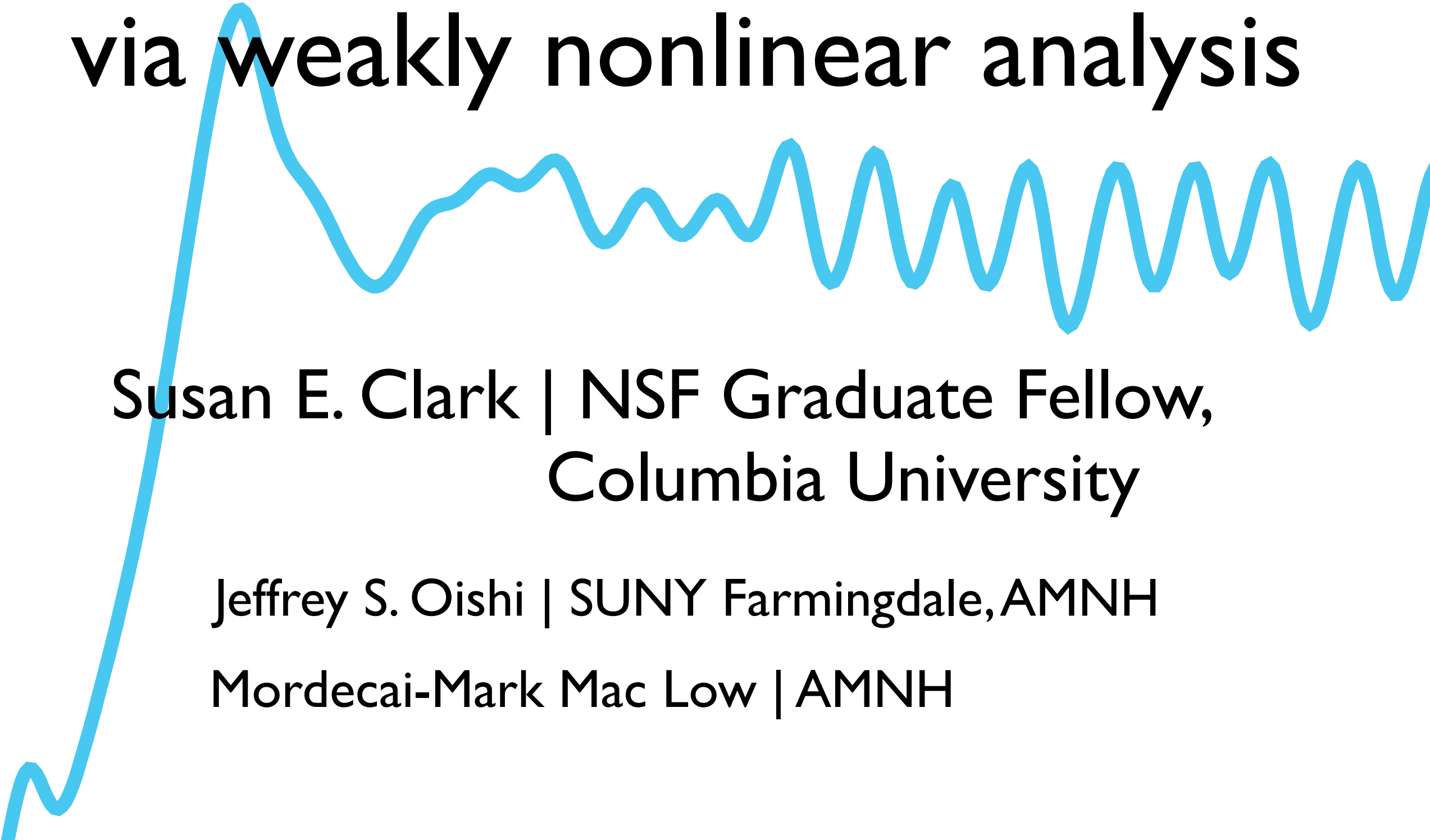


Exploring the saturation of the MRI via weakly nonlinear analysis

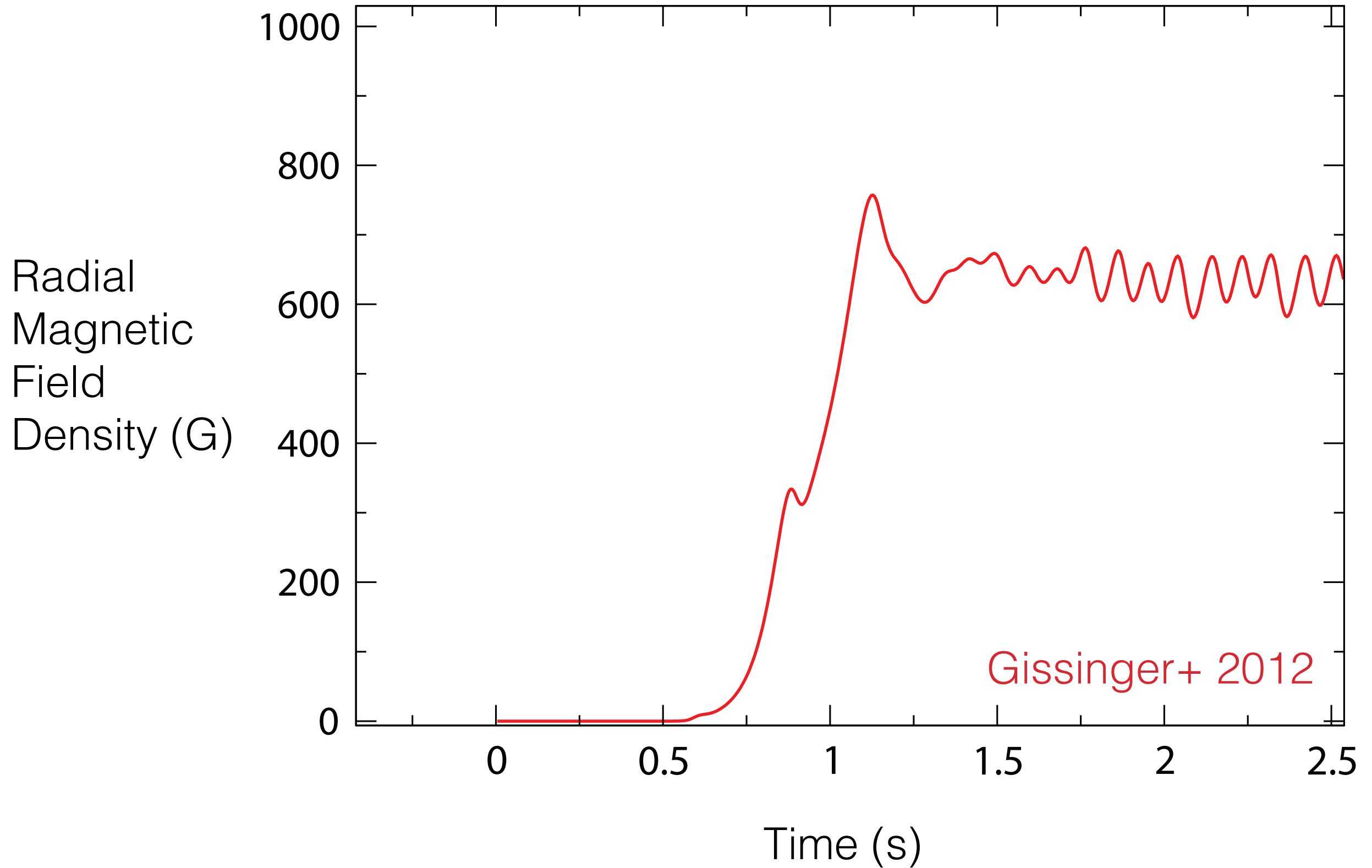


Susan E. Clark | NSF Graduate Fellow,
Columbia University

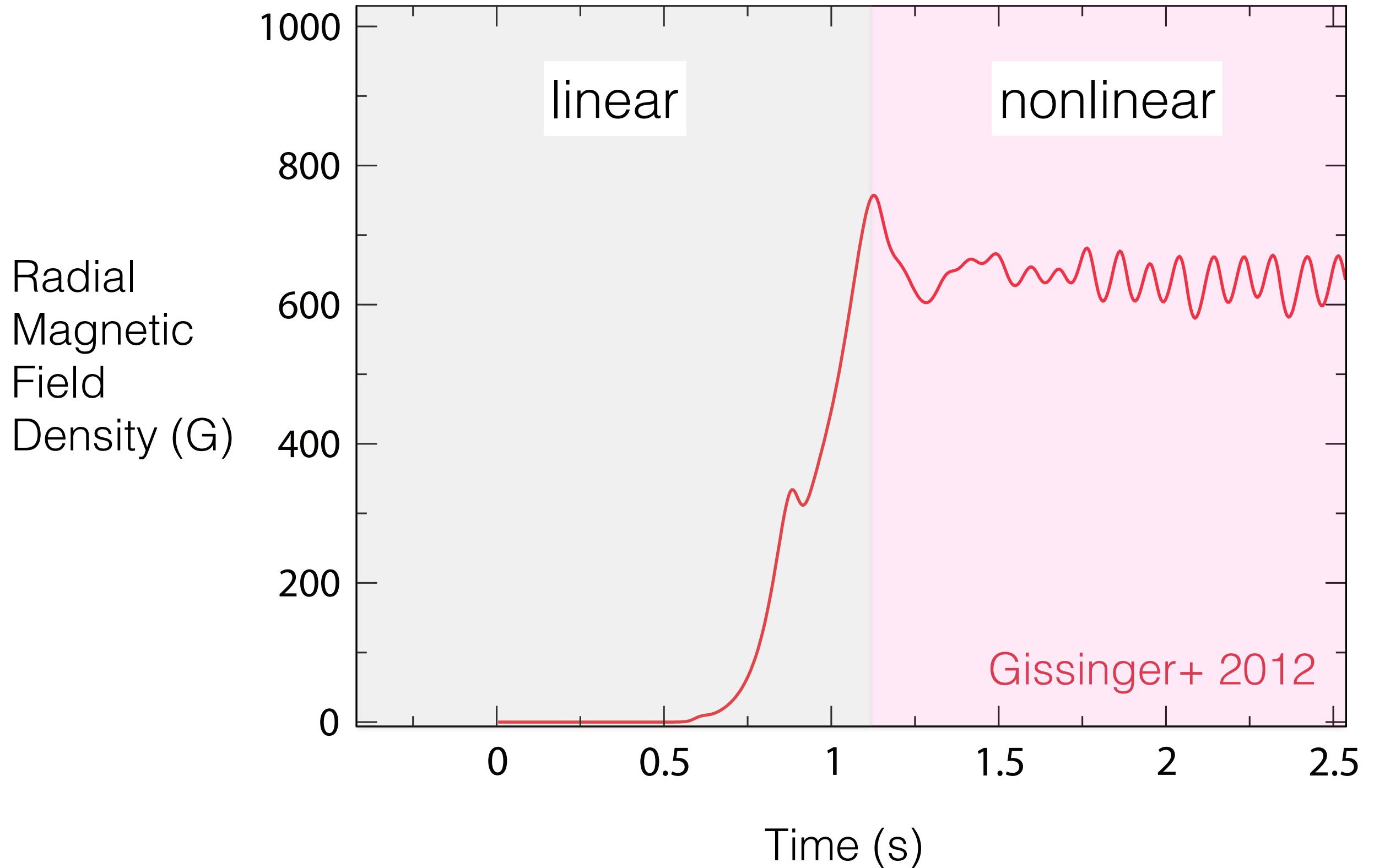
Jeffrey S. Oishi | SUNY Farmingdale, AMNH

Mordecai-Mark Mac Low | AMNH

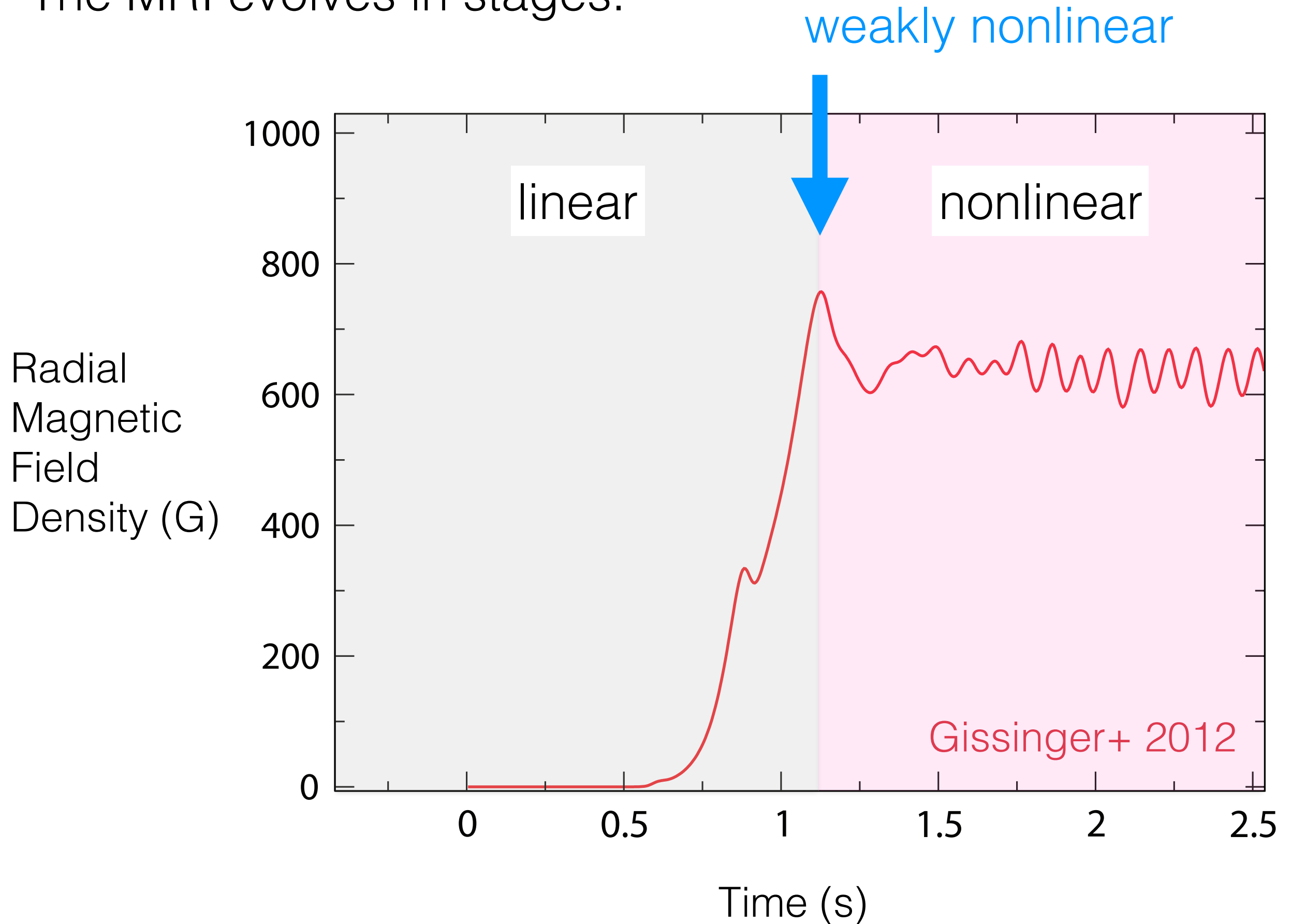
The MRI evolves in stages.



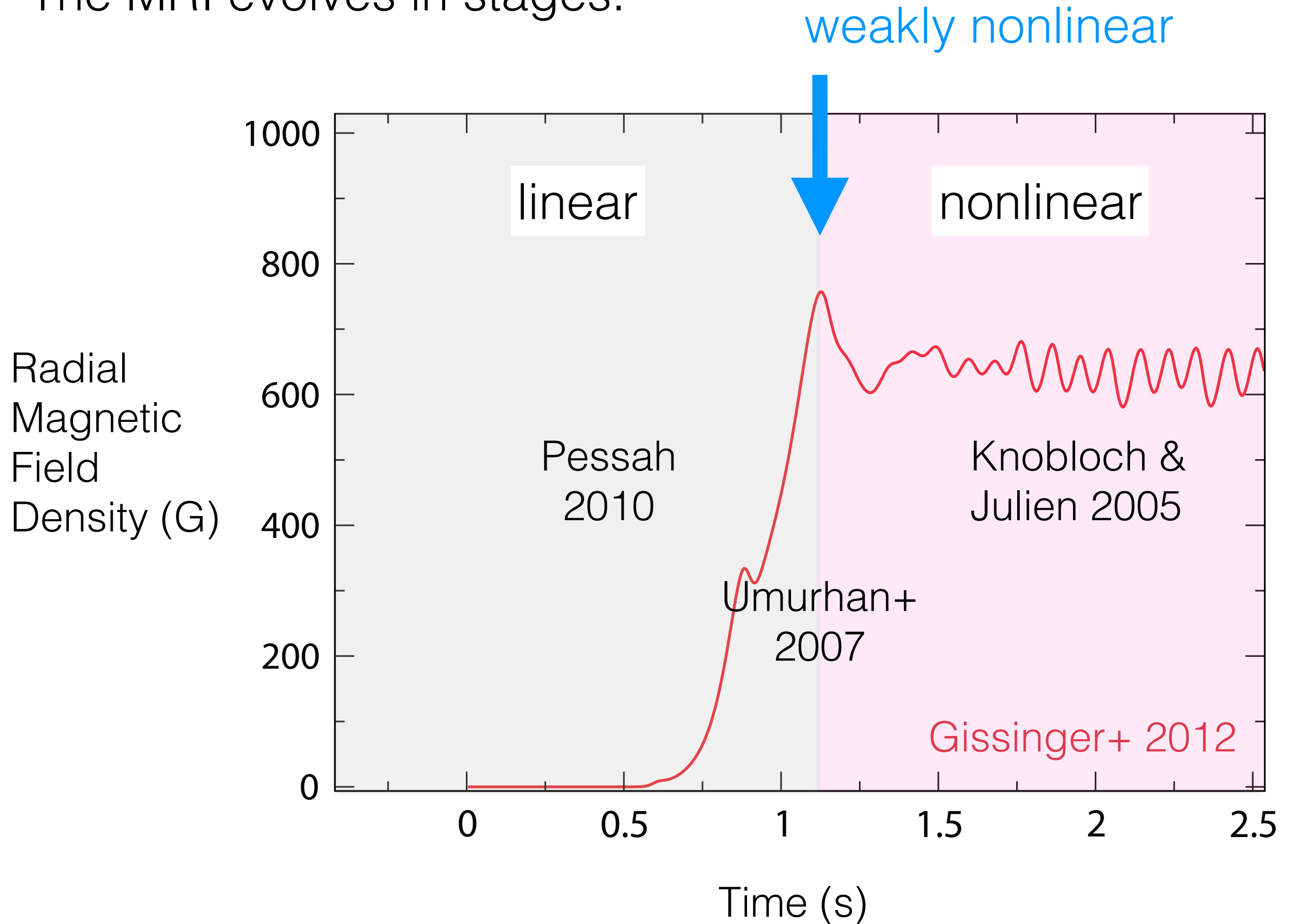
The MRI evolves in stages.



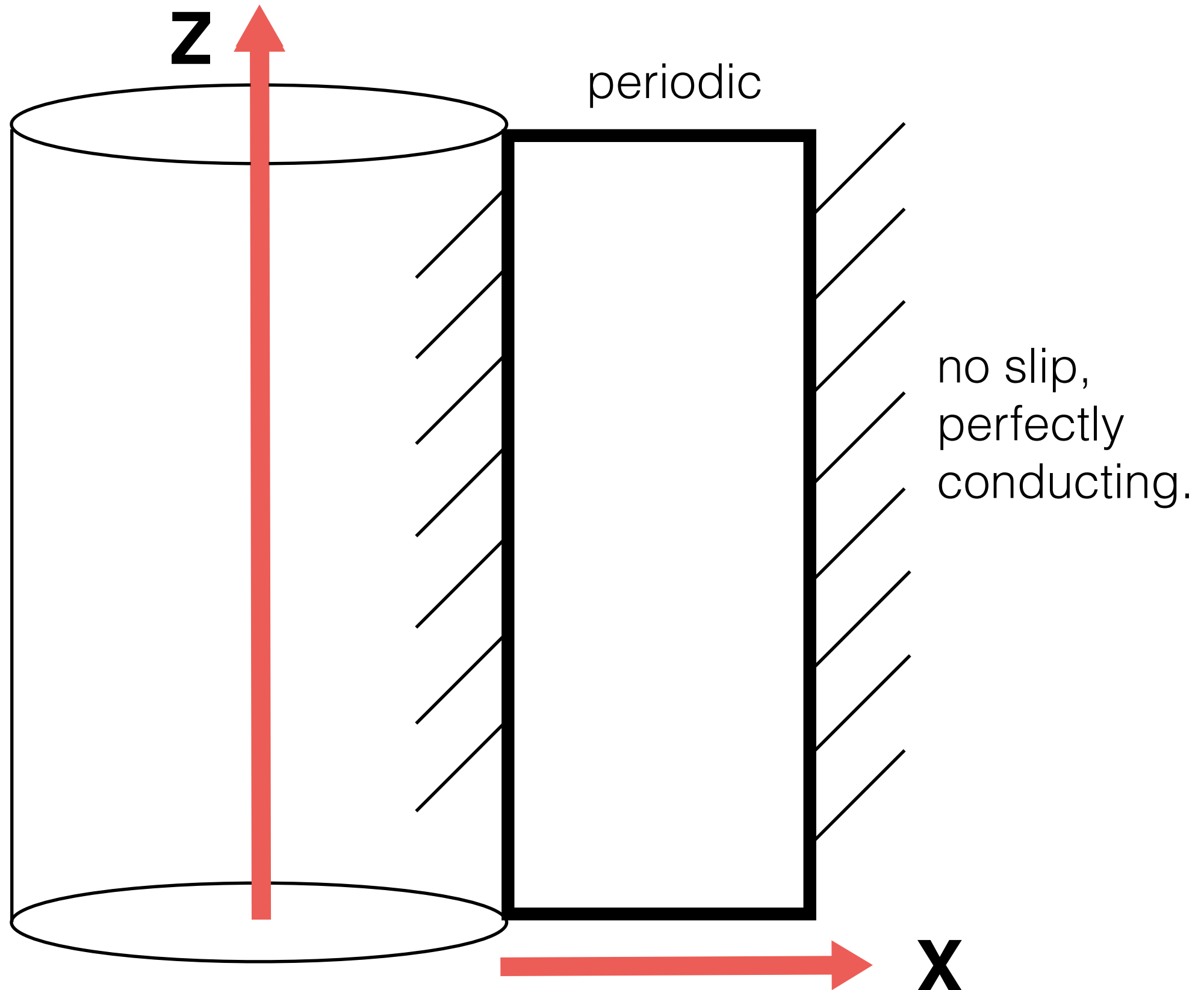
The MRI evolves in stages.



The MRI evolves in stages.



We use a thin-gap Taylor Couette setup.



We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

kinematic
viscosity



magnetic
resistivity



constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We nondimensionalize and perturb the nonlinear MRI equations.

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0} \right)^{-q}$$

shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$

background field

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$

Reynolds number

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$

magnetic Reynolds number

$$\beta \equiv \frac{8\pi \rho_0 \Omega_0^2 L^2}{B_0^2}$$

plasma beta

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \Psi \\ u_y \\ A \\ B_y \end{bmatrix}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2 - q) \Omega_0 \partial_z \Psi + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \frac{1}{Re} \nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2 - q) \Omega_0 \partial_z \Psi + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A} \quad \text{resistive}$$

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

viscous

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2 - q) \Omega_0 \partial_z \Psi} + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q \Omega_0 \partial_z A} + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \boxed{\frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

nonlinear

viscous

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2 - q) \Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J(A, B_y) - J(\Psi, u_y)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

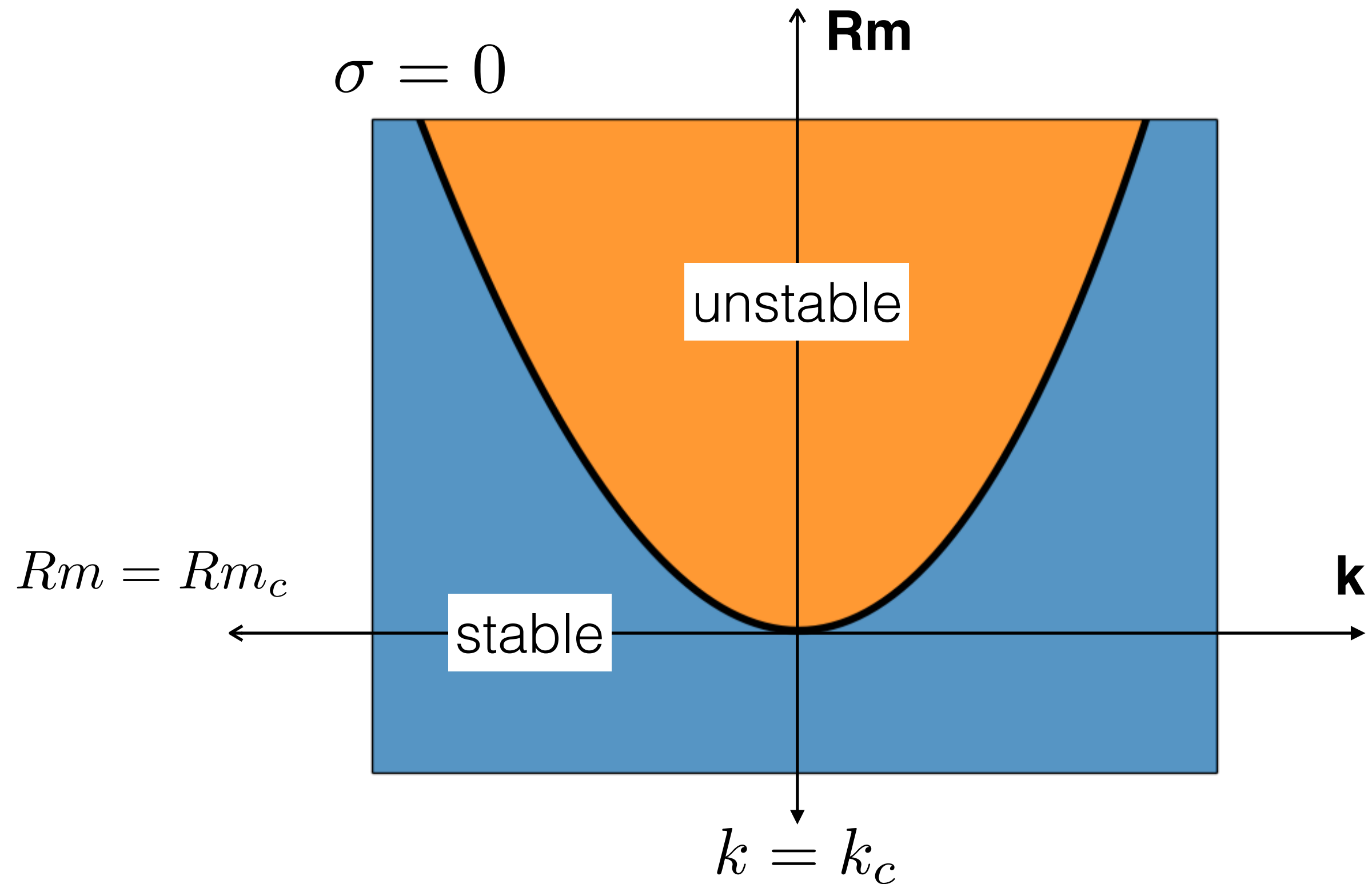
$$\partial_t A = B_0 \partial_z \Psi + \boxed{J(A, \Psi)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q \Omega_0 \partial_z A} + \boxed{J(A, u_y) - J(\Psi, B_y)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

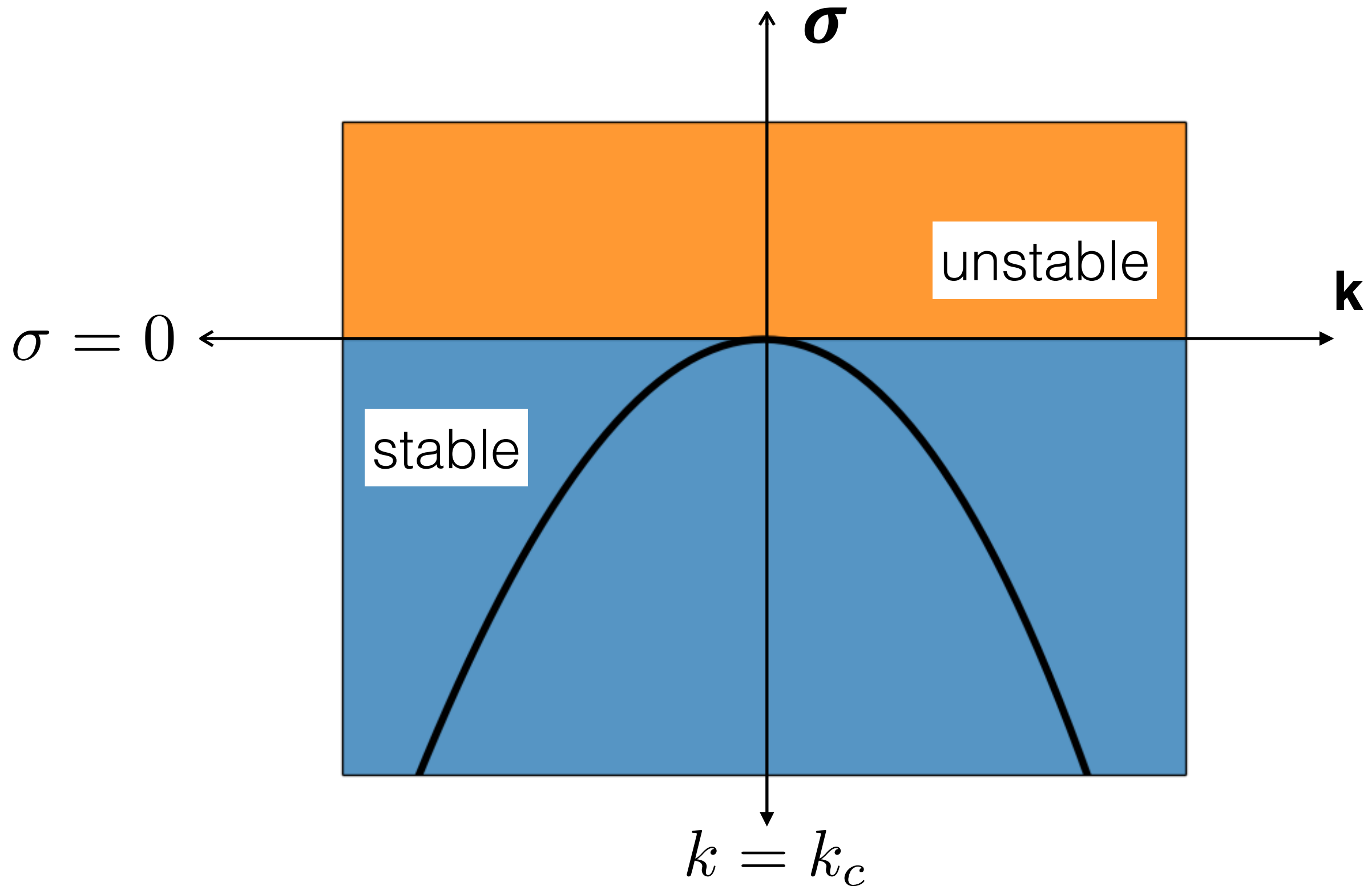
Weakly nonlinear analysis explores behavior at the margin of instability.

$$e^{ikz + \sigma t}$$

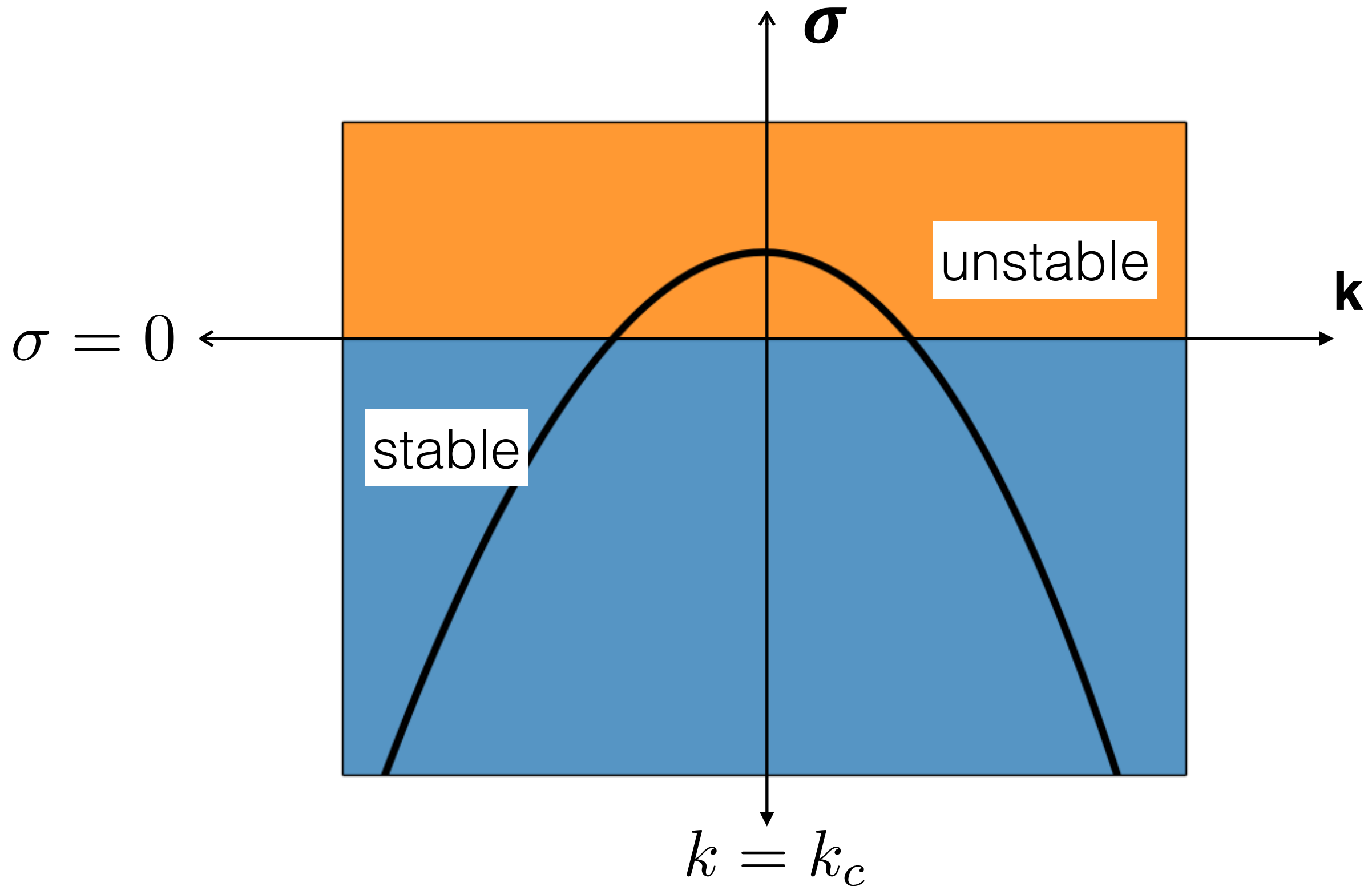


Weakly nonlinear analysis explores behavior at the margin of instability.

Fixed Rm



Tune the most unstable mode just over the threshold of instability.

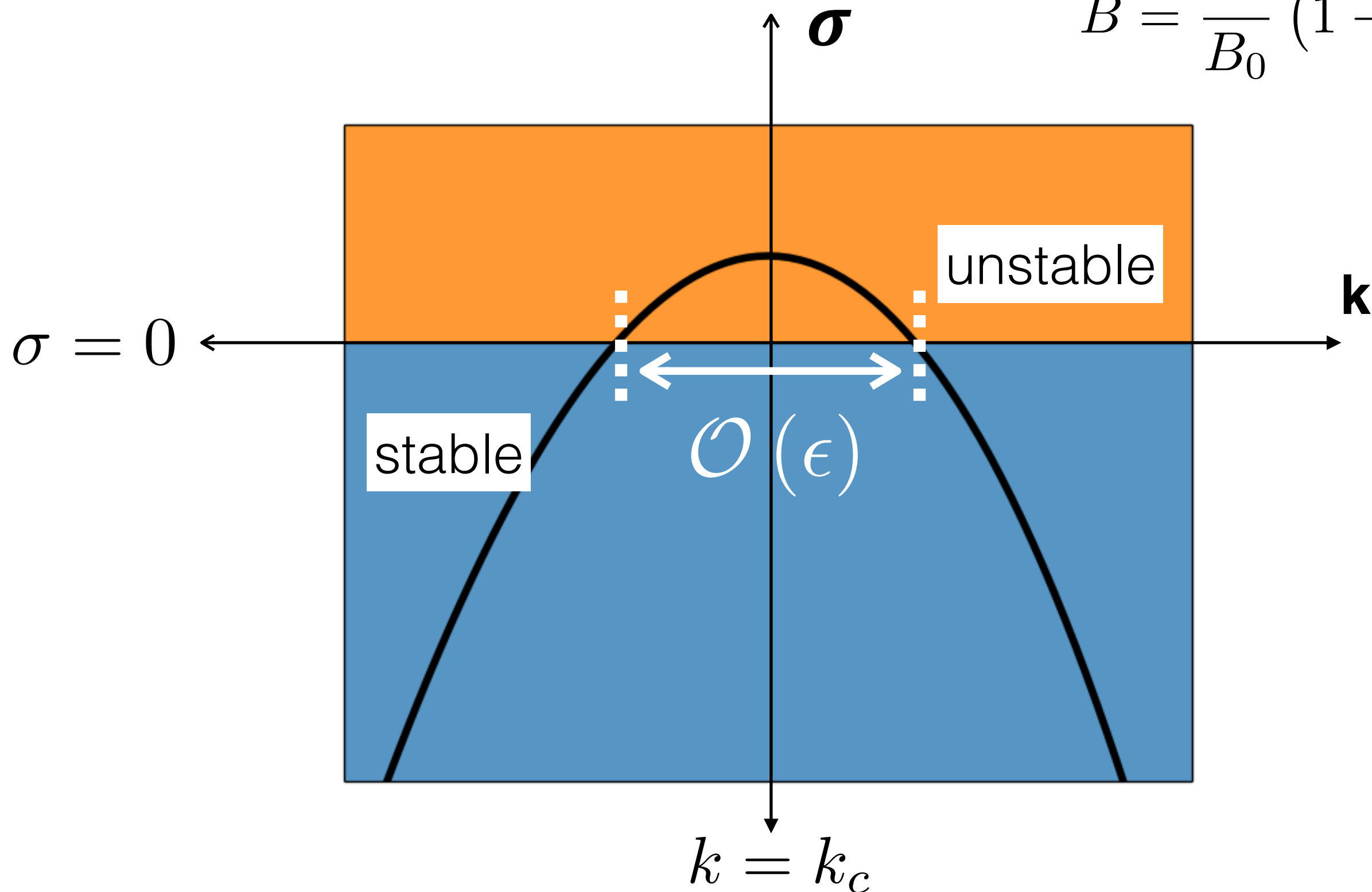


Tune the most unstable mode just over the threshold of instability.

small
parameter



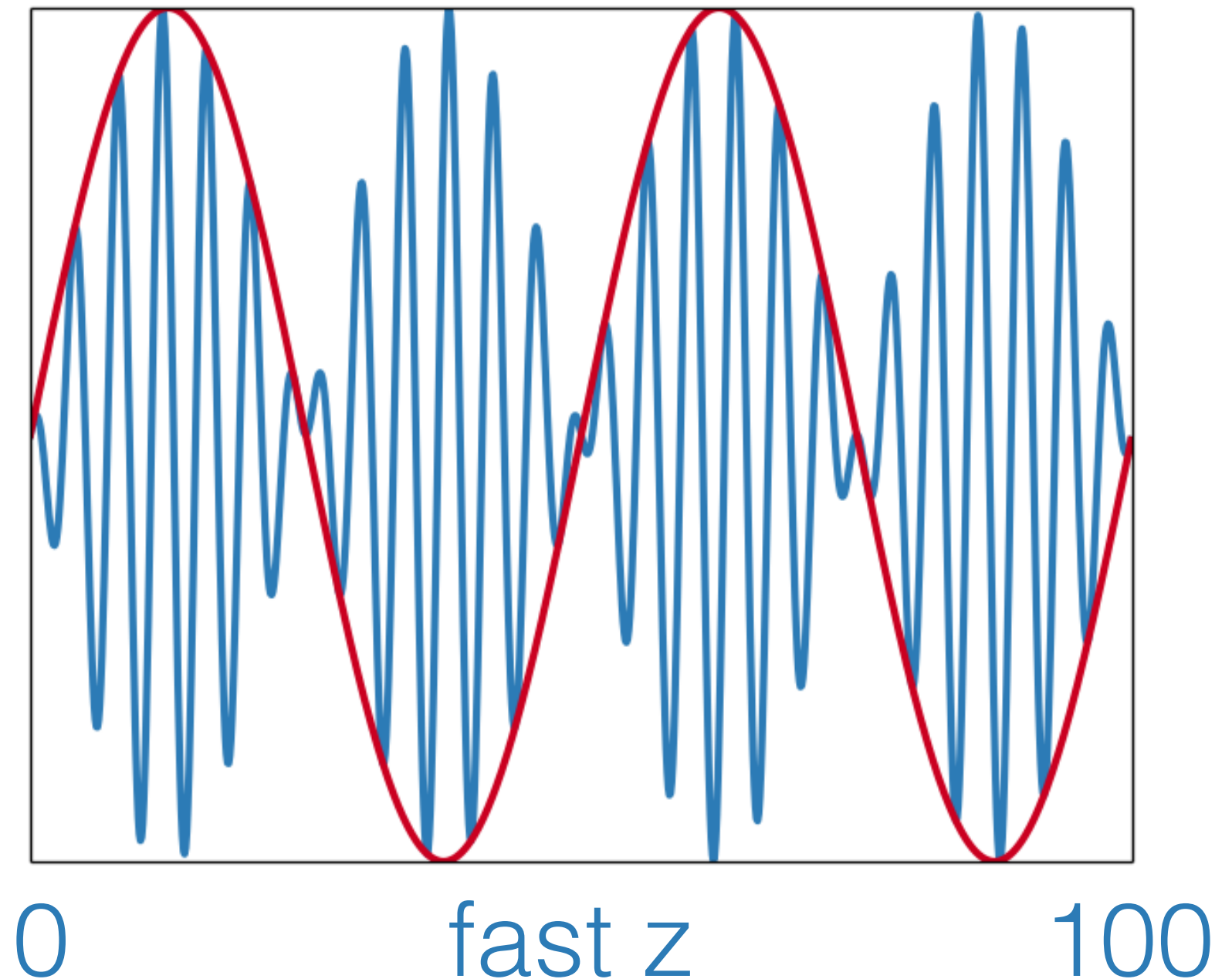
$$B = \frac{B}{B_0} (1 - \epsilon^2)$$



Multiscale analysis tracks the evolution of fast and slow variables.

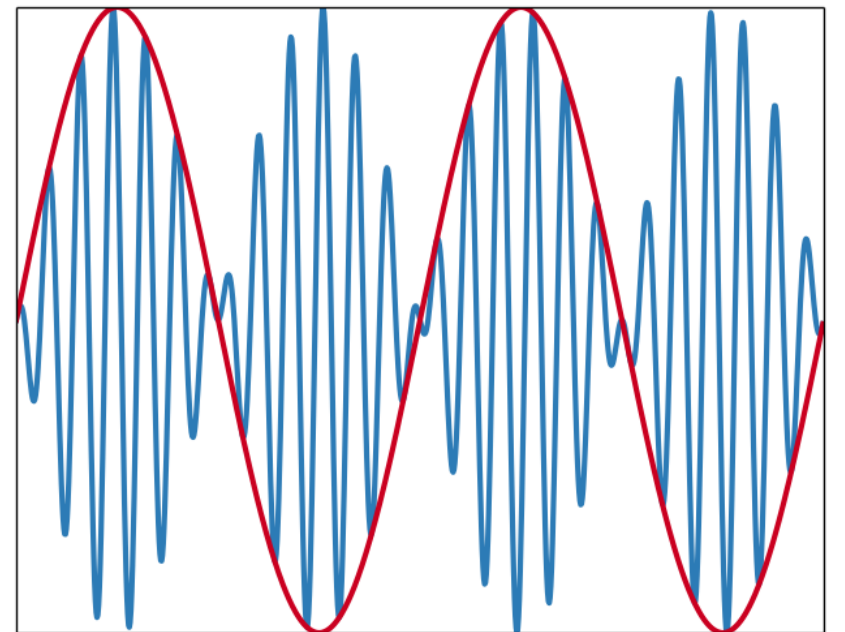
0 **slow Z** **10**

$$Z \equiv \epsilon z$$



We choose an ansatz state vector form.

$$\mathbf{V} = \alpha(Z, T) V(x) e^{ik_c z}$$



We choose an ansatz state vector form.

x dependence



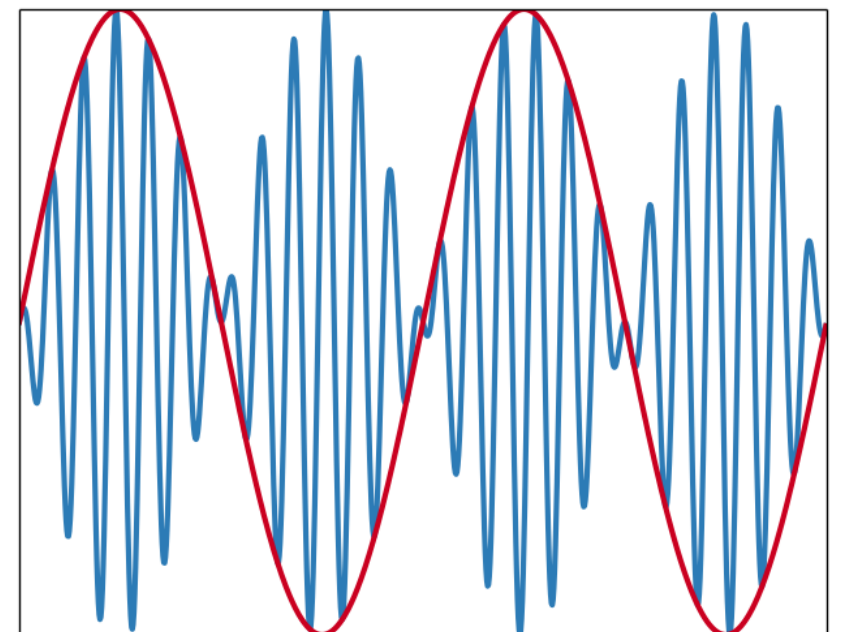
$$\mathbf{V} = \alpha(Z, T) V(x) e^{ik_c z}$$



vertical
periodicity



amplitude
function



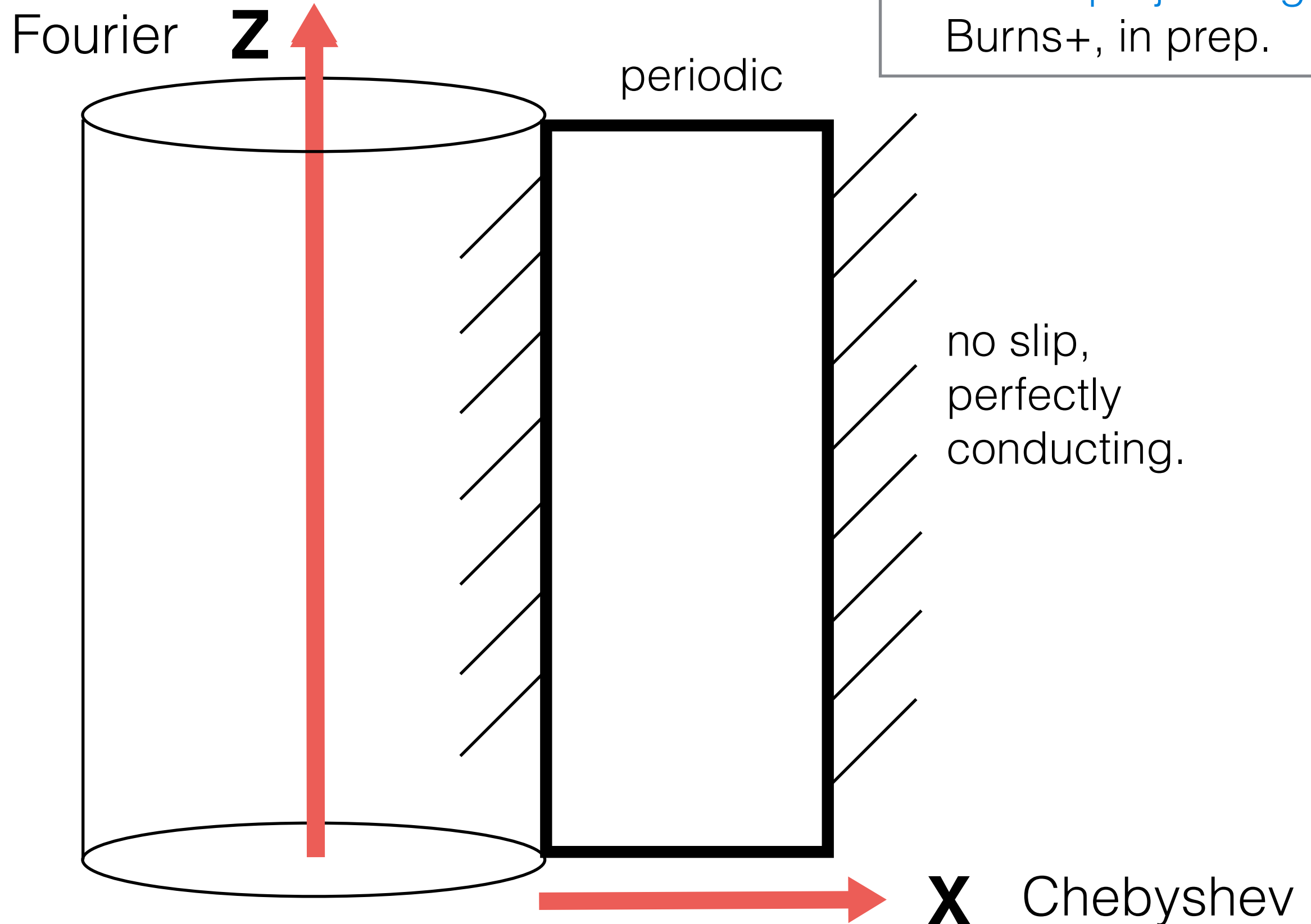
The fluid quantities are expanded in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + \epsilon^3 \mathbf{V}_3 + \dots$$

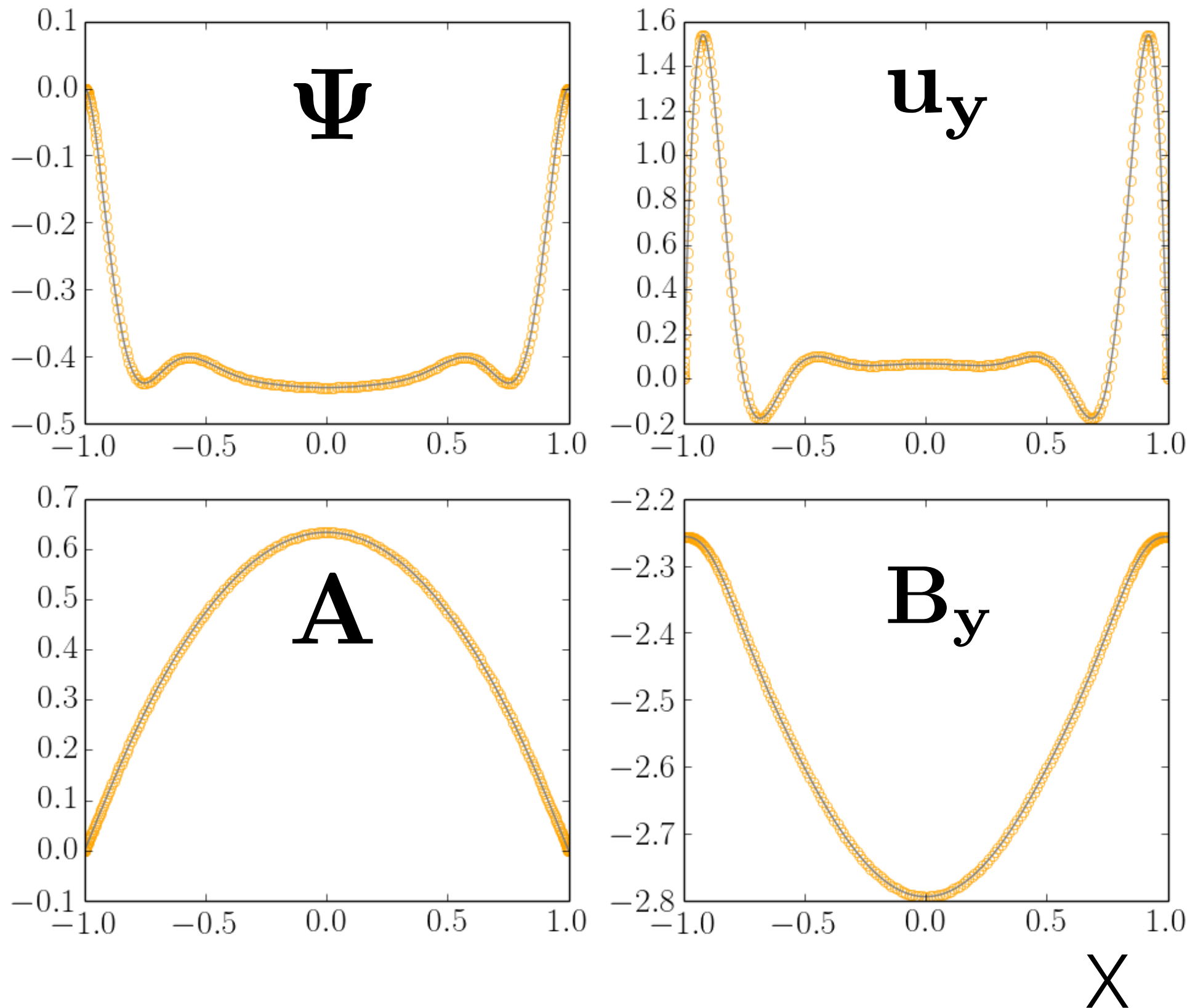
Dedalus is a general-purpose spectral code.

dedalus-project.org

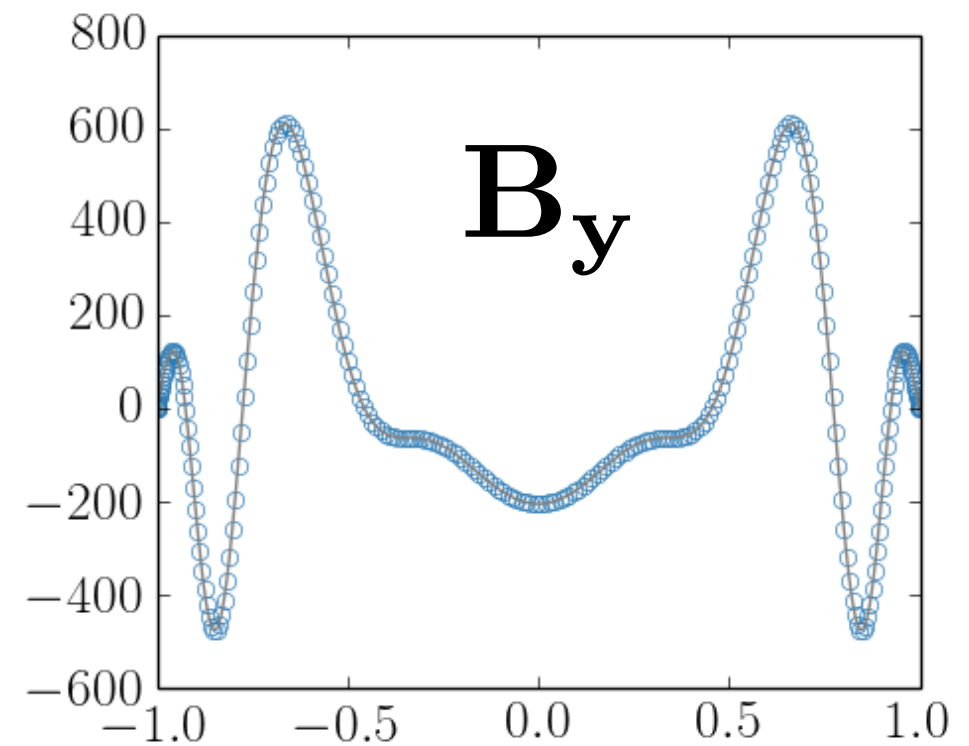
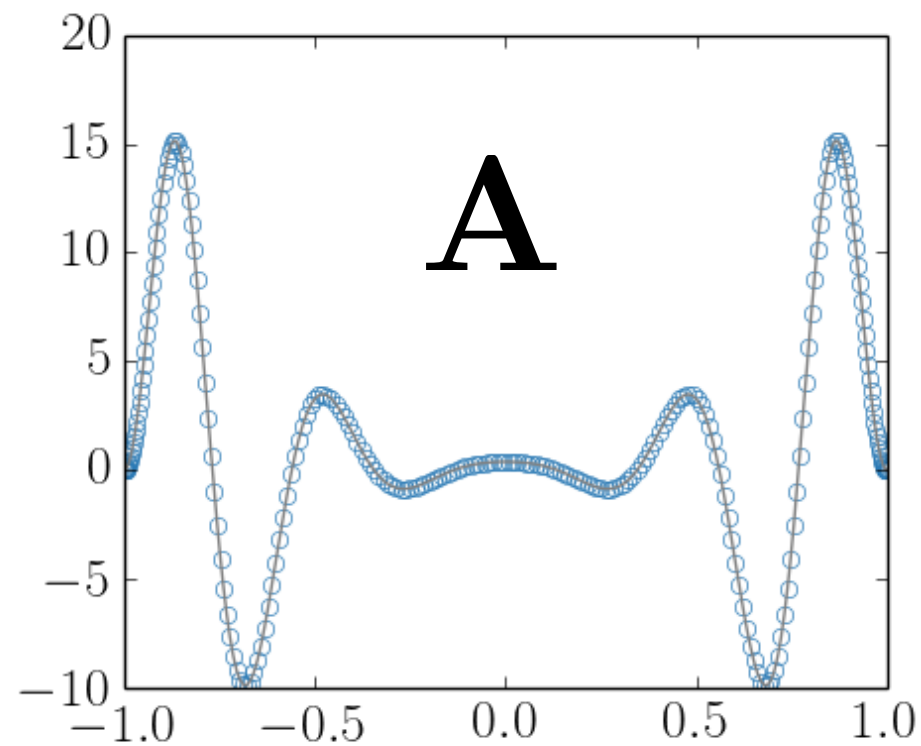
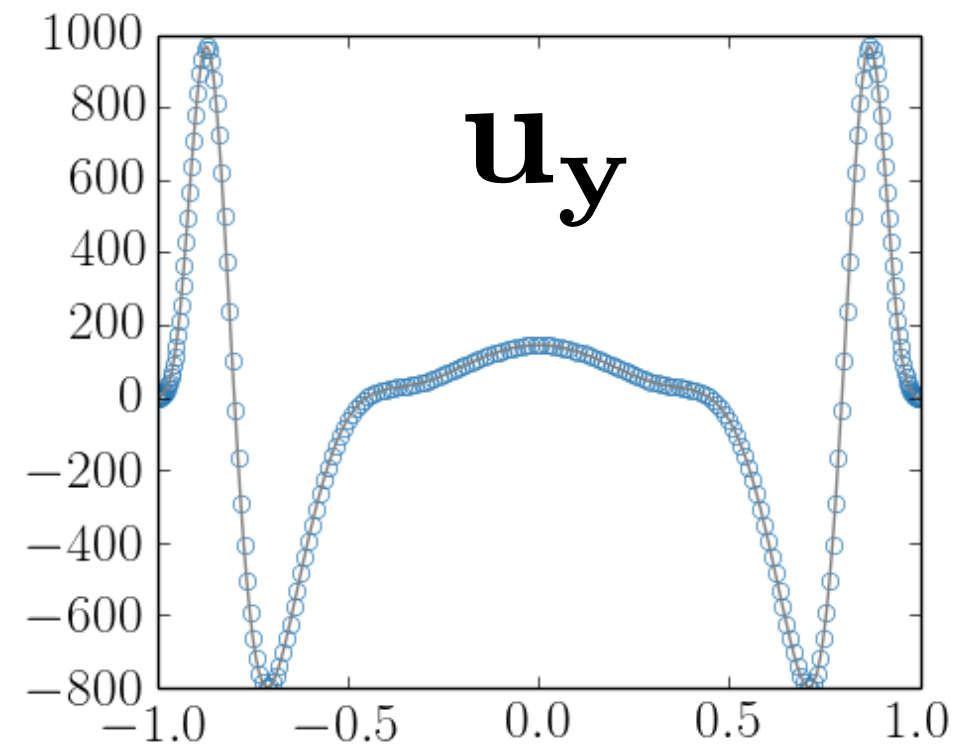
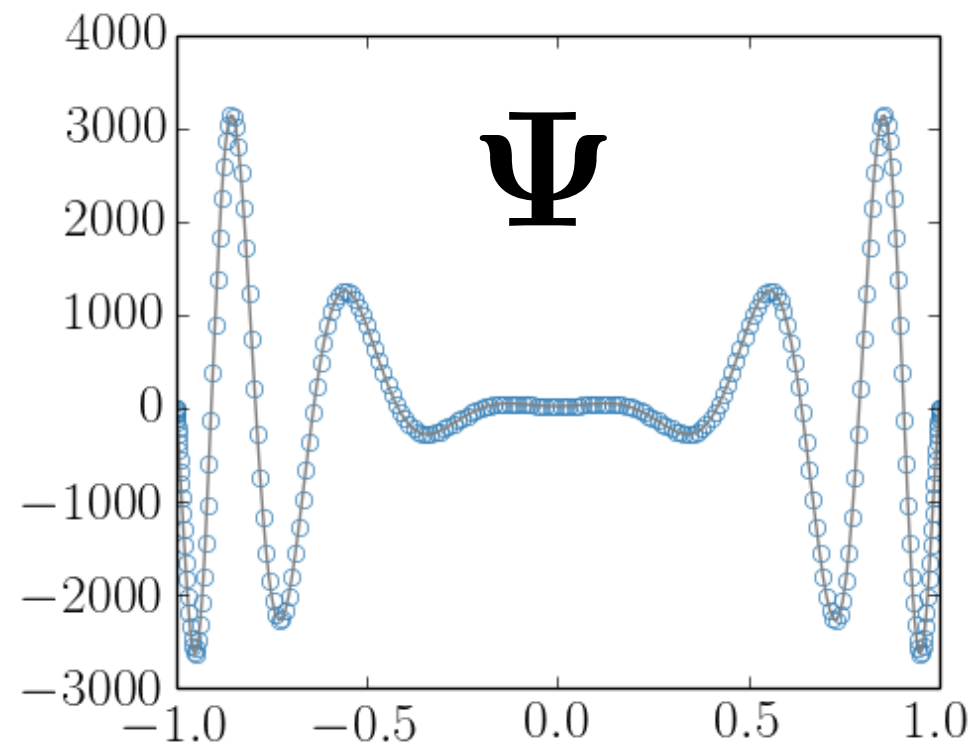
Burns+, in prep.



Spectrally solve the most unstable mode
of the linear MRI.

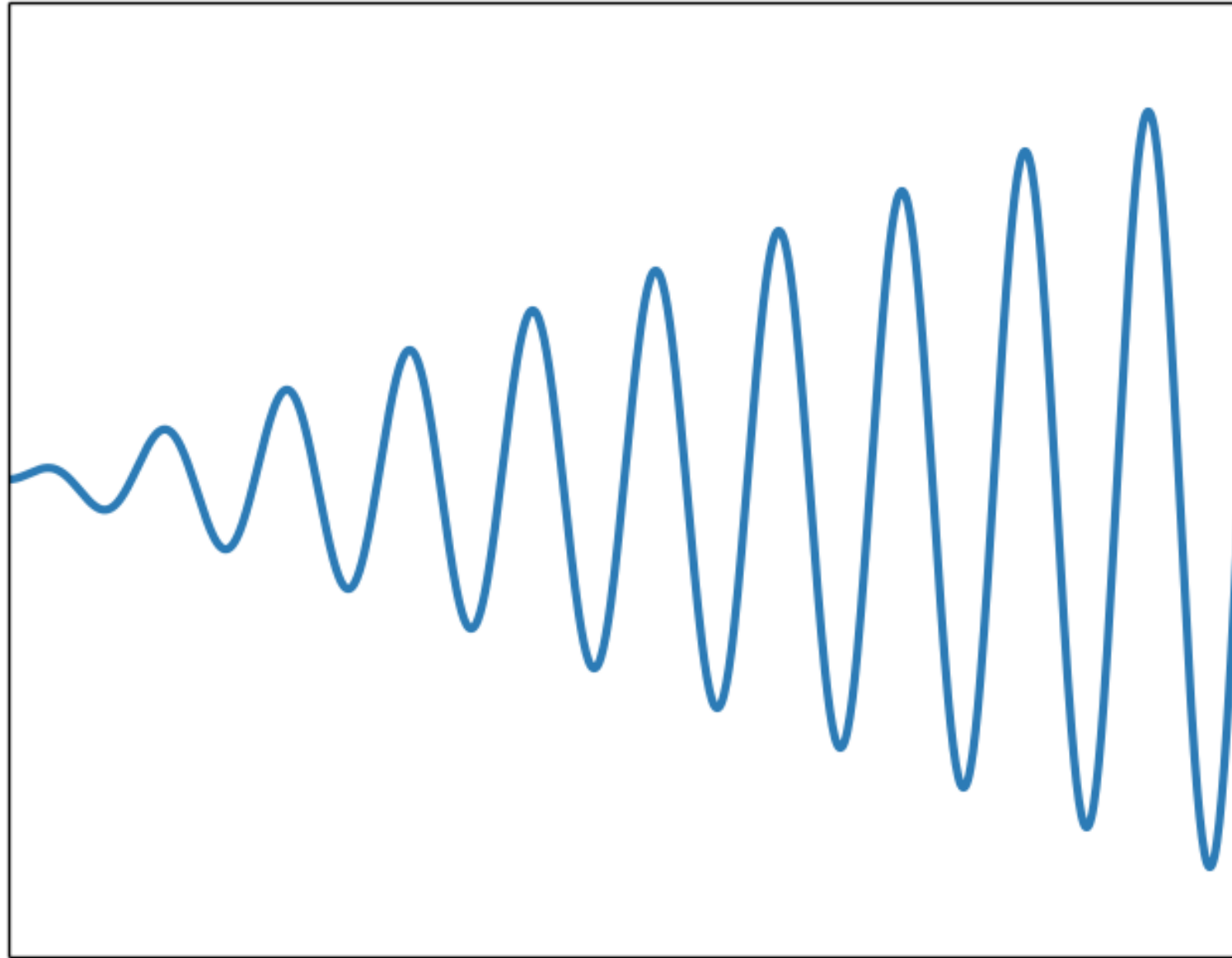


We solve each term in the expanded equations
at each order.

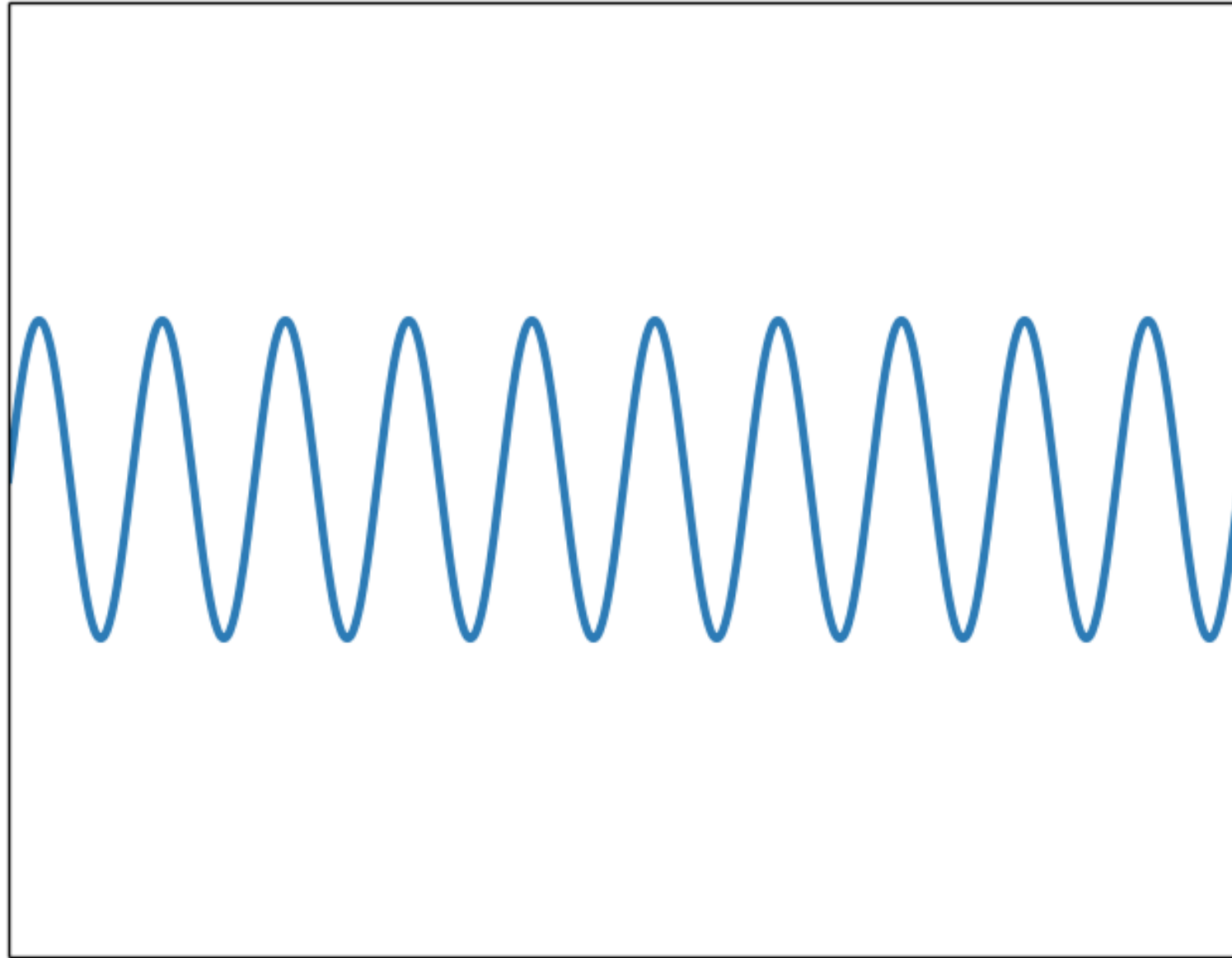


x

The removal of secular terms yields solvability criteria.



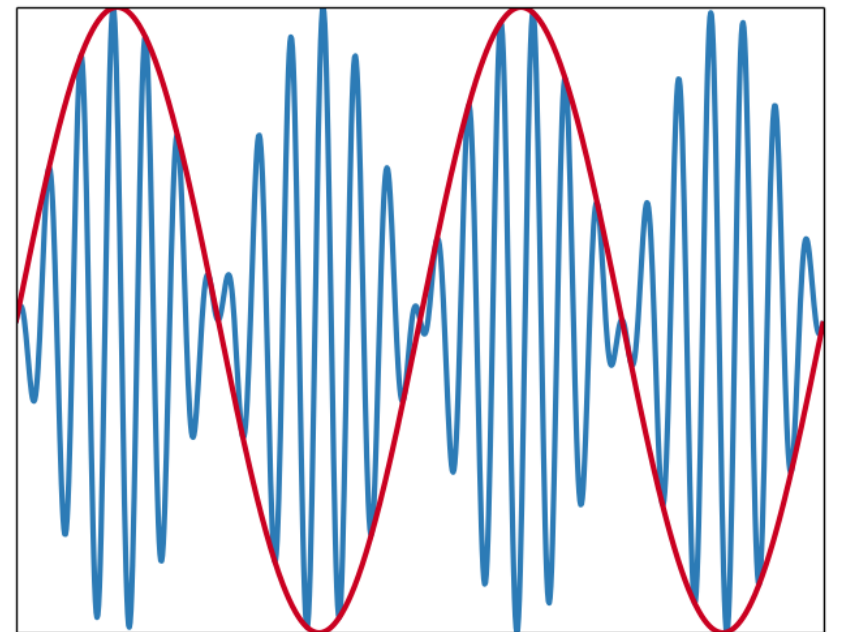
The removal of secular terms yields solvability criteria.



The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha|^2 + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

α



The result is an amplitude equation
for the most unstable mode.

diffusion term



$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha|^2 + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$



nonlinear term



linear growth

The result is an amplitude equation
for the most unstable mode.

?

diffusion term



$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha|^2 + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

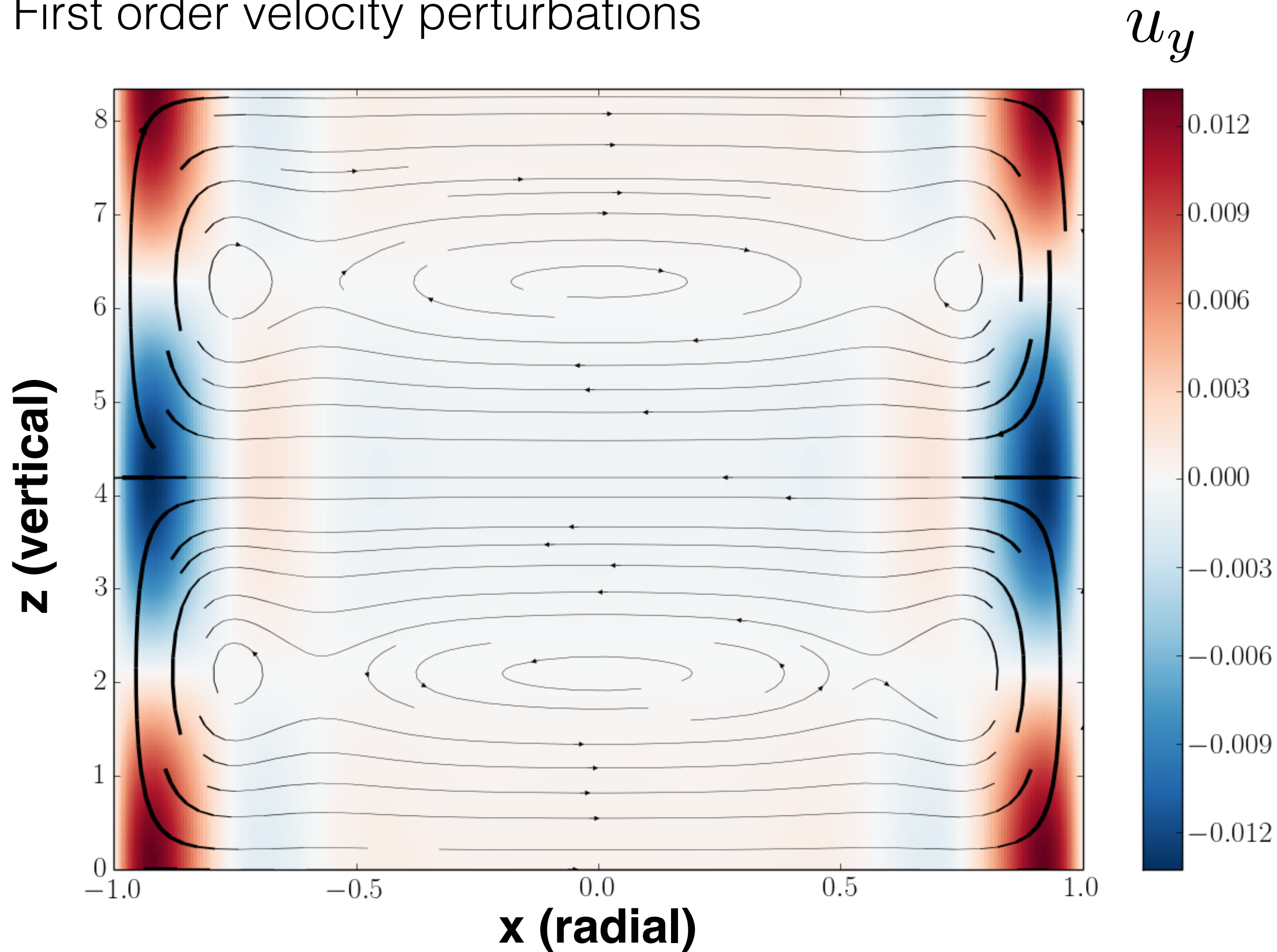


nonlinear term

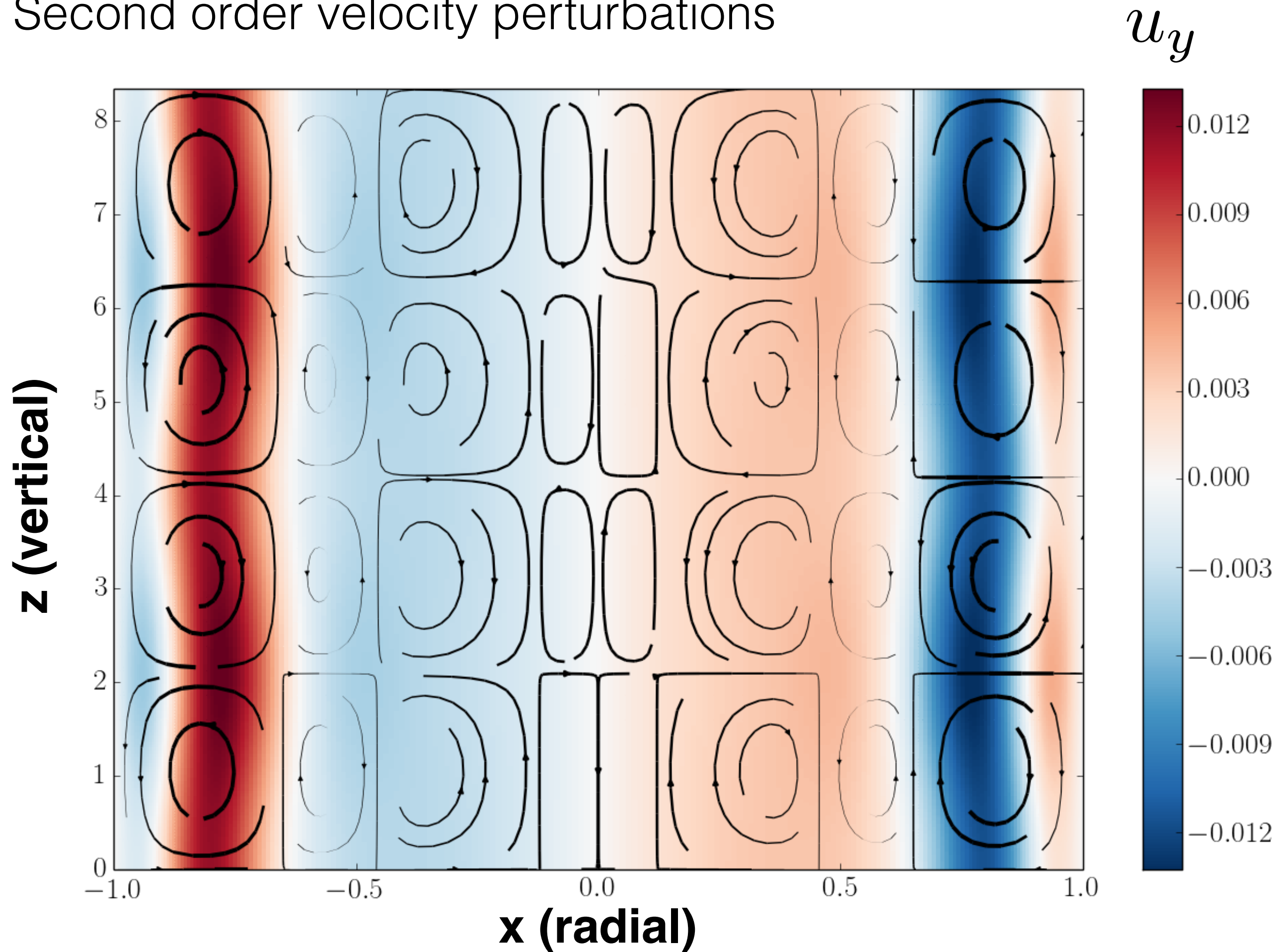


linear growth

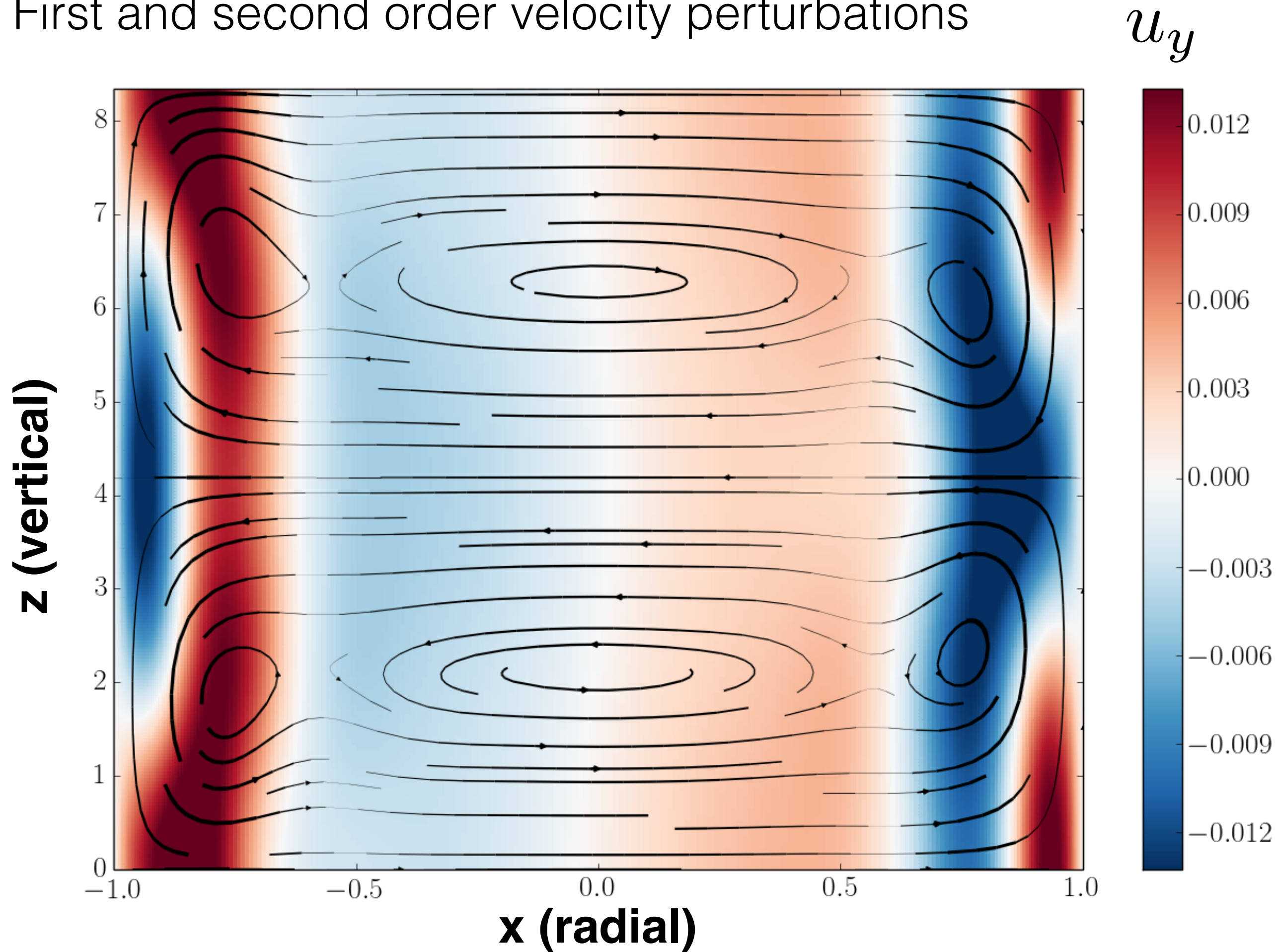
First order velocity perturbations



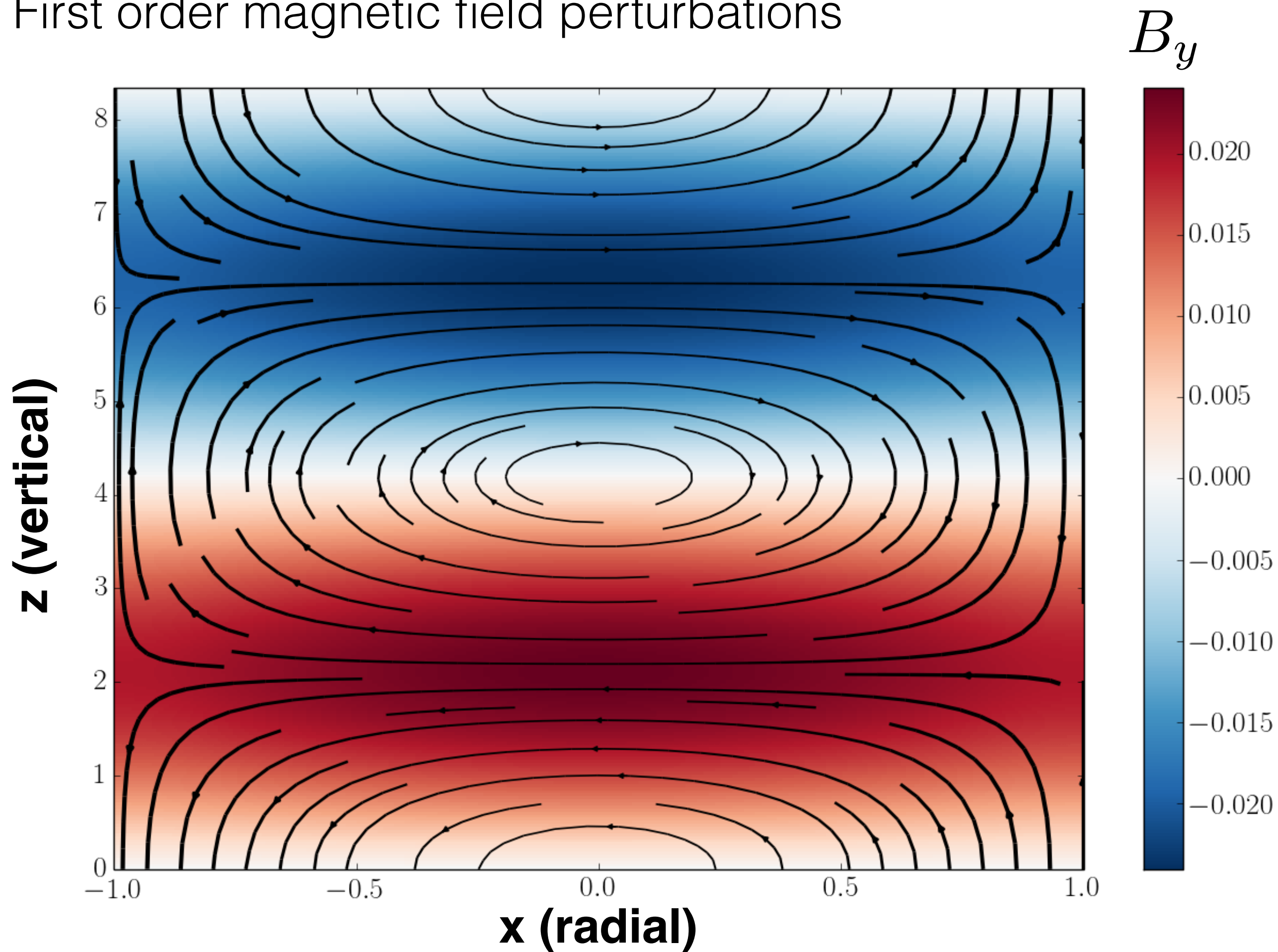
Second order velocity perturbations



First and second order velocity perturbations



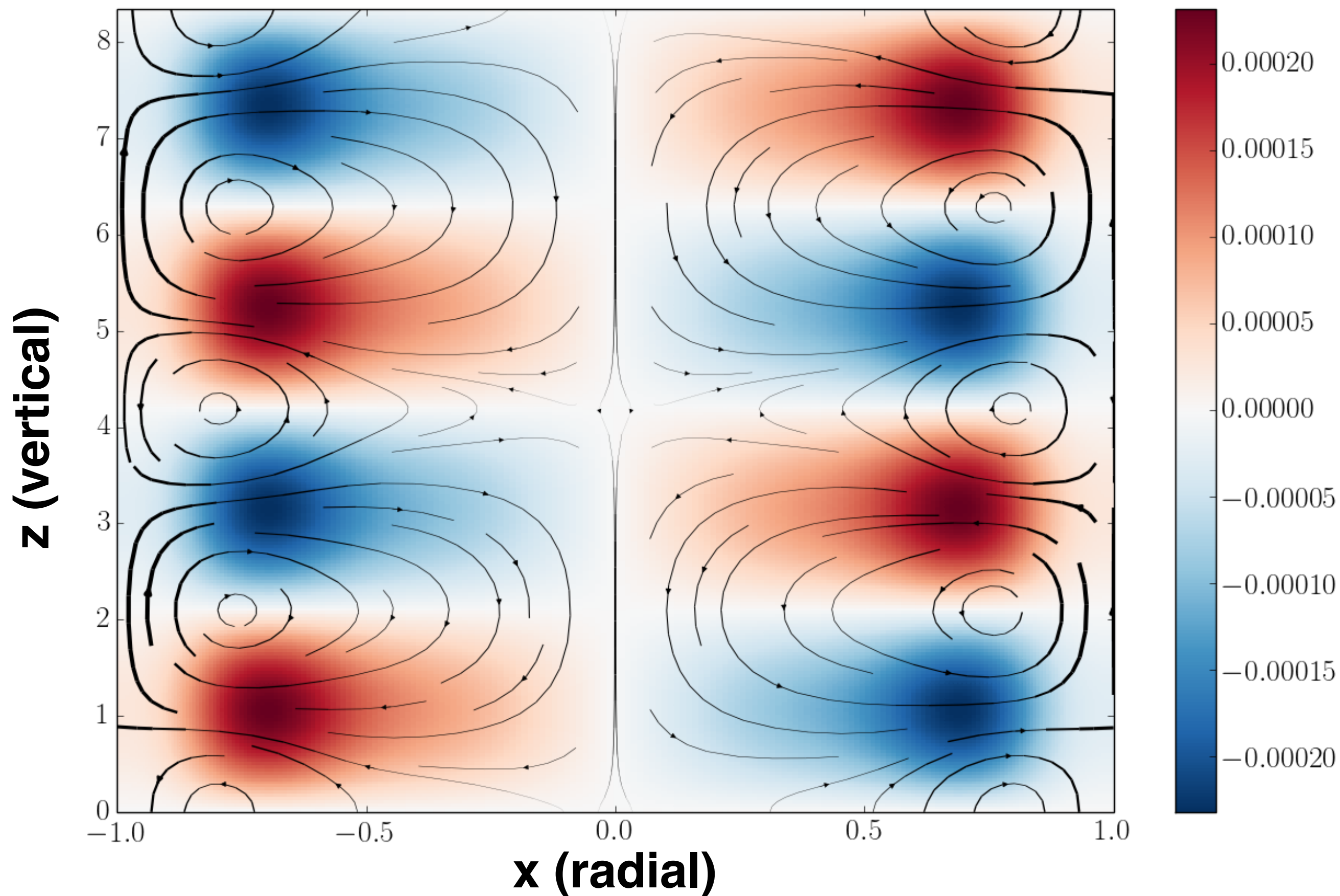
First order magnetic field perturbations



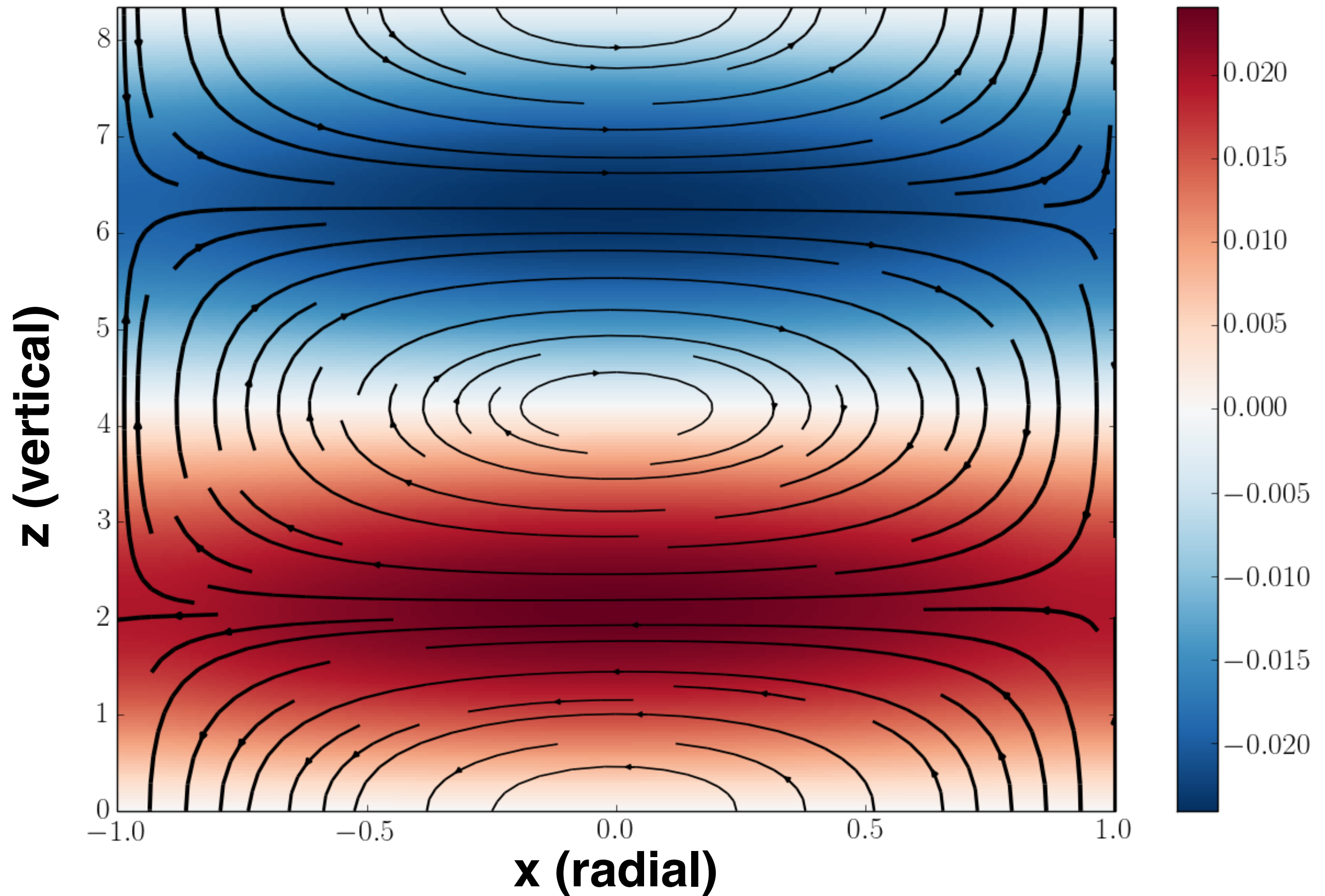
Second order magnetic field perturbations

two OOM smaller!

B_y



First and second order magnetic field perturbations B_y



Future work:

explore parameter space
relax thin gap approximation
comparison to experiment
helical MRI