A causal interpretation of target parameter in continuous time of Rytgaard et al. (2022)

Let us consider a setting similar to the one of Ryalen (2024). Specifically, we will work with an intervention that specifies the treatment decisions but not the timing of treatment visits. We consider death as the outcome of interest and are interested in the probability of death, had had we followed the regime of always treating. To simplify, we work without right-censoring, no covariates, and compliance to treatment at time 0. Let $(\Omega, \mathcal{F}, P)$ be a probability space. and consider (N^y, N^a), where

- N^y is a counting process on $[0, T]$ for death.
- N^a is a random measure for treatment on $[0, T] \times \{1, 0\}$, where 1 denotes treatment and 0 no treatment.

We consider the filtration generated by (N^y, N^a) and denote it by $(\mathcal{F}_t)_{t \geq 0}$, i.e.,

$$\mathcal{F}_t := \sigma(N^y(\mathrm{d}s), N^a(\mathrm{d}s \times \{x\}) \mid s \in (0, t], x \in \{0, 1\}).$$

Further, we assume that

- N^y and $N^a(\{(0, t] \times \{1, 0\}\})$ do not jump at the same time.
- $M^y = N^y - \Lambda^y$ denotes their P-$\mathcal{F}_t$ (local) martingale, where Λ^y is the P-$\mathcal{F}_t$-compensator of N^y.
- $M^a(\mathrm{d}t \times \{x\}) = N^a(\mathrm{d}t \times \{x\}) - (\pi_t)^{\mathbb{1}\{x=1\}}(1 - \pi_t)^{\mathbb{1}\{x=0\}} \Lambda^a(\mathrm{d}t)$ is the P-$\mathcal{F}_t$ (local) martingale for $x \in \{1, 0\}$, where π_t is the $\mathcal{F}_t$-predictable probability of treatment at time t (mark probability) and $\Lambda^a(\mathrm{d}t)$ is the total P-$\mathcal{F}_t$-compensator of $N^a(\mathrm{d}t \times \mathrm{d}x)$.

For this treatment regime, we see that

$$\tau^A = \inf\{t \geq 0 \mid N^a((0, t] \times \{0\}) > 0\}.$$

We are interested in the counterfactual mean outcome $\mathbb{E}_P\left[\tilde{Y}_t\right]$, where $\left(\tilde{Y}_t\right)_{t \geq 0}$ is the counterfactual outcome process of $Y := N^y$ under the intervention that sets treatment to 1 at all visitation times. Note the different exchangeability condition compared to Ryalen (2024), as Ryalen (2024) expresses exchangeability through the counting process $\mathbb{1}\{\tau^A \leq \cdot\}$. Let $\left(T_{(k)}, \Delta_{(k)}, A\left(T_{(k)}\right)\right)$ denote the ordered event times, event types, and treatment decisions at event k. Note that Equation 1 is the same likelihood ratio as in Rytgaard et al. (2022). We also impose the assumption that $N_t := N_t^y + N^a(\{(0, t] \times \{1, 0\}\})$ does not explode; we also assume that we work with a version of the compensator such that $\Lambda(\{t\} \times \{y, a\} \times \{1, 0\}) < \infty$ for all $t > 0$. We may generally also work with a compensator Λ that fulfills conditions (10.1.11)-(10.1.13) of Last & Brandt (1995). Let $\pi^*_{T_{(k)}}\left(\mathcal{F}_{T_{(k-1)}}\right)$ denote the interventional probability, which in this case we take to be 1. In this case,

$$\begin{aligned} \pi_t &= \sum_k \mathbb{1}\left\{T_{(k-1)} < t < T_{(k)}\right\} \pi_{T_{(k)}}\left(\mathcal{F}_{T_{(k-1)}}\right) \\ \pi^*_t &= \sum_k \mathbb{1}\left\{T_{(k-1)} < t < T_{(k)}\right\} \pi^*_{T_{(k)}}\left(\mathcal{F}_{T_{(k-1)}}\right) = 1. \end{aligned}$$

Theorem 0.1: Define

$$\zeta(t, m, a) := \mathbb{1}\{m = y\} + \mathbb{1}\{m = a\} \frac{\mathbb{1}\{a = 1\}}{\pi_t}$$

If *all* of the following conditions hold:

- **Consistency**: $\tilde{Y}_t \mathbb{1}\{\tau^a > \cdot\} = Y_t \mathbb{1}\{\tau^a > \cdot\} \quad P - \text{a.s.}$
- **Exchangeability**: Define $\mathcal{H}_t := \mathcal{F}_t \vee \sigma\left(\tilde{Y}\right)$. The P-$\mathcal{F}_t$ compensator for N^a is also the P-$\mathcal{H}_t$ compensator.
- **Positivity**: Let $N^{ax}(\mathrm{d}t) := N^a(\mathrm{d}(t) \times \{x\})$ for $x \in \{1, 0\}$.

$$W(t) := \prod_{j=1}^{N_t} \left(\left(\frac{\pi^*_{T_{(j)}}\left(\mathcal{F}_{T_{(j-1)}}\right)}{\pi_{T_{(j)}}\left(\mathcal{F}_{T_{(j-1)}}\right)} \right)^{\mathbb{1}\left\{A\left(T_{(k)}\right)=1\right\}} \left(\frac{1 - \pi^*_{T_{(j)}}\left(\mathcal{F}_{T_{(j-1)}}\right)}{1 - \pi_{T_{(j)}}\left(\mathcal{F}_{T_{(j-1)}}\right)} \right)^{\mathbb{1}\left\{A\left(T_{(k)}\right)=0\right\}} \right)^{\mathbb{1}\left\{\Delta_{(j)}=a\right\}} \tag{1}$$

fulfills that

$$\int_0^t W(s-)\left(\frac{\pi^*_s}{\pi_s} - 1\right) N^{a1}(\mathrm{d}s), \int_0^t W(s-)\left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) N^{a0}(\mathrm{d}s),$$

are zero mean square-integrable, P-$\mathcal{F}_t$-martingales.

Then,

$$\mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[Y_t W(t)]$$

Proof: We shall use that the likelihood ratio solves a specific stochastic differential equation. To this end, note that

$$\begin{aligned}
&W(t) \\
&= \prod_{s \le t} \left(1 + \left(\frac{\pi^*_s}{\pi_s} - 1\right) N^{a1}(\mathrm{d}s) + \left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) N^{a0}(\mathrm{d}s)\right) \\
&= \prod_{s \le t} \left(1 + \left(\frac{\pi^*_s}{\pi_s} - 1\right) N^{a1}(\mathrm{d}s) + \left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) N^{a0}(\mathrm{d}s) - (\pi^*_s - \pi_s)\Lambda^a(ds) - (\pi_s - \pi^*_s)\Lambda^a(ds)\right) \\
&= \prod_{s \le t} \left(1 + \left(\frac{\pi^*_s}{\pi_s} - 1\right) N^{a1}(\mathrm{d}s) + \left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) N^{a0}(\mathrm{d}s) - \frac{\pi^*_s - \pi_s}{\pi_s}\Lambda^{a1}(ds) - \frac{\pi_s - \pi^*_s}{1-\pi_s}\Lambda^{a0}(ds)\right) \\
&= \prod_{s \le t} \left(1 + \left(\frac{\pi^*_s}{\pi_s} - 1\right) M^{a1}(\mathrm{d}s) + \left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) M^{a0}(\mathrm{d}s)\right).
\end{aligned}$$

Thus, by properties of the product integral (e.g., Theorem II.6.1 of Andersen et al. (1993)),

$$W(t) = 1 + \int_0^t W(s-)\left(\frac{\pi^*_s}{\pi_s} - 1\right) M^{a1}(\mathrm{d}s) + \int_0^t W(s-)\left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) M^{a0}(\mathrm{d}s). \tag{2}$$

We have that

$$\zeta_t := \int_0^t W(s-)\left(\frac{\pi^*_s}{\pi_s} - 1\right) M^{a1}(\mathrm{d}s) + \int_0^t W(s-)\left(\frac{1-\pi^*_s}{1-\pi_s} - 1\right) M^{a0}(\mathrm{d}s)$$

is a zero mean$P$-$\mathcal{H}_t$-martingale. From this, we see that $\int_0^t \tilde{Y}_t \zeta(\mathrm{d}s)$ is also a zero mean P-$\mathcal{H}_t$-martingale. This implies that

$$\mathbb{E}_P[Y_t W(t)] \stackrel{*}{=} \mathbb{E}_P\left[\tilde{Y}_t W(t)\right] = \mathbb{E}_P\left[\tilde{Y}_t\right] + \mathbb{E}_P\left[\int_0^t \tilde{Y}_t \zeta(\mathrm{d}s)\right] = \mathbb{E}_P\left[\tilde{Y}_t\right],$$

where in $*$ we used consistency by noting that $W(t) \neq 0$ if and only if $\tau^a > t$. $\square$

Bibliography

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