1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi \ \hat{\mathbf{r}} \\ u_\phi \ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi \ \hat{\mathbf{z}} \end{bmatrix} ; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) = \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)$$
$$+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r}$$
$$+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$[\hat{\mathbf{r}}]: \partial_t\partial_z\Psi - \frac{1}{r^2}\partial_z^2\Psi + \frac{1}{r}\partial_z^2\partial_r\Psi - \frac{1}{r}\partial_r\partial_z^2\Psi - u_\phi^2 =$$
$$-\frac{1}{r}\partial_z^2\partial_r A + \frac{1}{r^2}\partial_z\partial_r^2 A - \frac{1}{r}\partial_z\partial_r^3 A + \frac{1}{\mathrm{Re}}\left[-\frac{1}{r}\partial_z\partial_r\Psi + \partial_z\partial_r^2\Psi + \partial_z^3\Psi\right] \tag{3}$$

$$[\hat{\mathbf{z}}]: \partial_t\partial_z\Psi - \frac{1}{r^3}\partial_z\partial_r\Psi + \frac{1}{r^2}\partial_z\partial_r^2\Psi - \frac{1}{r^2}\partial_r^2\partial_z\Psi =$$
$$B_\phi\partial_z B_\phi + \frac{1}{r^2}\partial_z^3 A - \frac{1}{r^3}\partial_r\partial_z A + \frac{1}{r^2}\partial_r^2\partial_z A$$
$$+ \frac{1}{\mathrm{Re}}\left[\frac{1}{r^3}\partial_r\Psi - \frac{1}{r^2}\partial_r^2\Psi + \frac{1}{r}\partial_r^2\Psi + \frac{1}{r}\partial_z^2\partial_r\Psi\right] \tag{4}$$

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{5}$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \eta\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{6}$$

Susan gets:

$$\partial_t A = \frac{1}{r}J(A, \Psi) + \frac{1}{Rm}\left[-\frac{1}{r}\partial_r A + \partial_r^2 A + \partial_z^2 A\right] \tag{7}$$

$$\partial_t B_\phi = \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi)$$
$$+ \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left(\nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \quad (8)$$

2   Recovery of Narrow Gap Equations

A   Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\phi} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}. \tag{9}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \tag{10}$$

and

$$\nabla \times \mathbf{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\phi} + \left(\frac{1}{r} \frac{\partial (r u_\phi)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right) \hat{\mathbf{z}}. \tag{11}$$

We also need the ϕ component of the convective derivative $\mathbf{u} \cdot \nabla \mathbf{u}$,

$$[\mathbf{u} \cdot \nabla \mathbf{u}]_\phi = \mathbf{u} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{12}$$

and finally, the vector Laplacian,

$$(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \tag{13}$$

$$(\nabla^2 \mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \tag{14}$$

$$(\nabla^2 \mathbf{u})_z = \nabla^2 u_z, \tag{15}$$

where ∇ on the vector components is given by equation (??).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$