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A note on the potential outcomes framework in continuous time

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Abstract

In this brief note, we consider the identification formulas of Ryalen (2024) and compare it with the identification formula given in Rytgaard et al. (2022), corresponding to their marked point process settings. It is shown that the resulting identification formulas are the same if and only if the probability of being treated given that you go to the doctor at time t is equal to 1 for Lebesgue-almost all t, provided that the transition hazards for dying are strictly positive for almost all t.

1 Introduction

The aim of this note is to clarify potential differences between the identification formulas of Helene and Pål. The target parameters are the same in both setups. The target parameters are thus the risk of death at time t, under the treatment regime which states that if you go to the doctor, you will be treated. Throughout we consider a simple setting without censoring, without time-varying covariates and without baseline covariates.

2 Causal framework

Let us say that we are interested in the effect of staying treated on being survival or death at time t. We suppose that the event time T is a positive continuous random variable $T \sim Q$ with say distribution function F. The event time T represents a counterfactual world in which a patient may go to the doctor, but the doctor cannot make the decision not to treat the patient.

However, in the real world, doctors can make the decision to not treat the patient. In the observed data $O = \left(T_{(2)}, T_{(1)}, D_{(1)}, A\left(T_{(1)}\right)\right) \sim P_{Q,G^*}$ and, as a shorthand, we let $P = P_{Q,G^*}$ be the probability law for both the observed data and the counterfactual world. Here $T_{(1)}$ is the first event time in the sample, $D_{(1)}$ is the event status (outcome y or gone to the doctor a) at time $T_{(1)}$ and $A\left(T_{(1)}\right)$ is the treatment assignment at time $T_{(1)}$ ($A\left(T_{(1)}\right) \in \{0,1\}$). $T_{(2)}$ is the terminal event time if the first event is not the terminal event. We take $T_{(2)} = \infty$ otherwise. We suppose that $T_{(1)}$ is also a continuous random variable and that $T_{(2)}$ is continuous if $D_{(1)} \neq y$.

We can then formulate consistency as follows:

$$T_{(1)} = T \text{ if } D_{(1)} = y \text{ or } T_{(2)} = T \text{ if } D_{(1)} = a \text{ and } A\left(T_{(1)}\right) = 1$$

Following Gill et al. (1997), we may then formulate Coarsening at Random as

$$P\Big(T_{(1)} \in \mathrm{d}\tilde{t}, D_{(1)} = a, A\Big(T_{(1)}\Big) = 0 \mid T = t\Big)$$

does not depend on t for $\tilde{t} < t$ (if either $D_{(1)} = y$ or $D_{(1)} = a$ and $A\Big(T_{(1)}\Big) = 1$, the variable is fully observed).

We can then let

$$G\Big(\tilde{t}\Big) = P\Big(T_{(1)} \leq \tilde{t}, D_{(1)} = a, A\Big(T_{(1)}\Big) = 0 \mid T = t\Big),$$

for $t > \tilde{t}$. Thus,

$$G(t\, -) = \lim_{\tilde{t} \to t-} G\Big(\tilde{t}\Big) = P\Big(D_{(1)} = a, A\Big(T_{(1)}\Big) = 0 \mid T = t\Big)$$

which means that

$$1 - G(t\, -) = P\Big(D_{(1)} = y \vee \Big(D_{(1)} = a \wedge A\Big(T_{(1)}\Big) = 1\Big) \mid T = t\Big)$$

Hence, we have that

$$P\Big(T_{(1)} \leq t, D_{(1)} = y \vee \Big(T_{(2)} \leq t \wedge D_{(1)} = a \wedge A\Big(T_{(1)}\Big) = 1\Big)\Big) = \int_0^t (1 - G(s\, -))F(\mathrm{d}s)$$

On the other hand,

$$P\Big(T_{(1)} \leq t, D_{(1)} = y\Big) = \int_0^t \exp\left(-\int_0^s \Lambda^y(u) + \Lambda^a(u) du\right)\Lambda^y(\mathrm{d}s)$$

$$P\Big(T_{(1)} \leq 2, D_{(1)} = a, A_1 = 1\Big) = \int_0^t \exp\left(-\int_0^s \Lambda^y(u) + \Lambda^a(u) du\right) \int_s^t \exp(-\Lambda^y(s,u))\Lambda^y(s,u)\pi(s)\Lambda^a(\mathrm{d}s)$$

What this suggests is that we may use Inverse Probability Weighting to obtain

$$\begin{aligned} Q(T \leq t) &= \mathbb{E}_P\left[\frac{1}{1 - G\Big(T_{(1)}\, -\Big)} \mathbb{1}\Big\{T_{(1)} \leq t, D_{(1)} = y \vee \Big(T_{(2)} \leq t \wedge D_{(1)} = a \wedge A\Big(T_{(1)}\Big) = 1\Big)\Big\}\right] \\ &= \mathbb{E}_P\left[\frac{1 - \mathbb{1}\Big\{T_{(1)} \leq t, D_{(1)} = a, A\Big(T_{(1)}\Big) = 0\Big\}}{1 - G\Big(T_{(1)}\, -\Big)} \mathbb{1}\Big\{T_{(1)} \leq t, D_{(1)} = y \vee T_{(2)} \leq t\Big\}\right]. \end{aligned} \quad (1)$$

Letting $\tau^A = T_{(1)}$ if $D_{(1)} = a, A\Big(T_{(1)}\Big) = 0$ and $\tau^A = \infty$ otherwise and $N_t(y) = \mathbb{1}\Big\{T_{(1)} \leq t, D_{(1)} = y \vee T_{(2)} \leq t\Big\}$ (observed outcome), we see that

$$Q(T \leq t) = \mathbb{E}_P\left[\frac{\mathbb{1}\{\tau^A > t\}}{1 - G\Big(T_{(1)}\, -\Big)} N^y(t)\right],$$

which coincides Equation (29) of Ryalen (2024) if we can show that

$$1 - G(t\, -) = \prod_{s \in (0,t)} (1 - \Lambda^a(t))$$

where Λ^a denotes the compensator of $\mathbb{1}\{\tau^A \leq t\}$. We can choose Λ^a on the form,

$$\Lambda^a(t) = \int_0^t \mathbb{1}\Big\{t \leq T_{(1)}\Big\}\Lambda$$

according to Last & Brandt (1995), which shows that they are the same.

Note that if we assunme additionally that the $\Lambda_t(\cdot)$ is locally independent of T, we are in trouble. The observed data requires that the treatment is always administred in the observed data.

3 Causal framework

Our overall goal is to estimate the interventional cumulative incidence function at time τ,

$$\Psi_\tau^g(P) = \mathbb{E}_P\left[\widetilde{N}^y(\tau)\right],$$

where $\widetilde{N}^y(t)$ is the potential outcome (a counting process with at most one jump) representing the counterfactual outcome $N^y(t) = \sum_{k=1}^K \mathbb{1}\left\{T_{(k)} \leq t, \Delta_{(k)} = y\right\}$ had the treatment regime g, possibly contrary to fact, been followed. For simplicity, we assume that the treatment regime specifies that $A(t) = 1$ for all $t \geq 0$. This means that treatment is administered at each visitation time. In terms of these data, this means that we must have $A(0) = 1$ and $A\left(T_{(k)}\right) = 1$ whenever $\Delta_{(k)} = a$ and $T_{(k)} < t$. We now define the càdlàg weight process $(W(t))_{t\geq 0}$ given by

$$W(t) = \prod_{k=1}^{N_t} \left(\frac{\mathbb{1}\left\{A\left(T_{(k)}\right) = 1\right\}}{\pi_k\left(T_{(k)}, \mathcal{F}_{T_{(k-1)}}\right)}\right)^{\mathbb{1}\left\{\Delta_{(k)}=a\right\}} \frac{\mathbb{1}\{A(0) = 1\}}{\pi_0(L(0))}, \tag{2}$$

where $N_t = \sum_k \mathbb{1}\left\{T_{(k)} \leq t\right\}$ is the number of events up to time t, and we consider the observed data target parameter $\Psi_\tau^{\text{obs}} : \mathcal{M} \to \mathbb{R}_+$[1] given by

$$\Psi_\tau^{\text{obs}}(P) = \mathbb{E}_P[N^y(\tau)W(\tau)]. \tag{3}$$

We provide both martingale and non-martingale conditions for the identification ($\Psi_\tau^g(P) = \Psi_\tau^{\text{obs}}(P)$) of the mean potential outcome in Theorem 3.1.1 and Theorem 3.2.1, respectively. One can also define a (stochastic) intervention with respect to a local independence graph (Røysland et al. (2024)) but we do not further pursue this here. While our theory provides a potential outcome framework, it is unclear at this point how graphical models can be used to reason about the conditions.[2]

3.1 Identification of the causal effect (martingale approach)

Let $N_t^a(\cdot) = N^\alpha((0,t] \times \{a\} \times \cdot \times \mathcal{L})$ be the random measure on $(\mathbb{R}_+ \times \{0,1\})$ for the treatment process and let $\Lambda_t^a(\cdot)$ be the corresponding P-$\mathcal{F}_t$ compensator. We adopt a martingale-based approach for identifying causal effects, following the methodology of Ryalen (2024)[3].

To this end, we define the stopping time T^a as the time of the first visitation event where the treatment plan is not followed, i.e.,

$$T^a = \inf_{t\geq 0}\{A(t) = 0\} = \begin{cases} \inf_{k>1}\left\{T_{(k)} \mid \Delta_{(k)} = a, A\left(T_{(k)}\right) \neq 1\right\} & \text{if } A(0) = 1 \\ 0 \text{ otherwise} \end{cases}$$

Overall T^a acts as a coarsening variable, limiting the ability to observe the full potential outcome process. An illustration of the consistency condition in Theorem 3.1.1 is provided in Figure 1. Informally, the consistency condition states that the potential outcome process $\widetilde{N}^y(t)$ coincides with $N^y(t)$ if the treatment plan has been adhered to up to time point t.

To fully phrase the causal inference problem as a missing data problem, we also need an exchangeability condition. The intuition behind the exchangeability condition in Theorem 3.1.1 is that the outcome process $\widetilde{N}^y$ should be independent of both the timing of treatment visits and treatment assignment, conditional on observed history.

[1] Note that by fifth equality of Appendix S1.2 of Rytgaard et al. (2022), this is the same as the target parameter in Rytgaard et al. (2022) with no competing event.

[2] see Richardson & Robins (2013) for the discrete time variant, i.e., single world intervention graphs.

[3] The overall difference between Ryalen (2024) and our exchangeability condition is that the P-$\mathcal{F}_t$ compensator $\Lambda_t^a(\{1\})$ is not required to be the P-$\left(\mathcal{F}_t \vee \sigma\left(\widetilde{N}^y\right)\right)$ compensator for N^a.

We also briefly discuss the positivity condition, which ensures that $(W(t))_{t \geq 0}$ is a uniformly integrable martingale with $\mathbb{E}_P[W(t)] = 1$ for all $t \in [0, \tau_{\text{end}}]$ by Equation 9. This guarantees that the observed data parameter $\Psi_\tau^{\text{obs}}(P)$ is well-defined.

Note that instead of conditioning on the entire potential outcome process in the exchangeability condition, we could have simply conditioned on a single potential outcome variable $\tilde{T}_y := \inf\left\{t > 0 \mid \widetilde{N}^y(t) = 1\right\} \in [0, \infty]$[4] and included that information at baseline[5].

We can also state the time-varying exchangeability condition of Theorem 3.1.1 explicitly in terms of the observed data: Let $\mathcal{H}_{T_{(k)}}$ be the corresponding stopping time σ-algebra for the k'th event with respect to the filtration $\{\mathcal{H}_t\}$ given in Theorem 3.1.1. In light of the canonical compensator, we see immediately that the exchangeability condition is fulfilled if $A\left(T_{(k)}\right) \perp \tilde{T}_y \mid T_{(k)}, \Delta_{(k)} = a, \mathcal{F}_{T_{(k-1)}}$ and the cause-specific cumulative hazards for $T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}, \tilde{T}_y$ for treatment visits only depend on $\mathcal{F}_{T_{(k-1)}}$ and not on $\tilde{T}_y$.

Further work is needed to cast this framework into a coarsening at random (CAR) framework (van der Vaart (2004)). In particular, it is currently unclear whether the parameter $\Psi_\tau(P)$ depends on the distribution of treatment visitation times and treatment assignment and whether the identification conditions impose restrictions on the distribution of the observed data process.

Theorem 3.1.1 (Martingale identification of mean potential outcome): Define

$$\zeta(t, m, a, l) := \sum_k \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \left(\frac{\mathbb{1}\{a = 1\}}{\pi_k\left(T_{(k)}, \mathcal{F}_{T_{(k-1)}}\right)}\right)^{\mathbb{1}\{m=a\}}, \tag{4}$$

If *all* of the following conditions hold:

- **Consistency**: $\widetilde{N}^y(t)\mathbb{1}\{T^a > t\} = N^y(t)\mathbb{1}\{T^a > t\} \quad P-\text{a.s.}$
- **Exchangeability**: Define $\mathcal{H}_t := \mathcal{F}_t \vee \sigma\left(\widetilde{N}^y\right)$. The P-$\mathcal{F}_t$ compensator for N^a $\Lambda_t^a(\cdot)$ is also the P-$\mathcal{H}_t$ compensator and
$$\widetilde{N}^y(t) \perp A(0) \mid L(0), \forall t \in (0, \tau_{\text{end}}].$$
- **Positivity**: $\mathbb{E}_P\left[\int \mathbb{1}\{t \leq \tau_{\text{end}}\} \left|\zeta(t, m, a, l) - 1\right| W(t-) N(d(t, m, a, l))\right] < \infty$ and $\mathbb{E}_P[W(0)] = 1$.

Then,

$$\Psi_t^g(P) = \Psi_t^{\text{obs}}(P)$$

for all $t \in (0, \tau_{\text{end}}]$.

Proof: We shall use that the likelihood ratio solves a specific stochastic differential equation (we use what is essentially Equation (2.7.8) of Andersen et al. (1993)), but present the argument using Theorem 10.2.2 of Last & Brandt (1995) as the explicit conditions are not stated in Andersen et al. (1993). First, let

[4] A competing event occuring corresponds to $\tilde{T}_y = \infty$

[5] Note that $\mathbb{1}\left\{\tilde{T}_y \leq t\right\} = \widetilde{N}^y(t)$ for all $t > 0$ because $\left(\widetilde{N}^y(t)\right)_{t \geq 0}$ jumps at most once.

$$
\begin{aligned}
\psi_{k,x}\left(t, \mathcal{F}_{T_{(k-1)}}, \mathrm{d}(m,a,l)\right) = \mathbb{1}\{x=a\}\Big(\delta_1(\mathrm{d}a)\pi_k\left(t,\mathcal{F}_{T_{(k-1)}}\right) + \delta_0(\mathrm{d}a)\Big(1-\pi_k\left(t,\mathcal{F}_{T_{(k-1)}}\right)\Big)\Big)\delta_{L\left(T_{(k-1)}\right)}(\mathrm{d}l) \\
+\mathbb{1}\{x=\ell\}\mu_k\left(dl, t, \mathcal{F}_{T_{(k-1)}}\right)\delta_{A\left(T_{(k-1)}\right)}(\mathrm{d}a) \qquad (5) \\
+\mathbb{1}\{x\in\{y,d\}\}\delta_{A\left(T_{(k-1)}\right)}(\mathrm{d}a)\delta_{L\left(T_{(k-1)}\right)}(\mathrm{d}l).
\end{aligned}
$$

We shall use that the P-$\mathcal{F}_t$ compensator of N^α is given by

$$
\Lambda^\alpha(d(t,m,a,l)) = \sum_k \mathbb{1}\left\{T_{(k-1)} < t \le T_{(k)}\right\} \sum_{x=a,\ell,y,d} \delta_x(\mathrm{d}m)\psi_{k,x}(t, d(a,l))\Lambda_k^x\left(\mathrm{d}t, \mathcal{F}_{T_{(k-1)}}\right) \qquad (6)
$$

Second, let $\Phi(d(t,x)) = \mathbb{1}\{t\le\tau_{\text{end}}\}N^\alpha(d(t,x))$ and $\nu(d(t,x)) = \mathbb{1}\{t\le\tau_{\text{end}}\}\Lambda^\alpha(d(t,x))$ be the restricted random measure and its compensator. We define P-$\mathcal{F}_t$ predictable, $\mu(d(t,x)) := \zeta(t,x)\nu(d(t,x))$. Here, we use the shorthand notation $x=(m,a,l)$. The likelihood ratio process $L(t)$ given in (10.1.14) of Last & Brandt (1995) is defined by

$$
\begin{aligned}
L(t) = \mathbb{1}\{t < T_\infty \wedge T_\infty(\nu)\}L_0 \prod_{n: T_{(n)}\le t} \zeta\left(T_{(n)}, \Delta_{(n)}, A\left(T_{(n)}\right), L\left(T_{(n)}\right)\right) \\
\prod_{\substack{s\le t \\ N_{s-}=N_s}} \frac{1-\bar{\nu}\{s\}}{1-\bar{\mu}\{s\}} \exp\left(\int \mathbb{1}\{s\le t\}(1-\zeta(s,x))\nu^c(d(s,x))\right) \qquad (7) \\
+\mathbb{1}\{t\ge T_\infty\wedge T_\infty(\nu)\} \liminf_{s\to T_\infty\wedge T_\infty(\nu)} L(s).
\end{aligned}
$$

Here $T_\infty := \lim_n T_n$, $T_\infty(\nu) := \inf\{t\ge 0 \mid \nu((0,t]\times\{y,d,a,\ell\}\times\{0,1\}\times\mathcal{L}) = \infty\}$, $\bar{\mu}(\cdot) := \mu(\cdot\times\{y,d,a,\ell\}\times\{0,1\}\times\mathcal{L})$, $\bar{\nu}(\cdot) := \nu(\cdot\times\{y,d,a,\ell\}\times\{0,1\}\times\mathcal{L})$, $\nu^c(\mathrm{d}(s,x)) := \mathbb{1}\{\bar{\nu}\{s\}=0\}\nu(d(s,x))$, and $L_0 := W(0) = \frac{\mathbb{1}\{A(0)=1\}}{\pi_0(L(0))}$.

First, we will show that $L(t) = W(t)$, where $W(t)$ is the weight process defined in Equation 2.

By our assumptions, $T_\infty = \infty$ P-a.s. and thus $T_\infty(\nu) = T_\infty = \infty$ in view Theorem 4.1.7 (ii) since $\bar{\nu}\{t\} < \infty$ for all $t > 0$.

Second, note that $\bar{\nu} = \bar{\mu}$. This follows since

$$
\begin{aligned}
\bar{\nu}(A) &= \int_{A\times\{y,d,a,\ell\}\times\{0,1\}\times\mathcal{L}} \zeta(t,m,a,l)\mu(d(t,m,a,l)) \\
&= \int_{A\times\{y,d,\ell\}\times\{0,1\}\times\mathcal{L}} \zeta(t,m,a,l)\mu(d(t,m,a,l)) + \int_{A\times\{a\}\times\{0,1\}\times\mathcal{L}} \zeta(t,m,a,l)\mu(d(t,m,a,l)) \\
&= \int_{A\times\{y,d,\ell\}\times\{0,1\}\times\mathcal{L}} 1\mu(d(t,m,a,l)) + \int_{A\times\{a\}\times\{0,1\}\times\mathcal{L}} \zeta(t,m,a,l)\mu(d(t,m,a,l)) \\
&= \mu(A\times\{y,d,\ell\}\times\{0,1\}\times\mathcal{L}) + \sum_k \mathbb{1}\left\{T_{(k-1)} < t\le T_{(k)}\right\}\Lambda^a\left(dt \mid \mathcal{F}_{T_{(k-1)}}\right) \\
&= \mu(A\times\{y,d,a,\ell\}\times\{0,1\}\times\mathcal{L}) = \bar{\mu}(A),
\end{aligned}
$$

for Borel measurable sets $A\subseteq\mathbb{R}_+$, where the last step follows from the form of the compensator (Equation 6). Thus

$$
\prod_{\substack{s\le t \\ N_{s-}=N_s}} \frac{1-\bar{\nu}\{s\}}{1-\bar{\mu}\{s\}} \exp\left(\int \mathbb{1}\{s\le t\}(1-\zeta(s,x))\nu^c(d(s,x))\right) = 1,
$$

and hence

$$L(t) = L_0 \prod_{n: T_{(n)} \leq t} \zeta\left(T_{(n)}, \Delta_{(n)}, A\left(T_{(n)}\right), L\left(T_{(n)}\right)\right)$$
$$\stackrel{\text{def.}}{=} W(t).$$

Let $V(s,x) = \zeta(s,x) - 1 + \frac{\bar{\nu}\{s\} - \bar{\mu}\{s\}}{1 - \bar{\mu}\{s\}} = \zeta(s,x) - 1$. $L(t)$ will fulfill that

$$L(t) = L_0 + \int \mathbb{1}\{s \leq t\} V(s,x) L(s-)[\Phi(d(s,x)) - \nu(d(s,x))]$$

if

$$\begin{gathered} \mathbb{E}_P[L_0] = 1, \\ \bar{\mu}\{t\} \leq 1, \\ \bar{\mu}\{t\} = 1 \text{ if } \bar{\nu}\{t\} = 1, \\ \bar{\mu}[T_\infty \wedge T_\infty(\mu)] = 0 \text{ and } \bar{\nu}[T_\infty \wedge T_\infty(\nu)] = 0. \end{gathered} \tag{8}$$

by Theorem 10.2.2 of Last & Brandt (1995).

The first condition holds by positivity. The second condition holds by the specific choice of compensator since $\sum_x \Lambda_k^x\left(\{t\}, \mathcal{F}_{T_{(k-1)}}\right) \leq 1$ for all $k = 1, ..., K$ and $t \in (0, \tau_{\text{end}}]$ (Theorem A5.9 of Last & Brandt (1995)). The third holds since $\bar{\mu} = \bar{\nu}$ and the fourth holds since $T_\infty = T_\infty(\nu) = T_\infty(\mu) = \infty$.

Thus,

$$W(t) = \frac{\mathbb{1}\{A(0) = 1\}}{\pi_0(L(0))} + \int_0^t W(s-) V(s,x) (\Phi(d(s,x)) - \nu(d(s,x)). \tag{9}$$

Then we shall show that

$$M_t^* := \int \widetilde{N}^y(t) \mathbb{1}\{s \leq t\} V(s,x) L(s-)[N(d(s,x)) - \Lambda(d(s,x))] \tag{10}$$

is a zero mean uniformly integrable martingale. This follows if

$$\mathbb{E}_P\left[\int \widetilde{N}^y(t) \ |V(s,x)| \ L(s-) \Phi(d(s,x))\right] < \infty.$$

and if $(\omega, s, x) \mapsto \widetilde{N}^y(t) \ |V(s,x)| \ L(s-)$ is P-$\mathcal{H}_s$ predictable by Exercise 4.1.22 of Last & Brandt (1995). Since

$$\mathbb{E}_P\left[\int \widetilde{N}^y(t) \ |V(s,x)| \ L(s-) \Phi(d(s,x))\right] \leq \mathbb{E}_P\left[\int \mathbb{1}\{s \leq \tau_{\text{end}}\} \ |V(s,x)| \ L(s-) N(d(s,x))\right] < \infty$$

and $(\omega, s) \mapsto \widetilde{N}^y(t)$ is predictable with respect to $\mathcal{H}_s$, $(\omega, s) \mapsto L(s-)$ is P-$\mathcal{H}_s$ predictable (càglàd and adapted; Theorem 2.1.10 of Last & Brandt (1995)), $(\omega, s, x) \mapsto V(s,x)$ is P-$\mathcal{H}_s$ predictable (Theorem 2.2.22 of Last & Brandt (1995)), so that $(\omega, s) \mapsto \widetilde{N}^y(t) \ |V(s,x)| \ L(s-)$ is P-$\mathcal{H}_s$ predictable, and the desired martingale result for Equation 10 follows. This in turn implies by Equation 9:

$$
\begin{aligned}
\mathbb{E}_P\left[\widetilde{N}_t^y W(t)\right] &= \mathbb{E}_P\left[\widetilde{N}_t^y W(0)\right] + \mathbb{E}_P[M_t^*] \\
&= \mathbb{E}_P\left[\widetilde{N}_t^y W(0)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left[\widetilde{N}_t^y \mid \mathcal{F}_0\right] W(0)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left[\widetilde{N}_t^y \mid L(0)\right] W(0)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left[\widetilde{N}_t^y \mid L(0)\right] \mathbb{E}_P[W(0) \mid L(0)]\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left[\widetilde{N}_t^y \mid L(0)\right] 1\right] \\
&= \mathbb{E}_P\left[\widetilde{N}_t^y\right],
\end{aligned}
$$

where we use the baseline exchangeability condition and the law of iterated expectation.
$\square$

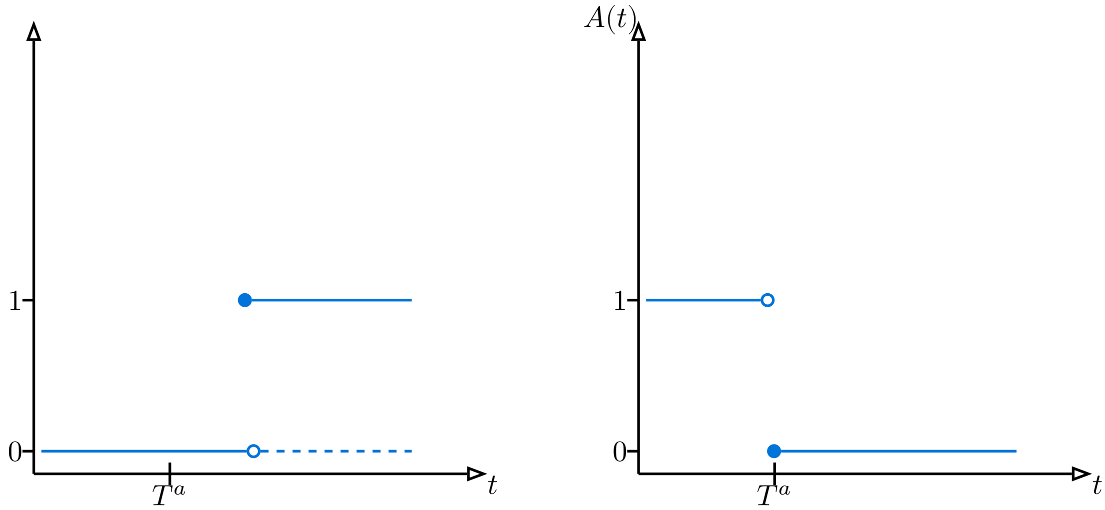

Figure 1: The figure illustrates the consistency condition for the potential outcome framework for single individual. The left panel shows the potential outcome process $\widetilde{N}^y(t)$ (dashed) and the observed process $N^y(t)$ (solid). The right panel shows the treatment process $A(t)$. At time T^a, the treatment is stopped and the processes may from some random future point diverge from each other.

3.2 Identification of the causal effect (non-martingale approach)

In this subsection, we present a non-martingale approach for the identification of causal effects, and the conditions are stated for identification at the time horizon of interest.

Theorem 3.2.1: Assume **Consistency** and **Positivity** as in Theorem 3.1.1 for a single timepoint τ (in the positivity condition replace τ_{end} with τ). Additionally, we assume that:

- **Exchangeability**: We have

$$\begin{aligned}
&\widetilde{N}^y(\tau)\mathbb{1}\left\{T_{(j)} \leq \tau < T_{(j+1)}\right\} \perp A\left(T_{(k)}\right) \mid \mathcal{F}_{T_{(k-1)}}, T_{(k)}, \Delta_{(k)} = a, \quad \forall j \geq k > 0, \\
&\widetilde{N}^y(\tau)\mathbb{1}\left\{T_{(j)} \leq \tau < T_{(j+1)}\right\} \perp A(0) \mid L(0), \quad \forall j \geq 0.
\end{aligned} \tag{11}$$

Then the estimand of interest is identifiable, i.e.,

$$\Psi_\tau^g(P) = \Psi_\tau^{\text{obs}}(P).$$

Proof: Write $\tilde{Y}_t = \sum_{k=1}^K \mathbb{1}\left\{T_{(k-1)} \leq \tau < T_{(k)}\right\}\widetilde{N}^y(\tau)$. The theorem is shown if we can prove that $\mathbb{E}_P\left[\mathbb{1}\left\{T_{(k-1)} \leq \tau < T_{(k)}\right\}\widetilde{N}^y(\tau)\right] = \mathbb{E}_P\left[\mathbb{1}\left\{T_{(k-1)} \leq \tau < T_{(k)}\right\}N_\tau^y W(\tau)\right]$ by linearity of expectation. We have that for $k \geq 1$,

$$
\begin{aligned}
\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} N^y_\tau W(\tau)\Big] &= \mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \mathbb{1}\{T^a > \tau\} N^y_\tau W(\tau)\Big] \\
&= \mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \mathbb{1}\{T^a > \tau\} \widetilde{N}^y(\tau) W(\tau)\Big] \\
&= \mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) W(\tau)\Big] \\
&= \mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) W\Big(T_{(k-1)}\Big)\Big] \\
&= \mathbb{E}_P\left[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \left(\frac{\mathbb{1}\Big\{A\Big(T_{(k-1)}\Big) = 1\Big\}}{\pi_{k-1}\Big(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\Big)}\right)^{\mathbb{1}\{\Delta_{(k-1)} = a\}} W\Big(T_{(k-2)}\Big)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \mid \mathcal{F}_{T_{(k-2)}}, \Delta_{(k-1)}, T_{(k-1)}, A\Big(T_{(k-1)}\Big)\Big]\right. \\
&\qquad \left. \times \left(\frac{\mathbb{1}\Big\{A\Big(T_{(k-1)}\Big) = 1\Big\}}{\pi_{k-1}\Big(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\Big)}\right)^{\mathbb{1}\{\Delta_{(k-1)} = a\}} W\Big(T_{(k-2)}\Big)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \mid \mathcal{F}_{T_{(k-2)}}, \Delta_{(k-1)}, T_{(k-1)}\Big]\right. \\
&\qquad \left. \times \left(\frac{\mathbb{1}\Big\{A\Big(T_{(k-1)}\Big) = 1\Big\}}{\pi_{k-1}\Big(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\Big)}\right)^{\mathbb{1}\{\Delta_{(k-1)} = a\}} W\Big(T_{(k-2)}\Big)\right] \\
&= \mathbb{E}_P\left[\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \mid \mathcal{F}_{T_{(k-2)}}, \Delta_{(k-1)}, T_{(k-1)}\Big]\right. \\
&\qquad \left. \times \mathbb{E}_P\left[\left(\frac{\mathbb{1}\Big\{A\Big(T_{(k-1)}\Big) = 1\Big\}}{\pi_{k-1}\Big(A\Big(T_{(k-1)}\Big), \mathcal{F}_{T_{(k-2)}}\Big)}\right)^{\mathbb{1}\{\Delta_{(k-1)} = a\}} \mid \mathcal{F}_{T_{(k-2)}}, \Delta_{(k-1)}, T_{(k-1)}\right] W\Big(T_{(k-2)}\Big)\right] \\
&= \mathbb{E}_P\Big[\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \mid \mathcal{F}_{T_{(k-2)}}, \Delta_{(k-1)}, T_{(k-1)}\Big] W\Big(T_{(k-2)}\Big)\Big] \\
&= \mathbb{E}_P\Big[\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau) \mid \mathcal{F}_{T_{(k-3)}}, \Delta_{(k-2)}, T_{(k-2)}, A\Big(T_{(k-2)}\Big)\Big] W\Big(T_{(k-2)}\Big)\Big]
\end{aligned}
$$

Iteratively applying the same argument, we get that $\mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} \widetilde{N}^y(\tau)\Big] = \mathbb{E}_P\Big[\mathbb{1}\Big\{T_{(k-1)} \leq \tau < T_{(k)}\Big\} N^y_\tau W(\tau)\Big]$ as needed. □

By the intersection property of conditional independence, we see that a sufficient condition for the first exchangeability condition in Equation 11 is that

$$
\begin{gathered}
\widetilde{N}^y(\tau) \perp A\Big(T_{(k)}\Big) \mid T_{(j)} \leq \tau < T_{(j+1)}, \mathcal{F}_{T_{(k-1)}}, T_{(k)}, \Delta_{(k)} = a, \quad \forall j \geq k > 0, \\
\mathbb{1}\Big\{T_{(j)} \leq \tau < T_{(j+1)}\Big\} \perp A\Big(T_{(k)}\Big) \mid \mathcal{F}_{T_{(k-1)}}, T_{(k)}, \Delta_{(k)} = a, \quad \forall j \geq k > 0.
\end{gathered}
$$

The second condition may in particular be too strict in practice as the future event times may be affected by prior treatment. While the overall exchangeability condition can be expressed in an alternative form, the consistency condition remains essentially the same. Specifically, let $\tilde{Y}_{\tau,k}$ be the potential outcome at event k corresponding to $\mathbb{1}\left\{T_{(k)} \leq \tau, \Delta_{(k)} = y\right\}$. Then the exchangeability condition is that $\tilde{Y}_{\tau,k} \perp A\left(T_{(j)}\right) \mid \mathcal{F}_{T_{(j-1)}}, T_{(j)}, \Delta_{(j)} = a)$ for $0 \leq j < k$ and $k = 1, ..., K$. However, it has been noted (Gill & Robins (2001)) in discrete time that the existence of multiple potential outcomes can be restrictive and that the resulting exchangeability condition may be too strong.

4 The intervened world

In a hypothetical world where the intervention is implemented all persons are treated until death or t years after the start of treatment, whatever comes first. We could imagine that a pump is inserted under the persons skin which injects the treatment and that this pump cannot be removed or stopped by a general practitioner.

We further assume that there is absolutely no effect on death of visiting the general practitioner in this hypothetical world. Hence, the hypothetical world can be described with a simple two-stage model and stochastic process $(X^*(s) \in \{0,1\})_{s\geq 0}(0 = \text{treated}, 1 = \text{death})$. The target parameter can be expressed as:

$$P(X^*(t) = 1) = \int_0^t \exp\left(-\int_0^s h^*(u)du\right) h^*(s)ds,$$

where h^* is the hazard rate of transitions from state 0 to state 1.

We can as well use an irreversible three state model where death is the only absorbing state and stochastic process $(X^{**}(s) \in \{0,1,2\})_{s\geq 0}(0 = \text{treated}, 1 = \text{has visited Tivoli}, 2 = \text{death})$. Here the irreversible intermediate state is ‘has visited Tivoli’ which should not change the likelihood of death. Note that since we assume absolutely no effect by visiting a general practitioner we could simply exchange ‘Tivoli’ with ‘visit to a general practitioner’ and the mathematical formula are not altered. Let $P_{12}^{**}(s,t) = P(X^{**}(t) = 2|X^{**}(s) = 1)$ and $P_{02}^{**}(s,t) = P(X^{**}(t) = 2|X^{**}(s) = 0)$ In this model the basic assumption is

$$P_{12}^{**}(s,t) = P_{02}^{**}(s,t)$$

for all $s < t$. Hence bunting processes of the transitions

$$N_t^{01,*} = N_t^{01,**} + N_t^{02,**}$$

Hence, we can always find the intensity for $N_t^{01,**}$ and $N_t^{02,**}$ from the transition intensities for the “Tivoli” model, i.e.,

$$h^*(t)\mathbb{1}\{t \leq T^*\} = h^{01,**}(t)\mathbb{1}\{t \leq T^*, \Delta = 2\} + h^{02,**}(t)\mathbb{1}\left\{\Delta = 1, \bar{T} < t \leq T^*)\right\}$$

where T^* denotes the terminal event time in the hypothetical world and Δ denotes initial transition from state 0 to state 1 or 2, $\bar{T} = \inf\left\{t > 0 \mid N_t^{01,**} + N_t^{02,**} = 0\right\}$. By writing up the target parameters in both settings, $h^{01,*}$ can easily be found in terms of $h^{01,**}, h^{02,**}$ and $h^{03,**}$. First note that

$$P_{12}^{**}(s,t) = \int_s^t \exp\left(-\int_s^w (h^{12,**}(u))du\right) h^{12,**}(w)dw$$

$$P_{02}^{**}(s,t) = \int_s^t \exp\left(-\int_s^w (h^{01,**}(u) + h^{02,**}(u))du\right) h^{02,**}(w)dw$$

so that

$$\int_s^t \exp\left(-\int_s^w (h^{12,**}(u))du\right) h^{12,**}(w)dw = \int_s^t \exp\left(-\int_s^w (h^{01,**}(u) + h^{02,**}(u))du\right) h^{02,**}(w)dw$$

by the basic assumption.

$$\begin{aligned}
P(X^{**}(t) = 2) &= \int_0^t S_0^{**}(s-)\left(h^{02,**}(s) + \int_s^t \exp\left(-\int_s^w h^{12,**}(u)du\right) h^{12,**}(w)dwh^{01}(s)\right) \mathrm{d}s \\
&= \int_0^t S_0^{**}(w-)h^{02,**}(w)dw + \int_0^t \int_s^t \exp\left(-\int_s^w h^{12,**}(u)du\right) h^{12,**}(w)dwS_0^{**}(s-)h^{01}(s)ds \\
&= \int_0^t S_0^{**}(w-)h^{02,**}(w)dw + \int_0^t \int_0^s \exp\left(-\int_s^w h^{12,**}(u)du\right) \frac{S_0^{**}(s-)}{S_0^{**}(w-)} h^{01}(s)dsS_0^{**}(w-)h^{12,**}(w \\
&= \int_0^t S_0^{**}(w-)h^{02,**}(w)dw + \int_0^t \int_0^s \exp\left(-\int_s^w h^{12,**}(u)du\right) S_0^{**}(s-)h^{01}(s)dsh^{12,**}(w)dw \\
&= \int_0^t S_0^{**}(s-)h^{02,**}(s)ds \\
&+ \int_0^t \exp\left(-\int_0^w h^{02,**}(u)du\right) \int_0^s \exp\left(-\int_0^s h^{12,**}(u)du\right) h^{01}(s)dsh^{02,**}(w)dw \\
P_{12}^{**}(s,t)h^{01,**}(s))ds &= \int_0^t S_0^{**}(w-)\left(\int_0^t h^{12,**}(s) + 1\right) h^{02,**}(s)ds
\end{aligned}$$

$$\exp\left(-\int_0^s h^*(u)du\right) h^*(s)ds$$

$$= 1 - \exp\left(-\int_0^t h^*(u)du\right),$$

What choice?

$$\int_0^t \int_0^s \exp\left(-\int_s^w h^{12,**}(u)du\right) \frac{S_0^{**}(s-)}{S_0^{**}(w-)} h^{01}(s)dsS_0^{**}(w-)h^{12,**}(w)dw$$

$$\int_0^t \int_0^s \exp\left(-\int_s^w h^{12,**}(u)du\right) \exp\left(\int_s^w h^{01(u)} + h^{02(u)}\right) h^{01}(s)dsS_0^{**}(w-)h^{12,**}(w)dw$$

$$= \int_0^t \int_0^s \exp\left(\int_0^w h^{01,**}(u)du\right) h^{01}(s)dsS_0^{**}(w-)h^{02,**}(w)dw$$

Now note that

$$P(X^{**}(t)=2) = \int_0^t S_0^{**}(s-)\left(h^{02,**}(s) + \int_s^t \frac{S(w-)}{S(s)} h^{02,**}(w) dw h^{01}(s)\right) \mathrm{d}s$$
$$= \int_0^t S_0^{**}(s-) h^{02,**}(s)\,\mathrm{d}s + \int_0^t \int_s^t S(w-) h^{02,**}(w) dw h^{01}(s)\,\mathrm{d}s$$
$$= \int_0^t S_0^{**}(s-) h^{02,**}(s)\,\mathrm{d}s + \int_0^t \int_0^w h^{01}(s)\,\mathrm{d}s S(w-) h^{02,**}(w)\,\mathrm{d}w$$
$$= \int_0^t S_0^{**}(s-)(1+H^{01}(s)) h^{02,**}(s)\,\mathrm{d}s$$
$$P_{12}^{**}(s,t) h^{01,**}(s))ds = \int_0^t S_0^{**}(w-)\left(\int_0^t h^{12,**}(s)+1\right) h^{02,**}(s) ds$$
$$\int_0^t \exp\left(-\int_0^s h^*(u)du\right) h^*(s)ds$$
$$= 1 - \exp\left(-\int_0^t h^*(u)du\right),$$
where
$$S_0^{**}(s) = \exp\left(-\int_0^s (h^{01,**}(u) + h^{02,**}(u))du\right)$$
Conversely, even under the basic assumption, there exist many choices of $h^{01,**}$ that will lead to the same $h^*(t)$, i.e., the basic assumption does not uniquely determine the transition intensities in the "Tivoli" model. For example, $h^{01,**}(t) = 0$ can always be a choice.

5 The observed world

The observed world is described by the four state multi-state model depicted in Figure 2. The model assumes that all persons are treated at time 0 and then allows that some persons visit a general practitioner without changing their treatment and others visit a general practitioner which leads to stopping the treatment. We allow for at most one visitation time per person that is the treatment can only be stopped at a single date in time. We observe the counting processes $N_t = (N_t^{01}, N_t^{02}, N_t^{03}, N_t^{13}, N_t^{23})$ on the canonical filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$, where $\mathcal{F}_t = \sigma(N_s \mid s \leq t)$. This means that we can represent the observed data as $O = \left(T_{(1)}, D_{(1)}, T_{(2)}\right)$, where $T_{(1)}$ is the first event time, $D_{(1)} \in \{01, 02, 03\}$ is the first event type, $A(T_1) = \mathbb{1}\{D_1 \neq 02\}$ is the treatment at the first event time, and $T_{(2)}$ is the second event time, possibly ∞. We will assume that the distributions of the jump times are continuous and that there are no jumps in common between the counting processes. By a well-known result for marked point processes (Proposition 3.1 of Jacod (1975)), we know there exists functions h^{ij}, such that the compensators Λ^{ij} of the counting processes N^{ij} with respect to $P - \mathcal{F}_t$ are given by
$$\Lambda^{0j}(dt) = \mathbb{1}\left\{t \leq T_{(1)}\right\} h^{0j}(t)dt, \quad j = 1,2,3$$
$$\Lambda^{i3}(dt) = \mathbb{1}\left\{T_{(1)} < t \leq T_{(2)}\right\} h^{i3}\left(T_{(1)}, t\right)dt, \quad i = 2,3$$

We let $S_0(t) = \prod_{s \in (0,t]} \left(1 - \sum_j h^{0j}(s) ds\right)$ and $S_1(t \mid d, s) = \prod_{u \in (s,t]} \left(1 - \sum_i h^{i3}(s,u) \mathbb{1}\{d = i\} du\right)$ be the survival functions for the first and second event times, respectively. Furthermore, denote by $P_{0j}(t) = \int_0^t S_{s-} h^{0j}(s) ds$ the probability of having an of type j at time by time t and $P_{i3}(s,t) = \int_s^t S_1(w- \mid d, s) h^{i3}(s,w) dw$ be the probability of having a terminal at time t given that the first event was of type d at time s.

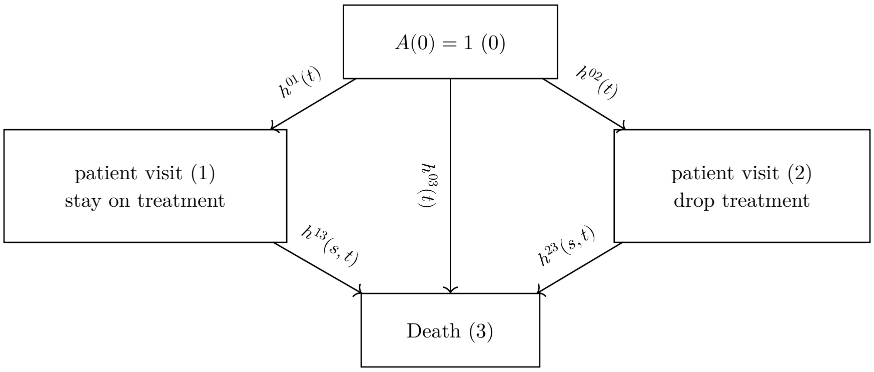

Figure 2: A multi-state model allowing one visitation time for the treatment with the possible treatment values 0/1.

6 The potential outcomes framework

To follow along Ryalen (2024), we restrict the observations to the interval $[0, \tau]$ for $\tau > 0$. We first need to define the intervention of interest, defining the counting processes that we would have like to have observed under the intervention. We can intervene on two components of N (N^{02}, N^{01}), defining the "interventional" processes as

$$N_t^{g,0} = 0$$
$$N_t^{g,1} = N_t^{01} + N_t^{02}$$

This treatment regime defines that the doctor always treats the patient at the visitation time and does not prevent the patient from visiting the doctor if they drop out of the treatment. This thus dictates that an individual that transitioned from 0 to 2 should instead transition to 1. We define $T^{a,g}$ as the first time where the observed and the interventional process deviate.

Define also the single "intervention" process

$$N_t^{g^*,0} = N_t^{g,0} = 0$$

where the interventional component is N^{02}. This dictates that an individual that transitioned from 0 to 2 should not transition to anything at that point. This intuitively thus means that a patient is prevented from visiting the doctor if they drop out of the treatment. The key issue in Ryalen (2024) is that we will not be able to differentiate between identification formulas for g and g^*. The reason is that the likelihood under the intervention only depends on the stopping time T^a and the problem that the stopping time T^a is the same under g and g^*.

To see this, let T^{a,g^*} be the first time where the observed and the interventional process (according to g^*) deviate. We have

$$\begin{aligned} T^{a,g} &= \inf_{t>0} \left\{ N_t^{g,0} \neq N_t^{01} \right\} \wedge \inf_{t>0} \left\{ N_t^{g,1} \neq N_t^{02} \right\} \\ &= \inf_{t>0} \{ N_t^{02} \neq 0 \} \wedge \inf_{t>0} \{ N_t^{02} \neq 0 \} \\ &= \inf_{t>0} \{ N_t^{02} \neq 0 \} \end{aligned}$$

Note that

$$T^{a,g^*} = \inf_{t>0}\left\{N_t^{g^*,0} \neq 0\right\} = \inf_{t>0}\left\{N_t^{g,0} \neq 0\right\}$$

so that $T^{a,g^*} = T^{a,g}$. Applying Theorem 6.1, we find that the identification formulas are the same because the weights W_t are the same under g and g^*. Also note that $\mathbb{1}\{T^a \leq t\} = N^{02}(t)$.

We now define the identification formula of interest in Ryalen (2024). The outcome of interest is death at time t, i.e.,

$$Y_t = N_t^{13} + N_t^{03} + N_t^{23} = \mathbb{1}\{T_1 \leq t, D_1 = y\} + \mathbb{1}\{T_2 \leq t\}$$

and we want to estimate $\mathbb{E}_P\left[\tilde{Y}_t\right]$ where $\tilde{Y}_t$ denotes the outcome at time t, had the treatment regime (staying on treatment), possibly contrary to fact, been followed.

Theorem 6.1 (Theorem 1 of Ryalen (2024)): We suppose that there exists a potential outcome process $\left(\tilde{Y}_t\right)_{t\in[0,\tau]}$ such that

1. Consistency: $\tilde{Y}_t\mathbb{1}\left\{T^A > t\right\} = Y_t\mathbb{1}\left\{T^A > t\right\}$ for all $t > 0$ P-a.s.
2. Exchangeability: The $P - \mathcal{F}_t$ compensators Λ^{01}, Λ^{02} are also compensators for $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(\tilde{Y}_s, \tau \geq s \geq 0\right)$. Here $\tilde{Y}_s$ is added at baseline, so that $\mathcal{G}_0 = \sigma\left(\tilde{Y}_s, \tau \geq s \geq 0\right)$.
3. Positivity: $W_t = \frac{\mathbb{1}\left\{T^A > t\right\}}{\exp(-\Lambda^{02}(t))} = \frac{1-\mathbb{1}\left\{T_{(1)} \leq t, D_{(1)} = a, A_{(1)} = 0\right\}}{\exp\left(-\int_0^t \mathbb{1}\left\{s \leq T_{(1)}\right\} h^a(s)\pi_s(0) ds\right)}$[6] is a uniformly integrable martingale or equivalently that $R^{\text{Pål}}$ given by $dR^{\text{Pål}} = W_\tau dP$ is a probability measure.

   Then the estimand of interest $\Psi_t^{\text{Ryalen}} : \mathcal{M} \to \mathbb{R}_+$ is identifiable by

$$\Psi_t^{\text{Ryalen}}(P) := \mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[Y_t W_t] = \mathbb{E}_{R^{\text{Pål}}}[Y_t]$$

From this, we can derive an alternate representation of the identification formula. We have that

[6] In the notation of Ryalen (2024), $\tau^A = T^a$, $\mathbb{N}_t = \mathbb{1}\left\{T^A \leq t\right\} = N_t^{02}$ and Λ_t^{02} is the compensator of this process.

$$
\begin{aligned}
\Psi_t^{\text{Ryalen}}(P) &= \mathbb{E}_P\left[\mathbb{1}\left\{T_{(1)} \le t\right\}Y_t W_t\right] + \mathbb{E}_P\left[\mathbb{1}\left\{T_{(2)} \le t\right\}Y_t W_t\right] \\
&= \mathbb{E}_P\left[\mathbb{1}\left\{T_{(1)} \le t\right\}Y_t \frac{1-\mathbb{1}\{T^a > t\}}{\exp\left(-\int_0^{T_{(1)}} h^{02}(s)ds\right)}\right] \\
&\quad + \mathbb{E}_P\left[\mathbb{1}\left\{T_{(2)} \le t\right\}Y_t \frac{1-\mathbb{1}\{T^a > t\}}{\exp\left(-\int_0^{T_{(1)}} h^{02}(s)ds\right)}\right] \\
&= \mathbb{E}_P\left[\mathbb{1}\left\{T_{(1)} \le t, D_{(1)} = 03\right\} \frac{1}{\exp\left(-\int_0^{T_{(1)}} h^{02}(s)ds\right)}\right] \\
&\quad + \mathbb{E}_P\left[\mathbb{1}\left\{T_{(2)} \le t, D_{(1)} = 01\right\} \frac{1}{\exp\left(-\int_0^{T_{(1)}} h^{02}(s)ds\right)}\right] \qquad (12) \\
&= \int_0^t \frac{1}{\exp\left(-\int_0^t h^{02}(s)ds\right)} P_{03}(dt) \\
&\quad + \int_0^t \frac{1}{\exp\left(-\int_0^t h^{02}(s)ds\right)} P_{13}(s,t) P_{01}(ds) \\
&= \int_0^t \exp\left(-\sum_{j \neq 2} \int_0^s h^{0j}(u)du\right) h^{03}(s)ds \\
&\quad + \int_0^t \exp\left(-\sum_{j \neq 2} \int_0^s h^{0j}(u)du\right) P_{13}(s,t) h^{01}(s)ds
\end{aligned}
$$

6.1 The identification formula in Rytgaard et al. (2022)

To discuss Rytgaard et al. (2022), additionally define

$$\Lambda^a(t) = (h^{01}(t) + h^{02}(t))\mathbb{1}\left\{T_{(1)} \le t\right\}$$

$$\pi_t(1) = \frac{h^{01}(t)}{h^{01}(t) + h^{02}(t)}$$

Here, we can interpret $\Lambda^a(t)$ as the cumulative intensity of the visitation times (i.e., $N_t^a = N_t^{01} + N_t^{02}$) and $\pi_t(1)$ as the probability of being treated given that you go to the doctor at time t. Furthermore, let $N_t^d = N_t^{03} + N_t^{13} + N_t^{23}$ be the counting process for the event of interest. Then, its compensator is given by

$$
\begin{aligned}
\Lambda^d(dt) &= \mathbb{1}\left\{t \le T_{(1)}\right\} h^{03}(t)dt \\
&\quad + \mathbb{1}\left\{T_{(1)} < t \le T_{(2)}\right\}\left(\mathbb{1}\left\{D_{(1)} = 01\right\} h^{13}\left(T_{(1)}, t\right) + \mathbb{1}\left\{D_{(1)} = 02\right\} h^{23}\left(T_{(1)}, t\right)\right)dt
\end{aligned}
$$

Furthermore, let $A(t) = \mathbb{1}\left\{T_{(1)} > t\right\} + \mathbb{1}\left\{T_{(1)} \le t, D_{(1)} \neq 02\right\}$ be the treatment process at time t. Notationwise, we also define $\Delta N(t) = N_t - N_{t-}$ for a cadlag process N. Rytgaard et al. (2022) give their likelihood as

$$
\begin{aligned}
dP(O) &= \prod_{t\in(0,\tau]} \Big(d\Lambda^a(t)(\pi_t(1))^{\mathbb{1}\{A(t)=1\}}(1-\pi_t(1))^{\mathbb{1}\{A(t)=0\}}\Big)^{\Delta N^a(t)}(1-d\Lambda^a(t))^{1-\Delta N^a(t)} \\
&\times \prod_{t\in(0,\tau]} \big(d\Lambda^d(t)\big)^{\Delta N^d(t)}\big(1-d\Lambda^d(t)\big)^{1-\Delta N^d(t)} \\
&= \prod_{t\in(0,\tau]} \Big((\pi_t(1))^{\mathbb{1}\{A(t)=1\}}(1-\pi_t(1))^{\mathbb{1}\{A(t)=0\}}\Big)^{\Delta N^a(t)} \\
&\times \prod_{t\in(0,\tau]} (d\Lambda^a(t))^{\Delta N^a(t)}(1-d\Lambda^a(t))^{1-\Delta N^a(t)}\big(d\Lambda^d(t)\big)^{\Delta N^d(t)}\big(1-d\Lambda^d(t)\big)^{1-\Delta N^d(t)} \\
&= \prod_{t\in(0,\tau]} dG_t dQ_t
\end{aligned}
$$

where

$$
dG_t = \Big((\pi_t(1))^{\mathbb{1}\{A(t)=1\}}(1-\pi_t(1))^{\mathbb{1}\{A(t)=0\}}\Big)^{\Delta N^a(t)}
$$

$$
dQ_t = (d\Lambda^a(t))^{\Delta N^a(t)}(1-d\Lambda^a(t))^{1-\Delta N^a(t)}\big(d\Lambda^d(t)\big)^{\Delta N^d(t)}\big(1-d\Lambda^d(t)\big)^{1-\Delta N^d(dt)}
$$

Let $dG_t^* = \big((1)^{\mathbb{1}\{A(t)=1\}}(0)^{\mathbb{1}\{A(t)=0\}}\big)^{\Delta N^a(t)} = \big((0)^{\mathbb{1}\{A(t)=0\}}\big)^{\Delta N^a(t)}$, corresponding to staying on treatment. Then define the interventional density as

$$
dP_{Q,G^*}(O) = \prod_{t\in(0,\tau]} dG_t^* dQ_t
$$

and their target estimand $\Psi_t^{\text{Rytgaard}} : \mathcal{M} \to \mathbb{R}_+$ as

$$
\Psi_\tau^{\text{Rytgaard}}(P) = \mathbb{E}_{P_{Q,G^*}}[N_\tau^d] = \int_{\mathcal{O}} y \prod_{t\in(0,\tau]} dG_t^* dQ_t \tag{13}
$$

We first need to define the integral in Equation 13. To get a fully rigorous result, consider Proposition 1 in Ryalen (2024) and Theorem 8.1.2 in Last & Brandt (1995).

First note that we have

$$
\prod_{t\in(0,\tau]} dG_t^* dQ_t = \prod_{t\in\left(0,t_{(1)}\right]} dG_t^* dQ_t \prod_{t\in\left(t_{(1)},\tau\right]} dG_t^* dQ_t \mathbb{1}\{t_1 < t_2\}
$$

Let $Y_\tau = \mathbb{1}\Big\{T_{(1)} \le \tau, D_{(1)} = 03\Big\} + \mathbb{1}\Big\{T_{(2)} \le \tau\Big\} := Y_\tau^{(1)} + Y_\tau^{(2)}$ be death at time τ. Then, note that

$$
y_\tau^{(1)}(t_1, d_1) \prod_{t\in\left(0,t_{(1)}\right]} dG_t^* dQ_t \prod_{t\in\left(t_{(1)},\tau\right]} dG_t^* dQ_t \mathbb{1}\{t_1 < t_2\} = y_\tau^{(1)} \prod_{t\in\left(0,t_{(1)}\right]} dG_t^* dQ_t
$$

The second product integral evaluates to 1 because death at event 1 implies that all intensities are 0 after the first event.

We find

$$
\begin{aligned}
\prod_{t \in (0, t_{(1)}]} dG_t^* dQ_t &= \left((0)^{\mathbb{1}\{d_1 = 02\}}\right)^{\mathbb{1}\{d_1 \in \{01,02\}\}} (d\Lambda^a(t_1))^{\mathbb{1}\{d_1 \in \{01,02\}\}} \left(d\Lambda^d(t_1)\right)^{\mathbb{1}\{d_1 = 03\}} \\
&\quad \times \prod_{t \in (0, t_{(1)})} \left(1 - d\Lambda^d(t)\right)(1 - d\Lambda^a(t)) \\
&= (d\Lambda^a(t_1))^{\mathbb{1}\{d_1 \in \{01\}\}} (0 dt_1)^{\mathbb{1}\{d_1 \in \{02\}\}} \left(d\Lambda^d(t_1)\right)^{\mathbb{1}\{d_1 = 03\}} S_0(t_1 -) \\
&= \left((h^{01}(t_1) + h^{02}(t_1)) dt_1\right)^{\mathbb{1}\{d_1 \in \{01\}\}} (0 dt_1)^{\mathbb{1}\{d_1 \in \{02\}\}} \left(h^{03}(t_1) dt_1\right)^{\mathbb{1}\{d_1 = 03\}} S_0(t_1 -) \\
&= S_0(t_1 -) \mathbb{1}\{d_1 = 01\} (h^{01}(t_1) + h^{02}(t_1)) dt_1 \\
&\quad + S_0(t_1 -) \mathbb{1}\{d_1 = 03\} h^{03}(t_1) dt_1
\end{aligned}
$$

(compare with Equation (11) in Ryalen (2024)). In the second equality, we used that the counting processes do not jump at the same time with probability one to get $S(t_1) = \prod_{t \in (0, t_{(1)}]} \left(1 - d\Lambda^d(t)\right)(1 - d\Lambda^a(t))$. But multiplying with $y_\tau^{(1)}$, we find

$$
y_\tau^{(1)} \prod_{t \in (0, t_{(1)}]} dG_t^* dQ_t = y_\tau^{(1)} S(t_1 -) \mathbb{1}\{d_1 = 03\} h^{03}(t_1) dt_1
$$

Therefore, we have

$$
\int y_\tau^{(1)} \prod_{t \in (0, t_{(1)}]} dG_t^* dQ_t = \int_0^\tau S(s -) h^{03}(s) ds
$$

Similarly, we may find

$$
\begin{aligned}
& y_\tau^{(2)} \prod_{t \in (0, t_{(1)}]} dG_t^* dQ_t \prod_{t \in (t_{(1)}, t_{(2)}]} dG_t^* dQ_t \mathbb{1}\{t_1 < t_2\} \\
&= y_\tau^{(2)} \mathbb{1}\{t_1 < t_2\} S(t_1 -) \mathbb{1}\{d_1 = 01\} (h^{01}(t_1) + h^{02}(t_1)) \\
&\quad \times S(t_2 - \mid 01, t_1) h^{13}(t_1, t_2) dt_2 dt_1
\end{aligned}
$$

Thus the target estimand is

$$
\begin{aligned}
\Psi_\tau^{\text{Rytgaard}}(P) &= \int_0^\tau S_0(s -) h^{03}(s) ds \\
&\quad + \int_0^\tau S_0(s -) P_{13}(s, \tau) (h^{01}(s) + h^{02}(s)) ds
\end{aligned}
\tag{14}
$$

6.2 Comparison of the approaches

We are now in a position, where we can readily compare the approaches in Rytgaard et al. (2022) and Ryalen (2024) by considering the difference between Equation 14 and Equation 12.

Suppose that $h^{02}(s) > 0$ and $h^{13}(s, w) > 0$ for Lebesgue almost all s, w. From this, we conclude that $\Psi_\tau^{\text{Rytgaard}}(P) = \Psi_\tau^{\text{Ryalen}}(P)$ if and only if $h^{02} \equiv 0$ a.e. if and only if $\pi_t(1) \equiv 1$ a.e. (with respect to the Lebesgue measure restricted to $[0, \tau]$). To see this, note that

$$
\begin{aligned}
\Psi_\tau^{\text{Ryalen}}(P) - \Psi_\tau^{\text{Rytgaard}}(P) = &\int_0^\tau \exp\left(-\sum_{j\neq 2}\int_0^s h^{0j}(u)du\right)\left(1-\exp\left(-\int_0^s h^{02}(u)du\right)\right)h^{03}(s)ds \\
&+ \int_0^\tau \exp\left(-\sum_{j\neq 2}\int_0^s h^{0j}(u)du\right)P_{13}(s,\tau) \\
&\qquad \times\left(1-\exp\left(-\int_0^s h^{02}(u)du\right)\right)h^{01}(s)ds \\
&+ \int_0^\tau S_0(s-)P_{13}(s,\tau)h^{02}(s)ds
\end{aligned} \tag{15}
$$

Since each term is non-negative, $\Psi_\tau^{\text{Rytgaard}}(P) = \Psi_\tau^{\text{Ryalen}}(P)$ implies that each term is equal to zero. Since each of the integrands are non-negative, we must have that the integrands are equal to zero (almost surely). By letting $m_{[0,\tau]}$ denote the Lebesgue measure on $[0,\tau]$, we have for the first term in Equation 15,

$$
\begin{aligned}
\exp\left(-\sum_{j\neq 2}\int_0^s h^{0j}(u)du\right)\left(1-\exp\left(-\int_0^s h^{02}(u)du\right)\right)h^{03}(s) = 0 \quad & m_{[0,\tau]} - \text{almost all } s \Leftrightarrow \\
\left(1-\exp\left(-\int_0^s h^{02}(u)du\right)\right)h^{03}(s) = 0 \quad & m_{[0,\tau]} - \text{almost all } s \Leftrightarrow \\
\left(1-\exp\left(-\int_0^s h^{02}(u)du\right)\right) = 0 \quad & m_{[0,\tau]} - \text{almost all } s \Leftrightarrow \\
h^{02}(s) = 0 \quad & m_{[0,\tau]} - \text{almost all } s
\end{aligned}
$$

with similar arguments for the second and third terms in Equation 15.

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