1) A causal interpretation in terms of potential outcomes of target parameter in Rytgaard et al. (2022)

We consider a setting similar to the one of Ryalen (2024) and Rytgaard et al. (2022). As in Rytgaard et al. (2022), we consider some measure P on a probability space $(\Omega, \mathcal{F}, P)$. We consider a setting in which we observe a multivariate random measure $N = \left(N^y, N^a, N^\ell\right)$ which is defined on $(\Omega, \mathcal{F})$, where any two of the components do not jump at the same time. These processes are observed in $[0, T]$ for some $T > 0$. Here, N^y denotes an outcome process of interest Y (e.g., death), random measure N^a on $[0, T] \times \mathcal{A}$ for treatment A, where $\mathcal{A}$ is a measurable space; N^ℓ denotes a random measure for covariates L on $[0, T] \times \mathcal{L}$, where $\mathcal{A}$ and $\mathcal{L}$ are measurable spaces; for instance finite subsets of $\mathbb{R}$ and $\mathbb{R}^d$. Numerating these options, we can take

$$\mathcal{A} = \{a_1, ..., a_k\}$$
$$\mathcal{L} = \{l_1, ..., l_m\}.$$

Equivalently (in the sense that the natural filtrations are the same), we may work with the multivariate counting process

$$N(t) = \left(N^y((0, t]), N^a((0, t] \times \{a_1\}), ..., N^a((0, t] \times \{a_k\}), N^\ell((0, t] \times \{l_1\}), ..., N^\ell((0, t] \times \{l_m\})\right).$$

This proces gives rise to a filtration $(\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t \coloneqq \sigma(N(s) \mid s \leq t)$. Further, we make the assumption of no explosion of N.

We concern ourselves with the hypothetical question if the treatment process N^a had been intervened upon such that treatment was given according to some treatment regime g^*. We will work with an intervention that specifies the treatment decisions but does not change timing of treatment visits. What this means precisely will be made clear below. We are interested in the outcome process Y under this intervention, which we denote by $\tilde{Y}$. Importantly, the intervention is defined as a static/dynamic intervention

$$N^{g^*}(\mathrm{d}t \times \mathrm{d}x) = \pi_t^*(\mathrm{d}x) N^a(\mathrm{d}t \times \mathcal{A})$$

where $\pi_t^*(\mathrm{d}x)$ is some kernel that specifies the treatment decision deterministically at time t in the sense that there are $\mathcal{F}_{T_{(k-1)}} \otimes \mathcal{B}([0, T])$-measurable functions g_k^* which return a treatment decision in $\mathcal{A}$ such that

$$\pi_t^*(\mathrm{d}x) = \sum_k \mathbb{1}\left\{T_{(k-1)} < t \leq T_{(k)}\right\} \delta_{g_k^*\left(\mathcal{F}_{T_{(k-1)}}, T_{(k)}\right)}(\mathrm{d}x),$$

This means that, critically, $N^{g^*}(\mathrm{d}t \times \mathrm{d}x)$ is also a random measure. Note that N^{g^*} has the compensator

$$\mathscr{L}(N)(\mathrm{d}t \times \mathrm{d}x) = \pi_t^*(\mathrm{d}x) \underbrace{\Lambda^a(\mathrm{d}t \times \mathcal{A})}_{=: \Lambda^a(\mathrm{d}t)},$$

where $\Lambda^a(\mathrm{d}t)$ is the P-$\mathcal{F}_t$-compensator of $N^a(\mathrm{d}t \times \mathcal{A})$ – also deemed the total P-$\mathcal{F}_t$-compensator of N^a. What this means is that

$$N^a((0, t] \times \mathrm{d}x) - \Lambda^a((0, t]) \times \mathrm{d}x)$$

is a local P-$\mathcal{F}_t$-martingale. We shall write similar notations for the other components of N. Let $\mathscr{L}$ denote the P-$\mathcal{F}_t$-canonical compensator of N^{g^*}. However, N^a generally has the compensator $\Lambda^a(\mathrm{d}t \times \mathrm{d}x) = \pi_t(\mathrm{d}x) \Lambda^a(\mathrm{d}t)$. Now define the time to deviation from the treatment regime as

$$\tau^{g^*} = \inf\{t \geq 0 \mid N^a((0,t] \times \{x\}) \neq N^{g^*}((0,t] \times \{x\}) \quad \text{for some} \quad x \in \mathcal{A}\},$$

when $\mathcal{A} = \{a_1, ..., a_k\}$ consists of a finite set of treatment options.

Definition 1.1: Let $\widetilde{\mathcal{F}}_t := \sigma\left(\widetilde{N}^y(\mathrm{d}s), \widetilde{N}^a(\mathrm{d}s \times \{x\}), \widetilde{N}^\ell(\mathrm{d}s \times \{y\}) \mid s \in (0,t], x \in \mathcal{A}, y \in \mathcal{L}\right)$. Let Λ denote the canonical P-$\mathcal{F}_t$-compensator of N.

A multivariate random measure $\widetilde{N} = \left(\widetilde{N}^y, \widetilde{N}^a, \widetilde{N}^\ell\right)$ is a **counterfactual random measure** under the intervention g^* if it satisfies the following conditions.

1. $\widetilde{N}^a$ has compensator $\mathscr{L}\left(\widetilde{N}\right)$ with respect to $\widetilde{\mathcal{F}}_t$.
2. $\widetilde{N}^x$, has the same compensator $\Lambda\left(\widetilde{N}\right)$ with respect to $\widetilde{\mathcal{F}}_t$ for $x \in \{y, \ell\}$.

Now let $\left(T_{(k)}\right)_k$ denote the ordered event times of N. The main outcome of interest is the counterfactual outcome process $\tilde{Y} := \widetilde{N}^y$; and we wish to identify $\mathbb{E}_P\left[\tilde{Y}_t\right]$.

Let $N^{a,x}(t) := N^a((0,t] \times \{x\})$ for $x \in \mathcal{A}$ and $M^{a,x}(t) := N^{a,x}(t) - \Lambda^{a,x}(t)$. Note that Equation 1 is the same likelihood ratio as in Rytgaard et al. (2022).

Theorem 1.1: If *all* of the following conditions hold:

- **Consistency**: $\tilde{Y}_{\cdot}\mathbb{1}\{\tau^{g^*} > \cdot\} = Y_{\cdot}\mathbb{1}\{\tau^{g^*} > \cdot\} \quad P - \text{a.s.}$
- **Exchangeability**: Define $\mathcal{H}_t := \mathcal{F}_t \vee \sigma\left(\tilde{Y}\right)$. Let $\Lambda^{a,a_j}(\mathrm{d}t) = \pi_t\left(\{a_j\}\right)\Lambda^a(\mathrm{d}t)$ denote the P-$\mathcal{F}_t$-compensator of N^{a,a_j} and $\Lambda_H^{a,a_j}(\mathrm{d}t) = \pi_t^H\left(\{a_j\}\right)\Lambda_H^a(\mathrm{d}t)$ denote the P-$\mathcal{H}_t$-compensator of N^{a,a_j}. π is indistinguishable from π^H, that is for all $j \in \{1, ..., k\}$ $P\left(\pi_t\left(\{a_j\}\right) = \pi_t^H\left(\{a_j\}\right), \forall t \in [0,T]\right) = 1$.
- **Positivity**:

$$W(t) := \prod_{j=1}^{N_t} \left(\prod_{i=1}^{k} \left(\frac{\pi^*_{T_{(j)}}\left(\{a_i\}; \mathcal{F}_{T_{(j-1)}}\right)}{\pi_{T_{(j)}}\left(\{a_i\}; \mathcal{F}_{T_{(j-1)}}\right)} \right)^{\mathbb{1}\{A(T_{(k)})=a_i\}} \right)^{\mathbb{1}\{\Delta_{(j)}=a\}} \tag{1}$$

  is a uniformly integrable P-$\mathcal{F}_t$-martingale.

  Furthermore, assume that $K(t) = \int_0^t \sum_{j=1}^k \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right) M^{a,a_j}(\mathrm{d}s)$ is a P-$\mathcal{F}_t$-martingale and that K is a process of **locally integrable variation**, meaning that $\mathbb{E}_P\left[\int_0^t |dK(s)|\right] < \infty$ for all $t > 0$.

Then,

$$\mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[Y_t W(t)]$$

and $W(t) = \mathcal{E}(K)_t$ is a uniformly integrable P-$\mathcal{F}_t$-martingale, where $\mathcal{E}$ denotes the Doléans-Dade exponential (Protter (2005)).

Proof: We shall use that the likelihood ratio solves a specific stochastic differential equation. To this end, note that

$$
\begin{aligned}
W(t) &= \mathcal{E}\left(\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)N^{a,a_j}(\mathrm{d}s)\right)_t \\
&\overset{(*)}{=} \mathcal{E}\left(\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)N^{a,a_j}(\mathrm{d}s)-\sum_{j=1}^{k}(\pi_s^*(\{a_j\})-\pi_s(\{a_j\}))\Lambda^{a}(\mathrm{d}s)\right)_t \\
&= \mathcal{E}\left(\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)N^{a,a_j}(\mathrm{d}s)-\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)\Lambda^{a,a_j}(\mathrm{d}s)\right)_t \\
&= \mathcal{E}\left(\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)M^{a,a_j}(\mathrm{d}s)\right)_t .
\end{aligned}
$$

In $(*)$, we use that $\sum_{j=1}^{k}\pi_s(\{a_j\})=\sum_{j=1}^{k}\pi_s^*(\{a_j\})=1$.

Thus, by properties of the product integral (e.g., Theorem II.6.1 of Andersen et al. (1993)),

$$
W(t)=1+\int_0^t W(s-)\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)M^{a,a_j}(\mathrm{d}s). \tag{2}
$$

We have that

$$
\zeta_t := \int_0^t W(s-)\sum_{j=1}^{k}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})}-1\right)M^{a,a_j}(\mathrm{d}s)
$$

is a zero mean P-$\mathcal{H}_t$-martingale by positivity. From this, we see that $\int_0^t \tilde{Y}_t\zeta(\mathrm{d}s)$ is also a uniformly integrable P-$\mathcal{H}_t$-martingale by Theorem 2.1.42 of Last & Brandt (1995). This implies that

$$
\mathbb{E}_P[Y_t W(t)] \overset{(**)}{=} \mathbb{E}_P\left[\tilde{Y}_t W(t)\right] = \mathbb{E}_P\left[\tilde{Y}_t \mathbb{E}_P[W(t)\mid \mathcal{H}_0]\right] = \mathbb{E}_P\left[\tilde{Y}_t W(0)\right] = \mathbb{E}_P\left[\tilde{Y}_t\right]
$$

where in $(**)$ we used consistency by noting that $W(t)\neq 0$ if and only if $\tau^{g^*}>t$. □

Note that in the proof, it suffices that $W(t)$ is uniformly bounded because then it will also be a P-$\mathcal{H}_t$-martingale since it is a local, bounded P-$\mathcal{H}_t$-martingale.

It is also natural to ask oneself: how does our conditions relate to the ones of Ryalen (2024)? The condition of consistency is the same. However, the exchangeability condition and the positivity condition are different in general. We present slightly strengthened versions of the conditions as these are easier to compare. Let $\mathbb{N}_t^a=\mathbb{1}\{\tau^{g^*}\le t\}$ and let $\mathbb{L}_t$ denote its P-$\mathcal{F}_t$-compensator.

- **Exchangeability**: Define $\mathcal{H}_t := \mathcal{F}_t \vee \sigma\left(\tilde{Y}\right)$. The P-$\mathcal{F}_t$ compensator for $\mathbb{N}^a$ is also the P-$\mathcal{H}_t$ compensator.
- **Positivity**:

$$
\widetilde{W}(t) := \frac{(\mathcal{E}(-\mathbb{N}^a))_t}{(\mathcal{E}(-\mathbb{L}^a))_t} = \mathcal{E}\left(\widetilde{K}\right)_t
$$

is uniformly integrable, where $\widetilde{K}_t=\int_0^t \frac{1}{1-\Delta\mathbb{L}_s^a}(\mathbb{N}^a(\mathrm{d}s)-\mathbb{L}^a(\mathrm{d}s))$. Furthermore, $\widetilde{K}$ is a process of **locally integrable variation** and a P-$\mathcal{F}_t$-martingale.

It is unclear at this point whether there exist potential outcomes processes which fulfill the exchangeability condition and the consistency condition for any observed data distribution of N. We leave this question for future research.

1.a) Comparison with Rytgaard et al. (2022)

In Rytgaard et al. (2022), both an exchangeability condition and a positivity condition are presented, but no proof is given that these conditions imply that their target parameter is identified. Our proposal shows that under the conditions of Theorem 1.1, the g-formula given in Rytgaard et al. (2022) causally identifies the counterfactual mean outcome under the assumption that the other martingales are orthogonal to the treatment martingale. Lemma 1 of Ryalen (2024) then gives the desired target parameter. Note that this is weaker than the assumptions in Rytgaard et al. (2022), as they implicitly require that *all* martingales are orthogonal due to their factorization of the likelihood. This is because $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y)$ if and only if $[X, Y] = 0$. This can be seen by applying Theorem 38, p. 130 of Protter (2005) and using that the stochastic exponential solves a specific stochastic differential equation.

Theorem 1.1.1 (g-formula): Let, further, $Q = W(T) \cdot P$ denote the probability measure defined by the likelihood ratio $W(T)$ given in Equation 1. Under positivity, then

1. The Q-$\mathcal{F}_t$ compensator of $N^a(\mathrm{d}t \times \mathrm{d}x)$ is $\pi_t^*(\mathrm{d}x)\Lambda_P^a(\mathrm{d}t)$.
2. The Q-$\mathcal{F}_t$ compensator of N^x is Λ_P^x for $x \in \{y, \ell\}$.

Proof: First note that for a local $\mathcal{F}_t$-martingale X in P, we have

$$\int_0^t \frac{1}{W_{s-}} \mathrm{d}\langle W, X\rangle_s^P = \langle K, X\rangle_t^P \tag{3}$$

since we have that $W_t = 1 + \int_0^t W_{s-} \,\mathrm{d}K_s$; whence

$$\langle W, X\rangle_t = \langle 1, X\rangle_t + \langle W_- \bullet K, X\rangle_t = W_{t-} \bullet \langle K, X\rangle_t$$

With $X = M^{a,x}$, we find

$$
\begin{aligned}
\langle K, M^{a,x}\rangle_t^P &= \int_0^t \sum_j \left(\frac{\pi_s^*(a_j)}{\pi_s(a_j)} - 1\right) \mathrm{d}\langle M_P^{a,a_j}, M_P^{a,x}\rangle_s^P \\
&= \int_0^t \left(\frac{\pi_s^*(x)}{\pi_s(x)} - 1\right) \mathrm{d}\langle M_P^{a,x}\rangle_s^P \\
&\quad + \sum_{j\neq x} \int_0^t \left(\frac{\pi_s^*(a_j)}{\pi_s(a_j)} - 1\right) \mathrm{d}\langle M_P^{a,a_j}, M_P^{a,x}\rangle_s^P \\
&= \int_0^t \left(\frac{\pi_s^*(x)}{\pi_s(x)} - 1\right) \pi_s(x) \Lambda_P^a(\mathrm{d}s) \\
&\quad - \int_0^t \left(\frac{\pi_s^*(x)}{\pi_s(x)} - 1\right) \Delta(\pi(x)\Lambda_P^a)_s \pi_s(x) \Lambda_P^a(\mathrm{d}s) \qquad (4) \\
&\quad - \sum_{j\neq x} \int_0^t \left(\frac{\pi_s^*(a_j)}{\pi_s(a_j)} - 1\right) \Delta(\pi(x)\Lambda_P^a)_s \pi_s(a_j) \Lambda_P^a(\mathrm{d}s) \\
&= \int_0^t (\pi_s^*(x) - \pi_s(x)) \Lambda_P^a(\mathrm{d}s) \\
&\quad - \sum_j \int_0^t (\pi_s^*(a_j) - \pi_s(a_j)) \Delta(\pi(x)\Lambda_P^a)_s \Lambda_P^a(\mathrm{d}s) \\
&= \int_0^t (\pi_s^*(x) - \pi_s(x)) \Lambda_P^a(\mathrm{d}s).
\end{aligned}
$$

Girsanov's theorem (Theorem 41, p. 136 of Protter (2005)) together with Equation 3 and Equation 4 gives that

$$
N^a(\mathrm{d}t \times \mathrm{d}x) - \pi_t(\mathrm{d}x)\Lambda_P^a(\mathrm{d}t) - (\pi_t^*(\mathrm{d}x) - \pi_t(\mathrm{d}x))\Lambda_P^a(\mathrm{d}t) = N^a(\mathrm{d}t \times \mathrm{d}x) - \pi_t^*(\mathrm{d}x)\Lambda_P^a(\mathrm{d}t)
$$

is a Q-$\mathcal{F}_t$-local martingale. The second statement follows by noting that

$$
\begin{aligned}
[M^y, K]_t &= \int_0^t \Delta N_t^y \sum_j \left(\frac{\pi_s^*(a_j)}{\pi_s(a_j)} - 1\right) N^{a,a_j}(\mathrm{d}s) \\
&\quad - \int_0^t \Delta \Lambda_P^y(s) \sum_j \left(\frac{\pi_s^*(a_j)}{\pi_s(a_j)} - 1\right) M^{a,a_j}(\mathrm{d}s)
\end{aligned}
$$

where we apply the trick with adding and subtracting the treatment compensators in the second term. The first term is zero because no two counting processes jump at the same time. The second term is a local martingale. This implies $\langle M^y, K\rangle_t^P = 0$. For $x = \ell$ the argument is the same. □

We now provide a sequential representation of the exchangeability condition. It aligns closely with the postulated exchangeability condition in Rytgaard et al. (2022); however, notably on the conditioning set, we include the k'th event time, which is not included in Rytgaard et al. (2022). We conclude that if we have independent marking for the treatment process, the condition in Rytgaard et al. (2022) is sufficient for causal identification.

Theorem 1.1.2: Suppose consistency and positivity holds as in Theorem 1.1. Then, we have

$$\mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[W_t Y_t],$$

for all $t \in [0, T]$, if for $k \in \mathbb{N}$ and $t \in [0, T]$ it holds that

$$\tilde{Y}_t \perp \mathbb{1}\left\{A\left(T_{(k)}\right) = g^*\left(\mathcal{F}_{T_{(k-1)}}, T_{(k)}\right)\right\} \mid \mathcal{F}^{g^*}_{T_{(k-1)}}, T_{(k)} \leq t, \Delta N^a\left(T_{(k)}\right) = 1,$$

where

$$\mathcal{F}^{g^*}_{T_{(k)}} = \sigma\left(L\left(T_{(k)}\right), \Delta_{(k)}, \mathbb{1}\left\{A\left(T_{(k)}\right) = g^*_k\left(\mathcal{F}_{T_{(k-1)}}, T_{(k)}\right)\right), ..., \mathbb{1}\{A(0) = g^*_0(L(0))\}, L(0)\right)$$

Proof: We see immediately that,

$$\begin{aligned}
&\int W_{s-} \mathbb{1}\left\{T_{(m)} < s < T_{(m+1)} \wedge t\right\} \mathrm{d}K_s \\
&= W_{T_{(m)}} \int \mathbb{1}\left\{T_{(m)} < s < T_{(m+1)} \wedge t\right\} \mathrm{d}K_s \\
&= W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} \left(\sum_j \frac{\pi^*_{T_{(m)}}\left(\left\{a_j\right\}\right)}{\pi_{T_{(m)}}\left(\left\{a_j\right\}\right)} - 1\right) N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \\
&= W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} \left(\frac{\mathbb{1}\left\{A\left(T_{(m)}\right) = g^*_m\left(\mathcal{F}_{T_{(m-1)}}, T_{(m)}\right)\right\}}{\pi_{T_{(m)}}\left(\left\{g^*_m\left(\mathcal{F}_{T_{(m-1)}}, T_{(m)}\right)\right\}\right)} - 1\right) N^{a,a_j}\left(T_{(m+1)} \wedge t\right)
\end{aligned}$$

By consistency and positivity, the desired result is equivalent to

$$\sum_{m=0}^{\infty} \mathbb{E}_P\left[\int W_{s-} \mathbb{1}\left\{T_{(m)} < s < T_{(m+1)} \wedge t\right\} \mathrm{d}K_s \tilde{Y}_t\right] = 0$$

by Lemma 4 of Ryalen (2024), so

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \mathbb{E}_P\left[\int W_{s-} \mathbb{1}\left\{T_{(m)} < s < T_{(m+1)} \wedge t\right\} \mathrm{d}K_s \tilde{Y}_t\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_P\left[W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\}\left(\frac{\mathbb{1}\left\{A\left(T_{(m)}\right) = g_m^*\left(\mathcal{F}_{T_{(m)}}, T_{(m+1)}\right)\right\}}{\pi_{T_{(m)}}\left(\left\{g_m^*\left(\mathcal{F}_{T_{(m)}}, T_{(m+1)}\right)\right\}\right)} - 1\right) N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \tilde{Y}_t\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_P\Bigg[W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \\
&\quad \times \mathbb{E}_P\left[\left(\frac{\mathbb{1}\left\{A\left(T_{(m)}\right) = g_m^*\left(\mathcal{F}_{T_{(m)}}, T_{(m+1)}\right)\right\}}{\pi_{T_{(m)}}\left(\left\{g_m^*\left(\mathcal{F}_{T_{(m-1)}}, T_{(m)}\right)\right\}\right)} - 1\right) \tilde{Y}_t \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq \tau, \Delta N^a\left(T_{(k)}\right) = 1\right]\Bigg] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_P\Bigg[W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \\
&\quad \times \mathbb{E}_P\left[\left(\frac{\mathbb{1}\left\{A\left(T_{(m)}\right) = g_m^*\left(\mathcal{F}_{T_{(m)}}, T_{(m+1)}\right)\right\}}{\pi_{T_{(m)}}\left(\left\{g_m^*\left(\mathcal{F}_{T_{(m-1)}}, T_{(m)}\right)\right\}\right)} - 1\right) \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq, \Delta N^a\left(T_{(k)}\right) = 1\right] \\
&\quad \times \mathbb{E}_P\left[\tilde{Y}_t \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq \tau, \Delta N^a\left(T_{(k)}\right) = 1\right]\Bigg] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_P[W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \\
&\quad \times \mathbb{E}_P\left[\mathbb{E}_P\left[\left(\frac{\mathbb{1}\left\{A\left(T_{(m)}\right) = g_m^*\left(\mathcal{F}_{T_{(m)}}, T_{(m+1)}\right)\right\}}{\pi_{T_{(m)}}\left(\left\{g_m^*\left(\mathcal{F}_{T_{(m-1)}}, T_{(m)}\right)\right\}\right)} - 1\right) \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)}, \Delta_{(m+1)} = a \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq \tau, \Delta N^a\left(T_{(k)}\right) = 1\right]\right. \\
&\quad \times \mathbb{E}_P\left[\tilde{Y}_t \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq \tau, \Delta N^a\left(T_{(k)}\right) = 1\right]\Bigg] \\
&= \sum_{m=0}^{\infty} \mathbb{E}_P\left[W_{T_{(m)}} \mathbb{1}\left\{T_{(m)} < t\right\} N^{a,a_j}\left(T_{(m+1)} \wedge t\right) \times (1-1) \mathbb{E}_P\left[\tilde{Y}_t \mid \mathcal{F}_{T_{(m)}}, T_{(m+1)} \leq \tau, \Delta N^a\left(T_{(k)}\right) = 1\right]\right] \\
&= 0.
\end{aligned}
$$

□

2) More general exchangeability conditions

We now consider more general exchangeability conditions.

Theorem 2.1: Let $Q_\kappa = \mathcal{E}(\kappa)_T \cdot P$ where κ is a local P-$\mathcal{F}_t$ martingale with $\Delta\kappa_t \geq -1$. If
1. Consistency holds as in Theorem 1.1.
2. $\mathcal{E}(\kappa)_t \mathcal{E}(-\mathbb{N}^a)_t = \mathcal{E}(\kappa)_t$ for all $t \in [0, T]$ P-a.s.
3. Q_κ is a uniformly integrable P-$\mathcal{F}_t$-martingale and P-$\mathcal{H}_t$-martingale, where $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\left(\tilde{Y}\right)$.

Then,

$$\mathbb{E}_P\left[\tilde{Y}_t\right] = \mathbb{E}_P[Y_t \mathcal{E}(\kappa)_t]$$

Proof: The proof is the same as in Theorem 1.1 – mutatis mutandis. □

We provide an equivalent characterization of condition 2 in the above theorem which gives direct interpretability of that condition in the sense that it should induce a probability measure Q_κ under which the time to deviation from the treatment regime is infinite almost surely.

Lemma 1: $Q_\kappa\left(\tau^{g^*} = \infty\right) = 1$ if and only if $\mathcal{E}(\kappa)_t \mathcal{E}(-\mathbb{N}^a)_t = \mathcal{E}(\kappa)_t$ for all $t \in [0, T]$ P-a.s.

Proof: "If" part:

$$\begin{aligned}
Q_\kappa\left(\tau^{g^*} = \infty\right) &= \mathbb{E}_P\left[\mathcal{E}(\kappa)_T \mathbb{1}\left\{\tau^{g^*} = \infty\right\}\right] \\
&= \mathbb{E}_P\left[\lim_{t \to \infty} \mathcal{E}(\kappa)_t \mathbb{1}\left\{\tau^{g^*} > t\right\}\right] \\
&= \mathbb{E}_P\left[\lim_{t \to \infty} \mathcal{E}(\kappa)_t\right] \\
&= \mathbb{E}_P[\mathcal{E}(\kappa)_T] = 1.
\end{aligned}$$

"Only if" part: Suppose that $Q_\kappa\left(\tau^{g^*} = \infty\right) = 1$. Then for every $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}_P\left[\mathcal{E}(\kappa)_t \mathbb{1}\left\{\tau^{g^*} > t\right\}\right] &= \mathbb{E}_P\left[\mathbb{E}_P[\mathcal{E}(\kappa)_T \mid \mathcal{F}_t] \mathbb{1}\left\{\tau^{g^*} > t\right\}\right] \\
&= \mathbb{E}_P\left[\mathcal{E}(\kappa)_T \mathbb{1}\left\{\tau^{g^*} > t\right\}\right] \\
&= Q\left(\tau^{g^*} > t\right) \\
&= 1.
\end{aligned}$$

On the other hand, $\mathbb{E}_P[\mathcal{E}(\kappa)_t] = \mathbb{E}_P[\mathbb{E}_P[\mathcal{E}(\kappa)_T \mid \mathcal{F}_t]] = \mathbb{E}_P[\mathcal{E}(\kappa)_T] = 1$. Conclude that

$$\mathbb{E}_P\left[\mathcal{E}(\kappa)_t\left(1 - \mathbb{1}\left\{\tau^{g^*} > t\right\}\right)\right] = 0.$$

The integrand on the left hand side is non-negative and so it must be zero P-a.s. □

Theorem 2.2: Let κ_t be a (local) martingale with $\Delta\kappa_t \geq -1$ and $\Delta\kappa_t^* > -1$ if $t < \tau^{g^*}$. Then,

$$W_t^* := \mathcal{E}(\kappa)_t = \mathcal{E}(\kappa)_t \mathcal{E}(-\mathbb{N}^a)_t \quad P - \text{a.s.}$$

if and only if

$$\mathbb{1}\left\{\tau^{g^*} < \infty\right\} \kappa_{\tau^{g^*}} = -\mathbb{1}\left\{\tau^{g^*} < \infty\right\} \quad P - \text{a.s.}$$

Proof: Using the well-known formula $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y])$, we have

$$\mathcal{E}(\kappa) = \mathcal{E}(\kappa - \mathbb{N}^a - [\kappa, \mathbb{N}^a])$$

This holds if and only if

$$1 + \int_0^t W_{t-} \, \mathrm{d}\kappa_s = 1 + \int_0^t W_{t-} \, \mathrm{d}\Big(\kappa_s - \mathbb{N}_s^a - [\kappa, \mathbb{N}^a]_s\Big)$$

if and only if

$$\int_0^t W_{t-} \Delta\kappa_s \, \mathrm{d}\mathbb{N}_s^a = -\int_0^t W_{t-} \, \mathrm{d}\mathbb{N}_s^a$$

and this is

$$\mathbb{1}\{\tau^{g^*} \leq t\} W_{\tau^{g^*}-} \Delta\kappa_{\tau^{g^*}} = -\mathbb{1}\{\tau^{g^*} \leq t\} W_{\tau^{g^*}-}$$

By assumption, $W_{\tau^{g^*}-} > 0$ (looking at the explicit solution of the SDE) and so the above holds if and only if

$$\mathbb{1}\{\tau^{g^*} \leq t\} \Delta\kappa_{\tau^{g^*}} = -\mathbb{1}\{\tau^{g^*} \leq t\}$$

Taking $t \to \infty$ gives the desired result. On the other hand, if the result holds then,

$$\begin{aligned}
\mathbb{1}\{\tau^{g^*} \leq t\} \Delta\kappa_{\tau^{g^*}} &= \mathbb{1}\{\tau^{g^*} \leq t\} \mathbb{1}\{\tau^{g^*} < \infty\} \Delta\kappa_{\tau^{g^*}} \\
&= \mathbb{1}\{\tau^{g^*} \leq t\} \mathbb{1}\{\tau^{g^*} < \infty\}(-1) = -\mathbb{1}\{\tau^{g^*} \leq t\}
\end{aligned}$$

□

Now, we consider only κ's of the form

$$\kappa(t) = \int_0^t \sum_{x \in \mathcal{A}} \mathbb{1}\{s \leq t\} \tilde{h}(s, x) M^{a,x}(\mathrm{d}s)$$

with $\tilde{h}(s, x)$ P-$\mathcal{F}_t$ predictable with the restriction stated in the above theorem. We make this restriction as any reasonable exchangeability conditions should be placed on the treatment process.

Theorem 2.3: Let $\kappa_t = \int_0^t \sum_{x \in \mathcal{A}} \mathbb{1}\{s \leq t\} \tilde{h}(s, x) M^{a,x}(\mathrm{d}s)$ for some P-$\mathcal{F}_t$-predictable process $\tilde{h}(s, x)$ and that $Q_\kappa\left(\tau^{g^*} = \infty\right) = 1$. Suppose that $\Delta\kappa_t \geq -1$ and $\Delta\kappa_t^* > -1$ if $t < \tau^{g^*}$ and that $\mathcal{E}(\kappa)_t$ is a uniformly integrable P-$\mathcal{F}_t$-martingale. Suppose that $\mathcal{A} = \{a_0, a_1\}$ and that $\pi_t^*(a_1) = 1$. Then any κ with $\tilde{h}(s, a_1)$ a P-$\mathcal{F}_t$-predictable process and

$$\tilde{h}(s, a_0) = \mathbb{1}\{s \leq \tau^{g^*}\} \frac{-1 + \mathbb{1}\{\Lambda^a(\{s\}) > 0\} h(s, a_1) \pi_s(a_1) \Delta\Lambda_s^a}{1 - \mathbb{1}\{\Lambda^a(\{s\}) > 0\}(1 - \pi_s(a_1)) \Delta\Lambda_s^a}$$

will satisfy the condition 2. of Theorem 2.1. Moreover, this solution is unique in the sense that whenever $\int_0^t \tilde{h}(s, a_0) M^{a,a_0}(\mathrm{d}s)$ is of finite variation, this is equal to

$$\int_0^t \mathbb{1}\{s \leq \tau^{g^*}\} \frac{-1 + \mathbb{1}\{\Lambda^a(\{s\}) > 0\} h(s, a_1) \pi_s(a_1) \Delta\Lambda_s^a}{1 - \mathbb{1}\{\Lambda^a(\{s\}) > 0\}(1 - \pi_s(a_1)) \Delta\Lambda_s^a} M^{a,a_0}(\mathrm{d}s).$$

Conclude that, under regularity conditions,

$$\begin{aligned}\mathcal{E}(\kappa)_t = 1 &+ \int_0^t \mathcal{E}(\kappa)_{s-} \mathbb{1}\{s \leq \tau^{g^*}\} \frac{-1 + \mathbb{1}\{\Lambda^a(\{s\}) > 0\} h(s, a_1) \pi_s(a_1) \Delta\Lambda_s^a}{1 - \mathbb{1}\{\Lambda^a(\{s\}) > 0\}(1 - \pi_s(a_1)) \Delta\Lambda_s^a} M^{a,a_0}(\mathrm{d}s) \\ &+ \int_0^t \mathcal{E}(\kappa)_{s-} \tilde{h}(s, a_1) M^{a,a_1}(\mathrm{d}s)\end{aligned}$$

are all the solutions to condition 2. of Theorem 2.1, where $\tilde{h}(s, a_1)$ is any P-$\mathcal{F}_t$-predictable process, ensuring that $\mathcal{E}(\kappa)_t$ is a uniformly integrable P-$\mathcal{F}_t$-martingale.

Proof:

The above theorem gives that we must have

$$\Delta\kappa_{\tau^{g^*}} = \sum_{x \in \mathcal{A}} \tilde{h}\left(\tau^{g^*}, x\right) \Delta M_{\tau^{g^*}}^{a,x} = -1$$

on the event that $\tau^{g^*} < \infty$. Suppose that $\mathcal{A} = \{a_0, a_1\}$ and that $\pi_t^*(a_1) = 1$. In this case, we can write the equation above as

$$h\left(\tau^{g^*}, a_1\right)\left(0 - \pi_{\tau^{g^*}}(a_1) \Delta\Lambda_{\tau^{g^*}}^a\right) + h\left(\tau^{g^*}, a_0\right)\left(1 - (1 - \pi_{\tau^{g^*}}(a_1)) \Delta\Lambda_{\tau^{g^*}}^a\right) = -1$$

or

$$\left(h\left(\tau^{g^*}, a_0\right) - h\left(\tau^{g^*}, a_1\right)\right) \pi_{\tau^{g^*}}(a_1) \Delta\Lambda_{\tau^{g^*}}^a + h\left(\tau^{g^*}, a_0\right)\left(1 - \Delta\Lambda_{\tau^{g^*}}^a\right) = -1$$

We consider various cases:
- Absolutely continuous case: $\Delta\Lambda^a \equiv 0$.
- $\bar{N}^a$ is $\mathcal{F}_t$-predictable.
- Jump times for $\bar{N}^a$ are discrete.
- General case.

2.a) Absolutely continuous case

In this case, conclude that $h\left(\tau^{g^*}, a_0\right) = -1$. However, nothing else can be said about $h\left(\tau^{g^*}, a_1\right)$ as the equation does not place any other restrictions than it being predictable. We can, however, conclude that $\int_0^{t \wedge \tau^{g^*}} h(s, a_0) M^{a,a_0}(\mathrm{d}s) = \int_0^{t \wedge \tau^{g^*}} (-1) M^{a,a_0}(\mathrm{d}s) = -\mathbb{N}^a(t) + \mathbb{L}^a(t)$ whenever that integral happens to be of finite variation. To see this, note that

$$
\begin{aligned}
\int_0^{t\wedge\tau^{g^*}} h(s,a_0)M^{a,a_0}(\mathrm{d}s) &= \int_0^{t\wedge\tau^{g^*}} h(s,a_0)N^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} h(s,a_0)\Lambda^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} (-1)N^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} h(s,a_0)\Lambda^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} (-1)M^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} (h(s,a_0)+1)\Lambda^{a,a_0}(\mathrm{d}s),
\end{aligned}
$$

meaning that $\int_0^{t\wedge\tau^{g^*}} (h(s,a_0)+1)\Lambda^{a,a_0}(\mathrm{d}s)$ is of finite variation, a local martingale, predicable and hence constant (and thus zero) by Theorem 15, p. 115 of Protter (2005).

2.b) $\bar{N}^a$ is $\mathcal{F}_t$-predictable

Im this case, $\Delta\Lambda_t^a = \Delta\bar{N}_t^a$ which is 1 at $t = \tau^{g^*}$. Therefore,

$$
\left(h\left(\tau^{g^*}, a_0\right) - h\left(\tau^{g^*}, a_1\right)\right)\pi_{\tau^{g^*}}(a_1) = -1
$$

or

$$
h\left(\tau^{g^*}, a_0\right) = h\left(\tau^{g^*}, a_1\right) - \frac{1}{\pi_{\tau^{g^*}}(a_1)}
$$

Thus, we have

$$
\begin{aligned}
K_t^h &= \int_0^{t\wedge\tau^{g^*}} h(s,a_0)M^{a,a_0}(\mathrm{d}s) + \int_0^{t\wedge\tau^{g^*}} h(s,a_1)M^{a,a_1}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} h(s,a_0)M^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} (h(s,a_1))M^{a,a_0}(\mathrm{d}s) + \int_0^{t\wedge\tau^{g^*}} (h(s,a_1))M^{a}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} (h(s,a_0) - h(s,a_1))M^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \left(-\frac{1}{\pi_s(a_1)}\right)N^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} (h(s,a_0) - h(s,a_1))\Lambda^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \left(-\frac{1}{\pi_s(a_1)}\right)M^{a,a_0}(\mathrm{d}s) - \int_0^{t\wedge\tau^{g^*}} \left((h(s,a_0) - h(s,a_1)) + \frac{1}{\pi_s(a_1)}\right)\Lambda^{a,a_0}(\mathrm{d}s)
\end{aligned}
$$

Assuming that $\int_0^{t\wedge\tau^{g^*}} (h(s,a_0) - h(s,a_1))M^{a,a_0}(\mathrm{d}s)$ is of finite variation, we have that $\int_0^{t\wedge\tau^{g^*}} ((h(s,a_0) - h(s,a_1))M^{a,a_0}(\mathrm{d}s) = \int_0^{t\wedge\tau^{g^*}} \left(-\frac{1}{\pi_s(a_1)}\right)M^{a,a_0}(\mathrm{d}s)$. We conclude that K_t^h if $\int_0^{t\wedge\tau^{g^*}} (h(s,a_0) - h(s,a_1))M^{a,a_0}(\mathrm{d}s)$ is of finite variation does not depend on the choice of h. Therefore, the stochastic exponential $\mathcal{E}\left(K^h\right)_t$ does not depend on the choice of h either, and we may conclude that $\mathcal{E}\left(K^h\right)_t = \mathcal{E}(K)_t$.

2.c) General case

Suppose that $(1 - \pi_t(a_1))\Delta\Lambda_t^a < 1$ for all $t > 0$. Otherwise, an argument similar to the one we will give will split into cases.

We have that

$$
\begin{aligned}
&\left(h\left(\tau^{g^*}, a_0\right) - h\left(\tau^{g^*}, a_1\right)\right)\pi_{\tau^{g^*}}(a_1)\Lambda^a\left(\left\{\tau^{g^*}\right\}\right)\mathbb{1}\left\{\Lambda^a\left(\left\{\tau^{g^*}\right\}\right) > 0\right\} \\
&\quad + h\left(\tau^{g^*}, a_0\right)\left(1 - \Lambda^a\left(\left\{\tau^{g^*}\right\}\right)\right)\mathbb{1}\left\{\Lambda^a\left(\left\{\tau^{g^*}\right\}\right) > 0\right\} \\
&\quad + h\left(\tau^{g^*}, a_0\right)\left(\mathbb{1}\left\{\Lambda^a\left(\left\{\tau^{g^*}\right\}\right) = 0\right\}\right) = -1
\end{aligned}
$$

By the same argument as in the absolutely continuous case, we have that

$$
\begin{aligned}
&\int_0^{t\wedge\tau^{g^*}} h(s, a_0)\mathbb{1}\{\Lambda^a(\{s\}) = 0\}M^{a,a_0}(\mathrm{d}s) \\
&= -\int_0^{t\wedge\tau^{g^*}} \mathbb{1}\{\Lambda^a(\{s\}) = 0\}M^{a,a_0}(\mathrm{d}s) \\
&= -\mathbb{N}^a(t)\mathbb{1}\left\{\Lambda^a\left(\left\{\tau^{g^*}\right\}\right) = 0\right\} + \mathbb{L}^{a,c}(t),
\end{aligned}
$$

where $\mathbb{L}^{a,c}$ is the continuous part of $\mathbb{L}^a$. Next whenever $\Lambda^a\left(\left\{\tau^{g^*}\right\}\right) > 0$, we find

$$
h\left(\tau^{g^*}, a_0\right) = \frac{-1 + h\left(\tau^{g^*}, a_1\right)\pi_{\tau^{g^*}}(a_1)\Delta\Lambda^a_{\tau^{g^*}}}{1 - (1 - \pi_{\tau^{g^*}}(a_1))\Delta\Lambda^a_{\tau^{g^*}}}
$$

Therefore, it will again be the case that

$$
\begin{aligned}
&\int_0^{t\wedge\tau^{g^*}} h(s, a_0)\mathbb{1}\{\Lambda^a(\{s\}) > 0\}M^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \frac{-1 + h(s, a_1)\pi_s(a_1)\Delta\Lambda^a_s}{1 - (1 - \pi_s(a_1))\Delta\Lambda^a_s}\mathbb{1}\{\Lambda^a(\{s\}) > 0\}M^{a,a_0}(\mathrm{d}s)
\end{aligned}
$$

Conclude that

$$
\begin{aligned}
&\int_0^{t\wedge\tau^{g^*}} h(s, a_0)M^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \left(\frac{-1 + h(s, a_1)\pi_s(a_1)\Delta\Lambda^a_s}{1 - (1 - \pi_s(a_1))\Delta\Lambda^a_s}\mathbb{1}\{\Lambda^a(\{s\}) > 0\} - \mathbb{1}\{\Lambda^a(\{s\}) = 0\}\right)M^{a,a_0}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \frac{-1 + \mathbb{1}\{\Lambda^a(\{s\}) > 0\}h(s, a_1)\pi_s(a_1)\Delta\Lambda^a_s}{1 - \mathbb{1}\{\Lambda^a(\{s\}) > 0\}(1 - \pi_s(a_1))\Delta\Lambda^a_s}M^{a,a_0}(\mathrm{d}s),
\end{aligned}
$$

and $h(\cdot, a_1)$ freely chosen, predictable satisfying some integrability criteria. Interestingly, this means that the stochastic exponential $\mathcal{E}\left(K^h\right)_t$ will depend on the choice of h in general, but only through $h(s, a_1)$ which can be freely chosen. □

3) Score operator calculations

Theorem 3.1: Let $K^*_t = K_{t \wedge \tau^{g^*}}$.

1. The score operator $S : \mathcal{M}^2(P) \to \mathcal{M}^2(Q)$ is given by

$$\Gamma \mapsto \Gamma - \langle \Gamma, K^* \rangle^P_t$$
$$- \int_0^{t \wedge \tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle^P_t}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} (M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle^P_s).$$

2. Let

$D : \mathcal{M}^2(P) \to \mathcal{M}^2(Q)$ be the linear operator defined by

$$\Gamma \mapsto \Gamma - \langle \Gamma, K^* \rangle^P_t.$$

Then its adjoint $D^* : \mathcal{M}^2(Q) \to \mathcal{M}^2(P)$ is given by

$$\Gamma' \mapsto \Gamma'_0 + \int_0^t W_{s-} \,\mathrm{d}(\Gamma' + [\Gamma', K^*])_s$$

The adjoint of the score operator $S^* : \mathcal{M}^2(Q) \to \mathcal{M}^2(P)$ is given by

$$\Gamma' \mapsto D^*\Gamma' - \int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_j \sum_k m^*_{s,k,j} W_{s-} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma' \rangle^Q_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \mathrm{d}\left(M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a\right)_s.$$

or $\left(\mathrm{Id} - \sum_j Y^*_j\right)(D^*)$ where $Y_j : \mathcal{M}^2(P) \to \mathcal{M}^2(P)$ is given by

$$Y_j \Gamma = \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_s - \pi_s(\{a_j\})(P) \,\mathrm{d}\langle \Gamma, M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} M^{a,a_j}(\mathrm{d}s)$$

and its adjoint $Y^*_j : \mathcal{M}^2(P) \to \mathcal{M}^2(P)$ is given by

$$Y^*_j \Gamma = \int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma' \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \mathrm{d}\left(M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a\right)_s$$

3. $\ker(S^*) = \{0\}$ if $\pi_s(\{a_j\})(P) > \eta$ for all $s \in [0, T]$ and j for some $\eta > 0$.
4. $K^* \in \ker(S)$.

1. Let $\bar{K}_t = K^*_t + \mathbb{N}^a_t$

for a given (local) martingale K^*_t with $\Delta K^*_t \geq -1$ and $\Delta K^*_t = -1$ if and only if $t = \tau^{g^*}$ and $\tau^{g^*} < \infty$. Then, it is the case that $\Delta \bar{K}_t = \mathbb{1}\{t \neq \tau^{g^*}\} \Delta K^*_t + \mathbb{1}\{t = \tau^{g^*}\}(0) > -1$ so $\mathcal{E}(\bar{K}) > 0$. First, we see that

$$\begin{aligned}
\mathcal{E}(K^*)_t &= \mathcal{E}(-\mathbb{N}^a)_t \mathcal{E}(K^*)_t \\
&= \mathcal{E}(K^* + \mathbb{N}^a - \mathbb{N}^a)_t \mathcal{E}(-\mathbb{N}^a)_t \\
&= \mathcal{E}(\bar{K})_t \mathcal{E}(-\mathbb{N}^a)_t.
\end{aligned}$$

Let $\bar{W} = \mathcal{E}(\bar{K})$ and $\partial_\varepsilon f(\varepsilon) = \frac{\partial}{\partial \varepsilon} f(\varepsilon)\big|_{\varepsilon=0}$. Then, let $\mathcal{L}$ denote the stochastic logarithm, so that

$$
\begin{aligned}
\frac{1}{\varepsilon}\mathcal{L}\left(\frac{\bar{W}^{\varepsilon}}{\bar{W}^{0}}\right)_t &= \frac{1}{\varepsilon}\mathcal{L}\left(\mathcal{E}(\bar{K}^{\varepsilon})\mathcal{E}\left(-\bar{K}^0 + \sum_{0<s\leq \cdot}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s}\right)\right)_t \\
&= \frac{1}{\varepsilon}\mathcal{L}\left(\mathcal{E}\left(\bar{K}^{\varepsilon} - \bar{K}^0 + \sum_{0<s\leq \cdot}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s} + \left[\bar{K}^{\varepsilon}, -\bar{K}^0 + \sum_{0<s\leq \cdot}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s}\right]\right)\right)_t \\
&= \frac{1}{\varepsilon}\left(\bar{K}^{\varepsilon}_t - \bar{K}^0_t + \sum_{0<s\leq t}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s} + \left[\bar{K}^{\varepsilon}, -\bar{K}^0 + \sum_{0<s\leq \cdot}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s}\right]_t\right) \\
&= \frac{1}{\varepsilon}\left(\bar{K}^{\varepsilon}_t - \bar{K}^0_t + \sum_{0<s\leq t}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s} - \left[\bar{K}^{\varepsilon}, \sum_{0<s\leq \cdot}\frac{\Delta \bar{K}^0_s}{1+\Delta \bar{K}^0_s}\right]_t\right) \\
&= \frac{1}{\varepsilon}\left(\bar{K}^{\varepsilon}_t - \bar{K}^0_t + \sum_{0<s\leq t}\frac{\left(\Delta \bar{K}^0_s\right)^2}{1+\Delta \bar{K}^0_s} - \sum_{0<s\leq t}\Delta \bar{K}^{\varepsilon}_s\frac{\Delta \bar{K}^0_s}{1+\Delta \bar{K}^0_s}\right) \\
&\to \partial_\varepsilon \bar{K}^{\varepsilon}_t - \sum_{0<s\leq t}\left(\partial_\varepsilon \Delta \bar{K}^{\varepsilon}_s\right)\frac{\Delta \bar{K}^0_s}{1+\Delta \bar{K}^0_s}
\end{aligned}
$$

as $\varepsilon \to 0$, where we use dominated convergence and L'Hopitals rule for the last step. The result is presented in Equation 8.

We will also need to calculate $\partial_\varepsilon\left(\pi_s\left(\left\{a_j\right\}\right)(P_\varepsilon)\right)$, which by definition fulfills that

$$
\Lambda^{a,a_j}(\mathrm{d}t)(P_\varepsilon) = \pi_t\left(\left\{a_j\right\}\right)(P_\varepsilon)\Lambda^a(\mathrm{d}t)(P_\varepsilon),
$$

where $M^a = \sum_{j=1}^K M^{a,a_j}$. Taking the derivative on both sides gives

$$
\langle \Gamma, M^{a,a_j}\rangle^P_t = \left(\partial_\varepsilon\left(\pi_t\left(\left\{a_j\right\}\right)(P_\varepsilon)\right)\Lambda^a(\mathrm{d}t)(P) + \pi_t\left(\left\{a_j\right\}\right)(P)\,\mathrm{d}\langle \Gamma, M^a\rangle^P_t\right)
$$

so we conclude that

$$
\begin{aligned}
\partial_\varepsilon\left(\pi_t\left(\left\{a_j\right\}\right)(P_\varepsilon)\right) &= \frac{\mathrm{d}\langle \Gamma, M^{a,a_j}\rangle^P_t - \pi_t\left(\left\{a_j\right\}\right)(P)\,\mathrm{d}\langle \Gamma, M^a\rangle^P_t}{\mathrm{d}\Lambda^a(t)(P)} \\
&= \frac{\mathrm{d}\langle \Gamma, \left(1 - \pi_\cdot\left(\left\{a_j\right\}\right)(P)\right)\bullet M^{a,a_j} - \pi_\cdot\left(\left\{a_j\right\}\right)(P)\bullet \sum_{i\neq j} M^{a,a_i}\rangle^P_t}{\mathrm{d}\Lambda^a(t)(P)}
\end{aligned}
$$

Here, we have used that using that $\partial_\varepsilon \Lambda_t(P_\varepsilon) = \langle \Gamma, M\rangle^P_t$ if $M = N - \Lambda(P)$.

Let $m^*_{s,k,j}$ predictable be given by $m^*_{s,k,j} = \mathbb{1}\left\{T_{(k-1)} < s \leq T_{(k)}\right\}\mathbb{1}\left\{j = g^*_k\left(\mathcal{F}_{T_{(k-1)}}, s\right)\right\}$. With $K^*_t = K_{t\wedge\tau^{g*}}$, we can take

$$
\begin{aligned}
\bar{K}_t &= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K \mathbb{1}\left\{T_{(k-1)} < s \le T_{(k)}\right\}\left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right) N^{a,a_j}(\mathrm{d}s) + \mathbb{N}_t^a \\
&= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K \mathbb{1}\left\{T_{(k-1)} < s \le T_{(k)}\right\}\mathbb{1}\left\{j \neq g_k^*\left(\mathcal{F}_{T_{(k-1)}}, s\right)\right\}(-1) N^{a,a_j}(\mathrm{d}s) + \mathbb{N}_t^a \\
&+ \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \left(\frac{1}{\pi_s(\{a_j\})} - 1\right) N^{a,a_j}(\mathrm{d}s) \\
&= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \left(\frac{1}{\pi_s(\{a_j\})} - 1\right) N^{a,a_j}(\mathrm{d}s)
\end{aligned}
\tag{5}
$$

This can also be written as

$$
\bar{K}_t = \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \left(\frac{1}{\pi_s(\{a_j\})} - 1\right) M^{a,a_j}(\mathrm{d}s) + \mathbb{L}_t^a \tag{6}
$$

Calculating the derivative of Equation 6 gives

$$
\begin{aligned}
\partial_\varepsilon \bar{K}_t^\varepsilon &= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \frac{1}{\pi_s(\{a_j\})(P)^2} \partial_\varepsilon(\pi_s(\{a_j\})(P_\varepsilon)) M^{a,a_j}(\mathrm{d}s) \\
&\quad - \int_0^{\tau\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \left(\frac{1}{\pi_s(\{a_j\})} - 1\right) \partial_\varepsilon(\Lambda^{a,a_j}(\mathrm{d}s)(P_\varepsilon)) + \partial_\varepsilon \mathbb{L}_t^a(P_\varepsilon) \\
&= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \frac{1}{\pi_s(\{a_j\})(P)^2} \partial_\varepsilon(\pi_s(\{a_j\})(P_\varepsilon)) M^{a,a_j}(\mathrm{d}s) \\
&\quad - \int_0^{\tau\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \left(\frac{1}{\pi_s(\{a_j\})} - 1\right) \mathrm{d}\langle \Gamma, M^{a,a_j}\rangle_s^P + \langle \Gamma, \mathbb{L}^a\rangle_t^P \\
&= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \frac{1}{\pi_s(\{a_j\})(P)^2} \partial_\varepsilon(\pi_s(\{a_j\})(P_\varepsilon)) M^{a,a_j}(\mathrm{d}s) - \langle \Gamma, K^*\rangle_t^P,
\end{aligned}
$$

again using that $\partial_\varepsilon \Lambda_t(P_\varepsilon) = \langle \Gamma, M\rangle_t^P$ if $M = N - \Lambda(P)$ is a P-martingale. On the other hand also calculating it for Equation 5 gives

$$
\partial_\varepsilon \bar{K}_t^\varepsilon = -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m_{s,k,j}^* \frac{1}{\pi_s(\{a_j\})(P)^2} \partial_\varepsilon(\pi_s(\{a_j\})(P_\varepsilon)) N^{a,a_j}(\mathrm{d}s)
$$

Also note that

$$
\frac{\Delta \bar{K}_s^0}{1 + \Delta \bar{K}_s^0} = \sum_k \sum_{j=1}^K m_{s,k,j}^* (1 - \pi_s(\{a_j\})) \Delta N^{a,a_j}(s)
$$

Thus,

$$
\begin{aligned}
\sum_{0<s\leq t} \left(\partial_\varepsilon \Delta \bar{K}_s^\varepsilon\right) \frac{\Delta \bar{K}_s^0}{1+\Delta \bar{K}_s^0} &= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \partial_\varepsilon\left(\pi_s\left(\left\{a_j\right\}\right)(P_\varepsilon)\right) \frac{1}{\left(\pi_s\left(\left\{a_j\right\}\right)\right)^2} \left(1-\pi_s\left(\left\{a_j\right\}\right)\right) N^{a,a_j}(\mathrm{d}s) \\
&= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\left\{a_j\right\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\left\{a_j\right\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) N^{a,a_j}(\mathrm{d}s) \\
&= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\left\{a_j\right\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\left\{a_j\right\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) M^{a,a_j}(\mathrm{d}s) \\
&\quad -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\left\{a_j\right\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\left\{a_j\right\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) \Lambda^{a,a_j}(\mathrm{d}s)
\end{aligned}
$$

Conclude that, if $\pi_s\left(\left\{a_j\right\}\right)(P) > 0$ for all $s \in [0,T]$ and j,

$$
\begin{aligned}
&\partial_\varepsilon \bar{K}_t^\varepsilon - \sum_{0<s\leq t} \left(\partial_\varepsilon \Delta \bar{K}_s^\varepsilon\right) \frac{\Delta \bar{K}_s^0}{1+\Delta \bar{K}_s^0} \\
&= -\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)^2} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) M^{a,a_j}(\mathrm{d}s) - \langle \Gamma, K^* \rangle_t^P \\
&\quad + \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\{a_j\}\right)} - 1\right) M^{a,a_j}(\mathrm{d}s) \\
&\quad + \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\{a_j\}\right)} - 1\right) \Lambda^{a,a_j}(\mathrm{d}s) \\
&= -\langle \Gamma, K^* \rangle_t^P \\
&\quad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) M^{a,a_j}(\mathrm{d}s) \\
&\quad + \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) \left(\frac{1}{\pi_s\left(\{a_j\}\right)} - 1\right) \Lambda^{a,a_j}(\mathrm{d}s) \\
&= -\langle \Gamma, K^* \rangle_t^P \qquad (7) \\
&\quad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) \left(M^{a,a_j}(\mathrm{d}s) - \sum_v \sum_i m^*_{s,v} \left(\frac{1}{\pi_s(\{a_i\})} - 1\right) \Lambda^{a,a_i}(\mathrm{d}s)\right) \\
&= -\langle \Gamma, K^* \rangle_t^P \\
&\quad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) (M^{a,a_j}(\mathrm{d}s) - \sum_v \sum_i \Bigg(m^*_{s,v} \left(\frac{1}{\pi_s(\{a_i\})} - 1\right) \\
&\qquad - \left(\frac{0}{\pi_s\left(\{a_j\}\right)} - 1\right) \mathbb{1}\left\{T_{(v-1)} < s \leq T_{(v)}\right\} \mathbb{1}\left\{j \neq g_v^*\left(\mathcal{F}_{T_{(v-1)}}, s\right)\right\} \Lambda^{a,a_i}(\mathrm{d}s)\Bigg) \\
&= -\langle \Gamma, K^* \rangle_t^P \\
&\quad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s\left(\{a_j\}\right)(P)} \partial_\varepsilon\left(\pi_s\left(\{a_j\}\right)(P_\varepsilon)\right) \left(M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle_s^P\right) \\
&= -\langle \Gamma, K^* \rangle_t^P \\
&\quad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle_t^P - \pi_t\left(\{a_j\}\right)(P)\, \mathrm{d}\langle \Gamma, M^a \rangle_t^P}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} \left(M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle_s^P\right)
\end{aligned}
$$

Note that $\frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle_t^P - \pi_t\left(\{a_j\}\right)(P)\, \mathrm{d}\langle \Gamma, M^a \rangle_t^P}{\mathrm{d}\Lambda^{a,a_j}(t)(P)}$ can be be chosen predictable so that the corresponding term is a (local) martingale and that the last two terms in Equation 7 can be written as $\langle \Gamma, Z \rangle_t^P$ for some (local) martingale Z not specified (here). Conclude that the Score operator S is given by

$$
\begin{aligned}
&\mathcal{L}^2(P) \ni \Gamma \mapsto \Gamma - \langle \Gamma, K^* \rangle_t^P \\
&\qquad - \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot\left(\{a_j\}\right)(P) \bullet M^a \rangle_t^P}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} \left(M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle_s^P\right) \in \mathcal{L}^2(Q)
\end{aligned}
\tag{8}
$$

2. Assume that we have found the adjoint operator of $D : \mathcal{L}^2(P) \to \mathcal{L}^2(Q)$

$$
\mathcal{L}^2(P) \ni \Gamma \mapsto \Gamma - \langle \Gamma, K^* \rangle_t^Q \in \mathcal{L}^2(Q)
$$

say $D^* : \mathrm{Range}(D) \subset \mathcal{L}^2(Q) \to \mathcal{L}^2(P)$. Let $H_j : \mathcal{L}^2(P) \to \mathcal{L}^2(Q)$ be given by

$$H_j \Gamma = \int_0^{t \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_t - \pi_t(\{a_j\})(P) \, \mathrm{d}\langle \Gamma, M^a \rangle^P_t}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} (M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle^P_s)$$

Then, we have that

$$\begin{aligned}
\langle H_k \Gamma, \Gamma \rangle_Q &:= \mathbb{E}_Q \left[\langle H_k \Gamma, \Gamma' \rangle^Q \right] \\
&= \mathbb{E}_Q \left[\langle \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_s - \pi_s(\{a_j\})(P) \, \mathrm{d}\langle \Gamma, M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} D M^{a,a_j}(\mathrm{d}s), \Gamma' \rangle^Q_s \right] \\
&= \mathbb{E}_Q \left[\langle D \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_s - \pi_s(\{a_j\})(P) \, \mathrm{d}\langle \Gamma, M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} M^{a,a_j}(\mathrm{d}s), \Gamma' \rangle^Q_s \right] \\
&= \mathbb{E}_P \left[\langle \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_s - \pi_s(\{a_j\})(P) \, \mathrm{d}\langle \Gamma, M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} M^{a,a_j}(\mathrm{d}s), D^* \Gamma' \rangle^P_s \right] \\
&:= \mathbb{E}_P \left[\langle Y_j \Gamma, D^* \Gamma' \rangle^P_s \right] \\
&= \langle \Gamma, Y_j^* D^* \Gamma' \rangle_P,
\end{aligned}$$

so if we have found D^* and Y_j^*, we have found the adjoint of H_j. Here, we let $Y_j : \mathcal{L}^2(P) \to \mathcal{L}^2(P)$ be given by

$$Y_j \Gamma = \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} \rangle^P_s - \pi_s(\{a_j\})(P) \, \mathrm{d}\langle \Gamma, M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} M^{a,a_j}(\mathrm{d}s)$$

Then, we may calculate directly that

$$\begin{aligned}
\langle Y_j \Gamma, \Gamma' \rangle_P &:= \mathbb{E}_P \left[\langle Y_j \Gamma, \Gamma' \rangle^P \right] \\
&= \mathbb{E}_P \left[\langle \int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \, \mathrm{d}M^{a,a_j}(s), \Gamma' \rangle^P \right] \\
&= \mathbb{E}_P \left[\int_0^{\cdot \wedge \tau^{g^*}} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \, \mathrm{d}\langle M^{a,a_j}, \Gamma' \rangle^P_s \right] \\
&= \mathbb{E}_P \left[\langle \Gamma, \mathbb{1}\{\tau^{g^*} \leq \cdot\} \sum_k m^*_{\cdot,k,j} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma' \rangle^P_\cdot}{\mathrm{d}\Lambda^{a,a_j}(\cdot)(P)} \bullet \left(M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \right) \rangle^P \right] \\
&= \mathbb{E}_P \left[\langle \Gamma, Y_j^* \Gamma' \rangle^P \right] \\
&= \langle \Gamma, Y_j^* \Gamma' \rangle_P,
\end{aligned}$$

Now note that

$$
\begin{aligned}
D^*\Gamma' &= \Gamma'_0 + [\Gamma', W]_t - \int_0^t W_{s-} \,\mathrm{d}\Gamma'_s \\
&= \Gamma'_0 + \int_0^t W_{s-} \,\mathrm{d}(\Gamma' + [\Gamma', K^*])_s \\
&= \Gamma'_t W_t - \int_0^t \Gamma'_{s-} \,\mathrm{d}W_s \\
&= \Gamma'_t W_t - \int_0^t \Gamma'_{s-} W_{s-} \,\mathrm{d}K^*_s
\end{aligned}
$$

by the arguments in “Projection Notes” and integration by parts for semimartingales. This composition yields,

$$
\begin{aligned}
&\int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} \frac{\mathrm{d}\langle M^{a,a_j}, D^*\Gamma'\rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}\big(M^{a,a_j} - \pi_{\cdot}(\{a_j\})(P) \bullet M^a\big)_s \\
&= \int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} W_{s-} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma' + [\Gamma', K^*]\rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}\big(M^{a,a_j} - \pi_{\cdot}(\{a_j\})(P) \bullet M^a\big)_s
\end{aligned}
$$

FACT:

$$
\langle X, Y\rangle^Q = \langle X + [X, K^*], Y\rangle^P = \langle X, Y + [Y, K^*]\rangle^P,
$$

so it follows that

$$
\begin{aligned}
&\int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} W_{s-} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma' + [\Gamma', K^*]\rangle^P_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}\big(M^{a,a_j} - \pi_{\cdot}(\{a_j\})(P) \bullet M^a\big)_s \\
&= \int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} W_{s-} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma'\rangle^Q_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}\big(M^{a,a_j} - \pi_{\cdot}(\{a_j\})(P) \bullet M^a\big)_s \\
&= \int_0^t \mathbb{1}\{\tau^{g^*} \leq s\} \sum_k m^*_{s,k,j} W_{s-} \frac{\mathrm{d}\langle M^{a,a_j}, \Gamma'\rangle^Q_s}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}\big(N^{a,a_j} - \pi_{\cdot}(\{a_j\})(P) \bullet N^a\big)_s.
\end{aligned}
$$

Note that this operator sends to piecewise constant functions.
3.

Note that the adjoint operator can be written as $\left(\mathrm{Id} - \sum_j Y^*_j\right) D^*$. If $\mathrm{Id} - \sum_j Y^*_j$ is injective, it holds that

$$
\ker\ (S^*) = \ker\ \left(\left(\mathrm{Id} - \sum_j Y^*_j\right) D^*\right) = \ker\ (D^*) = \{0\}
$$

where the last equality follows by “Projection Notes”. So this will follow, if we can show that $\mathrm{Id} - \sum_j Y^*_j$ is injective. To this end, we shall show that $\mathrm{Id} - \sum_j Y_j$ has dense range in $\mathcal{M}^2(P)$ (Corollaries to Theorem 4.12 of Rudin, Functional Analysis, 2nd edition).

Take $\Gamma^* \in \mathcal{M}^2(P)$ such that

$$
\Gamma^* = \int_0^\cdot \sum_x \gamma^*_x(s) M^x(\mathrm{d}s)
$$

where $\gamma^*_x(s)$ is bounded and predictable We shall find $\Gamma \in \mathcal{M}^2(P)$ such that

$$\Gamma^* = \Gamma - \sum_j Y_j \Gamma.$$

We will find our solution as

$$\Gamma = \int_0^{\cdot} \sum_x \gamma_x(s) M^x(\mathrm{d}s),$$

where $\gamma_x(s)$ is bounded and predictable. We now explain how this results in dense range: The lemma on p. 173 of Protter (2005) shows that any $\Gamma^* \in \mathcal{M}^2(P)$ can be approximated in $\mathcal{L}^2(P)$ by a sequence of such processes.

We will need to find the $\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_t^P$ term.

$$\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_t^P = \int_0^t \sum_x h_x(s) \,\mathrm{d} \langle M^x, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_s^P$$

Now note that

$$\begin{aligned}
\langle M^x, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_s^P &= \Lambda_s^{*,x,j} - \int_0^s \Delta \langle M^x \rangle_s \,\mathrm{d}\Lambda^{a,a_j}(s) + \int_0^s \pi_s(\{a_j\})(P) \Delta \langle M^x \rangle_s \,\mathrm{d}\Lambda^a(s) \\
&= \Lambda_s^{*,x,a_j} - \int_0^s \Delta \langle M^x \rangle_s \pi_s(\{a_j\})(P) \,\mathrm{d}\Lambda^a(s) + \int_0^s \pi_s(\{a_j\})(P) \Delta \langle M^x \rangle_s \,\mathrm{d}\Lambda^a(s) \\
&= \Lambda_s^{*,x,a_j}
\end{aligned}$$

where Λ^{*,x,a_j} is the compensator of $\int_0^t \Delta N_s^x \,\mathrm{d}N^{a,a_j}(s) - \int_0^t \pi_s(\{a_j\})(P) \Delta N^x(s)) \,\mathrm{d}N_s^a$. If $x \notin \mathcal{A}$, then $\Lambda^{*,x,a_j} = 0$. If $x = a_i \in \mathcal{A}$, then

$$\begin{aligned}
&\int_0^t \Delta N_s^x \,\mathrm{d}N^{a,a_j}(s) - \int_0^t \pi_s(\{a_j\})(P) \Delta N^x(s)) \,\mathrm{d}N_s^a \\
&= \mathbb{1}\{j = i\} N^{a,a_j}(t) - \int_0^t \pi_s(\{a_j\})(P) \,\mathrm{d}N^{a_i}(s)
\end{aligned}$$

This has the compensator

$$\begin{aligned}
\Lambda_t^{*,x,a_j} &= \mathbb{1}\{j = i\} \Lambda^{a,a_j}(t) - \int_0^t \pi_s(\{a_j\})(P) \,\mathrm{d}\Lambda^{a,a_i}(s) \\
&= \mathbb{1}\{j = i\} \Lambda^{a,a_j}(t) - \int_0^t \pi_s(\{a_i\})(P) \,\mathrm{d}\Lambda^{a,a_j}(s) \\
&= \int_0^t \mathbb{1}\{j = i\} - \pi_s(\{a_i\})(P) \,\mathrm{d}\Lambda^{a,a_j}(s)
\end{aligned}$$

where the second properties works by properties of Radon-Nikodym derivatives and positivity. Now, we have that

$$\begin{aligned}
Y_j \Gamma - \Gamma &= \int_0^{\cdot \wedge \tau^{g^*}} \sum_j \sum_k m_{s,k,j}^* \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_s^P}{\mathrm{d}\Lambda^{a,a_j}(s)(P)} \,\mathrm{d}(M^{a,a_j})_s - \int_0^{\cdot} \sum_x \gamma_x(s) M^x(\mathrm{d}s) \\
&= \int_0^{\cdot \wedge \tau^{g^*}} \sum_j \sum_k m_{s,k,j}^* \sum_{x \in \mathcal{A}} \gamma_x(s) \big(\mathbb{1}\{a_j = x\} - \pi_s(\{x\})(P)\big) \,\mathrm{d}M_s^{a,a_j} - \int_0^{\cdot} \sum_x \gamma_x(s) M^x(\mathrm{d}s)
\end{aligned}$$

Whenever $x \notin \mathcal{A}$, we can choose $\gamma_x(s) = -\gamma_x^*(s)$. Otherwise, we may pick

$$\gamma_x(s) = \mathbb{1}\{s \leq \tau^{g^*}\} \sum_k \mathbb{1}\left\{T_{(k-1)} < s \leq T_{(k)}\right\} \mathbb{1}\left\{x \neq g_k^*\left(\mathcal{F}_{T_{(k-1)}}, s\right)\right\}(-\gamma_x^*(s))$$
$$+\mathbb{1}\{s \leq \tau^{g^*}\} \sum_k \mathbb{1}\left\{T_{(k-1)} < s \leq T_{(k)}\right\} \mathbb{1}\left\{x = g_k^*\left(\mathcal{F}_{T_{(k-1)}}, s\right)\right\}\left(-\frac{\sum_{y \in \mathcal{A}, y \neq x} \pi_s(\{y\})(P)\gamma_y^*(s)}{\pi_s(\{x\})(P)}\right)$$
$$+\mathbb{1}\{s > \tau^{g^*}\}(-\gamma_x^*(s)).$$

which is bounded by assumption and predictable.

4.

To this end, we will need to calculate $\langle K^* \rangle_t^P$ and $\langle K^*, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle_t^P$.

$$\begin{aligned}
\langle K^* \rangle_t^P &= \int_0^{t \wedge \tau^{g^*}} \sum_i \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)\left(\frac{\pi_s^*(\{a_i\})}{\pi_s(\{a_i\})} - 1\right) \mathrm{d}\langle M^{a,a_i}, M^{a,a_j} \rangle_s^P \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_i \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)\left(\frac{\pi_s^*(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i = j\} - \Delta\Lambda^{a,a_i})\, \mathrm{d}\Lambda^{a,a_j}(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)^2 \mathrm{d}\Lambda^{a,a_j}(s) \\
&\quad - \int_0^{t \wedge \tau^{g^*}} \sum_i \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)\left(\frac{\pi_s^*(\{a_i\})}{\pi_s(\{a_i\})} - 1\right) \Delta\Lambda^{a,a_i}\, \mathrm{d}\Lambda^{a,a_j}(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)^2 \mathrm{d}\Lambda^{a,a_j}(s) \\
&\quad - \int_0^{t \wedge \tau^{g^*}} \sum_i \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)\left(\frac{\pi_s^*(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)\pi_s(\{a_i\})\pi_s(\{a_j\})\Delta\Lambda^a(s)\, \mathrm{d}\Lambda^a(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)^2 \mathrm{d}\Lambda^{a,a_j}(s) \\
&\quad - \int_0^{t \wedge \tau^{g^*}} \sum_i \sum_j (\pi_s^*(\{a_j\}) - \pi_s(\{a_j\}))(\pi_s^*(\{a_i\}) - \pi_s(\{a_i\}))\Delta\Lambda^a(s)\, \mathrm{d}\Lambda^a(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_j \left(\frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right)^2 \mathrm{d}\Lambda^{a,a_j}(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \sum_j \frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} \mathrm{d}\Lambda^a(s) - 2\int_0^{t \wedge \tau^{g^*}} \sum_j (\pi_s^*(\{a_j\}))\, \mathrm{d}\Lambda^a(s) + \int_0^{t \wedge \tau^{g^*}} \sum_j \pi_s(\{a_j\})(P)\, \mathrm{d}\Lambda^a(s) \\
&= \int_0^{t \wedge \tau^{g^*}} \left(\sum_j \frac{\pi_s^*(\{a_j\})}{\pi_s(\{a_j\})} - 1\right) \mathrm{d}\Lambda^a(s).
\end{aligned}$$

For the next calculation, we have that

$$
\begin{aligned}
\langle K^*, M^{a,a_j} - \pi_{\cdot}\big(\{a_j\}\big)(P) \bullet M^a\rangle^P_t &= \int_0^{t\wedge\tau^{g^*}} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right) \mathrm{d}\langle M^{a,a_i}, M^{a,a_j} - \pi_{\cdot}\big(\{a_j\}\big)(P) \bullet M^a\rangle^P_s \\
&= \int_0^{t\wedge\tau^{g^*}} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right) (\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))\, \mathrm{d}\Lambda^{a,a_j}(s).
\end{aligned}
$$

by the previous calculations. Therefore, it holds that

$$\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{\mathrm{d}\langle \Gamma, M^{a,a_j} - \pi_\cdot(\{a_j\})(P) \bullet M^a \rangle^P_t}{\mathrm{d}\Lambda^{a,a_j}(t)(P)} (M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle^P_s)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))(M^{a,a_j}(\mathrm{d}s) - \mathrm{d}\langle M^{a,a_j}, K^* \rangle^P_s)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times \left(M^{a,a_j}(\mathrm{d}s) - \left(\sum_v \frac{\pi^*_s(\{a_v\})}{\pi_s(\{a_v\})} - 1\right) \mathrm{d}\langle M^{a,a_j}, M^{a,v} \rangle^P_s\right)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times \left(M^{a,a_j}(\mathrm{d}s) - \left(\sum_v \frac{\pi^*_s(\{a_v\})}{\pi_s(\{a_v\})} - 1\right) \pi_s(\{a_v\})(\mathbb{1}\{j=v\} - \pi(\{a_j\}\Delta\Lambda^a)\,\mathrm{d}\Lambda^a(s))\right)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times \left(M^{a,a_j}(\mathrm{d}s) - \left(\sum_v \frac{\pi^*_s(\{a_v\})}{\pi_s(\{a_v\})} - 1\right) \pi_s(\{a_v\}) \mathbb{1}\{j=v\}\,\mathrm{d}\Lambda^a(s)\right)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times \left(M^{a,a_j}(\mathrm{d}s) - \left(\frac{\pi^*_s(\{a_j\})}{\pi_s(\{a_j\})} - 1\right) \pi_s(\{a_j\})\,\mathrm{d}\Lambda^a(s)\right)$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i \left(\frac{\pi^*_s(\{a_i\})}{\pi_s(\{a_i\})} - 1\right)(\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times (M^{a,a_j}(\mathrm{d}s) - (\pi^*_s(\{a_j\}) - \pi_s(\{a_j\}))\,\mathrm{d}\Lambda^a(s))$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \frac{1}{\pi_s(\{a_j\})}(1 - \pi_s(\{a_i\})(P))$$
$$\times (N^{a,a_j}(\mathrm{d}s) - \pi^*_s(\{a_j\})\,\mathrm{d}\Lambda^a(s))$$
$$- \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \sum_i (\mathbb{1}\{i=j\} - \pi_s(\{a_i\})(P))$$
$$\times (N^{a,a_j}(\mathrm{d}s) - \pi^*_s(\{a_j\})\,\mathrm{d}\Lambda^a(s))$$

$$= \int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K m^*_{s,k,j} \left(\frac{1}{\pi_s(\{a_j\})} - 1\right)(N^{a,a_j}(\mathrm{d}s) - 1)\,\mathrm{d}\Lambda^a(s))$$

This we find the score operator is given by evaluated at K^* by combining the results,

$$-\int_0^{t\wedge\tau^{g^*}} \sum_k \sum_{j=1}^K \mathbb{1}\{T_{(k-1)} < s \le T_{(k)}\} \mathbb{1}\{j \neq g^*_k(\mathcal{F}_{T_{(k-1)}}, s)\} N^{a,a_j}(\mathrm{d}s)$$

which is zero Q-a.s.

3.a) Comparisons of the positivity assumptions in Ryalen (2024)

One may ask oneself if positivity holds in Ryalen (2024); under what assumptions does positivity in Theorem 1.1 hold? In general, however, it would appear that the two positivity conditions are different and neither implies the other.

Can we find a process φ such that $\mathcal{E}(K) = \frac{\mathcal{E}(-\mathbb{N}^a)}{\mathcal{E}(-\mathbb{L}^a)}\mathcal{E}(\varphi)$?

Theorem 3.1.1: φ is given by

$$\varphi_t = K_t - \mathbb{L}^a_t + \mathbb{N}^a_t - [K, \mathbb{L}^a]_t,$$

where $[\cdot, \cdot]$ denotes the quadratic covariation process (Protter (2005)), where

$$[K, \mathbb{L}^a]_t = \int_0^{t \wedge \tau^{g^*}} \sum_v \mathbb{1}\left\{T_{(v-1)} < s \leq T_{(v)}\right\} \sum_{i \neq g^*_v\left(\mathcal{F}_{T_{(v-1)}}, T_{(v)}\right)} \pi_s\left(\left\{a_j\right\}\right) \Delta\Lambda^a(s)$$

$$\times \sum_{j=1}^k \left(\frac{\pi^*_s\left(\left\{a_j\right\}\right)}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) N^{a, a_j}(\mathrm{d}s)$$

In the absolutely continuous case, $[K, \mathbb{L}^a]_t = 0$ as $\Delta\Lambda^a_t = 0$ for all $t > 0$. If, further, $\pi^*_t\left(\left\{a_j\right\}\right) = 1$ for some j, then

$$\varphi_t = \int_0^{t \wedge \tau^{g^*}} \left(\frac{1}{\pi_s} - 1\right) M^{a, a_j}(\mathrm{d}s).$$

Proof: To this end, let $v := \mathbb{1}\left\{W(t) > 0, \widetilde{W}(t) > 0\right\} = \mathbb{1}\left\{\tau^{g^*} > t\right\} = \mathcal{E}(-\mathbb{N}^a)$ and calculate

$$\begin{aligned}
\mathcal{E}(\varphi)v &= \frac{\mathcal{E}(K)\mathcal{E}(-\mathbb{L}^a)}{\mathcal{E}(-\mathbb{N}^a)} v \\
&= \mathcal{E}(K)\mathcal{E}(-\mathbb{L}^a) v \\
&= \mathcal{E}(K - \mathbb{L}^a - [K, \mathbb{L}^a]) v \\
&= \mathcal{E}(K - \mathbb{L}^a + \mathbb{N}^a - [K, \mathbb{L}^a]) v,
\end{aligned}$$

where the last equality follows since $\mathbb{N}^a v \equiv 0$. Note that

$$\begin{aligned}
[K, \mathbb{L}^a]_t &= \int_0^t \Delta\mathbb{L}^a_s \sum_{j=1}^k \left(\frac{\pi^*_s\left(\left\{a_j\right\}\right)}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) N^{a, a_j}(\mathrm{d}s) \\
&\overset{*}{=} \int_0^{t \wedge \tau^{g^*}} \sum_v \mathbb{1}\left\{T_{(v-1)} < s \leq T_{(v)}\right\} \sum_{i \neq g^*_v\left(\mathcal{F}_{T_{(v-1)}}, T_{(v)}\right)} \pi_s\left(\left\{a_j\right\}\right) \Delta\Lambda^a(s) \\
&\qquad \times \sum_{j=1}^k \left(\frac{\pi^*_s\left(\left\{a_j\right\}\right)}{\pi_s\left(\left\{a_j\right\}\right)} - 1\right) N^{a, a_j}(\mathrm{d}s)
\end{aligned}$$

In the case where $\Delta\Lambda^a_s \equiv 0$ and $\pi^*_s\left(\left\{a_j\right\}\right) = 1$ for some j, then

$$v(K_t - \mathbb{L}^a_t + \mathbb{N}^a_t) = v \int_0^{t \wedge \tau^{g^*}} \left(\frac{1}{\pi_s} - 1 \right) M^{a, a_j}(\mathrm{d}s)$$

and $[K, \mathbb{L}^a]_t = 0$. □

A simple consequence of this is the following. Assume that $Q_{\text{ryalen}} \ll P$ with $Q_{\text{ryalen}} = \frac{\mathcal{E}(-\mathbb{N}^a)}{\mathcal{E}(-\mathbb{L}^a)} \bullet P$. and, say, $\mathcal{E}(\varphi)$ is a uniformly integrable Q_{ryalen}-$\mathcal{F}_t$-martingale, i.e., that $Q \ll Q_{\text{ryalen}}$, then $Q_{\text{ryalen}} \ll P$ implies that $Q \ll P$. This happens for example if $\mathcal{E}(\varphi)$ is uniformly bounded by a constant.

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