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$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)$

$\uparrow x_3 \quad \beta = -\frac{d\Theta}{dz}$

Thermal Instability

$\partial_t \rho + \frac{\partial(\rho u_j)}{\partial x_j} = 0$ continuity

$\rho \frac{D}{Dt} u_i = \rho\left(\frac{\partial}{\partial t} + u\cdot\nabla\right) u_i = -g\rho\delta_{i3} + \frac{\partial \sigma_{ij}}{\partial x_j}$ Navier-Stokes EOM

stress tensor: $\sigma_{ij} = -p\delta_{ij} + \underset{\substack{\uparrow \\ \text{dynamic viscosity}}}{\mu}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) + \underset{\substack{\uparrow \\ \text{bulk viscosity}}}{\lambda}\frac{\partial u_k}{\partial x_k}\delta_{ij}$

write this before as

$\partial_t u + u\cdot\nabla u = -\frac{\nabla P}{\rho} + \nu\nabla^2 u$

Heat conduction:

$\rho\frac{DE}{Dt} = \rho\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\underset{\substack{\uparrow \\ \text{internal E per unit mass}}}{E} = \frac{\partial}{\partial x_j}\left(\underset{\substack{\uparrow \\ \text{thermal conductivity}}}{k}\frac{\partial \overset{\text{temp}}{\Theta}}{\partial x_j}\right) - p\frac{\partial u_j}{\partial x_j} + \Phi$

rate of viscous dissipation per unit volume of fluid: $\Phi = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \left(\lambda - \frac{2}{3}\mu\right)\left(\frac{\partial u_k}{\partial x_k}\right)^2$

perfect liquid: $E = c\Theta$ (c is specific heat)

Boussinesq: density variation neglected everywhere except in buoyancy.

$\Theta = \Theta_0,$
$\rho = \rho_0$

$\therefore \rho = \rho_0\{1 - \alpha(\Theta - \Theta_0)\}$

$\hookrightarrow \mu, k, c$ are constants

$\hookrightarrow$ neglect λ

$\hookrightarrow \frac{\partial u_j}{\partial x_j} = 0$, as for an incompressible fluid $(\nabla\cdot u = 0)$

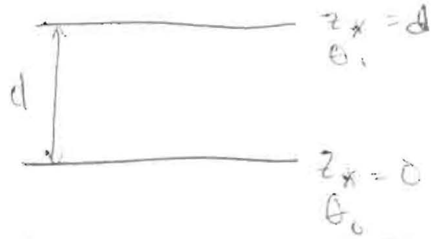
$\therefore \sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$\frac{D}{Dt}u_i = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + gz\right) - \alpha g(\Theta_0 - \Theta)\delta_{i3} + \nu\frac{\partial^2}{\partial x_j^2}u_i$

Heat eqn $\rightarrow$

$\frac{D}{Dt}\Theta = \kappa\Delta\Theta$

$\kappa = \frac{k}{\rho_0 c}$ ratio of $\frac{\text{thermal conductivity}}{\text{bottom density} * \text{specific heat}}$



temp gradient
$\beta = \dfrac{(\theta_0 - \theta_1)}{d}$

Boussinesq eqns:

instability when $\theta_0 > \theta_1$

$$\begin{cases} \dfrac{\partial u_i}{\partial x_j} = 0 \\ \left(\dfrac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right) u_i = -\dfrac{\partial}{\partial x_i}\left(\dfrac{p}{\rho_0} + gz\right) - \alpha g(\theta_0 - \theta)\delta_{i3} + \nu\dfrac{\partial^2}{\partial x_j^2} u_i \\ \left(\dfrac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\theta = \kappa \Delta \theta \end{cases}$$

Laplacian

$u_* = 0$
$\Theta_* = \theta_0 - \beta z_*$
$P_* = p_0 - g\rho_0 (z_* + \frac{1}{2}\alpha\beta z_*^2)$ for $0 \le z_* \le d$

small (primed) perturbations:
$u_* = u_*'(\vec{x}_*, t_*)$
$\theta_* = \Theta_*(z_*) + \theta_*'(\vec{x}_*, t_*)$
$p_* = P_*(z_*) + p_*'(\vec{x}_*, t_*)$

① $\nabla \cdot u = 0 \longrightarrow \nabla_* \cdot u_*' = 0$

$$\frac{\partial}{\partial t} u_* + \cancel{\vec{u}\cdot\nabla u_*}^{\,0} = -\nabla\left(\frac{p}{\rho_0}\right) - \nabla(gz) - \alpha g(\theta_0 - \theta)\delta_{i3} + \nu\nabla^2 u$$

$$\frac{\partial u_*'}{\partial t} = -\frac{1}{\rho_0}\nabla_*\left(P_*(z_*) + p_*'(\vec{x}_*, t_*)\right) - g\nabla_* z_* - \alpha g\left(\theta_0 - (\Theta_*(z_*) + \theta_*'(\vec{x}_*, t_*))\right)\delta_{i3} + \nu\nabla^2 u$$

$$\frac{\partial u_*'}{\partial t} = -\frac{1}{\rho_0}\nabla_* p_*' - \frac{1}{\rho_0}\nabla_* P_* - g\nabla_* z_* - \alpha g(\theta_0 - \Theta_* - \theta_*')\delta_{i3} + \nu\nabla_*^2 u_*'$$

$$\frac{\partial u_*'}{\partial t} = -\frac{1}{\rho_0}\nabla_* p_*' - \frac{1}{\rho_0}\nabla_*(\cancel{p_0}^{\,0} - g\rho_0(z_* + \tfrac{1}{2}\alpha\beta z_*^2)) - \alpha g(\cancel{\theta_0} - \cancel{\theta_0} + \beta z_* - \theta_*')\delta_{i3} + \nu\nabla_*^2 u_*'$$

$$= -\frac{1}{\rho_0}\nabla_* p_*' + \cancel{g\nabla_* z_*}^{\,0} + \tfrac{1}{2}g\alpha\beta\cancel{\nabla_*} z_*^2 - \alpha g\cancel{\beta} z_* + \alpha g\theta_*'\delta_{i3} + \nu\nabla_*^2 u_*'$$

② $\dfrac{\partial u_*}{\partial t} = -\dfrac{1}{\rho_0}\nabla_* p_*' + \alpha g\theta_*'\hat{k} + \nu\nabla_*^2 u_*'$

$$\frac{\partial}{\partial t}(\cancel{\theta_0}^{\,0} - \beta z_* + \theta_*') + u_* \cdot \nabla_*(\cancel{\theta_0}^{\,0} - \beta z_* + \theta_*') = \kappa\nabla_*^2(\cancel{\theta_0} - \cancel{\beta z_*} + \theta_*')$$

$$-\beta u_{z_*} + \frac{\partial\theta_*'}{\partial t} + \cancel{u_*'\cdot\nabla_*\theta_*'}^{\,0\ \text{(2nd order)}} = \kappa\nabla_*^2\theta_*'$$

③ $\dfrac{\partial\theta_*'}{\partial t} - \beta w_*' = \kappa\nabla_*^2\theta_*'$

So now we have the linearized, dimensional eqns.

$$\begin{cases} \nabla_* \cdot u_*' = 0 \\ \dfrac{\partial u_*'}{\partial t_*} = -\dfrac{1}{\rho_0}\nabla_* p_*' + \alpha g \theta_*' \hat{k} + \nu \nabla_*^2 u_*' \\ \dfrac{\partial \theta_*'}{\partial t_*} - \beta w_*' = \kappa \nabla_*^2 \theta_*' \end{cases}$$

scales: length: d time: $\frac{d^2}{\kappa}$ temp diff: $\theta_0 - \theta_1 = \beta d$

$$\therefore \quad \vec{x} = \frac{\vec{x}_*}{d} \qquad t = \frac{\kappa t_*}{d^2} \qquad \vec{u} = \frac{d\vec{u}_*'}{\kappa} \qquad \theta = \frac{\theta_*'}{\beta d} \qquad p = \frac{d^2 p_*'}{\rho_0 \kappa^2}$$

(1) SO: $\nabla \cdot \left(\frac{d u_*'}{\kappa}\right) = 0 \rightarrow \nabla \cdot u = 0$ $\qquad \nabla_* \rightarrow \frac{1}{d}\nabla$?

$$\vec{x}_* = d\vec{x} \qquad \frac{d^2 t}{\kappa} = t_* \qquad \frac{\kappa \vec{u}}{d} = \vec{u}_*' \qquad \beta d \theta = \theta_*' \qquad \frac{\rho_0 \kappa^2 p}{d^2} = p_*'$$

$$\frac{\partial\left(\frac{\kappa u}{d}\right)}{\partial\left(\frac{d^2 t}{\kappa}\right)} = \frac{\left(\frac{\kappa}{d}\right)}{\left(\frac{d^2}{\kappa}\right)}\frac{\partial u}{\partial t} = \frac{\kappa^2}{d^3}\frac{\partial u}{\partial t} = \frac{-1}{\rho_0}\nabla\left(\frac{1}{d}\right)\left(\frac{\rho_0 \kappa^2}{d^2}\right)p + \alpha g \beta d \theta \hat{k} + \nu \frac{1}{d^2}\nabla^2 \frac{\kappa u}{d}$$

$$\frac{\kappa^2}{d^3}\frac{\partial u}{\partial t} = -\frac{\kappa^2}{d^3}\nabla p + \alpha g \beta d \theta \hat{k} + \nu \frac{\kappa}{d^3}\nabla^2 u$$

$$\frac{\partial u}{\partial t} = -\nabla p + \underbrace{\frac{\alpha g \beta d^4}{\kappa^2}}\theta\hat{k} + \underbrace{\frac{\nu}{\kappa}}_{\equiv Pr \text{ (Prandtl number)}}\nabla^2 u$$

Rayleigh number $R = \frac{g\alpha\beta d^4}{\kappa\nu}$

(2) $$\therefore \frac{\partial u}{\partial t} = -\nabla p + R\,Pr\,\theta\hat{k} + Pr\nabla^2 u$$

$$\frac{\partial(\beta d\theta)}{\partial\left(\frac{d^2}{\kappa}t\right)} = \frac{\beta\kappa}{d}\frac{\partial\theta}{\partial t} - \beta\frac{\kappa}{d}w = \kappa\left(\frac{1}{d}\right)^2\nabla^2\beta d\theta = \frac{\kappa\beta^2}{d}\nabla^2\theta$$

$$\frac{\partial\theta}{\partial t} - w = \frac{\cancel{d}}{\cancel{\beta\kappa}}\left(\frac{\cancel{\kappa\beta^2}}{\cancel{d}}\right)\nabla^2\theta$$

(3) $$\frac{\partial\theta}{\partial t} - w = \nabla^2\theta$$

Now we have perturbed, linearized, dimensionless eqns:

$$\begin{cases} \nabla \cdot u = 0 & (1) \\ \frac{\partial u}{\partial t} = -\nabla p + RPr\,\theta \hat{k} + Pr \nabla^2 u & (2) \\ \frac{\partial \theta}{\partial t} - w = \nabla^2 \theta & (3) \end{cases}$$

Take the curl of (2):

$$\nabla \times \left(\frac{\partial u}{\partial t}\right) = \nabla \times (-\cancelto{0}{\nabla p} + RP_r\,\theta \hat{k} + Pr \nabla^2 u)$$

$$\frac{\partial}{\partial t}(\nabla \times u) = RP_r(\nabla\theta \times \hat{k}) + Pr \nabla^2(\nabla \times u)$$

$\vec{\omega} = \vec{\nabla} \times \vec{u}$: vorticity

$$\frac{\partial}{\partial t}(\omega) = RPr(\nabla\theta \times \hat{k}) + Pr \nabla^2 \omega$$

→ take the curl again...

$$\nabla \times \left(\frac{\partial \omega}{\partial t}\right) = \nabla \times (RPr(\nabla\theta \times \hat{k})) + \nabla \times (Pr \nabla^2 \omega)$$

LHS: $\frac{\partial}{\partial t}(\nabla \times (\nabla \times u)) = \frac{\partial}{\partial t}(\cancelto{0}{(\nabla \cdot u)}\nabla - \nabla \cdot \nabla u) = -\frac{\partial}{\partial t}(\nabla^2 u)$

RHS: 1st term $RP_r(\nabla \times (\nabla\theta \times \hat{k})) = (\nabla \cdot \hat{k} - \nabla \cdot \nabla\theta\,\hat{k})RPr$

2nd term $\nabla \times (Pr \nabla^2 \omega) = Pr(\nabla \times \nabla^2(\nabla \times u)) = Pr(\nabla^2)^2 u$ ← check

$$\therefore \quad -\frac{\partial}{\partial t}(\nabla^2 u) = RPr(\nabla \cdot \hat{k} - \nabla \cdot \nabla\theta\,\hat{k}) + Pr(\nabla^2)^2 u$$