Let us consider the setting of Ryalen (2024) and their regularity conditions. Specifically, we will work with an intervention that specifies the treatment decisions but not the timing of treatment visits. We work with Example 4 of Ryalen (2024), in which

$$\pi^*(\varphi, t, \mathrm{d}x) = \delta_{a_0}(\mathrm{d}x),$$

i.e., treatment is always assigned to a_0. To simplify, we work without right-censoring and no covariates. This means that (N^y, N^a), where N^y denotes the counting process on $[0, T]$ for death and N^a random measure for treatment on $[0, T] \times \{a_0, a_1\}$. For this treatment regime, we see that

$$\tau^A = \inf\{t \geq 0 \mid N^a((0, t] \times \{a_1\}) > 0\}.$$

We can associate each of the random measures N^y and N^a with the random measure

$$N(\mathrm{d}(t, m, a)) = N^y(\mathrm{d}t)\delta_y(\mathrm{d}m) + \delta_a(\mathrm{d}m)\left\{N^a(\mathrm{d}(t) \times \{a_0\})\delta_{a_0}(\mathrm{d}m) + N^a(\mathrm{d}(t) \times \{a_1\})\delta_{a_1}(\mathrm{d}m)\right\}.$$

This gives rise to a counting process filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by N. We can then find that N has the compensator

$$\Lambda(\mathrm{d}(t, m, a)) = \Lambda^y(\mathrm{d}t)\delta_y(\mathrm{d}m) + \delta_a(\mathrm{d}m)\left\{\pi_t(\mathcal{F}_{t-})\delta_{a_0}(\mathrm{d}m) + (1 - \pi_t(\mathcal{F}_{t-}))\delta_{a_1}(\mathrm{d}m)\right\}\Lambda^a(\mathrm{d}(t)),$$

where we can choose π_t to be $\mathcal{F}_t$-predictable. We are interested in the counterfactual mean outcome $\mathbb{E}_P\left[\tilde{Y}_t\right]$, where $\left(\tilde{Y}_t\right)_{t \geq 0}$ is the counterfactual outcome process of $Y := N^y$ under the intervention that sets treatment to a_0 at all visitation times. Note the different exchangeability condition compared to Ryalen (2024), as Ryalen (2024) expresses exchangeability through the counting process $\mathbb{1}\{\tau^A \leq \cdot\}$; this is actually a weaker condition. Let $\left(T_{(k)}, \Delta_{(k)}, A\left(T_{(k)}\right)\right)$ denote the ordered event times, event types, and treatment decisions. Note that Equation 1 is the same likelihood ratio as in Rytgaard et al. (2022). We also impose the assumption that $N_t := N^y_t + N^{a(\{(0,t] \times \{a_0, a_1\}\})}$ does not explode; we also assume that we work with a version of the compensator such that $\Lambda(\{t\} \times \{y, a\} \times \{a_0, a_1\}) < \infty$ for all $t > 0$. We may generally also work with a compensator Λ that fulfills conditions (10.1.11)-(10.1.13) of Last & Brandt (1995).

NOTE: So the issue is that **Positivity** might not actually hold. If we look at $W(t)$, then it is piecewise constant and only jumps at event times. If it were generally a likelihood ratio, then it would solve Equation 4. However, the second term is not generally piecewise constant, so we have placed restrictions (in this case $V(s, x)$ would have to be purely discontinuous $\overset{?}{\Rightarrow}$ predictable visitation times (note: special case discrete compensator for treatment)). In that case, local independence cannot even motivate this estimand.

- **What about pointwise identification**?
- $\mathbb{E}_P[W(\tau)] = 1$ but not necessarily $\mathbb{E}_P[W(t)] \neq 1$ for all t, so that $W(t)$ is not generally a martingale, then it still be possible to reweight as follows $\mathbb{E}_P\left[\tilde{Y}_\tau\right] = \mathbb{E}_P[Y_\tau W(\tau)]$.
- **Might be relevant:** https://pmc.ncbi.nlm.nih.gov/articles/PMC3857358/pdf/nihms529556.pdf

Theorem 0.1: Define

$$\zeta(t, m, a) \coloneqq \mathbb{1}\{m = y\} + \mathbb{1}\{m = a\} \frac{\mathbb{1}\{a = a_0\}}{\pi_t}$$

If *all* of the following conditions hold:

- **Consistency**: $\tilde{Y}_t \mathbb{1}\{T^a > \cdot\} = Y_t \mathbb{1}\{T^a > \cdot\} \quad P - \text{a.s.}$
- **Exchangeability**: Define $\mathcal{H}_t \coloneqq \mathcal{F}_t \vee \sigma\left(\tilde{Y}\right)$. The P-$\mathcal{F}_t$ compensator for N^a is also the P-$\mathcal{H}_t$ compensator.
- **Positivity**:

$$W_t \coloneqq \prod_{j=1}^{N_t} \left(\frac{\mathbb{1}\left\{A\left(T_{(j)}\right) = a_0\right\}}{\pi_{T_{(j)}}\left(\mathcal{F}_{T_{(j-1)}}\right)} \right)^{\mathbb{1}\left\{\Delta_{(j)} = a\right\}} \tag{1}$$

  fulfills that $\int_0^t W(s-) V(s, m, a) (N(d(s, m, a)) - \Lambda(d(s, m, a))$ is a zero mean square-integrable, P-$\mathcal{F}_t$-martingale.

Then,

$$\mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[Y_t W_t]$$

Proof: We shall use that the likelihood ratio solves a specific stochastic differential equation. To this end, we define the random measure, $\mu(d(t, m, a)) \coloneqq \zeta(t, m, a) \nu(d(t, m, a))$, where $\nu \coloneqq \Lambda$. The likelihood ratio process $L(t)$ given in (10.1.14) of Last & Brandt (1995) is defined by

$$\begin{aligned} L(t) = {} & \mathbb{1}\{t < T_\infty \wedge T_\infty(\nu)\} L_0 \prod_{n : T_{(n)} \leq t} \zeta\left(T_{(n)}, \Delta_{(n)}, A\left(T_{(n)}\right)\right) \\ & \prod_{\substack{s \leq t \\ N_{s-} = N_s}} \frac{1 - \bar{\nu}\{s\}}{1 - \bar{\mu}\{s\}} \exp\left(\int \mathbb{1}\{s \leq t\} (1 - \zeta(s, m, a)) \nu^c(d(s, m, a)) \right) \\ & + \mathbb{1}\{t \geq T_\infty \wedge T_\infty(\nu)\} \liminf_{s \to T_\infty \wedge T_\infty(\nu)} L(s). \end{aligned} \tag{2}$$

Here $T_\infty \coloneqq \lim_n T_n$, $T_\infty(\nu) \coloneqq \inf\{t \geq 0 \mid \nu((0, t] \times \{y, a\} \times \{a_0, a_1\}) = \infty\}$, $\bar{\mu}(\cdot) \coloneqq \mu(\cdot \times \{y, a\} \times \{a_0, a_1\})$, $\bar{\nu}(\cdot) \coloneqq \nu(\cdot \times \{y, a\} \times \{a_0, a_1\})$, $\nu^c(\mathrm{d}(s, m, a)) \coloneqq \mathbb{1}\{\bar{\nu}\{s\} = 0\} \nu(d(s, m, a))$, and $L_0 \coloneqq W(0) = 1$.

By our assumptions, $T_\infty = \infty \quad P$-a.s. and thus $T_\infty(\nu) = T_\infty = \infty$ in view Theorem 4.1.7 (ii) of Last & Brandt (1995) since $\bar{\nu}\{t\} < \infty$ for all $t > 0$.

Second, note that $\bar{\nu} = \bar{\mu}$. This follows since

$$
\begin{aligned}
\bar{\mu}(A) &= \int_{A\times\{y,a\}\times\{a_0,a_1\}} \zeta(t,m,a)\nu(d(t,m,a)) \\
&= \int_{A\times\{y\}\times\{a_0,a_1\}} \zeta(t,m,a)\nu(d(t,m,a)) + \int_{A\times\{a\}\times\{a_0,a_1\}} \zeta(t,m,a)\nu(d(t,m,a)) \\
&= \int_{A\times\{y\}\times\{a_0,a_1\}} 1\nu(d(t,m,a)) + \int_{A\times\{a\}\times\{a_0,a_1\}} \left(\frac{\mathbb{1}\{a=a_0\}}{\pi_t}\right)\nu(d(t,m,a)) \\
&= \nu(A\times\{y\}\times\{a_0,a_1\}) + \int_A \Lambda^a(dt) \\
&= \nu(A\times\{y\}\times\{a_0,a_1\}) + \nu(A\times\{a\}\times\{a_0,a_1\}) \\
&= \nu(A\times\{y,a\}\times\{a_0,a_1\}) = \bar{\nu}(A).
\end{aligned}
$$

Thus

$$
\prod_{\substack{s\le t\\ N_{s-}=N_s}} \frac{1-\bar{\nu}\{s\}}{1-\bar{\mu}\{s\}} \exp\left(\int \mathbb{1}\{s\le t\}(1-\zeta(s,m,a))\nu^c(d(s,m,a))\right) = 1,
$$

and hence

$$
\begin{aligned}
L(t) &= \prod_{n:T_{(n)}\le t} \zeta\left(T_{(n)}, \Delta_{(n)}, A\left(T_{(n)}\right)\right) \\
&\overset{\text{def.}}{=} W(t).
\end{aligned}
$$

Let $V(s,m,a) = \zeta(s,m,a) - 1 + \frac{\bar{\nu}\{s\}-\bar{\mu}\{s\}}{1-\bar{\mu}\{s\}} = \zeta(s,m,a) - 1$. $L(t)$ will fulfill that

$$
L(t) = L_0 + \int \mathbb{1}\{s\le t\}V(s,m,a)L(s-)[\Phi(d(s,m,a)) - \nu(d(s,m,a))]
$$

if

$$
\begin{gathered}
\mathbb{E}_P[L_0] = 1, \\
\bar{\mu}\{t\} \le 1, \\
\bar{\mu}\{t\} = 1 \;\text{ if }\; \bar{\nu}\{t\} = 1, \\
\bar{\mu}[T_\infty \wedge T_\infty(\mu)] = 0 \;\text{ and }\; \bar{\nu}[T_\infty \wedge T_\infty(\nu)] = 0.
\end{gathered} \tag{3}
$$

by Theorem 10.2.2 of Last & Brandt (1995). These can be easily verified.

Thus,

$$
W(t) = 1 + \int_0^t W(s-)V(s,m,a)(\Phi(d(s,m,a)) - \nu(d(s,m,a)). \tag{4}
$$

and it follows that $\int_0^t W(s-)V(s,m,a)(\Phi(d(s,m,a)) - \nu(d(s,m,a))$ is a zero mean$P$-$\mathcal{H}_t$-martingale. From this, we see that $\int_0^t \tilde{Y}_t W_{s-} V(s,m,a)(\Phi(d(s,m,a)) - \nu(d(s,m,a))$ is also a zero mean P-$\mathcal{H}_t$-martingale. This implies that

$$
\begin{aligned}
\mathbb{E}_P[Y_t W_t] &\overset{\text{(consistency)}}{=} \mathbb{E}_P\left[\tilde{Y}_t W_t\right] \\
&= \mathbb{E}_P\left[\tilde{Y}_t\right] + \mathbb{E}_P[\int_0^t \tilde{Y}_t W_{s-} V(s,m,a)(\Phi(d(s,m,a)) - \nu(d(s,m,a))] \\
&= \mathbb{E}_P\left[\tilde{Y}_t\right].
\end{aligned}
$$

□

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