✓ $\partial_t \nabla^2\psi + J(\psi, \nabla^2\psi) - \frac{2}{\beta} J(A, \nabla^2 A) = \overset{\Omega_0}{2}\partial_z u_y + \frac{2}{\beta} B_0 \partial_z \nabla^2 A + \frac{1}{Re}\nabla^4\psi$

✓ $\partial_t u_y + J(\psi, u_y) - \frac{2}{\beta} J(A, B_y) + (2-q)\Omega_0 \partial_z \psi = \frac{2}{\beta} B_0 \partial_z B_y + \frac{1}{Re}\nabla^2 u_y$

✓ $\partial_t A = B_0 \partial_z \psi + J(A, \psi) + \frac{1}{Rm}\nabla^2 A$

✓ $\partial_t B_y = B_0 \partial_z u_y + J(A, u_y) - J(\psi, B_y) + \frac{1}{Rm}\nabla^2 B_y - q\Omega_0\partial_z A$

These eqns agree w/ Umurhan eq 8-11. Only missing that Ω_0 (corrected)

$$N = \begin{bmatrix} J(\psi, \nabla^2\psi) - \frac{2}{\beta} J(A, \nabla^2 A) \\ J(\psi, u_y) - \frac{2}{\beta} J(A, B_y) \\ -J(A, \psi) \\ J(\psi, B_y) - J(A, u_y) \end{bmatrix}$$

(w/ all on LHS of eqns)

$$D = \begin{bmatrix} \nabla^2 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$RHS = \begin{bmatrix} \frac{1}{Re}\nabla^4 & & & 0 \\ & \frac{1}{Re}\nabla^2 & & \\ & & \frac{1}{Rm}\nabla^2 & \\ 0 & & & \frac{1}{Rm}\nabla^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{2}{\beta}B_0\partial_z\nabla^2 & 0 \\ 0 & 0 & 0 & \frac{2}{\beta}B_0\partial_z \\ B_0\partial_z & 0 & 0 & 0 \\ 0 & B_0\partial_z & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 2\Omega_0\partial_z & 0 & 0 \\ (2-q)\Omega_0\partial_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q\Omega_0\partial_z & 0 \end{bmatrix}$$

$\nabla^4 = \partial_x^4 + \partial_z^4 + 2\partial_x^2\partial_z^2 \rightarrow$ expansion of 1st mat. correct in notes.

fixed L_1.

Nonlinear term check. $\quad J = \partial_z a \, \partial_x b - \partial_x a \, \partial_z b$

$$N^{(\psi)} = J(\psi, \nabla^2 \psi) - \frac{2}{\beta} J(A, \nabla^2 A)$$

$$\partial_z \to \partial_z + \epsilon \partial_Z$$

$$\nabla^2 \to \nabla^2 + \epsilon^2 \partial_Z^2 + 2\epsilon \partial_z \partial_Z$$

$$J(\psi, \nabla^2\psi) = \partial_z \psi \partial_x \nabla^2 \psi - \partial_x \psi \partial_z \nabla^2 \psi$$

$$= (\partial_z + \epsilon \partial_Z)\psi \, \partial_x (\nabla^2 + \epsilon^2 \partial_Z^2 + 2\epsilon \partial_z \partial_Z)\psi - \partial_x \psi \partial_z \nabla^2 \psi$$

$$= \partial_z \psi \partial_x \nabla^2 \psi + \partial_z \psi \partial_x \epsilon^2 \partial_Z^2 \psi + \partial_z \psi \partial_x 2\epsilon \partial_z \partial_Z \psi$$

$$+ \epsilon \partial_Z \psi \partial_x \nabla^2 \psi + \epsilon \partial_Z \psi \partial_x \epsilon^2 \partial_Z^2 \psi + \epsilon \partial_Z \psi \partial_x 2\epsilon \partial_z \partial_Z \psi$$

$$- \partial_x \psi (\partial_z + \epsilon \partial_Z)(\nabla^2 + \epsilon^2 \partial_Z^2 + 2\epsilon \partial_z \partial_Z)\psi$$

$$- \partial_x \psi \partial_z \nabla^2 \psi - \partial_x \psi \partial_z \epsilon^2 \partial_Z^2 \psi - \partial_x \psi \partial_z 2\epsilon \partial_z \partial_Z \psi$$

$$- \partial_x \psi \epsilon \partial_Z \nabla^2 \psi - \partial_x \psi \epsilon \partial_Z \epsilon^2 \partial_Z^2 \psi - \partial_x \psi \epsilon \partial_Z 2\epsilon \partial_z \partial_Z \psi$$

$$\partial_z \psi \partial_x \nabla^2 \psi + \epsilon^2 \partial_z \psi \partial_x \partial_Z^2 \psi + 2\epsilon \partial_z \psi \partial_x \partial_z \partial_Z \psi$$

$$+ \epsilon \partial_Z \psi \partial_x \nabla^2 \psi + \epsilon^3 \partial_Z \psi \partial_x \partial_Z^2 \psi + 2\epsilon^2 \partial_Z \psi \partial_x \partial_z \partial_Z \psi$$

$$- \partial_x \psi \partial_z \nabla^2 \psi - \epsilon^2 \partial_x \psi \partial_z \partial_Z^2 \psi - 2\epsilon \partial_x \psi \partial_z^2 \partial_Z \psi$$

$$- \epsilon \partial_x \psi \partial_Z \nabla^2 \psi - \epsilon^3 \partial_x \psi \partial_Z^3 \psi - 2\epsilon^2 \partial_x \psi \partial_Z^2 \partial_z \psi$$

$$= J(\psi, \nabla^2 \psi) + \epsilon^2 J(\psi, \partial_Z^2 \psi) + 2\epsilon J(\psi, \partial_z \partial_Z \psi) \quad ✓$$

$$+ \epsilon \tilde{J}(\psi, \nabla^2 \psi) + \epsilon^3 \tilde{J}(\psi, \partial_Z^2 \psi) + 2\epsilon^2 \tilde{J}(\psi, \partial_z \partial_Z \psi)$$

$$N^{(\psi)} = J(\psi, \nabla^2 \psi) + \cancel{\epsilon^2 J(\psi, \partial_Z^2 \psi)} + 2\epsilon J(\psi, \partial_z \partial_Z \psi)$$

$$+ \epsilon \tilde{J}(\psi, \nabla^2 \psi) + \cancel{\epsilon^3 \tilde{J}(\psi, \partial_Z^2 \psi)} + \cancel{2\epsilon^2 \tilde{J}(\psi, \partial_z \partial_Z \psi)}$$

$$- \frac{2}{\beta} J(A, \nabla^2 A) - \cancel{\epsilon^2 \frac{2}{\beta} J(A, \partial_Z^2 A)} - 2\epsilon \frac{2}{\beta} J(A, \partial_z \partial_Z A)$$

$$- \epsilon \frac{2}{\beta} \tilde{J}(A, \nabla^2 A) - \cancel{\epsilon^3 \frac{2}{\beta} \tilde{J}(A, \partial_Z^2 A)} - \cancel{2\epsilon^2 \frac{2}{\beta} \tilde{J}(A, \partial_z \partial_Z A)}$$

ignore $\mathcal{O}(\epsilon^2)$ or greater.

multiscale nonlin. expan:

$$J(a+b, c+d) = J(a,c) + J(a,d) + J(b,c) + J(b,d)$$

$$J(\varepsilon\psi_1 + \varepsilon^2\psi_2, \varepsilon\nabla^2\psi_1 + \varepsilon^2\nabla^2\psi_2)$$

$$= \varepsilon^2 J(\psi_1, \nabla^2\psi_1) + \varepsilon^3 J(\psi_1, \nabla^2\psi_2) + \varepsilon^3 J(\psi_2, \nabla^2\psi_1) + \cancel{\varepsilon^4 J(\psi_2, \nabla^2\psi_2)}$$

$$2\varepsilon J(\varepsilon\psi_1 + \varepsilon^2\psi_2, \partial_z\partial_{\bar z}\varepsilon\psi_1 + \partial_z\partial_{\bar z}\varepsilon^2\psi_2)$$

$$= 2\varepsilon^3 J(\psi_1, \partial_z\partial_{\bar z}\psi_1) + 2\varepsilon^4 J(\psi_1, \partial_z\partial_{\bar z}\psi_2) + 2\varepsilon^4 J(\psi_2, \partial_z\partial_{\bar z}\psi_1)$$

$$+ 2\varepsilon^5 J(\psi_2, \partial_z\partial_{\bar z}\psi_2)$$

$$\varepsilon\tilde J(\varepsilon\psi_1 + \varepsilon^2\psi_2, \varepsilon\nabla^2\psi_1 + \varepsilon^2\nabla^2\psi_2)$$

$$= \varepsilon^3\tilde J(\psi_1, \nabla^2\psi_1) + \varepsilon^4\tilde J(\psi_1, \nabla^2\psi_2) + \varepsilon^4\tilde J(\psi_2, \nabla^2\psi_1) + \cancel{\varepsilon^5\tilde J(\psi_2, \nabla^2\psi_2)}$$

$-\frac{2}{\beta} J(\varepsilon A_1 + \varepsilon^2 A_2, \varepsilon\nabla^2 A_1 + \varepsilon^2\nabla^2 A_2)$ .. see above.

Finally:

$$N(\psi) = \varepsilon^2 J(\psi_1, \nabla^2\psi_1) + \varepsilon^3 J(\psi_1, \nabla^2\psi_2) + \varepsilon^3 J(\psi_2, \nabla^2\psi_1)$$

$$- \varepsilon^2\frac{2}{\beta} J(A_1, \nabla^2 A_1) - \varepsilon^3\frac{2}{\beta} J(A_1, \nabla^2 A_2) - \varepsilon^3\frac{2}{\beta} J(A_2, \nabla^2 A_1)$$

$$+ 2\varepsilon^3 J(\psi_1, \partial_z\partial_{\bar z}\psi_1) - 2\varepsilon^3\frac{2}{\beta} J(A_1, \partial_z\partial_{\bar z}A_1)$$

$$+ \varepsilon^3\tilde J(\psi_1, \nabla^2\psi_1) - \frac{2}{\beta}\varepsilon^3\tilde J(A_1, \nabla^2 A_1)$$

$$N_2^{(\psi)} = J(\psi_1, \nabla^2\psi_1) - \frac{2}{\beta} J(A_1, \nabla^2 A_1)$$

$$N_3^{(\psi)} = J(\psi_1, \nabla^2\psi_2) + J(\psi_2, \nabla^2\psi_1) - \frac{2}{\beta} J(A_1, \nabla^2 A_2) - \frac{2}{\beta} J(A_2, \nabla^2 A_1)$$

$$+ 2J(\psi_1, \partial_z\partial_{\bar z}\psi_1) - 2\frac{2}{\beta} J(A_1, \partial_z\partial_{\bar z}A_1)$$

$$+ \tilde J(\psi_1, \nabla^2\psi_1) - \frac{2}{\beta}\tilde J(A_1, \nabla^2 A_1)$$

Confirmed: Umurhan's E19 has a sign error.

$N(u)$

$$= J(\Psi, u_y) - \frac{2}{\beta} J(A, B_y)$$

$$= J(\varepsilon\Psi_1 + \varepsilon^2\Psi_2, \varepsilon u_1 + \varepsilon^2 u_2) - \frac{2}{\beta} J(\varepsilon A_1 + \varepsilon^2 A_2, \varepsilon B_1 + \varepsilon^2 B_2)$$

$$= \varepsilon^2 J(\Psi_1, u_1) + \varepsilon^3 J(\Psi_1, u_2) + \varepsilon^3 J(\Psi_2, u_1) + \varepsilon^4 J(\Psi_2, u_2)$$
$$- \varepsilon^2 \frac{2}{\beta} J(A_1, B_1) - \varepsilon^3 \frac{2}{\beta} J(A_1, B_2) - \varepsilon^3 \frac{2}{\beta} J(A_2, B_1) - \varepsilon^4 \frac{2}{\beta} J(A_2, B_2)$$

$$N_2(u) = J(\Psi_1, u_1) - \frac{2}{\beta} J(A_1, B_1)$$

$$N_3(u) = J(\Psi_1, u_2) + J(\Psi_2, u_1) - \frac{2}{\beta} J(A_1, B_2) - \frac{2}{\beta} J(A_2, B_1)$$

$$N(u) = (\partial_z + \varepsilon\partial_{\underline{Z}})\Psi\, \partial_x u_y + \frac{2}{\beta} \partial_x A (\partial_z + \varepsilon\partial_{\underline{Z}}) B_y$$
$$- \partial_x \Psi (\partial_z + \varepsilon\partial_{\underline{Z}}) u_y - \frac{2}{\beta} (\partial_z + \varepsilon\partial_{\underline{Z}}) A\, \partial_x B_y$$

$$= \partial_z \Psi\, \partial_x u_y + \varepsilon \partial_{\underline{Z}} \Psi\, \partial_x u_y + \frac{2}{\beta} \partial_x A\, \partial_z B_y + \varepsilon \frac{2}{\beta} \partial_x A\, \partial_{\underline{Z}} B_y$$
$$- \partial_x \Psi\, \partial_z u_y - \varepsilon \partial_x \Psi\, \partial_{\underline{Z}} u_y - \frac{2}{\beta} \partial_z A\, \partial_x B_y - \varepsilon \frac{2}{\beta} \partial_{\underline{Z}} A\, \partial_x B_y$$

$$= J(\Psi, u_y) + \varepsilon \tilde{J}(\Psi, u_y) + \frac{2}{\beta} J(A, B_y) + \varepsilon \frac{2}{\beta} \tilde{J}(A, B_y)$$

$$= J(\varepsilon\Psi_1 + \varepsilon^2\Psi_2, \varepsilon u_1 + \varepsilon^2 u_2) + \varepsilon \tilde{J}(\varepsilon\Psi_1 + \varepsilon^2\Psi_2, \varepsilon u_1 + \varepsilon^2 u_2)$$
$$- \frac{2}{\beta} J(\varepsilon A_1 + \varepsilon^2 A_2, \varepsilon B_1 + \varepsilon^2 B_2) - \varepsilon \frac{2}{\beta} \tilde{J}(\varepsilon A_1 + \varepsilon^2 A_2, \varepsilon B_1 + \varepsilon^2 B_2)$$

$$= \varepsilon^2 J(\Psi_1, u_1) + \varepsilon^3 J(\Psi_1, u_2) + \varepsilon^3 J(\Psi_2, u_1) + \varepsilon^4 J(\Psi_2, u_2)$$
$$+ \varepsilon^3 \tilde{J}(\Psi_1, u_1) + \cancel{\varepsilon^4 \tilde{J}(\Psi_1, u_2)} + \varepsilon^4 \tilde{J}(\Psi_2, u_1) + \varepsilon^5 \tilde{J}(\Psi_2, u_2)$$
$$- \varepsilon^2 \frac{2}{\beta} J(A_1, B_1) - \varepsilon^3 \frac{2}{\beta} J(A_1, B_2) - \varepsilon^3 \frac{2}{\beta} J(A_2, B_1) - \cancel{\varepsilon^4 \frac{2}{\beta} J(A_2, B_2)}$$
$$- \varepsilon^3 \frac{2}{\beta} \tilde{J}(A_1, B_1) - \cancel{\varepsilon^4 \frac{2}{\beta} \tilde{J}(A_1, B_2)} - \cancel{\varepsilon^4 \frac{2}{\beta} \tilde{J}(A_2, B_1)} - \varepsilon^5 \frac{2}{\beta} \tilde{J}(A_2, B_2)$$

$$N_2^{(u)} = J(\psi_1, u_1) - \frac{2}{\beta} J(A_1, B_1)$$

$$N_3^{(u)} = J(\psi_1, u_2) + J(\psi_2, u_1) + \tilde{J}(\psi_1, u_1)$$
$$- \frac{2}{\beta} J(A_1, B_2) - \frac{2}{\beta} J(A_2, B_1) - \frac{2}{\beta} \tilde{J}(A_1, B_1)$$

(B20 is correct. where it says "$-+$" should be "$+$".)

$$N^{(A)} = -J(A, \psi)$$
$$= -(\partial_z + \varepsilon \partial_{\underline{Z}}) A \partial_x \psi + \partial_x A (\partial_z + \varepsilon \partial_{\underline{Z}}) \psi$$
$$= -\partial_z A \partial_x \psi - \varepsilon \partial_{\underline{Z}} A \partial_x \psi + \partial_x A \partial_z \psi + \partial_x A \varepsilon \partial_{\underline{Z}} \psi$$
$$= -\partial_z (\varepsilon A_1 + \varepsilon^2 A_2) \partial_x (\varepsilon \psi_1 + \varepsilon^2 \psi_2) - \varepsilon \partial_{\underline{Z}} (\varepsilon A_1 + \varepsilon^2 A_2) \partial_x (\varepsilon \psi_1 + \varepsilon^2 \psi_2)$$
$$+ \partial_x (\varepsilon A_1 + \varepsilon^2 A_2) \partial_z (\varepsilon \psi_1 + \varepsilon^2 \psi_2) + \varepsilon \partial_x (\varepsilon A_1 + \varepsilon^2 A_2) \partial_{\underline{Z}} (\varepsilon \psi_1 + \varepsilon^2 \psi_2)$$
$$= -\varepsilon^2 \partial_z A_1 \partial_x \psi_1 - \varepsilon^3 \partial_z A_1 \partial_x \psi_2 - \varepsilon^3 \partial_z A_2 \partial_x \psi_1 \cancel{- \varepsilon^4 \partial_z A_2 \partial_x \psi_2}$$
$$- \varepsilon^3 \partial_{\underline{Z}} A_1 \partial_x \psi_1 - \varepsilon^4 \_\_\_\_ - \varepsilon^4 \_\_\_\_ - \varepsilon^5 \_\_\_\_$$
$$+ \varepsilon^2 \partial_x A_1 \partial_z \psi_1 + \varepsilon^3 \partial_x A_1 \partial_z \psi_2 + \varepsilon^3 \partial_x A_2 \partial_z \psi_1 + \varepsilon^4 \_\_\_\_$$
$$+ \varepsilon^3 \partial_x A_1 \partial_{\underline{Z}} \psi_1 + \varepsilon^4 \_\_\_\_ + \varepsilon^4 \_\_\_\_ + \varepsilon^5 \_\_\_\_$$

$$N^{(A)} = -\varepsilon^2 J(A_1, \psi_1) - \varepsilon^3 J(A_1, \psi_2) - \varepsilon^3 J(A_2, \psi_1) - \varepsilon^3 \tilde{J}(A_1, \psi_1)$$

$$N_2^{(A)} = -J(A_1, \psi_1)$$
$$N_3^{(A)} = -J(A_1, \psi_2) - J(A_2, \psi_1) - \tilde{J}(A_1, \psi_1)$$

$$N^{(B)} = J(\Psi, B_y) - J(A, u_y)$$

Similarly to $N_2^{(\psi)}$, we find

$$\boxed{\begin{aligned} N_2^{(B)} &= J(\Psi_1, B_1) - J(A_1, u_1) \\ N_3^{(B)} &= J(\Psi_1, B_2) + J(\Psi_2, B_1) + \tilde{J}(\Psi_1, B_1) \\ &\quad - J(A_1, u_2) - J(A_2, u_1) - \tilde{J}(A_1, u_1) \end{aligned}}$$