

## 1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r} \partial_z \psi \hat{\mathbf{r}} \\ u_\phi \hat{\phi} \\ -\frac{1}{r} \partial_r \psi \hat{\mathbf{z}} \end{bmatrix}; \quad (1)$$

here we define  $A$  in the same way.

Using the definitions in

$$\begin{aligned} \partial_t \left[ \frac{1}{r} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] + \frac{1}{r^2} J(\psi, \nabla^2 A - \frac{2\partial_r \psi}{r}) &= \frac{\partial_z A}{r^3} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \\ &+ \frac{1}{r} J \left( A, \frac{1}{r} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \right) - \frac{2B_\phi \partial_z B_\phi}{r} \\ &+ \nu \left\{ \nabla^2 \left[ \frac{1}{r} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] - \frac{1}{r^2} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right\} \end{aligned} \quad (2)$$

$$\partial_t u_\phi + J(\psi, u_\phi) + \frac{u_\phi \partial_r \psi}{r^2} = J(B_\phi, A) - \frac{B_\phi \partial_z A}{r^2} + \nu \left( \nabla^2 u_\phi - \frac{u_\phi}{r} \right) \quad (3)$$

$$\partial_t A = \frac{1}{r^2} J(\psi, A) - \eta \left[ \frac{1}{r} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \right] \quad (4)$$

$$\begin{aligned} \partial_t B_\phi &= \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ &+ \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left( \nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \end{aligned} \quad (5)$$

## 2 Recovery of Narrow Gap Equations

### A Cylindrical derivatives

Everything here follows <http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl>.

For a scalar field  $\psi$ ,

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\phi} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}. \quad (6)$$

However, for a *vector* field  $\mathbf{u}$ ,

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \quad (7)$$

and

$$\nabla \times \mathbf{u} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\phi} + \left( \frac{1}{r} \frac{\partial(r u_\phi)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right) \hat{\mathbf{z}}. \quad (8)$$

We also need the  $\phi$  component of the convective derivative  $\mathbf{u} \cdot \nabla \mathbf{u}$ ,

$$[\mathbf{u} \cdot \nabla \mathbf{u}]_\phi = \mathbf{u} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}, \quad (9)$$

and finally, the vector Laplacian,

$$(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \quad (10)$$

$$(\nabla^2 \mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \quad (11)$$

$$(\nabla^2 \mathbf{u})_z = \nabla^2 u_z, \quad (12)$$

where  $\nabla$  on the vector components is given by equation (6).