxed to allow for a dividend paid continuously and proportional to the value of the firm. See Merton (1973) for such an adjustment. (A dividend adjustment would allow these assets to be existing buildings which are perfect substitutes for those which could be constructed on the parcel of land.)

[21]For earlier work on secured debt, see Smith and Warner (1979) and Scott (1978).

other claimants than those already discussed. At date T, the holders of the secured bond receive $\min\{F, \max\{V, S\}\}$. This implies that the holders of the secured bond receive at least the value of the collateral in the event of default; they receive the value of the assets of the firm if, after all claimants of higher priority have been paid off, that value exceeds the value of the collateral. The value of the secured bond at maturity can be written as

$$D(V, S, F, 0) = F - \max\{F - \max(V, S), 0\}. \tag{32}$$

It follows from eq. (32) that the value of the secured debt at maturity is equal to the value of a default-free discount-bond with maturity T and facevalue F minus the value at maturity T of a put option on the maximum of V and S with exercise price F. In section 3, a formula for a put option on the maximum of two risky assets is given. To apply this formula to value $D(V, S, F, \tau)$, it is necessary that V and S are jointly lognormally distributed. It does not seem realistic to assume that V is lognormally distributed when S is lognormally distributed, because V is equal to $W - G$ and W is a sum of random variables, one of which is S. Should one assume that V is lognormally distributed, however, one would obtain the interesting result that the lower the coefficient of correlation between the dynamics of the price of the collateral and the dynamics of the value of the firm, the more valuable is the secured debt.

5. Conclusion and summary

This paper presents analytical formulas for the pricing of a European put and call option on the maximum or the minimum of two risky assets. It is shown that a call option on the minimum of two risky assets with an exercise price equal to zero can be evaluated by using the formula for the pricing of an option to exchange one asset for another. A call option on the minimum of two risky assets is an increasing function of the price of each risky asset and a decreasing function of the exercise price. It is an increasing function of the coefficient of correlation between the dynamics of the two risky assets. An increase in time to maturity or in the standard deviation of the return on one of the risky assets has an ambiguous effect on the value of the option.

This paper contains a number of applications of options on the minimum or the maximum of two risky assets. Foreign currency bonds, default-free currency option-bonds and risky currency option-bonds are priced. It is shown that a wide variety of risk-sharing and incentive contracts have a payoff at maturity which contains the payoff at maturity of an option on the minimum or maximum of two risky assets. Investment opportunities which correspond to complex projects in which a firm chooses among various cash-

flows at a future date can also be valued using the techniques developed in this paper. Finally, secured debt has a payoff at maturity which corresponds to the payoff of a default-free discount bond minus the payoff of a put option on the maximum of two risky assets.

Appendix 1

In this appendix, the derivation of eq. (11) is sketched. For the purpose of computing the expected value of the option at maturity in a risk-neutral world, there is no loss in generality if it is assumed that V and H each follow a lognormal distribution. Let $v = \ln V$ and $h = \ln H$. Define $k = \min(v, h)$. The probability that k is smaller than some value K is

$$F(K) = 1 - \int_K^\infty \int_K^\infty n_2(v, h) \, \mathrm{d}h \, \mathrm{d}v, \tag{A.1}$$

where $n_2(v, h)$ is the bivariate normal density function. To obtain the density function of the minimum, take the derivative of (A.1) with respect to K. Let $\bar{v}$ be the expected value of v, $\sigma(v)$ the standard deviation of v, $\sigma(h)$ the standard deviation of h and ρ the coefficient of correlation between v and h. If $f(k)$ is the probability density function of k, then

$$f(k) = N_1\left(\frac{-k + \bar{v} + \rho(\sigma(v)/\sigma(h))(k - \bar{h})}{\sqrt{1 - \rho^2}\,\sigma(v)}\right) n_1\left(\frac{k - \bar{h}}{\sigma(h)}\right)$$

$$+ N_1\left(\frac{-k + \bar{h} + \rho(\sigma(h)/\sigma(v))(k - \bar{v})}{\sqrt{1 - \rho^2}\,\sigma(h)}\right) n_1\left(\frac{k - \bar{v}}{\sigma(v)}\right). \tag{A.2}$$

$N_1(a)$ is the cumulative standard normal distribution evaluated at a and $n_1(a)$ is the standard normal density function evaluated at a.

The expected payoff of the option at maturity is

$$E(C(V, H, F, 0)) = \int_{\ln F}^\infty e^k f(k) \, \mathrm{d}k - \int_{\ln F}^\infty F f(k) \, \mathrm{d}k. \tag{A.3}$$

(A.3) can be rewritten as the sum of three terms. The first term is the expected value of H at maturity if H is the maximum and exceeds F. The second term is the expected value of V at maturity if V is the minimum and exceeds F. The third term is minus F times the probability that the option

will be exercised at maturity. The first term can be written as

$$
A = \int_{\ln(F/H)}^{\infty} e^{R\tau} H N_1 \left(\frac{-a + \ln(V/H) + (R - \frac{1}{2}\sigma_V^2)\tau}{\sigma_V \sqrt{\tau} \sqrt{1 - \rho_{VH}^2}} \right.
$$

$$
\left. + \frac{\rho_{VH}(\sigma_V/\sigma_H)(a - (R - \frac{1}{2}\sigma_H^2)\tau)}{\sigma_V \sqrt{\tau} \sqrt{1 - \rho_{VH}^2}} \right) n_1 \left(\frac{a - (R + \frac{1}{2}\sigma_H^2)\tau}{\sigma_H \sqrt{\tau}} \right) da. \tag{A.4}
$$

Rewrite this in double integral form with Q being the upper limit of integration of the cumulative normal distribution in (A.4),

$$
A = e^{R\tau} \int_{\ln(F/H)}^{\infty} \int_{-\infty}^{Q} n_1(Z) n_1 \left(\frac{a - (R + \frac{1}{2}\sigma_H^2)\tau}{\sigma_H \sqrt{\tau}} \right) dZ \, da. \tag{A.5}
$$

Rewrite Q as $aX + B$. Use the change of variables $Y = -X$ and $U = aY + Z$ and evaluate the integrals in (A.5). Premultiplying A by $e^{-R\tau}$ yields the first term in (11). Following the same procedure for the two other terms defined earlier yields eq. (11).

Appendix 2

This appendix gives the partial derivatives of the price of the call option on the minimum of two risky assets with respect to H, F, and R,

$$
\partial c / \partial H = N_2(\alpha_1, \alpha_2, \rho_c) + N_1 \left(\frac{\alpha_1 - \rho_c \alpha_2}{\sqrt{1 - \rho_c^2}} \right) e^{-\frac{1}{2}\alpha_2^2} \frac{1}{k^{\frac{1}{2}}}
$$

$$
- \left(\frac{V}{H} \right) N_1 \left(\frac{\beta_1 - \rho_c \beta_2}{\sqrt{1 - \rho_c^2}} \right) e^{-\frac{1}{2}\beta_2^2} \frac{1}{k^{\frac{1}{2}}}, \tag{A.6}
$$

$$
\partial C / \partial F = -e^{-R\tau} N_2(\gamma_1, \gamma_2, \rho_{VH}) < 0, \tag{A.7}
$$

$$
\partial C / \partial R = \tau e^{-R\tau} N_2(\gamma_1, \gamma_2, \rho_{VH}) > 0. \tag{A.8}
$$

Appendix 3

Eq. (20) can be rewritten as

$$\frac{\partial C}{\partial \rho_{VH}} = FN_1\left(\frac{\alpha_1 - \rho_c \alpha_2}{\sqrt{1-\rho_c^2}}\right)e^{\frac{1}{2}\alpha_2^2}e^{-\frac{1}{2}\alpha_1^2}e^{-\frac{1}{2}\gamma_1^2}\frac{1}{\sqrt{2\pi}}\left[\frac{\sigma_V\sigma_H\sqrt{\pi}}{\sigma} + \frac{\sigma_V\sigma_H\alpha_2}{\sigma^2}\right]$$

$$+ FN_1\left(\frac{\beta_1 - \rho_c \beta_2}{\sqrt{1-\rho_c^2}}\right)e^{-\frac{1}{2}\beta_2^2}e^{-\frac{1}{2}\beta_1^2}e^{-\frac{1}{2}\gamma_2^2}\frac{1}{\sqrt{2\pi}}\left[\frac{\sigma_V\sigma_H\sqrt{\tau}}{\sigma} + \frac{\sigma_V\sigma_H\beta_2}{\sigma^2}\right].$$

(A.9)

Note now that

$$\alpha_2 = (\gamma_2\sigma_V - \gamma_1\sigma_H\sqrt{\tau} + (\rho_{VH}\sigma_V/\sigma_H - 1)\sigma_H^2\tau)/\sqrt{\tau}\sigma$$

$$= (\ln(V/H) - \tfrac{1}{2}\sigma_V^2\tau - \tfrac{1}{2}\sigma_H^2\tau + \rho_{VH}\sigma_V\sigma_H\tau)/\sqrt{\tau}\sigma$$

$$= \ln(V/H)/\sqrt{\tau}\sigma - \tfrac{1}{2}\sigma\sqrt{\tau}.$$

Similarly,

$$\beta_2 = \ln(H/V)/\sqrt{\tau}\sigma - \tfrac{1}{2}\sigma\sqrt{\tau}.$$

This implies that

$$\beta_2^2 = \alpha_2^2 - 2\ln(V/H).$$

It can also be shown that

$$(\alpha_1 - \rho_c\alpha_2)/\sqrt{1-\rho_c^2} = (\beta_1 - \rho_c\beta_2)/\sqrt{1-\rho_c^2}.$$

By substituting these results in (A.9), it follows that

$$\frac{\partial C}{\partial \rho_{VH}} = VN_1\left(\frac{\beta_1 - \rho_c\beta_2}{\sqrt{1-\rho_c^2}}\right)\frac{e^{-\frac{1}{2}\beta_2^2}}{\sqrt{2\pi}}\left[\frac{2\sigma_V\sigma_H\sqrt{\tau}}{\sigma} + \frac{\sigma_V\sigma_H(\beta_2 + \alpha_2)}{\sigma^2}\right]$$

$$= VN_1\left(\frac{\beta_1 - \rho_c\beta_2}{\sqrt{1-\rho_c^2}}\right)\frac{e^{-\frac{1}{2}\beta_2^2}}{\sqrt{2\pi}} \cdot \frac{\sigma_V\sigma_H\sqrt{\tau}}{\sigma} > 0.$$

Note that to obtain (A.9), it is necessary to use a result on partial moments of normally distributed variables which can be found in Winkler et al. (1972).

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