A Novel Approach to the Estimation of Causal Effects of Multiple Event Point Interventions in Continuous Time

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Abstract

In medical research, causal effects of treatments that may change over time on an outcome can be defined in the context of an emulated target trial. We are concerned with estimands that are defined as contrasts of the absolute risk that an outcome event occurs before a given time horizon τ under prespecified treatment regimens. Most of the existing estimators based on observational data require a projection onto a discretized time scale Rose & van der Laan (2011). We consider a recently developed continuous-time approach to causal inference in this setting Rytgaard et al. (2022), which theoretically allows preservation of the precise event timing on a subject level. Working on a continuous-time scale may improve the predictive accuracy and reduce the loss of information. However, continuous-time extensions of the standard estimators comes at the cost of increased computational burden. We will discuss a new sequential regression type estimator for the continuous-time framework which estimates the nuisance parameter models by backtracking through the number of events. This estimator significantly reduces the computational complexity and allows for efficient, single-step targeting using machine learning methods from survival analysis and point processes, enabling robust continuous-time causal effect estimation.

1 Introduction

In medical research, the estimation of causal effects of treatments over time is often of interest. We consider a longitudinal continuous-time setting that is very similar to Rytgaard et al. (2022) in which patient characteristics can change at subject-specific times. This is the typical setting of registry data, which usually contains precise information about when events occur, e.g., information about drug purchase history, hospital visits, and laboratory measurements. This approach offers an advantage over discretized methods, as it eliminates the need to select a time grid mesh for discretization, which can affect both the bias and variance of the resulting estimator. A continuous-time approach would adapt to the events in the data. Furthermore, continuous-time data captures more precise information about when events occur, which may be valuable in a predictive sense. Let τ_{end} be the end of the observation period. We will focus on the estimation of the interventional cumulative incidence function in the presence of time-varying confounding at a specified time horizon $\tau < \tau_{\text{end}}$.

Assumption 1 (Bounded number of events): In the time interval $[0, \tau_{\text{end}}]$ there are at most $K - 1 < \infty$ many changes of treatment and covariates in total for a single individual. Without loss of register data applications, we assume that the maximum number of treatment and covariate changes of an individual is bounded by $K = 10,000$. Practically, we shall adapt K to our data and our target parameter. We let $K - 1$ be given by the maximum number of non-terminal events for any individual in the data.

Assumption 2 (No simultaneous jumps): The counting processes N^a, N^ℓ, N^y, N^d, and N^c have with probability 1 no jump times in common.

Let $\kappa_i(\tau)$ be the number of events for individual i up to time τ. In Rytgaard et al. (2022), the authors propose a continuous-time LTMLE for the estimation of causal effects in which a single step of the targeting procedure must update each of the nuisance estimators $\sum_{i=1}^n \kappa_i(\tau)$ times. We propose an estimator where the number of nuisance parameters is reduced to $\sim \max_i \kappa_i(\tau)$ in total, and, in principle, only one step of the targeting procedure is needed to update all nuisance parameters. We provide an iterative conditional expectation formula that, like Rytgaard et al. (2022), iteratively updates the nuisance parameters. The key difference is that the estimation of the nuisance parameters can be performed by going back in the number of events instead of going back in time. The different approaches are illustrated in Figure 2 and Figure 3 for an outcome Y of interest. Moreover, we argue that the nuisance components can be estimated with existing machine learning algorithms from the survival analysis and point process literature. As always let $(\Omega, \mathcal{F}, P)$ be a probability space on which all processes and random variables are defined.

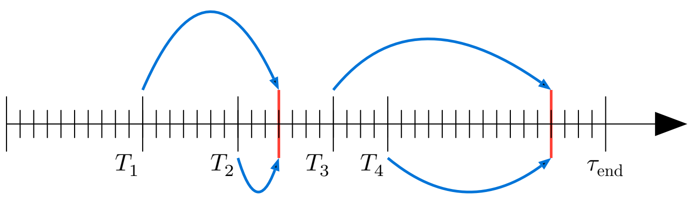


Figure 1: The "usual" approach where time is discretized. Each event time and its corresponding mark is rolled forward to the next time grid point, that is the values of the observations are updated based on the on the events occuring in the previous time interval.

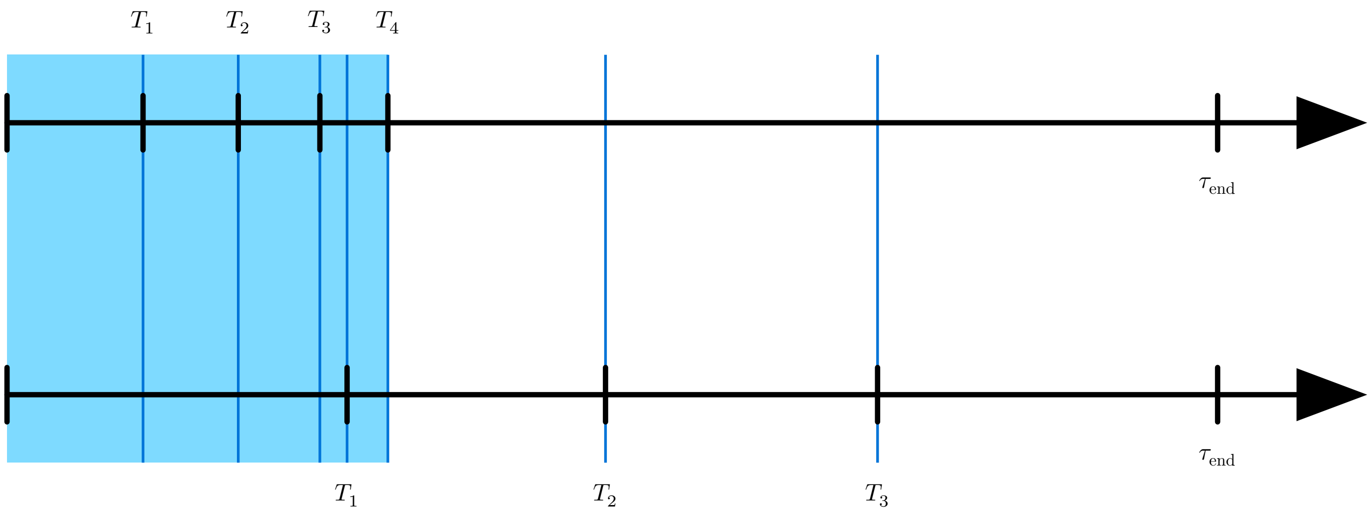


Figure 2: The figure illustrates the sequential regression approach given in Rytgaard et al. (2022) for two observations: Let $t_1 < \ldots < t_m$ be all the event times in the sample. Then, given $\mathbb{E}_Q\left[Y \mid \mathcal{F}_{t_r}\right]$, we regress back to $\mathbb{E}_Q\left[Y \mid \mathcal{F}_{t_{r-1}}\right]$ (through multiple regressions).

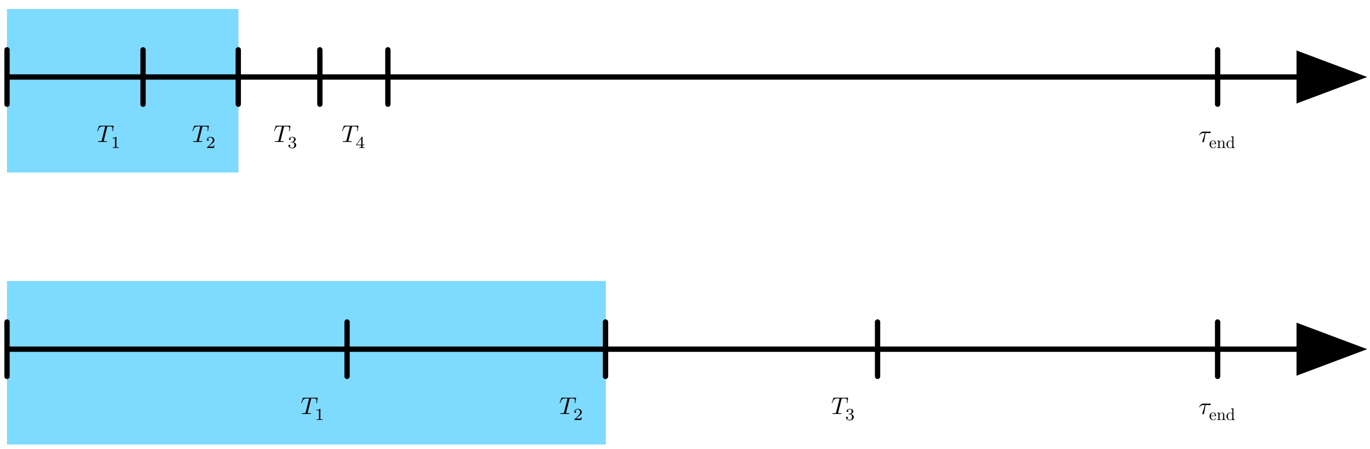


Figure 3: The figure illustrates the sequential regression approach proposed in this article. For each event k in the sample, we regress back on the history $\mathcal{F}_{T_{(k-1)}}$. That is, given $\mathbb{E}_Q\left[Y \mid \mathcal{F}_{T_{(k)}}\right]$, we regress back to $\mathbb{E}_Q\left[Y \mid \mathcal{F}_{T_{(k-1)}}\right]$. In the figure, $k = 3$.

2 Setting and Notation

First, we assume that at baseline, we observe the treatment A_0 and the time-varying confounders at time 0, L_0. The time-varying confounders can in principle include covariates which do not change over time, but for simplicity of notation, we will include them among those that do change over time. Also, we assume that we have two treatment options, $A(t) = 0, 1$ (e.g., placebo and active treatment). The time-varying confounders and treatment are assumed to take values in $\mathbb{R}^m$ and $\mathbb{R}$, and that $L(t) : \Omega \to \mathbb{R}^m$ and $A(t) : \Omega \to \mathbb{R}$ are measurable for each $t \geq 0$, respectively. These processes are assumed to be càdlàg, i.e., right-continuous with left limits. Furthermore, the times at which the treatment and covariates may only change at the jump times of the counting processes N^a and N^ℓ, respectively which makes $L(t)$ and $A(t)$ into jump processes (Last & Brandt (1995)). The jump times of these counting processes thus represent visitation times.

We are interested in the cumulative incidence function, so we also observe N^y and N^d corresponding to the counting processes for the primary and competing event, respectively. Finally, let N^c be the counting process for the censoring counting process. Our the outcome of interest is $Y_\tau = I(T \leq \tau, \Delta = y)$, where T is the time of the terminal event and $\Delta \in \{y, d\}$ is the indicator for which terminal event occurred. We assume that the jump times differ with probability 1 (Assumption 2). Moreover, we assume that only a bounded number of events occur for each individual in the time interval $[0, \tau_{\text{end}}]$ (Assumption 1).

We consider the framework in Rytgaard et al. (2022) and cast it into the framework of marked point processes. To this end, we can define the jump process M as

$$M(s) = \left(N^y(s), N^d(s), N^c(s), L(s), N^\ell(s), A(s), N^a(s)\right) \tag{1}$$

and consider its corresponding natural filtration by

$$\mathcal{F}_t^M = \sigma\left(N^y(s), N^d(s), N^c(s), L(s), N^\ell(s), A(s), N^a(s) \mid s \leq t\right) \tag{2}$$

and the corresponding point process given by

$$(\pi_n(M), k_n(M)) \tag{3}$$

where $\pi_n(M) = T_{(n)}$ is the n'th jump time of M and

$$k_n(M) = \begin{cases} \left(N^y\left(T_{(n)}\right), N^d\left(T_{(n)}\right), N^c\left(T_{(n)}\right), L\left(T_{(n)}\right), N^\ell\left(T_{(n)}\right), A\left(T_{(n)}\right), N^a\left(T_{(n)}\right)\right) \text{ if } T_{(n)} < \infty \\ \nabla \text{ otherwise} \end{cases} \tag{4}$$

Consider the counting process N of M given by

$$N(dt, dx) = \sum_{n=1}^{K} \delta_{(\pi_n(M), k_N(M))}(dt, dx) \tag{5}$$

By reparametrization and Assumption 2, we can essentially use the random measure

$$\widetilde{N}(dt, dx) = \sum_{n=1}^{K} \delta_{\pi_n(M)}(dt) \delta_{\left(D_n, L\left(T_{(n)}\right), A\left(T_{(n)}\right)\right)}(dx) \tag{6}$$

instead, since their histories are the same $\left(\mathcal{F}_t = \sigma(N((0, s], \cdot) \mid s \leq t) \vee \mathcal{F}_0 = \widetilde{\mathcal{F}}_t = \sigma\left(\widetilde{N}((0, s], \cdot) \mid s \leq t\right)\right) \vee \mathcal{F}_0$. Moreover its natural filtration (Theorem 2.5.10 of Last & Brandt (1995) under so-called *minimality* which we will just assume) satisfies,

$$\mathcal{F}_t^N = \sigma\left(\widetilde{N}((0, s], \cdot) \mid s \leq t\right) \vee \mathcal{F}_0 = \mathcal{F}_t^M \tag{7}$$

for $\mathcal{F}_0 = \sigma(L_0, A_0)$. Since N is a marked point process, we may assume the filtration to be right-continuous. Then $\mathcal{F}_{T_{(k)}} = \left(T_{(k)}, \Delta_{(k)}, L_{(k)}, A_{(k)}\right) \vee \mathcal{F}_{T_{(k-1)}}$ is the history up to the k'th event. Our observations can thus be assumed to be on the form $O = \mathcal{F}_{T_{(K)}}$.

Assumption 3 (Conditional distributions of jumps and marks): We assume that the conditional distributions $P\left(T_{(k)} \in \cdot \mid \mathcal{F}_{T_{(k-1)}}\right) \ll m$ P-a.s., and $P\left(A\left(T_{(k)}\right) \in \cdot \mid T_{(k)} = t, \Delta_{(k)} = a, \mathcal{F}_{T_{(k-1)}}\right) \ll \nu_a$ P-a.s. and $P\left(L\left(T_{(k)}\right) \in \cdot \mid T_{(k)} = t, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right) \ll \nu_\ell$ P-a.s., where m is the Lebesgue measure on $\mathbb{R}_+$, ν_a is a measure on $\mathcal{A}$, and ν_ℓ is a measure on $\mathcal{L}$.

Theorem 1 (Existence of compensator): Let $\mathbb{F}_k = (\mathbb{R}_+ \times \{a, \ell, c, d, y\} \times \mathcal{A} \times \mathcal{L})^k$. Under Assumption 1, Assumption 2, and Assumption 3, there exists functions for $k = 1, ..., K$, functions $\lambda_k^x(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{F}_k \to \mathbb{R}_+$, $\pi_k(\cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathcal{A} \times \mathbb{F}_k \to \mathbb{R}_+$, and $\mu_k(\cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathcal{L} \times \mathbb{F}_k \to \mathbb{R}_+$ such that

$$\begin{aligned}
\Lambda(dt, dm, da, dl) = \sum_{k=1}^K & \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_a(dm) \lambda_{k-1}^a\left(t, \mathcal{F}_{T_{(k-1)}}\right) \pi_{k-1}\left(t, da, \mathcal{F}_{T_{(k-1-1)}}\right) \\
& + \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_\ell(dm) \lambda_{k-1}^\ell\left(t, \mathcal{F}_{T_{(k-1)}}\right) \mu_{k-1}\left(t, dl, \mathcal{F}_{T_{(k-1-1)}}\right) \\
& + \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_y(dm) \lambda_{k-1}^y\left(t, \mathcal{F}_{T_{(k-1)}}\right) \\
& + \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_d(dm) \lambda_{k-1}^d\left(t, \mathcal{F}_{T_{(k-1)}}\right) \\
& + \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_c(dm) \lambda_{k-1}^c\left(t, \mathcal{F}_{T_{(k-1)}}\right)
\end{aligned} \tag{8}$$

is a P-$\mathcal{F}_t$-compensator measure of N. As a consequence we have,

$$\begin{aligned}
& P\left(T_{(k)} \leq s, \Delta_{(k)} = x, L\left(T_{(k)}\right) \in dl, A\left(T_{(k)}\right) \in da \mid \mathcal{F}_{T_{(k-1)}}\right) \\
& = \int_0^s \underbrace{\exp\left(- \sum_{x=y,d,\ell,a,c} \int_0^t \lambda_{k-1}^x\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)}_{\text{probability of surviving up to } t.} \underbrace{\lambda_k^x\left(t, \mathcal{F}_{T_{(k)}}\right)}_{\text{probability that it was an event of type } x} \\
& \Bigg(\underbrace{\int_{\mathbb{L}} \mu_{k-1}\left(t, x, \mathcal{F}_{T_{(k-1-1)}}\right) \nu_\ell(\mathrm{d}x)}_{\text{probability of } L\left(T_{(k)}\right) \in \mathbb{L} \text{ given } \Delta_{(k)} = \ell \text{ and } T_{(k)} = t} \delta_{\left(\ell, A\left(T_{(k-1)}\right)\right)}(\{x\}, da) + \\
& + \underbrace{\int_{\mathbb{A}} \pi_{k-1}\left(t, x, \mathcal{F}_{T_{(k-1-1)}}\right) \nu_a(\mathrm{d}x)}_{\text{probability of } A\left(T_{(k)}\right) \in \mathbb{A} \text{ given } \Delta_{(k)} = a \text{ and } T_{(k)} = t} \delta_{\left(a, L\left(T_{(k-1)}\right)\right)}(\{x\}, dl) + \delta_{(\{x\}, \emptyset, \emptyset)}(\{d, y, c\}, dl, da) \Bigg) \mathrm{d}t.
\end{aligned} \tag{9}$$

on the event $T_{(k-1)} < s$.

Proof: Simply use Theorem 4.1.11 of Last & Brandt (1995) which states that

$$\Lambda(dt, dm, da, dl) = \sum_{k: T_{(k-1)} < \infty} \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \frac{P\left(\left(T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right) \in (dt, dm, da, dl) \mid \mathcal{F}_{T_{(k-1)}}\right)}{P\left(T_{(k)} \geq t \mid \mathcal{F}_{T_{(k-1)}}\right)} \tag{10}$$

is a P-$\mathcal{F}_t$-compensator measure of N. Now rewrite $P\left(\left(T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right) \in (dt, dm, da, dl) \mid \mathcal{F}_{T_{(k-1)}}\right) = P\left(\left(\Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right) \in (dm, da, dl) \mid T_{(k)} = t, \mathcal{F}_{T_{(k-1)}}\right) P\left(T_{(k)} = dt \mid \mathcal{F}_{T_{(k-1)}}\right)$. Under Assumption 3, we can write

$$P\left(\Delta_{(k)} = x \mid T_{(k)} = t, \mathcal{F}_{T_{(K-1)}}\right) \frac{P\left(T_{(k)} \in dt \mid \mathcal{F}_{T_{(k-1)}}\right)}{P\left(T_{(k)} \geq t \mid \mathcal{F}_{T_{(k-1)}}\right)} = \lambda_{k-1}^x\left(t, \mathcal{F}_{T_{(k-1)}}\right) dt \tag{11}$$

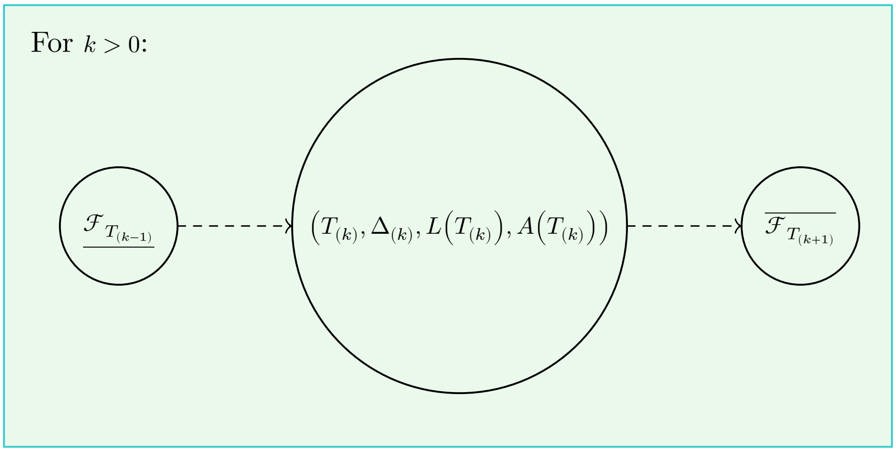
which is simply the cause-specific hazard function of the k'th event. Also, we can define

$$\begin{aligned}
\pi_{k-1}\left(t, da, \mathcal{F}_{T_{(k-1-1)}}\right) &= P\left(\left(L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right) \in \left(\left\{L_{(k-1)}\right\}, da\right) \mid T_{(k)} = t, \Delta_{(k)} = a, \mathcal{F}_{T_{(k-1)}}\right) \\
&= P\left(A\left(T_{(k)}\right) \in da \mid T_{(k)} = t, \Delta_{(k)} = a, \mathcal{F}_{T_{(k-1)}}\right) \\
\mu_{k-1}\left(t, dl, \mathcal{F}_{T_{(k-1-1)}}\right) &= P\left(\left(L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right) \in \left(dl, \left\{A_{(k-1)}\right\}\right) \mid T_{(k)} = t, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right) \\
&= P\left(L\left(T_{(k)}\right) \in dl \mid T_{(k)} = t, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right)
\end{aligned} \tag{12}$$

and the result follows. The latter conclusion can be seen by Theorem 4.3.8 of Last & Brandt (1995). □

3 A pragmatic approach to continuous-time causal inference

One classical causal inference perspective requires that we know how the data was generated up to unknown parameters (NPSEM) (Pearl, 2009). This approach has only to a small degree been discussed in continous-time causal inference literature (Røysland et al. (2024)). This is initially considered in the uncensored case, but is later extended to the censored case. For a moment, we ponder how the data was generated given a DAG. The DAG given in the section provides a useful tool for simulating continuous-time data, but not for drawing causal inference conclusions. For the event times, we can draw a figure representing the data generating mechanism which is shown in Figure 4. Some, such as Chamapiwa (2018), write down this DAG, but with an arrow from $T_{(k)}$ to $L(T_{(k)})$ and $A(T_{(k)})$ instead of displaying a multivariate random variable which they deem the "time-as-confounder" approach to allow for irregularly measured data (see Figure 5). Fundamentally, this arrow would only be meaningful if the event time was known prior to the treatment and covariate value measurements, which they might not be. This can make sense if the event is scheduled ahead of time, but for, say, a stroke the time is not measured prior to the event. Because a cause must precede an effect, this makes the arrow invalid from a philosophical standpoint. On the other hand, DAGs such as the one in Figure 4, are not informative about the causal relationships between the variables are. This issue with simultaneous events is likely what has led to the introduction of local independence graphs (Didelez (2008)) but is also related to the notion that the treatment times are not predictable (that is knowable just prior to the event) as in Ryalen (2024).



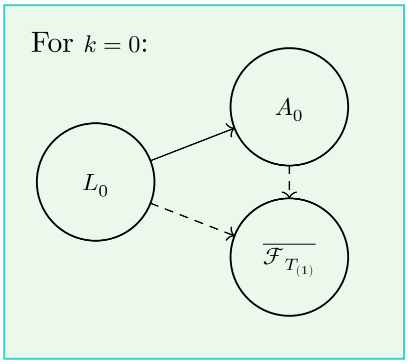


Figure 4: A DAG representing the relationships between the variables of O. The dashed lines indicate multiple edges from the dependencies in the past and into the future.

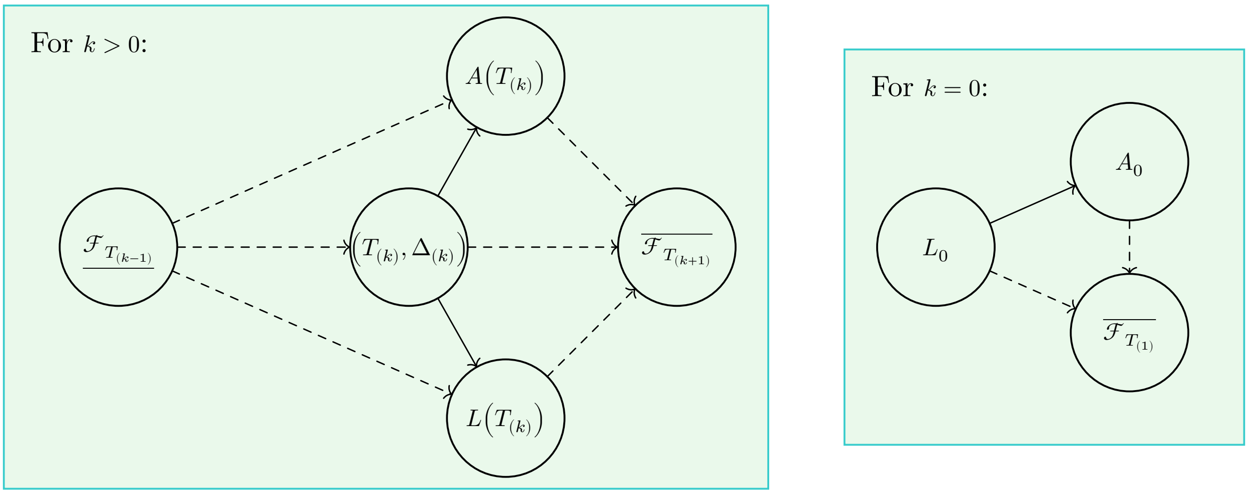


Figure 5: A DAG for simulating the data generating mechanism or such as those that may be found in Chamapiwa (2018). The dashed lines indicate multiple edges from the dependencies in the past and into the future. Here $\underline{\mathcal{F}_{T_{(k)}}}$ is the history up to and including the k'th event and $\overline{\mathcal{F}_{T_{(k)}}}$ is the history after and including the k'th event.

We now take an interventionalist stance to causal inference such as the one given in Ryalen (2024). In the interventionalist school of thought, one tries to emulate a randomized controlled trial. In the continuous-time longitudinal setting, this can e.g., correspond to a trial in which there is perfect compliance. We reformulate the conditions of Ryalen (2024) to our setting, stating the conditions directly in terms of the events instead of using martingales. For simplicity, we presuppose that there are two treatment levels (0/1). As in randomized trials, we suppose that there is a treatment plan g_k at each event point which specifies the treatment that the person observation should have at teach event point which is a treatment event point that is $g_k : \mathbb{R}_+ \times \mathbb{F}_{k-1} \to \{0, 1\}$. Specifically, the plan specifies that $A(T_{(k)}) = g_k\left(T_{(k)}, \mathcal{F}_{T_{(k-1)}}\right)$ if $T_{(k)} < \infty$ and $\Delta_{(k)} = a$.

We choose not to index the random variables with g when it can clearly be inferred from the context. Let

$$\begin{aligned} T_k^y &:= \begin{cases} T_{(k)} \text{ if } \Delta_{(k)} = y \\ \infty \text{ otherwise} \end{cases} \\ \tilde{T}_k^{g,y} &:= \begin{cases} T_{(k)} \text{ if } \Delta_{(k)} = y \\ \infty \text{ otherwise} \end{cases}. \end{aligned} \tag{13}$$

Then our outcome is $\left(\tilde{T}_1^{g,y}, ..., \tilde{T}_K^{g,y}\right)$.

We are then interested in estimating the causal parameter given in Definition 1.

Definition 1 (Target parameter): Our target parameter $\Psi_\tau^g : \mathcal{M} \to \mathbb{R}$ is the mean interventional potential outcome at time τ given the intervention plan g,

$$\Psi_\tau^g(P) = \mathbb{E}_P\left[\sum_{k=1}^{K} \mathbb{1}\left\{\tilde{T}_k^{g,y} \leq \tau\right\}\right] \tag{14}$$

Let $T^a = \inf\left\{T_{(k)} \mid \Delta_{(k)} = a, A\left(T_{(k)}\right) \neq g_k\left(T_{(k)}, \mathcal{F}_{T_{(k-1)}}\right)\right\}$ be the time of the first treatment event where the treatment plan is not followed. The three identifying conditions for the target parameter are as follows:

For each $k \in \{1, ..., K\}$, we need:

- **Consistency**:

$$\tilde{T}_k^{g,y} \mathbb{1}\left\{T^a > T_{(k-1)}, A(0) = g(L_0)\right\} = T_k^y \mathbb{1}\left\{T^a > T_{(k-1)}, A(0) = g(L_0)\right\} \tag{15}$$

- **Exchangeability**:

$$A\left(T_{(k)}\right) \perp \left(\tilde{T}^{g,y}_{k+1}, ..., \tilde{T}^{g,y}_{K}\right) \mid \Delta_{(k)} = a, \mathcal{F}_{T_{(k-1)}} \tag{16}$$

  Wrong! Why would to state conditional independence for multiple of the outcomes. Go back to poster.
- **Positivity**: The weights

$$w_k(f_{k-1}, t_k) = \frac{\mathbb{1}\{a_0 = g_0(l_0)\}}{\pi_0(g(l_0))} \prod_{j=1}^{k-1} \left(\frac{\mathbb{1}\{a_j = g_j(t_j, f_{j-1})\}}{\pi_j(t_j, g_j(t_j, f_{j-1}))}\right)^{\mathbb{1}\{\delta_j = a\}} \mathbb{1}\{t_1 < ... < t_k\} \tag{17}$$

  fulfill $\mathbb{E}_P\left[w_k\left(\mathcal{F}_{T_{(k-1)}}, T_{(k)}\right)\right] = 1$.

Then we have the following theorem

Theorem 2 (Identification via inverse probability weights): Under the conditions of consistency, exchangeability, and consistency, the target parameter is identified by

$$\Psi^g_\tau(P) = \mathbb{E}_P\left[\sum_{k=1}^K w_{k-1} \mathbb{1}\{T_k \leq \tau, D_k = y\}\right] \tag{18}$$

Proof: We will show this by proving that $\psi_{k,\tau}(P) = \mathbb{E}_P[w_{k-1}\mathbb{1}\{T_k \leq \tau, D_k = y\}] = \mathbb{E}_P\left[\mathbb{1}\left\{\tilde{T}_k \leq \tau, \tilde{D}_k = y\right\}\right]$. Let $Y^*_{k,j} = \mathbb{E}_P\left[\mathbb{1}\left\{\tilde{T}_k \leq \tau, \tilde{D}_k = y\right\} \mid A\left(T_{(j-1)}\right), T_{(j-1)}, \Delta_{(j-1)}, L\left(T_{(j-1)}\right), \mathcal{F}_{T_{(j-1)}}\right]$. By assumption, this is a function of $T_{(j-1)}, \Delta_{(j-1)}, L\left(T_{(j-1)}\right), \mathcal{F}_{T_{(j-1)}}$ only. Let $g^*_k = g_k$ if $\Delta_{(k)} = a$ and $T_{(k)} < \infty$ and $g^*_k = g^*_{k-1}$ otherwise. We use the law of iterated expectations to find that

$$\begin{aligned}
\psi_{k,\tau}(P) &= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\left\{T_{(k)} \leq \tau, T_{(k)} = y\right\} \mid \mathcal{F}_{T_{(k-1)}}\right]\right] \\
&= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\{T_k \leq \tau, D_k = y\} \mid \mathcal{F}_{T_{(k-1)}}\right]\right] \\
&= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\{T_k \leq \tau, D_k = y\} \mid \left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), A\left(T_{(k-1)}\right) = g^*_k\left(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\right), ..., A_0 = g_0(L_0), L_0\right)\right]\right] \\
&= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\{T^y_k \leq \tau\}\mathbb{1}\left\{T^a > T_{(k-1)}, A(0) = g(L_0)\right\} \mid \left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), A\left(T_{(k-1)}\right) = g^*_k\left(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\right), ..., A_0 = g_0(L_0), L_0\right)\right]\right] \\
&= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\left\{\tilde{T}^{g,y}_k \leq \tau\right\}\mathbb{1}\left\{T^a > T_{(k-1)}, A(0) = g(L_0)\right\} \mid \left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), A\left(T_{(k-1)}\right) = g^*_k\left(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\right), ..., A_0 = g_0(L_0), L_0\right)\right]\right] \\
&= \mathbb{E}_P\left[w_{k-1}\mathbb{E}_P\left[\mathbb{1}\left\{\tilde{T}^{g,y}_k \leq \tau\right\} \mid \left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), A\left(T_{(k-1)}\right) = g^*_k\left(T_{(k-1)}, \mathcal{F}_{T_{(k-2)}}\right), ..., A_0 = g_0(L_0), L_0\right)\right]\right] \\
&= \mathbb{E}_P[w_{k-2}Y^*_{k,k}\left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), \mathcal{F}_{T_{(k-2)}}\right)\mathbb{E}_P\left[\left(\frac{\mathbb{1}\left\{A\left(T_{(k-1)}\right) = g_{k-1}\left(T_{(k-1)}, \mathcal{F}^g_{T_{(k-1)}}\right)\right\}}{\pi_{k-1}\left(T_{(k-1)}, g_{k-1}\left(T_{(k-1)}, \mathcal{F}_{T_{(k-1)}}\right)\right)}\right)^{\mathbb{1}\left\{\Delta_{(k-1)} = a\right\}} \mid \left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), \mathcal{F}_{T_{(k-2)}}\right)\right] \qquad (19) \\
&= \mathbb{E}_P\left[w_{k-2}Y^*_{k,k}\left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), \mathcal{F}_{T_{(k-2)}}\right)\right] \\
&= \mathbb{E}_P[w_{k-2}Y^*_{k,k-1}\left(T_{(k-2)}, \Delta_{(k-2)}, L\left(T_{(k-2)}\right), \mathcal{F}_{T_{(k-3)}}\right)\mathbb{E}_P\left[\left(\frac{\mathbb{1}\left\{A\left(T_{(k-2)}\right) = g_{k-2}\left(T_{(k-2)}, \mathcal{F}^g_{T_{(k-2)}}\right)\right\}}{\pi_{k-2}\left(T_{(k-2)}, g_{k-2}\left(T_{(k-2)}, \mathcal{F}_{T_{(k-2)}}\right)\right)}\right)^{\mathbb{1}\left\{\Delta_{(k-2)} = a\right\}} \mid \left(T_{(k-2)}, \Delta_{(k-2)}, L\left(T_{(k-2)}\right), \mathcal{F}_{T_{(k-3)}}\right)\right] \\
&= \mathbb{E}_P\left[w_{k-2}Y^*_{k,k-1}\right] \\
&\dots \\
&= \mathbb{E}_P\left[\mathbb{1}\left\{\tilde{T}_k \leq \tau, \tilde{D}_k = y\right\}\right]
\end{aligned}$$

The third and fifth last equality can be arrived at by applying the law of iterated expectations twice: First by conditioning on $\left(A\left(T_{(k-1)}\right), T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), \mathcal{F}_{T_{(k-2)}}\right)$ and then using exchangeability. Then use the law of iterated conditional expectations again on $\left(T_{(k-1)}, \Delta_{(k-1)}, L\left(T_{(k-1)}\right), \mathcal{F}_{T_{(k-2)}}\right)$, where the first factor no longer depends on $A\left(T_{(k-1)}\right)$. □

Similar conditions have been given in Ryalen (2024). They require, in addition, to our conditions

$$
\begin{aligned}
&\lambda^a\left(t \mid \mathcal{F}_{T_{(k-1)}} \vee \left(\tilde{Y}_t\right)_{t\in[0,\tau_{\text{end}}]}\right) \\
&= \lim_{h\to 0} \frac{P\left(t \le T_{(k)} < t+h, \Delta_k = a \mid T_{(k)} \ge t, \mathcal{F}_{T_{(k-1)}}, \left(\tilde{Y}_t\right)_{t\in[0,\tau_{\text{end}}]}\right)}{h}
\end{aligned}
\tag{20}
$$

does not depend on $\left(\tilde{Y}_t\right)_{t\in[0,\tau_{\text{end}}]}$ for the potential outcome process $\left(\tilde{Y}_t\right)_{t\in[0,\tau_{\text{end}}]}$.

In the appendix, it will be shown that the identification formulas are the same in our specific setting.

3.1 Censoring

Let $C > 0$ be the censoring time. The censoring time in our setting is defined so that $C = \inf\left\{T_{(k)} \mid \Delta_{(k)} = c\right\}$. Censoring changes what is observed in the following way:

$$
\begin{aligned}
\bar{T}_k &= C \wedge T_{(k)} \\
\bar{D}_k &= \begin{cases} \Delta_{(k)} \text{ if } C > T_{(k)} \\ \text{c otherwise} \end{cases} \\
\bar{A}_k &= \begin{cases} A\left(T_{(k)}\right) \text{ if } C > T_{(k)} \\ A\left(T_{(k-1)}\right) \text{ otherwise} \end{cases} \\
\bar{L}_k &= \begin{cases} L\left(T_{(k)}\right) \text{ if } C > T_{(k)} \\ L\left(T_{(k-1)}\right) \text{ otherwise} \end{cases}
\end{aligned}
\tag{21}
$$

so that we really observe $\bar{O} = \left(\bar{T}_1, \bar{D}_1, \bar{A}_1, \bar{L}_1, ..., \bar{T}_K, \bar{D}_K, \bar{A}_K, \bar{L}_K\right)$. Let $\mathcal{F}_{t\wedge C}$ denote the corresponding filtration for the censored (potentially unobserved process).

For the censoring, one has the conditions,

Independent censoring: $C \perp \left(T_k, D_k, A\left(T_{(k)}\right), L\left(T_{(k)}\right)\right) \mid \mathcal{F}_{T_{(k-1)}}$.

Positivity: The weights $w_k^c(f_{k-1}, t_k, d_k) = \frac{\mathbb{1}\{d_k \neq c\}}{\prod_{j=1}^k S^c(t_j \mid f_{j-1})}$ fulfill $\mathbb{E}_P\left[w_k^c\left(\mathcal{F}_{T_{(k-1)}}, T_{(k)}, \Delta_{(k)}\right)\right] = 1$. Here $S^c(t \mid f_{k-1}) = \exp\left(-\int_{t_{k-1}}^t \lambda_{k-1}^c\left(s, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}s\right)$.

Then by Theorem 1, Λ is also the P-compensating measure for $\mathcal{F}_t \vee C$, where C is included as a baseline covariate. By the innovation theorem,

$$
\begin{aligned}
\Lambda^*(dt, dm, da, dl) = \sum_{k=1}^K &\mathbb{1}\left\{T_{(k-1)} \wedge C < t \le T_{(k)} \wedge C\right\}\delta_a(dm)\lambda_{k-1}^a\left(t, \mathcal{F}_{T_{(k-1)}}\right)\pi_{k-1}\left(t, da, \mathcal{F}_{T_{(k-1-1)}}\right) \\
&+\mathbb{1}\left\{T_{(k-1)} \wedge C < t \le T_{(k)} \wedge C\right\}\delta_\ell(dm)\lambda_{k-1}^\ell\left(t, \mathcal{F}_{T_{(k-1)}}\right)\mu_{k-1}\left(t, dl, \mathcal{F}_{T_{(k-1-1)}}\right) \\
&+\mathbb{1}\left\{T_{(k-1)} \wedge C < t \le T_{(k)} \wedge C\right\}\delta_y(dm)\lambda_{k-1}^y\left(t, \mathcal{F}_{T_{(k-1)}}\right) \\
&+\mathbb{1}\left\{T_{(k-1)} \wedge C < t \le T_{(k)} \wedge C\right\}\delta_d(dm)\lambda_{k-1}^d\left(t, \mathcal{F}_{T_{(k-1)}}\right) \\
&+\mathbb{1}\left\{T_{(k-1)} \wedge C < t \le T_{(k)} \wedge C\right\}\delta_c(dm)\lambda_{k-1}^c\left(t, \mathcal{F}_{T_{(k-1)}}\right)
\end{aligned}
\tag{22}
$$

is the P-$(\mathcal{F}_{t\wedge C})$-compensator measure of N. This means that every component can be identified.

3.2 A simple lemma

We first state and prove a formula for at target parameter that is not causal, but we will use it to identify the causal parameter. This will be useful for the derivation of the efficient influence function.

Lemma 1: Let $\bar{Q}_K = I\big(T_{(K)} \leq \tau, \Delta_{(K)} = y\big)$ and $\bar{Q}_k = \mathbb{E}_P\Big[\sum_{j=k+1}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k)}}\Big]$. Then,

$$\begin{aligned}\bar{Q}_{k-1} = \mathbb{E}_P\Big[&I\big(T_{(k)} \leq \tau, \Delta_{(k)} = \ell\big)\bar{Q}_k\Big(A\big(T_{(k-1)}\big), L\big(T_{(k)}\big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big)\\ &+I\big(T_{(k)} \leq \tau, \Delta_{(k)} = a\big)\mathbb{E}_P\Big[\bar{Q}_k\Big(A\big(T_{(k)}\big), L\big(T_{(k-1)}\big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big]\\ &+I\big(T_{(k)} \leq \tau, \Delta_{(k)} = y\big) \Big| \mathcal{F}_{T_{(k-1)}}\Big]\end{aligned} \tag{23}$$

for $k = K, ..., 1$. Thus, $\mathbb{E}_P\Big[\sum_{k=1}^K I\big(T_{(k)} \leq \tau, \Delta_{(k)} = y\big)\Big] = \mathbb{E}_P\big[\bar{Q}_0\big]$.

Proof: We find

$$\begin{aligned}\bar{Q}_k &= \mathbb{E}_P\left[\sum_{j=k+1}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k)}}\right]\\ &= \mathbb{E}_P\left[\mathbb{E}_P\left[\mathbb{E}_P\left[\sum_{j=k+1}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k+1)}}\right] \mid T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right] \mid \mathcal{F}_{T_{(k)}}\right]\\ &= \mathbb{E}_P\Bigg[I\big(T_{(k+1)} \leq \tau, \Delta_{(k+1)} = y\big)\\ &\qquad +\mathbb{E}_P\left[\mathbb{E}_P\left[\sum_{j=k+2}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k+1)}}\right] \mid T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right] \Bigg| \mathcal{F}_{T_{(k)}}\Bigg]\\ &= \mathbb{E}_P\Bigg[I\big(T_{(k+1)} \leq \tau, \Delta_{(k+1)} = y\big)\\ &+I\big(T_{(k+1)} \leq \tau, \Delta_{(k+1)} = a\big)\mathbb{E}_P\left[\mathbb{E}_P\left[\sum_{j=k+2}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k+1)}}\right] \mid T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right] \mid \mathcal{F}_{T_{(k)}}\Bigg]\\ &+\mathbb{E}_P\left[I\big(T_{(k+1)} \leq \tau, \Delta_{(k+1)} = \ell\big)\mathbb{E}_P\left[\sum_{j=k+2}^K I\big(T_{(j)} \leq \tau, \Delta_{(j)} = y\big) \mid \mathcal{F}_{T_{(k+1)}}\right] \mid \mathcal{F}_{T_{(k)}}\right]\end{aligned} \tag{24}$$

by the law of iterated expectations and that

$$\big(T_{(k)} \leq \tau, \Delta_{(k)} = y\big) \subseteq \big(T_{(j)} \leq \tau, \Delta_{(j)} \in \{a, \ell\}\big) \tag{25}$$

for all $j = 1, ..., k-1$ and $k = 1, ..., K$. □

Theorem 3 (Identification via g-formula): Let $\bar{Q}^a_{k,\tau} = \bar{Q}_k(Q)$ be defined as in the previous theorem for Q. Let

$$
\begin{aligned}
&p_{ka}\left(t \mid \mathcal{F}_{T_{(k-1)}}\right) \\
&= \int_{T_{(k)}}^{t} \exp\left(- \sum_{x \in \{\ell, a, d, y\}} \int_{T_{(k)}}^{s} \lambda^x_k\left(u, \mathcal{F}_{T_{(k)}}\right) \mathrm{d}u\right) \lambda^a_k\left(s, \mathcal{F}_{T_{(k)}}\right) \\
&\qquad \times \left(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k-1)}\right), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\right) \pi^*_k\left(s, a_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k)\right) \mathrm{d}s \\
&p_{k\ell}\left(t \mid \mathcal{F}_{T_{(k-1)}}\right) \\
&= \int_{T_{(k)}}^{t} \exp\left(- \sum_{x \in \{\ell, a, d, y\}} \int_{T_{(k)}}^{s} \lambda^x_k\left(u, \mathcal{F}_{T_{(k)}}\right) \mathrm{d}u\right) \lambda^\ell_k\left(s, \mathcal{F}_{T_{(k)}}\right) \\
&\qquad \times \left(\int_{\mathcal{L}} \bar{Q}^a_{k+1,\tau}\left(l_k, A\left(T_{(k-1)}\right), s, \ell, \mathcal{F}_{T_{(k-1)}}\right) \mu_k\left(s, l_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_L(\mathrm{d}l_k)\right) \mathrm{d}s \\
&= \int_{T_{(k)}}^{t} \exp\left(- \sum_{x \in \{\ell, a, d, y\}} \int_{T_{(k)}}^{s} \lambda^x_k\left(u, \mathcal{F}_{T_{(k)}}\right) \mathrm{d}u\right) \lambda^\ell_k\left(s, \mathcal{F}_{T_{(k)}}\right) \\
&\qquad \times \left(\mathbb{E}_P\left[\bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right]\right) \mathrm{d}s \\
&p_{ky}\left(t \mid \mathcal{F}_{T_{(k-1)}}\right) \\
&= \int_{T_{(k)}}^{t} \exp\left(- \sum_{x \in \{\ell, a, d, y\}} \int_{T_{(k)}}^{s} \lambda^x_k\left(u, \mathcal{F}_{T_{(k)}}\right) \mathrm{d}u\right) \lambda^y_k\left(s, \mathcal{F}_{T_{(k)}}\right) \mathrm{d}s
\end{aligned}
\tag{26}
$$

Then, we can identify $\bar{Q}^a_{k,\tau}$ via the intensities as

$$
\bar{Q}^a_{k,\tau} = p_{ka}\left(\tau \mid \mathcal{F}_{T_{(k-1)}}\right) + p_{k\ell}\left(\tau \mid \mathcal{F}_{T_{(k-1)}}\right) + p_{ky}\left(\tau \mid \mathcal{F}_{T_{(k-1)}}\right)
\tag{27}
$$

Alternatively, we can apply inverse probability of censoring weighting to obtain

$$
\begin{aligned}
\bar{Q}^a_{k-1,\tau} = \mathbb{E}_P\Bigg[&\frac{I\left(T_{(k)} \le \tau, \Delta_{(k)} = \ell\right)}{\exp\left(-\int_{T_{(k-1)}}^{T_{(k)}} \lambda^c\left(s \mid \mathcal{F}_{T_{(k)}}\right) \mathrm{d}s\right)} \bar{Q}^a_{k,\tau}\left(A\left(T_{(k-1)}\right), L\left(T_{(k)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \\
&+ \frac{I\left(T_{(k)} \le \tau, \Delta_{(k)} = a\right)}{\exp\left(-\int_{T_{(k-1)}}^{T_{(k)}} \lambda^c\left(s \mid \mathcal{F}_{T_{(k)}}\right) \mathrm{d}s\right)} \\
&\quad \times \int \bar{Q}^a_{k,\tau}\left(a_k, L\left(T_{(k-1)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \pi^*_{k-1}\left(T_{(k)}, a_k, \mathcal{F}_{T_{(k-1-1)}}\right) \nu_A(\mathrm{d}a_k) \\
&+ \frac{I\left(T_{(k)} \le \tau, \Delta_{(k)} = y\right)}{\exp\left(-\int_{T_{(k-1)}}^{T_{(k)}} \lambda^c\left(s \mid \mathcal{F}_{T_{(k)}}\right) \mathrm{d}s\right)} \,\Bigg|\, \mathcal{F}_{T_{(k-1)}} \Bigg]
\end{aligned}
\tag{28}
$$

for $k = K-1, ..., 1$. This is Method 3. Then,

$$
\Psi_\tau(Q) = \mathbb{E}_P\left[\int \bar{Q}^a_{0,\tau}(a, L_0) \pi^*_0\left(0, a, \mathcal{F}_{T_{(0-1)}}\right) \nu_A(\mathrm{d}a)\right].
\tag{29}
$$

Proof: The theorem is an immediate consequence of Theorem 1 and Lemma 1 (the sets $\left(T_{(k)} \le t, \Delta_{(k)} = x, L\left(T_{(k)}\right) \in \mathbb{L}, A\left(T_{(k)}\right) \in \mathbb{A}\right)$ fully determine the regular conditional distribution of $\left(T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right)$ given $\mathcal{F}_{T_{(k-1)}}$). □

Interestingly, Equation 27 corresponds exactly with the target parameter of Rytgaard et al. (2022) and Gill & Robins (2023) by plugging in the definitions of $\bar{Q}^a_{k,\tau}$ and simplifying (to be shown). A simple implementation of the IPCW is provided below in the simple case of a static treatment plan. The other representations of the target parameter in terms of the intensities are useful directly, but we may, as in the discrete, estimate the target parameter by Monte Carlo integration (i.e., direct simulation from the estimated intensities/densities).

3.3 Implementation of Method 3

- For each event point $k = K, K-1, ..., 1$ (starting with $k = K$):
  1. Obtain $\hat{S}^c\left(t \mid \mathcal{F}_{T_{(k-1)}}\right)$ by fitting a cause-specific hazard model for the censoring via the interevent time $S_{(k)} = T_{(k)} - T_{(k-1)}$, regressing on $\mathcal{F}_{T_{(k-1)}}$ (among the people who are still at risk after $k-1$ events).
  2. Define the subject-specific weight:
  $$\hat{\eta}_k = \frac{\mathbb{1}\left\{T_{(k)} \leq \tau, \Delta_k \in \{a, \ell\}, k < K\right\} \hat{\nu}_k\left(\mathcal{F}^{-A}_{T_{(k)}}, \mathbf{1}\right)}{\hat{S}^c\left(T_{(k)} \mid \mathcal{F}^{-A}_{T_{(k-1)}}, \mathbf{1}\right)} \tag{30}$$
  Then calculate the subject-specific pseudo-outcome
  $$\hat{R}_k = \frac{\mathbb{1}\left\{T_{(k)} \leq \tau, \Delta_k = y\right\}}{\hat{S}^c\left(T_{(k)} \mid \mathcal{F}^{-A}_{T_{(k-1)}}, \mathbf{1}\right)} + \hat{\eta}_k \tag{31}$$
  Regress $\hat{R}_k$ on $\mathcal{F}_{T_{(k-1)}}$ on the data with $T_{(k-1)} < \tau$ and $\Delta_k \in \{a, \ell\}$ to obtain a prediction function $\hat{\nu}_{k-1} : \mathcal{H}_{k-1} \to \mathbb{R}_+$.
- At baseline, we obtain the estimate $\hat{\Psi}_n = \frac{1}{n} \sum_{i=1}^n \hat{\nu}_0(L_i(0), 1)$.

4 Implementation of the Iterative Conditional Expectations formula

We assume that K_τ is the 1 + the maximal number of non-terminal events that occur before time τ. For now, we assume that this is number is fixed and does not depend on the sample. Let $\tilde{Y}_k(t) = I\left(T_{(k-1)} < t \leq T_{(k)}\right)$.

For $k = K_\tau - 1, ..., 0$:
- We want a prediction function $\bar{Q}^a_{k,\tau}$ of the history up to the k'th event, that is $\bar{Q}^a_{k,\tau} : \mathcal{H}_k \longrightarrow \mathbb{R}$, given that we have one for the $(k+1)$'th event, i.e., $\bar{Q}^a_{k+1,\tau} : \mathcal{H}_k \longrightarrow \mathbb{R}$ (note that for $k = K_\tau$, we have $\bar{Q}^a_{K_\tau,\tau} = I\left(T_{(K_\tau)} \leq \tau, \Delta_{(K_\tau)} = y\right).$). We consider the data set $\mathcal{D}_{k,n}$ that is obtained from the original data $\mathcal{D}_n$ by only considering the observations that have had k non-terminal events, that is $\Delta_{(k)} \in \{a, \ell\}$ for $j = 1, ..., k$. On this data:
  - We estimate $\lambda^c_{k+1}\left(\cdot, \mathcal{F}_{T_{(k+1)}}\right)$ by using $T_{(k+1)}$ as the time-to-event and $\Delta_{(k+1)}$ as the event indicator on the data set $\mathcal{D}_{k,n}$, regressing on $\mathcal{F}_{T_{(k)}} = \left(L\left(T_{(k)}\right), A\left(T_{(k)}\right), T_{(k)}, \Delta_{(k)}, ..., L_0, A_0\right)$[1]

  We are now able provide estimated values for the integrand in Equation 28. These values are provided on the smaller data set $\mathcal{D}_{k,n,\tau}$ of $\mathcal{D}_{k,n}$ where we only consider the observations with $T_{(k)} \leq \tau$. This is done as follows:
  1. For observations in $\mathcal{D}_{k,n,\tau}$ with $\Delta_{(k+1)} = \ell$ and $T_{(k+1)} \leq \tau$, use the previous function to predict values $\bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k+1)}\right), A\left(T_{(k)}\right), T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right)$.
  2. For observations in $\mathcal{D}_{k,n,\tau}$ with $\Delta_{(k+1)} = a$ and $T_{(k+1)} \leq \tau$, integrate using the previous function $\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), a_k, T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right) \pi^*_k\left(T_{(k+1)}, a_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k)$. If for example the intervention sets the treatment to 1, then $\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), a_k, T_{(k+1)}, \Delta_{(k+1)}, \mathcal{F}_{T_{(k)}}\right) \pi^*_k\left(T_{(k)}, 1, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k) = \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), 1, T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k)}}\right)$. This gives predicted values for this group.
  3. For observations in $\mathcal{D}_{k,n,\tau}$ with $\Delta_{(k+1)} = y$ and $T_{(k+1)} \leq \tau$, simply put the values equal to 1.
  4. For all other observations put their values equal to 0.

  For all the observations, divide the corresponding values by estimates of censoring survival function $\exp\left(-\int_{T_{(k)}}^{T_{(k+1)}} \lambda^c\left(s \mid \mathcal{F}_{T_{(k)}}\right) \mathrm{d}s\right)$. We then regress the values on $\mathcal{F}_{T_{(k)}} = \left(L\left(T_{(k)}\right), A\left(T_{(k)}\right), T_{(k)}, \Delta_{(k)}, ..., L_0, A_0\right)$. From this regression, we set $\bar{Q}^a_{k,\tau}$ to be the predicted values of the function from the regression.

[1] We abuse the notation a bit by writing $\mathcal{F}_{T_{(k)}}$ here, but it is actually a σ-algebra.

- If $k=0$: We estimate the target parameter via $\mathbb{P}_n\left[\sum_{k=1}^{K_\tau} \bar{Q}^a_{0,\tau}(\cdot, a_0)\nu_A(\mathrm{d}a_0)\right]$.

Note: The $\bar{Q}^a_{k,\tau}$ have the interpretation of the heterogenous causal effect after k events.

For now, we recommend Equation 28 for estimating $\bar{Q}^a_{k,\tau}a$: For estimators of the hazard that are piecewise constant, we would need to compute integrals for each unique pair of history and event times occurring in the sample at each event k. On the other hand, the IPCW approach is very sensitive to the specification of the censoring distribution. Something very similar can be written down when we use Equation 27.

4.1 Alternative nuisance parameter estimators

An alternative is to estimate the entire cumulative hazards Λ^x at once instead of having K separate parameters: There are very few methods for marked point process estimation but see Liguori et al. (2023) for methods mostly based on neural networks or Weiss & Page (2013) for a forest-based method. As a final alternative, we can use temporal difference learning to avoid iterative estimation of $\bar{Q}^a, \tilde{Q}$ Shirakawa et al. (2024). Most point process estimators are actually on the form given in terms of ref:intensity.

5 The efficient influence function

We want to use machine learning estimators of the nuisance parameters, so to get inference we need to debias our estimate with the efficient influence function, e.g., double/debiased machine learning Chernozhukov et al. (2018) or targeted minimum loss estimation van der Laan & Rubin (2006). We use Equation 28 for censoring to derive the efficient influence function, because it will contain fewer martingale terms. Let $N^c_k(t) = N_t(\{c\} \times \mathcal{L} \cup \{\emptyset\} \times \mathcal{A} \cup \{\emptyset\})$.

Theorem 4 (Efficient influence function): Let $N^x_k = N_t(\{x\} \times \mathcal{L} \cup \{\emptyset\} \times \mathcal{A} \cup \{\emptyset\})$ and $\tilde{Y}_{k-1}(t) = I\left(T_{(k-1)} < t \leq T_{(k)}\right)$. The efficient influence function is given by

$$
\begin{aligned}
\varphi^*_\tau(P) = \sum_{k=1}^{K} \prod_{j=0}^{k-1} & \left(\frac{\pi^*_{j-1}\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1-1)}}\right)}{\pi_{j-1}\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1-1)}}\right)} \right)^{I(\Delta_{(j)}=a)} \frac{I\left(\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} \leq \tau\right)}{\exp\left(-\sum_{1 \leq j < k} \int_{T_{(j-1)}}^{T_{(j)}} \lambda^c_{j-1}\left(s, \mathcal{F}_{T_{(j-1)}}\right) \mathrm{d}s\right)} \\
& \times \Bigg[\left(\bar{Z}^a_{k,\tau} - \bar{Q}^a_{k-1,\tau}\left(\tau, \mathcal{F}_{T_{(k-1)}}\right)\right) \\
& + \int_{T_{(k-1)}}^{\tau} \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)\right) \frac{1}{\exp\left(-\int_{T_{(k-1)}}^{u} \sum_{x=a,\ell,d,y,c} \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^c_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^c_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right) \Bigg] \\
& + \int \bar{Q}^a_{1,\tau}(a, L_0) \pi^*_0\left(0, a, \mathcal{F}_{T_{(0-1)}}\right) \nu_A(\mathrm{d}a) - \Psi_\tau(P)
\end{aligned}
\tag{32}
$$

(we take the empty sum to be zero and define $T_0 = 0$, $\Delta_{(0)} = a$ and $\mathcal{F}_{T_{(-1)}} = L(0)$.)

Proof: Define (sorry about the notation!)

$$\bar{Z}^a_{k,\tau}(s,t_k,d_k,l_k,a_k,f_{k-1}) = \frac{I(t_k \leq s, d_k = \ell)}{\exp\left(-\int_{t_{k-1}}^{t_k} \lambda^c(s \mid f_{k-1}) \mathrm{d}s\right)} \bar{Q}^a_{k,\tau}(a_{k-1},l_k,t_k,d_k,f_{k-1})$$
$$+ \frac{I(t_k \leq s, d_k = a)}{\exp\left(-\int_{t_{k-1}}^{t_k} \lambda^c(s \mid f_{k-1}) \mathrm{d}s\right)}$$
$$\times \int \bar{Q}^a_{k,\tau}(\tilde{a}_k, l_{k-1}, t_k, d_k, f_{k-1}) \pi^*_{k-1}\left(t_k, \tilde{a}_k, \mathcal{F}_{T_{(k-1-1)}}\right) \nu_A(\mathrm{d}\tilde{a}_k)$$
$$+ \frac{I(t_k \leq s, d_k = y)}{\exp\left(-\int_{t_{k-1}}^{t_k} \lambda^c(s \mid f_{k-1}) \mathrm{d}s\right)}, s \leq \tau \tag{33}$$

and let

$$\bar{Q}^a_{k-1,\tau}(s) = \mathbb{E}_P\left[\bar{Z}^a_{k,s}\left(\tau, T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right), \mathcal{F}_{T_{(k-1)}}\right) \mid \mathcal{F}_{T_{(k-1)}}\right], s \leq \tau \tag{34}$$

We compute the efficient influence function by taking the Gateaux derivative of the above with respect to P, by discretizing the time. We will use two well-known "results" for the efficient influence function.

$$\frac{\partial}{\partial \varepsilon} \int_{T_{(k-1)}}^t \lambda^x_\varepsilon\left(s \mid \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s$$
$$= \frac{\delta_{\mathcal{F}_{T_{(k-1)}}}(f_{k-1})}{P\left(\mathcal{F}_{T_{(k-1)}} = f_{k-1}\right)} \int_{T_{(k-1)}}^t \frac{1}{\exp\left(-\sum_{x=a,\ell,c,d,y} \int_{T_{(k-1)}}^s \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^x_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right) \tag{35}$$

and

$$\frac{\partial}{\partial \varepsilon} \mathbb{E}_{(1-\varepsilon)P + \varepsilon \delta_{(Y,X)}}[Y \mid X = x]\Big|_{\varepsilon=0} = \frac{\delta_X(x)}{P(X=x)}(Y - \mathbb{E}_P[Y \mid X = x]) \tag{36}$$

We will recursively calculate the derivative,

$$\frac{\partial}{\partial \varepsilon} \bar{Q}^{a,\varepsilon}_{k-1,\tau}(a_{k-1}, l_{k-1}, t_{t-1}, d_{k-1}, f_{k-2})\left((1-\varepsilon)P + \varepsilon \delta_{\mathcal{F}_{T_{(k-1)}}}\right)\Big|_{\varepsilon=0} \tag{37}$$

where we have introduced the notation for the dependency on P. Then, taking the Gateaux derivative of the above yields,

$$\frac{\partial}{\partial \varepsilon} \bar{Q}^{a,\varepsilon}_{k-1,\tau}(a_{k-1}, l_{k-1}, t_{t-1}, d_{k-1}, f_{k-2})\left((1-\varepsilon)P + \varepsilon \delta_{\mathcal{F}_{T_{(k-1)}}}\right)\Big|_{\varepsilon=0}$$
$$= \frac{\delta_{\mathcal{F}_{T_{(k-1)}}}(f_{k-1})}{P\left(\mathcal{F}_{T_{(k-1)}} = f_{k-1}\right)} \Bigg(\bar{Z}^a_{k,\tau} - \bar{Q}^a_{k-1,\tau}\left(\tau, \mathcal{F}_{T_{(k-1)}}\right) +$$
$$+ \int_{T_{(k-1)}}^\tau \bar{Z}^a_{k,\tau}(\tau, t_k, d_k, l_k, a_k, f_{k-1}) \int_{T_{(k-1)}}^{t_k} \frac{1}{\exp\left(-\sum_{x=a,\ell,c,d,y} \int_{T_{(k-1)}}^s \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^c_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^c_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)$$
$$P_{\left(T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right)}\left(\mathrm{d}t_k, \mathrm{d}d_k, \mathrm{d}l_k, \mathrm{d}a_k \mid \mathcal{F}_{T_{(k-1)}} = f_{k-1}\right) \Bigg) \tag{38}$$
$$+ \int_{T_{(k-1)}}^\tau \Bigg(\frac{I(t_k \leq \tau, d_k \in \{a, \ell\})}{\exp\left(-\int_{T_{(k-1)}}^{t_k} \lambda^c(s \mid f_{k-1}) \mathrm{d}s\right)} \cdot \left(\frac{\pi^*_{k-1}\left(t_k, a_k, \mathcal{F}_{T_{(k-1-1)}}\right)}{\pi_{k-1}\left(t_k, a_k, \mathcal{F}_{T_{(k-1-1)}}\right)}\right)^{I(d_k=a)} \frac{\partial}{\partial \varepsilon} \bar{Q}^{a,\varepsilon}_{k-1,\tau}(a_k, l_k, t_k, d_k, f_{k-1})\left((1-\varepsilon)P + \varepsilon \delta_{\mathcal{F}_{T_{(k)}}}\right)\Big|_{\varepsilon=0}$$
$$P_{\left(T_{(k)}, \Delta_{(k)}, L\left(T_{(k)}\right), A\left(T_{(k)}\right)\right)}\left(\mathrm{d}t_k, \mathrm{d}d_k, \mathrm{d}l_k, \mathrm{d}a_k \mid \mathcal{F}_{T_{(k-1)}} = f_{k-1}\right)$$

Now note for the second term, we can write

$$
\begin{aligned}
&\int_{T_{(k-1)}}^{\tau} \bar{Z}^a_{k,\tau}(\tau, t_k, d_k, l_k, a_k, f_{k-1}) \int_{T_{(k-1)}}^{t_k} \frac{1}{\exp\left(-\sum_{x=a,\ell,c,d,y} \int_{T_{(k-1)}}^{s} \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^c_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^c_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right) \\
&\qquad P_{(T_{(k)}, \Delta_{(k)}, L(T_{(k)}), A(T_{(k)}))}\left(\mathrm{d}t_k, \mathrm{d}d_k, \mathrm{d}l_k, \mathrm{d}a_k \mid \mathcal{F}_{T_{(k-1)}} = f_{k-1}\right)) \\
&= \int_{T_{(k-1)}}^{\tau} \int_{s}^{\tau} \bar{Z}^a_{k,\tau}(\tau, t_k, d_k, l_k, a_k, f_{k-1}) P_{(T_{(k)}, \Delta_{(k)}, L(T_{(k)}), A(T_{(k)}))}\left(\mathrm{d}t_k, \mathrm{d}d_k, \mathrm{d}l_k, \mathrm{d}a_k \mid \mathcal{F}_{T_{(k-1)}} = f_{k-1}\right) \\
&\qquad \frac{1}{\exp\left(-\sum_{x=a,\ell,c,d,y} \int_{T_{(k-1)}}^{s} \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^c_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^c_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)) \\
&= \int_{T_{(k-1)}}^{\tau} \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(s)\right) \\
&\qquad \frac{1}{\exp\left(-\sum_{x=a,\ell,c,d,y} \int_{T_{(k-1)}}^{s} \lambda^x_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right)} \left(N^c_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s) \lambda^c_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}s\right))
\end{aligned} \tag{39}
$$

by an exchange of integerals. Combining the results iteratively gives the result. □

For now, we recommend using the one step estimator and not the TMLE because the martingales are computationally intensive to estimate. This means that multiple TMLE updates may not be a good idea.

5.1 Comparison with the EIF in Rytgaard et al. (2022)

Let $B_{k-1}(u) = \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)\right) \frac{1}{\exp\left(-\sum_{x=a,\ell,d,y} \int_{T_{(k-1)}}^{u} \lambda^x_{k-1}\left(w, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}w\right)}$ and $S\left(u \mid \mathcal{F}_{T_{(k-1)}}\right) = \exp\left(-\sum_{x=a,\ell,d,y} \int_{T_{(k-1)}}^{u} \lambda^x_{k-1}\left(w, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}w\right)$ and $S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right) = \exp\left(-\int_{T_{(k-1)}}^{u} \lambda^c_{k-1}\left(w, \mathcal{F}_{T_{(k-1)}}\right) \mathrm{d}w\right)$. We claim that the efficient influence function can also be written as:

$$
\begin{aligned}
\varphi^*_\tau(P) = &\sum_{k=1}^{K} \prod_{j=0}^{k-1} \left(\frac{\pi^*_{j-1}\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1-1)}}\right)}{\pi_{j-1}\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1-1)}}\right)}\right)^{I\left(\Delta_{(j)} = a\right)} \frac{I\left(\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} \leq \tau\right)}{\exp\left(-\sum_{1 \leq j < k} \int_{T_{(j-1)}}^{T_{(j)}} \lambda^c_{j-1}\left(s, \mathcal{F}_{T_{(j-1)}}\right) \mathrm{d}s\right)} \Bigg[\\
&\int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \left(\int_{\mathcal{A}} \bar{Q}^a_{k,\tau}\left(L\left(T_{(k-1)}\right), a_k, s, a, \mathcal{F}_{T_{(k)}}\right) \pi^*_k\left(s, a_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k) - B_{k-1}(u)\right) M^a_k(du) \\
&+ \int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \left(\mathbb{E}_P\left[\bar{Q}^a_{k,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right] - B_{k-1}(u)\right) M^\ell_k(du) \\
&+ \int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} (1 - B_{k-1}(u)) M^y_k(du) + \int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} (0 - B_{k-1}(u)) M^d_k(du) \\
&+ \frac{1}{S^c\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right)} I\left(T_{(k)} \leq \tau, \Delta_{(k)} = \ell, k < K\right) \left(\bar{Q}^a_{k,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), T_{(k)}, \ell, \mathcal{F}_{T_{(k-1)}}\right)\right. \\
&\qquad \left. - \mathbb{E}_P\left[\bar{Q}^a_{k-1,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \mid \widetilde{T_{(k)}} = T_{(k)}, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right]\right) \Bigg] \\
&+ \int \bar{Q}^a_{1,\tau}(a, L_0) \pi^*_0\left(0, a, \mathcal{F}_{T_{(0-1)}}\right) \nu_A(\mathrm{d}a) - \Psi_\tau(P)
\end{aligned} \tag{40}
$$

We find immediately that

$$
\begin{aligned}
\varphi^*_\tau(P) = \sum_{k=1}^{K} \prod_{j=0}^{k-1} & \left(\frac{\pi^*_{j-1}\Big(T_{(j)}, A\big(T_{(j)}\big), \mathcal{F}_{T_{(j-1-1)}}\Big)}{\pi_{j-1}\Big(T_{(j)}, A\big(T_{(j)}\big), \mathcal{F}_{T_{(j-1-1)}}\Big)} \right)^{I\left(\Delta_{(j)}=a\right)} \frac{I\Big(\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} \leq \tau\Big)}{\exp\Big(-\sum_{1\leq j<k} \int_{T_{(j-1)}}^{T_{(j)}} \lambda^c_{j-1}\Big(s, \mathcal{F}_{T_{(j-1)}}\Big)\mathrm{d}s\Big)} \Bigg[\\
& - \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}^a_{k,\tau}\Big(L\big(T_{(k-1)}\big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big) \pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k) \right) \Lambda^a_k(du) \\
& - \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\mathbb{E}_P\Big[\bar{Q}^a_{k,\tau}\Big(L\big(T_{(k)}\big), A\big(T_{(k-1)}\big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}} \Big] \right) \Lambda^\ell_k(du) \\
& - \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} (1) \Lambda^y_k(du) - \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} (0) \Lambda^d_k(du) \\
& - \int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} B_{k-1}(u) M^\bullet_k(\mathrm{d}u) \\
& + \bar{Z}_{k,\tau}\Big(T_{(k)}, \Delta_{(k)}, L\big(T_{(k)}\big), A\big(T_{(k)}\big), \mathcal{F}_{T_{(k-1)}}\Big) + \int_{T_{(k-1)}}^{\tau} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} B_{k-1}(u) M^c_k(du) \Bigg] \\
& + \int \bar{Q}^a_{1,\tau}(a, L_0) \pi^*_0\Big(0, a, \mathcal{F}_{T_{(0-1)}}\Big)\nu_A(\mathrm{d}a) - \Psi_\tau(P)
\end{aligned}
\tag{41}
$$

Now note that

$$
\begin{aligned}
& \int_{T_{(k-1)}}^{\tau} \big(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)\big) \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(N^\bullet_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s)\lambda^\bullet_{k-1}\Big(s, \mathcal{F}_{T_{(k-1)}}\Big)\mathrm{d}s\Big) \\
& = \Big(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}\big(T_{(k)}\big)\Big) \frac{1}{S^c\Big(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big)} \mathbb{1}\Big\{T_{(k)} \leq \tau\Big\} \\
& - \bar{Q}^a_{k-1,\tau}(\tau) \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^\bullet_{k-1}\Big(s, \mathcal{F}_{T_{(k-1)}}\Big)\mathrm{d}s \\
& + \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{\bar{Q}^a_{k-1,\tau}(u)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^\bullet_{k-1}\Big(s, \mathcal{F}_{T_{(k-1)}}\Big)\mathrm{d}s
\end{aligned}
\tag{42}
$$

Let us calculate the second integral

$$
\begin{aligned}
& \bar{Q}^a_{k-1,\tau}(\tau) \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^\bullet_{k-1}\Big(s, \mathcal{F}_{T_{(k-1)}}\Big)\mathrm{d}s \\
& = \bar{Q}^a_{k-1,\tau}(\tau) \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{\Big(S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\Big)^2} \lambda^\bullet_{k-1}\Big(s, \mathcal{F}_{T_{(k-1)}}\Big)\mathrm{d}s \\
& = \bar{Q}^a_{k-1,\tau}(\tau) \left(\frac{1}{S^c\Big(T_{(k)} \wedge \tau \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(T_{(k)} \wedge \tau \mid \mathcal{F}_{T_{(k-1)}}\Big)} - 1 \right)
\end{aligned}
\tag{43}
$$

where the last line holds by the Duhamel equation (or using that the antiderivative of $-\frac{f'}{f^2}$ is $\frac{1}{f}$). The first of these integrals is equal to

$$
\begin{aligned}
&\int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{\bar{Q}^a_{k+1,\tau}(u)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^{\bullet}_{k-1}\Big(u, \mathcal{F}_{T_{(k-1)}}\Big) \mathrm{d}u \\
&= \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \Bigg[\int_0^u S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^a_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Bigg(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\Big) \pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big) \nu_A(\mathrm{d}a_k)\Bigg) \\
&+ \int_0^u S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^\ell_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Big(\mathbb{E}_P\Big[\bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big) \\
&+ \int_0^u S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^y_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big)\Bigg] \\
&\times \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^{\bullet}_{k-1}\Big(u, \mathcal{F}_{T_{(k-1)}}\Big) \mathrm{d}u \\
&= \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \int_s^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^{\bullet}_{k-1}\Big(u, \mathcal{F}_{T_{(k-1)}}\Big) \mathrm{d}u \Bigg[S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^a_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Bigg(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\Big) \pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big) \nu_A(\mathrm{d}a_k)\Bigg) \\
&+ S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^\ell_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Big(\mathbb{E}_P\Big[\bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big) \\
&+ S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) \Lambda^y_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big)\Bigg]
\end{aligned} \tag{44}
$$

Now note that

$$
\begin{aligned}
&\int_s^{\tau \wedge T_{(k)}} \frac{1}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \lambda^{\bullet}_{k-1}\Big(u, \mathcal{F}_{T_{(k-1)}}\Big) \mathrm{d}u \\
&= \frac{1}{S^c\Big(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big)} - \frac{1}{S^c\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(s \mid \mathcal{F}_{T_{(k-1)}}\Big)}
\end{aligned} \tag{45}
$$

Setting this into the previous integral, we get

$$
\begin{aligned}
&- \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c(s)} \lambda^{\bullet}_{k-1}\Big(u, \mathcal{F}_{T_{(k-1)}}\Big) \mathrm{d}u \Bigg[\Lambda^a_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Bigg(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\Big) \pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big) \nu_A(\mathrm{d}a_k)\Bigg) \\
&+ \Lambda^\ell_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big) \\
&\qquad \times \Big(\mathbb{E}_P\Big[\bar{Q}^a_{k+1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big) \\
&+ \Lambda^y_{k-1}\Big(ds, \mathcal{F}_{T_{(k-1)}}\Big)\Bigg] \\
&+ \frac{1}{S^c\Big(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big) S\Big(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\Big)} \bar{Q}^a_{k-1,\tau}\Big(\tau \wedge T_{(k)}\Big)
\end{aligned} \tag{46}
$$

Thus, we find

$$
\begin{aligned}
&\int_{T_{(k-1)}}^{\tau} \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)\right) \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \left(N^\bullet_k(\mathrm{d}s) - \tilde{Y}_{k-1}(s)\lambda^\bullet_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}s\right) \\
&= \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}\left(T_{(k)}\right)\right) \frac{1}{S^c\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right)} \mathbb{1}\left\{T_{(k)} \leq \tau\right\} \\
&- \bar{Q}^a_{k-1,\tau}(\tau) \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \lambda^\bullet_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}s \\
&+ \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{\bar{Q}^a_{k-1,\tau}(u)}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \lambda^\bullet_{k-1}\left(s, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}s \\
&= \left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}\left(T_{(k)}\right)\right) \frac{1}{S^c\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right)} \mathbb{1}\left\{T_{(k)} \leq \tau\right\} \\
&- \left(\bar{Q}^a_{k-1,\tau}(\tau) \left(\frac{1}{S^c\left(T_{(k)} \wedge \tau \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(T_{(k)} \wedge \tau \mid \mathcal{F}_{T_{(k-1)}}\right)}\right) - 1\right) \\
&- \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c(s)} \lambda^\bullet_{k-1}\left(u, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}u \Bigg[\Lambda^a_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right) \\
&\qquad\qquad \times \left(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k-1)}\right), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\right) \pi^*_k\left(s, a_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k)\right) \\
&+ \Lambda^\ell_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right) \\
&\qquad\qquad \times \left(\mathbb{E}_P\left[\bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right]\right) \\
&+ \Lambda^y_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right)\Bigg] \\
&+ \frac{1}{S^c\left(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(\tau \wedge T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right)} \bar{Q}^a_{k-1,\tau}\left(\tau \wedge T_{(k)}\right) \\
&= - \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}} \frac{1}{S^c(s)} \lambda^\bullet_{k-1}\left(u, \mathcal{F}_{T_{(k-1)}}\right)\mathrm{d}u \Bigg[\Lambda^a_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right) \\
&\qquad\qquad \times \left(\int_{\mathcal{A}} \bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k-1)}\right), a_k, s, a, \mathcal{F}_{T_{(k-1)}}\right) \pi^*_k\left(s, a_k, \mathcal{F}_{T_{(k-1)}}\right) \nu_A(\mathrm{d}a_k)\right) \\
&+ \Lambda^\ell_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right) \\
&\qquad\qquad \times \left(\mathbb{E}_P\left[\bar{Q}^a_{k+1,\tau}\left(L\left(T_{(k)}\right), A\left(T_{(k-1)}\right), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\right) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\right]\right) \\
&+ \Lambda^y_{k-1}\left(ds, \mathcal{F}_{T_{(k-1)}}\right)\Bigg] + \bar{Q}^a_{k-1,\tau}(\tau)
\end{aligned} \tag{47}
$$

6 Remainder term for $K = 2$

Taking the mean of the EIF with respect to P_0 gives

$$
\begin{aligned}
\varphi^*_\tau(P)] = \mathbb{E}_{P_0}\Bigg[\sum_{k=1}^{K}\prod_{j=0}^{k-1}\left(\frac{\pi^*_{j-1}\Big(T_{(j)}, A\Big(T_{(j)}\Big), \mathcal{F}_{T_{(j-1-1)}}\Big)}{\pi_{j-1}\Big(T_{(j)}, A\Big(T_{(j)}\Big), \mathcal{F}_{T_{(j-1-1)}}\Big)}\right)^{I\left(\Delta_{(j)}=a\right)} \frac{I\Big(\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} \leq \tau\Big)}{\exp\Big(-\sum_{1\leq j<k}\int_{T_{(j-1)}}^{T_{(j)}} \lambda^c_{j-1}\Big(s, \mathcal{F}_{T_{(j-1)}}\Big)\mathrm{d}s\Big)}\Bigg[\\
&\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\left(\int_{\mathcal{A}} \bar{Q}^a_{k,\tau}\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k) - B_{k-1}(u)\right)\Big(\Lambda^a_{k,0}(du) - \Lambda^a_k(du)\Big) \\
&+ \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\Big(\mathbb{E}_P\Big[\bar{Q}^a_{k,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] - B_{k-1}(u)\Big)\Big(\Lambda^\ell_{k,0}(du) - \Lambda^\ell_k \\
&+ \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}(1 - B_{k-1}(u))\Big(\Lambda^y_{k,0}(du) - \Lambda^y_k(du)\Big) + \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} - B_{k-1}(u)\Big(\Lambda^d_{k,0}(du) - \Lambda^d_k(du \\
&+ \mathbb{1}\{k < K\}\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\Big(\mathbb{E}_{P_0}\Big[\bar{Q}^a_{k-1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] \\
&\qquad - \mathbb{E}_P\Big[\bar{Q}^a_{k-1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda^c_{k,0}(du)\Bigg]\Bigg] \\
&+ \mathbb{E}_{P_0}\Big[\int \bar{Q}^a_{1,\tau}(a, L_0)\pi^*_0\Big(0, a, \mathcal{F}_{T_{(0-1)}}\Big)\nu_A(\mathrm{da})\Big] - \Psi_\tau(P)
\end{aligned}
\tag{48}
$$

We need to calculate

$$
\begin{aligned}
P, P_0) &= \Psi_\tau(P) - \Psi_\tau(P_0) + P_0(\varphi^*_\tau(P)) \\
&= \mathbb{E}_{P_0}\Bigg[\sum_{k=1}^{K}\prod_{j=0}^{k-1}\left(\frac{\pi^*_{j-1}\Big(T_{(j)}, A\Big(T_{(j)}\Big), \mathcal{F}_{T_{(j-1-1)}}\Big)}{\pi_{j-1}\Big(T_{(j)}, A\Big(T_{(j)}\Big), \mathcal{F}_{T_{(j-1-1)}}\Big)}\right)^{I\left(\Delta_{(j)}=a\right)} \frac{I\Big(\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} \leq \tau\Big)}{\exp\Big(-\sum_{1\leq j<k}\int_{T_{(j-1)}}^{T_{(j)}} \lambda^c_{j-1}\Big(s, \mathcal{F}_{T_{(j-1)}}\Big)\mathrm{d}s\Big)}\Bigg[\\
&\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\left(\int_{\mathcal{A}} \bar{Q}^a_{k,\tau}\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi^*_k\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k) - B_{k-1}(u)\right)\Big(\Lambda^a_{k,0}(du) - \Lambda^a_k(du)\Big) \\
&+ \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\Big(\mathbb{E}_P\Big[\bar{Q}^a_{k,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] - B_{k-1}(u)\Big)\Big(\Lambda^\ell_{k,0}(du) - \Lambda^\ell_k(\\
&+ \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}(1 - B_{k-1}(u))\Big(\Lambda^y_{k,0}(du) - \Lambda^y_k(du)\Big) + \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} - B_{k-1}(u)\Big(\Lambda^d_{k,0}(du) - \Lambda^d_k(du) \\
&+ \mathbb{1}\{k < K\}\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)\frac{S^c_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}\Big(\mathbb{E}_{P_0}\Big[\bar{Q}^a_{k-1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] \\
&\qquad - \mathbb{E}_P\Big[\bar{Q}^a_{k-1,\tau}\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda^\ell_{k,0}(du)\Bigg]\Bigg] \\
&+ \mathbb{E}_{P_0}\Big[\int \bar{Q}^a_{1,\tau}(a, L_0) - \bar{Q}^a_{1,\tau_0}(a, L_0)\pi^*_0\Big(0, a, \mathcal{F}_{T_{(0-1)}}\Big)\nu_A(\mathrm{da})\Big]
\end{aligned}
\tag{49}
$$

By the Duhamel equation,

$$
\begin{aligned}
&\bar{Q}^a_{1,\tau}(a, L_0) - \bar{Q}^a_{1,\tau_0} \\
&= S_0(s)(\mathrm{bla} - \mathrm{bla}_0) + (S(s) - S_0(s))\ \mathrm{bla} \\
&= S_0(s)(\mathrm{bla}\ Q - \mathrm{bla}_0 Q + \mathrm{bla}_0 Q - \mathrm{bla}_0 Q_0) - \int S_0 B_{k-1}\sum_x(\Lambda_{0,x} - \Lambda_x)
\end{aligned}
\tag{50}
$$

The second term gives that we can ignore B_k:

$$
\begin{aligned}
&\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi_k^*\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k)\right)\big(\Lambda_{k,0}^a(du) - \Lambda_k^a(du)\big) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_P\Big[\bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\big(\Lambda_{k,0}^\ell(du) - \Lambda_k^\ell(du)\big) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \big(\Lambda_{k,0}^y(du) - \Lambda_k^y(du)\big) \\
&+\mathbb{1}\{k<K\}\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_{P_0}\Big[\bar{Q}_{k-1,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] \\
&\qquad -\mathbb{E}_P\Big[\bar{Q}_{k-1,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda_{k,0}^\ell(du) \\
&= \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi_k^*\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k)\right)\big(\Lambda_{k,0}^a(du) - \Lambda_k^a(du)\big) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_P\Big[\bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\big(-\Lambda_k^\ell(du)\big) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \big(\Lambda_{k,0}^y(du) - \Lambda_k^y(du)\big) \\
&+\mathbb{1}\{k<K\}\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_{P_0}\Big[\bar{Q}_{k-1,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), \widetilde{T_{(k)}}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid \widetilde{T_{(k)}} = u, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda_{k,0}^\ell(du) \qquad (51) \\
&= \int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}_{k,\tau_0}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi_k^*\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k)\right)\Lambda_{k,0}^a(du) \\
&-\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi_k^*\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k)\right)\Lambda_k^a(du) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \left(\int_{\mathcal{A}} \bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big) - \bar{Q}_{k,\tau_0}^a\Big(L\Big(T_{(k-1)}\Big), a_k, s, a, \mathcal{F}_{T_{(k)}}\Big)\pi_k^*\Big(s, a_k, \mathcal{F}_{T_{(k-1)}}\Big)\nu_A(\mathrm{d}a_k)\right)\Lambda_{k,0}^a(du) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_{P_0}\Big[\bar{Q}_{k,\tau}^a \cdot \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big] - \mathbb{E}_{P_0}\Big[\bar{Q}_{k,\tau_0}^a \cdot \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda_{k,0}^\ell(du) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_{P_0}\Big[\bar{Q}_{k,\tau_0}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda_{k,0}^\ell(du) \\
&-\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \Big(\mathbb{E}_P\Big[\bar{Q}_{k,\tau}^a\Big(L\Big(T_{(k)}\Big), A\Big(T_{(k-1)}\Big), T_{(k)}, \Delta_{(k)}, \mathcal{F}_{T_{(k-1)}}\Big) \mid T_{(k)} = s, \Delta_{(k)} = \ell, \mathcal{F}_{T_{(k-1)}}\Big]\Big)\Lambda_k^\ell(du) \\
&+\int_{T_{(k-1)}}^{\tau} S_0\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big) \frac{S_0^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)}{S^c\Big(u \mid \mathcal{F}_{T_{(k-1)}}\Big)} \big(\Lambda_{k,0}^y(du) - \Lambda_k^y(du)\big)
\end{aligned}
$$

Adding the first term together in the sum with the last term, we have

$$
\begin{aligned}
{}^*_\tau(P)] &= \mathbb{E}_{P_0^*}\Bigg[\int_0^\tau S_0(u \mid \mathcal{F}_0)\left(\frac{\pi_0(L(0))}{\pi(L(0))}\frac{S_0^c(u \mid \mathcal{F}_0)}{S^c(u \mid \mathcal{F}_0)} - 1\right)\left(\bar{Q}^a_{0,\tau_0}(ds, \mathcal{F}_0) - \bar{Q}^a_{0,\tau}(ds, \mathcal{F}_0)\right) \\
&+ \int_0^\tau S_0(u \mid \mathcal{F}_0)\frac{\pi_0(L(0))}{\pi(L(0))}\frac{S_0^c(u \mid \mathcal{F}_0)}{S^c(u \mid \mathcal{F}_0)}\left(\sum_{x=a,\ell}\mathbb{E}_{P_0^*}\left[\bar{Q}^a_{1,\tau} - \bar{Q}^a_{1,\tau_0} \mid T_{(1)} = s, \Delta_{(1)} = x, \mathcal{F}_0\right]\Lambda^x_{1,0}(du)\right)\Bigg] \\
&= \mathbb{E}_{P_0^*}\left[\int_0^\tau S_0(u \mid \mathcal{F}_0)\left(\frac{\pi_0(L(0))}{\pi(L(0))}\frac{S_0^c(u \mid \mathcal{F}_0)}{S^c(u \mid \mathcal{F}_0)} - 1\right)\left(\bar{Q}^a_{0,\tau_0}(ds, \mathcal{F}_0) - \bar{Q}^a_{0,\tau}(ds, \mathcal{F}_0)\right)\right] \\
&+ \mathbb{E}_{P_0^*}[\int_{T_{(k-1)}}^\tau S_0\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)\frac{\pi_0(L(0))}{\pi(L(0))}\frac{S_0^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)} \\
&\times \left(\sum_{x=a,\ell}\mathbb{E}_{P_0^*}\left[\int_0^\tau S_0\left(s \mid \mathcal{F}_{T_{(1)}}\right)\left(\frac{\pi_{1,0}\left(T_{(1)}, A\left(T_{(1)}\right), \mathcal{F}_0\right)}{\pi_1\left(T_{(1)}, A\left(T_{(1)}\right), \mathcal{F}_0\right)}\frac{S_0^c\left(s \mid \mathcal{F}_{T_{(1)}}\right)}{S^c\left(u \mid \mathcal{F}_{T_{(1)}}\right)} - 1\right)\left(\bar{Q}^a_{1,\tau_0}\left(ds, \mathcal{F}_{T_{(1)}}\right) - \bar{Q}^a_{1,\tau}\left(ds, \mathcal{F}_{T_{(1)}}\right)\right) \mid T_{(1)} = u, \Delta_{(1)} = x, \mathcal{F}_0\right]\Lambda^x_{1,0}(
\end{aligned}
\tag{52}
$$

6.1 Coupled ICE one-step estimator

We provide a special estimator for the purpose of one-step estimation. This iterative regression is a little bit different from the ICE IPCW estimator, because we also integrate over the covariate distribution from the previous step.

- For each event point $k = K, K-1, ..., 1$ (starting with $k = K$):
  1. Obtain $\hat{S}^c\left(t \mid \mathcal{F}_{T_{(k-1)}}\right)$ by fitting a cause-specific hazard model for the censoring via the interevent time $S_{(k)} = T_{(k)} - T_{(k-1)}$, regressing on $\mathcal{F}_{T_{(k-1)}}$ (among the people who are still at risk after $k-1$ events).
  2. Define the subject-specific weight:
  $$\hat{\eta}_k = \frac{\mathbb{1}\left\{T_{(k)} \leq \tau, \Delta_k \in \{a, \ell\}, k < K\right\}\hat{\nu}_k\left(\mathcal{F}^{-A,L}_{T_{(k)}}, \mathbf{1}\right)}{\hat{S}^c\left(T_{(k)} \mid \mathcal{F}^{-A}_{T_{(k-1)}}, \mathbf{1}\right)} \tag{53}$$
  Then calculate the subject-specific pseudo-outcome
  $$\hat{R}_k = \frac{\mathbb{1}\left\{T_{(k)} \leq \tau, \Delta_k = y\right\}}{\hat{S}^c\left(T_{(k)} \mid \mathcal{F}^{-A}_{T_{(k-1)}}, \mathbf{1}\right)} + \hat{\eta}_k \tag{54}$$
  If $k > 1$: Regress $\hat{R}_k$ on $\mathcal{F}^{-L}_{T_{(k-1)}}$ on the data with $T_{(k-1)} < \tau$ and $\Delta_k \in \{a, \ell\}$ to obtain a prediction function $\hat{\nu}_{k-1} : \mathcal{H}_{k-1} \to \mathbb{R}_+$.
  3. Calculate the corresponding term in the efficient influence function based on $\hat{\nu}_k$. This is the "coupling" step: Given cause-specific estimators $\hat{\Lambda}^x_{k-1}$ for $x = a, l, d, y$, estimate $\bar{Q}^a_{k-1,\tau}\left(u, \mathcal{F}_{T_{(k-1)}}\right)$ by
  $$
  \begin{aligned}
  \tilde{\nu}_k\left(u, \mathcal{F}_{T_{(k-1)}}\right) = \int_{T_{(k-1)}}^u &\hat{S}\left(u \mid \mathbf{1}, \mathcal{F}^{-A}_{T_{(k-1)}}\right)\left(\hat{\Lambda}^y\left(du \mid \mathbf{1}, \mathcal{F}^{-A}_{T_{(k-1)}}\right)\right. \\
  &+ \hat{\nu}_k\left(\mathbf{1}, u, a, \mathcal{F}^{-A}_{T_{(k-1)}}\right)\mathcal{F}^{-A}_{T_{(k-1)}}\Big)\hat{\Lambda}^a_{k-1}\left(du \mid \mathbf{1}, \mathcal{F}^{-A}_{T_{(k-1)}}\right) \\
  &+ \hat{\nu}_k\left(\mathbf{1}, u, \ell, \mathcal{F}_{T_{(k-1)}}\right)\hat{\Lambda}^\ell_{k-1}\left(du \mid \mathbf{1}, \mathcal{F}^{-A}_{T_{(k-1)}}\right))
  \end{aligned}
  \tag{55}
  $$
  We then debias the corresponding term in the efficient influence function, i.e.,
  $$
  \begin{aligned}
  &\frac{\pi_0^*(L(0))}{\pi_0(L(0))}\prod_{j=1}^{k-1}\left(\frac{\pi_j^*\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1)}}\right)}{\pi_j\left(T_{(j)}, A\left(T_{(j)}\right), \mathcal{F}_{T_{(j-1)}}\right)}\right)^{\mathbb{1}\left\{\Delta_{(j)} = a\right\}}\frac{1}{S^c\left(T_{(j)} \mid \mathcal{F}_{T_{(j-1)}}\right)}\mathbb{1}\left\{\Delta_{(k-1)} \in \{\ell, a\}, T_{(k-1)} < \tau\right\} \\
  &\times \left(\bar{Z}^a_{k,\tau} - \bar{Q}^a_{k-1,\tau}\left(\tau, \mathcal{F}_{T_{(k-1)}}\right) + \int_{T_{(k-1)}}^{\tau \wedge T_{(k)}}\left(\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)\right)\frac{1}{S^c\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)S\left(u \mid \mathcal{F}_{T_{(k-1)}}\right)}M^c_k(\mathrm{d}s)\right)
  \end{aligned}
  \tag{56}
  $$
  by also estimating π.
- At baseline, we obtain the estimate $\hat{\Psi}_n = \frac{1}{n}\sum_{i=1}^n \bar{Q}^a_{0,\tau}(L_i(0), 1)$.

If for example, we use a cause-specific models, for the computation of the integral. This requires then for combination of a point in the time grid and each observation, estimates from $\hat{\nu}_{k-1}$. If we assume that we use empirical estimates of the cumulative incidence function at that point, we can estimate $\bar{Q}^a_{k-1,\tau}(\tau) - \bar{Q}^a_{k-1,\tau}(u)$ by

$$\tilde{\nu}_k(s) = \widehat{F}_y\left(s, \mathcal{F}_{T_{(k-1)}}\right) + \int_0^s \hat{\nu}_k\left(\mathbf{1}, u, a, \mathcal{F}_{T_{(k-1)}}\right)\widehat{F}_a\left(du, \mathcal{F}_{T_{(k-1)}}\right) + \int_0^s \hat{\nu}_k\left(\mathbf{1}, u, \ell, \mathcal{F}_{T_{(k-1)}}\right)\widehat{F}_\ell\left(du, \mathcal{F}_{T_{(k-1)}}\right) \tag{57}$$

If we assume that the cumulative hazard for the censoring jumps at the points $T_{(k-1)} = t_0 < t_1 < t_2 < \dots < t_c = \tau$. Let $k^* = \sup\left\{k \mid t_k < T_{(k)}\right\}$, then we can estimate the martingale integral by

$$\mathbb{1}\left\{T_{(k)} \le \tau, \Delta_{(k)} = c\right\} \frac{\tilde{\nu}_k(\tau) - \tilde{\nu}_k\left(T_{(k)}\right)}{S^c\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(T_{(k)} \mid \mathcal{F}_{T_{(k-1)}}\right)} - \sum_{j=1}^{k^*} \frac{\tilde{\nu}_k(\tau) - \tilde{\nu}_k\left(t_j\right)}{S^c\left(t_j \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(t_j \mid \mathcal{F}_{T_{(k-1)}}\right)} \left(\Lambda^c\left(t_j\right) - \Lambda^c\left(t_{j-1}\right)\right) \tag{58}$$

If we write this down in terms of the interevent times instead. Let $0 = s_0 < s_1 < s_2 < \dots < s_c = \tau - \min_i T_{(k-1)}$ be the event grid times and $k^* = \sup\left\{k \mid s_k \le \min\left(S_k, \tau - T_{(k-1)}\right)\right\}$. Then we can write

$$\mathbb{1}\left\{S_k \le \tau - T_{(k-1)}, \Delta_{(k)} = c\right\} \frac{\tilde{\nu}_k\left(\tau - T_{(k-1)}\right) - \tilde{\nu}_k\left(T_{(k-1)} + S_k\right)}{S^c\left(S_k \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(S_k \mid \mathcal{F}_{T_{(k-1)}}\right)} - \sum_{j=1}^{k^*} \frac{\tilde{\nu}_k\left(\tau - T_{(k-1)}\right) - \tilde{\nu}_k\left(s_j + T_{(k-1)}\right)}{S^c\left(s_j \mid \mathcal{F}_{T_{(k-1)}}\right) S\left(s_j \mid \mathcal{F}_{T_{(k-1)}}\right)} \left(\Lambda^c\left(s_j\right) - \Lambda^c\left(s_{j-1}\right)\right) \tag{59}$$

The evaluation of $\tilde{\nu}$ for each term in $\tilde{\nu}$ here would be done on another grid for each term $0 = s_0^* < s_1^* < s_2^* < \dots < s_m^* = \tau - \min_i T_{(k-1)}$. For s_j, the corresponding term would be s_l^* with $l = \sup\left\{k \mid s_k^* \le s_j\right\}$.

This tactic for debiasing does not directly yield a TMLE alternative.

7 Data-adaptive choice of K

In practice, we will want to use K_τ to be equal to 1 + maximum number of non-terminal events up to τ in the sample. It turns out, under the boundedness condition of the number of events, that an estimator that is asymptotically linear with efficient influence function $\varphi_\tau^*(P)(\max_i \kappa_i(\tau))$ is also asymptotically linear with efficient influence function $\varphi_\tau^*(P)(K_\tau)$ where K_τ is the last event point such that $P(\kappa_i(\tau) = K_\tau) > 0$.

Sketch: We want to use $K = K_n = \max_i \kappa_i(\tau)$. If we can do asymptotically and efficient inference for K_n, then we can also do it for a limiting $K_n \le K$. Assume that the estimator is asymptotically linear with efficient influence function $\varphi_\tau^*(P)(K_n)$. Then by Assumption 1, there exists a K_{lim} which is the last point such that $P(K_n = K_{\text{lim}}) > 0$. Then, K_n converges to K_{lim} (by independence), and moreover, under standard regularity conditions such as strict positivity,

$$(\mathbb{P}_n - P)(\varphi_\tau^*(P)(K_n) - \varphi_\tau^*(P)(K)) \tag{60}$$

is $o_P\left(n^{-\frac{1}{2}}\right)$, so if have asymptotic linearity in terms of $\varphi_\tau^*(P)(K_n)$, then we automatically have it for the original estimator for $\varphi_\tau^*(P)(K_{\text{lim}})$

8 Issues relating to rare patient histories

In the case of irregular data, we may have few people with many events. In that case, the iterative conditional expectations estimator will fail, because there are not enough people at each event point (see Table 1). We are then left with three options:

- Pooling
- A data-adaptive target parameter (Hubbard et al. (2016)), where the number of events considered (not all events) for the parameter is chosen based on the data, that is $\Psi_\tau(P) = \sum_{k=1}^{K_n} \mathbb{E}_P\left[\mathbb{1}\left\{\widetilde{T_{(k)}} \le \tau, \widetilde{\Delta_{(k)}} = y\right\}\right]$, where K_n is selected based on the data. This is essentially done by sample splitting/cross-validation.
- Event-adaptive model selection, where the complexity of the models for each event point is based on how many data points are available at that event point (parametric models for very few data points, nonparametric models for many data points). By sample splitting, we should be able to take into account the data-adaptive model selection.

k	0	1	2	3	4	5
$\tilde{Y}_k(\tau)$	10000	8540	5560	2400	200	4
$\Delta A(T_{(k)})$	6000	3560	1300	100	2	NA
$\Delta L(T_{(k)})$	2540	2000	1100	100	2	NA

8.1 Pooling

Some people have complex histories. There may be very few of these people in the sample, so how do we do estimate the cause-specific hazard for the censoring in, say, the first step? In the artificial data example, there are only 4 people at the last time point.

We propose to pool the regressions across event points: Let us say that we want to estimate the cause-specific hazard for the censoring at event $k+1$ among people who are at risk of being censored at the $k+1$'th event, that is they either had a treatment change or a covariate change at their k event. If this population in the sample is very small, then we could do as follows. We delete the first event for these observations. Then the number of covariates is reduced by one, so we have the same number of covariates as we did for the people who are at risk of having an event at the k'th event. We combine these two data sets into one and regress the cause-specific hazard for the censoring at event "k". This provides a data set with correlated observations, which likely is not biased as we are not interested in variance estimation for parameters appearing in the regression.

To estimate the regression for the time-varying covariates, one could do:

- Not intervene on the last two or three time points, letting certain parts of the data generating mechanism be observational, that is $\pi_j^*\left(t, \cdot, \mathcal{F}_{T_{(j-1)}}\right) = \pi_j\left(t, \cdot, \mathcal{F}_{T_{(j-1)}}\right)$ for $j = 4, 5$.
- Another is to make a Markov-like assumption in the interventional world, i.e.,
$$\mathbb{E}_Q\left[\sum_{j=1}^{3} I\left(T_{(j)} \leq \tau, \Delta_{(j)} = y\right) \mid \mathcal{F}_0\right] = \mathbb{E}_Q\left[\sum_{j=6}^{8} I\left(T_{(j)} \leq \tau, \Delta_{(j)} = y\right) \mid \mathcal{F}_{T_{(5)}}\right] \tag{61}$$
So we separately estimate the target parameter on the left hand side and use it to estimate the one on the right when we need to, pooling the data from the last three events with the data from the first three events.

Other possible methods are:

- Use an estimation procedure that is similar to Shirakawa et al. (2024) or use hazards which are estimated all at once.
- Bayesian methods may be useful since they do not have issues with finite sample size. They are also a natural way of dealing with the missing data problem. However, nonparametric Bayesian methods are not (yet) able to deal with a large number of covariates.

8.2 Other ideas

Some other issues are that the covariates are (fairly) high dimensional. This may yield issues with regression-based methods.

- Use Early-stopping cross-validation described as follows: First fit models with no covariates. Then we fit a model with the covariates from the last event. Determine if this is improves model fit via cross-validation and then we move on to the two latest changes and so on. Stop when the model fit does not improve. Theorem 2 of Schuler & van der Laan (2022) states that the convergence rates for an empirical risk minimizer are preserved. CTMLE also does something very similar (van der Laan & Gruber, 2010). This way, we may only select variables that are important in the specification of the treatment and outcome mechanism.

8.3 Topics for further research

Interestingly, $\int \bar{Q}^a_{0,\tau}(a, L_0)\nu_A(\mathrm{d}a)$ is a heterogenous causal effect. Can we estimate heterogenous causal effects in this way?

Time-fixed time-varying treatment could probably be interesting within a register-based study since it may be easier to define treatment in an interval rather than two define on, each time point, if the patient is on the treatment or not.

It may also sometimes be the case that some time-varying covariates are measured regularly instead of at subject-specific times. In this case, we may be able to do something similar to the above.

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