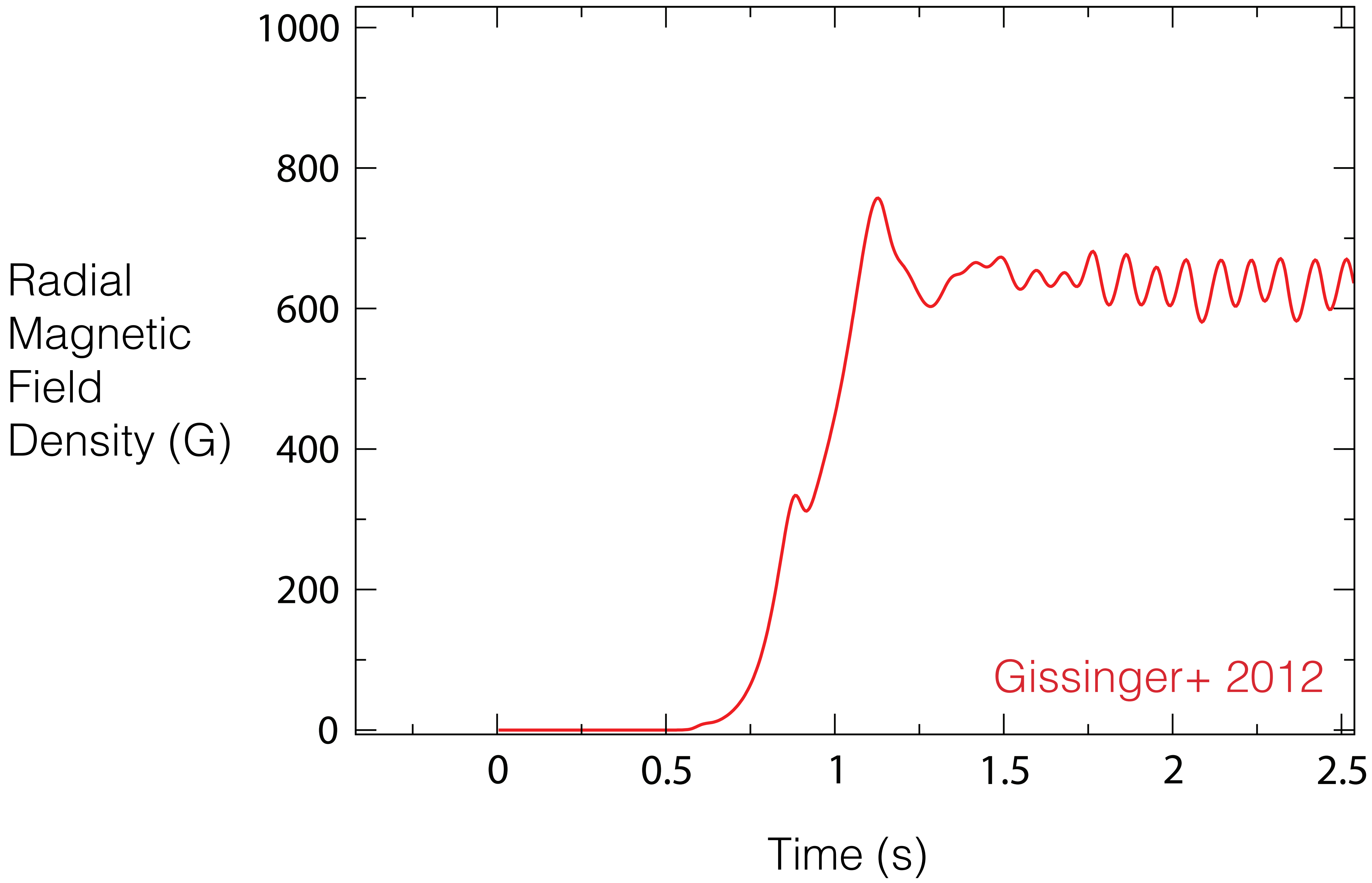
# Exploring the saturation of the MRI via weakly nonlinear analysis

Susan E. Clark | NSF Graduate Fellow,
Columbia University

Jeffrey S. Oishi | SUNY Farmingdale, AMNH

Mordecai-Mark Mac Low | AMNH

# The MRI evolves in stages.



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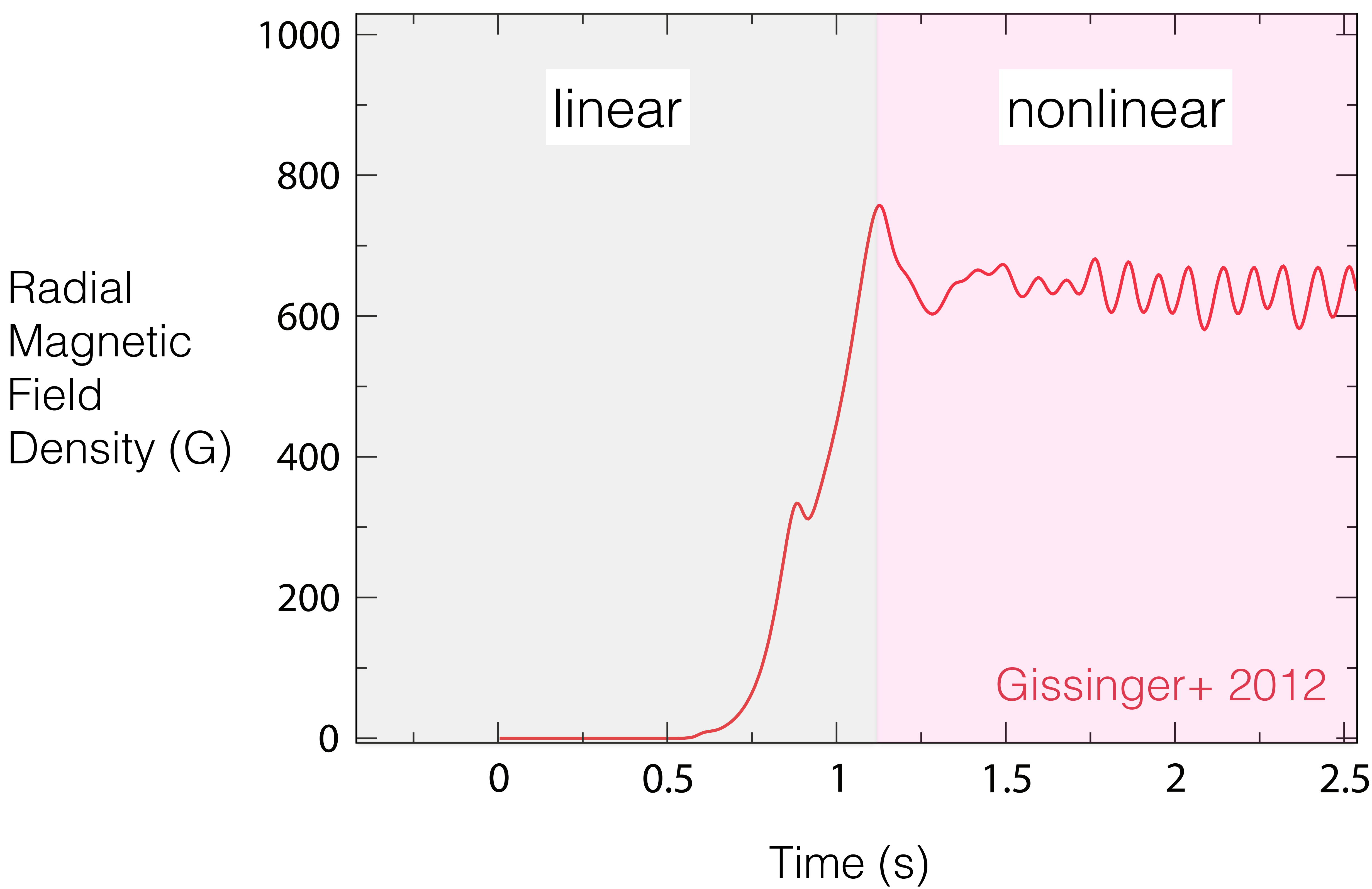

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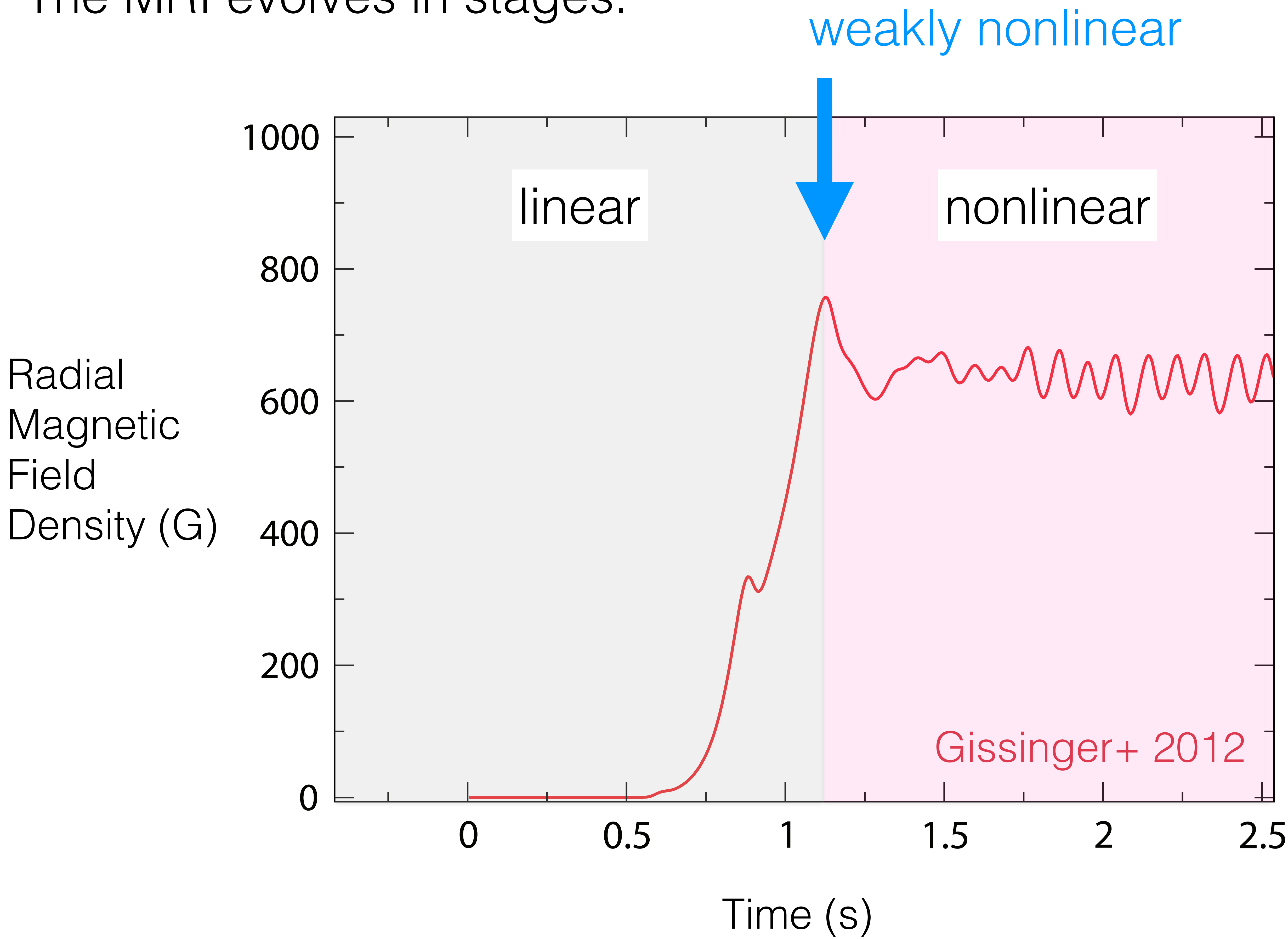

We solve the non-ideal MRI equations.

## momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

## induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

## constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal MRI equations.

**momentum**

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}\left(\mathbf{J}\times\mathbf{B}\right) - 2\Omega\times\mathbf{u} - \Omega\times\left(\Omega\times\mathbf{r}\right) + \nu\nabla^2\mathbf{u}$$

microscopic
viscosity

**induction**

$$\partial_t \mathbf{B} = \nabla\times\left(\mathbf{u}\times\mathbf{B}\right) + \eta\nabla^2\mathbf{B}$$

magnetic
resistivity

**constraints**

$$\nabla\cdot\mathbf{u} = 0$$

$$\nabla\cdot\mathbf{B} = 0$$

We nondimensionalize and perturb the nonlinear MRI equations.

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$

Reynolds number

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$

magnetic Reynolds number

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0}\right)^{-q}$$

shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$

background field

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{u_y} \\ \mathbf{A} \\ \mathbf{B_y} \end{bmatrix}$$

We work in terms of flux and stream functions.

## momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \frac{1}{Re} \nabla^2 u_y$$

## induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

**momentum**

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re}\nabla^4 \Psi}$$

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**induction**

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resistive

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm}\nabla^2 B_y}$$

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**momentum**

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shear

**induction**

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

**momentum**

nonlinear viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \boxed{\frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

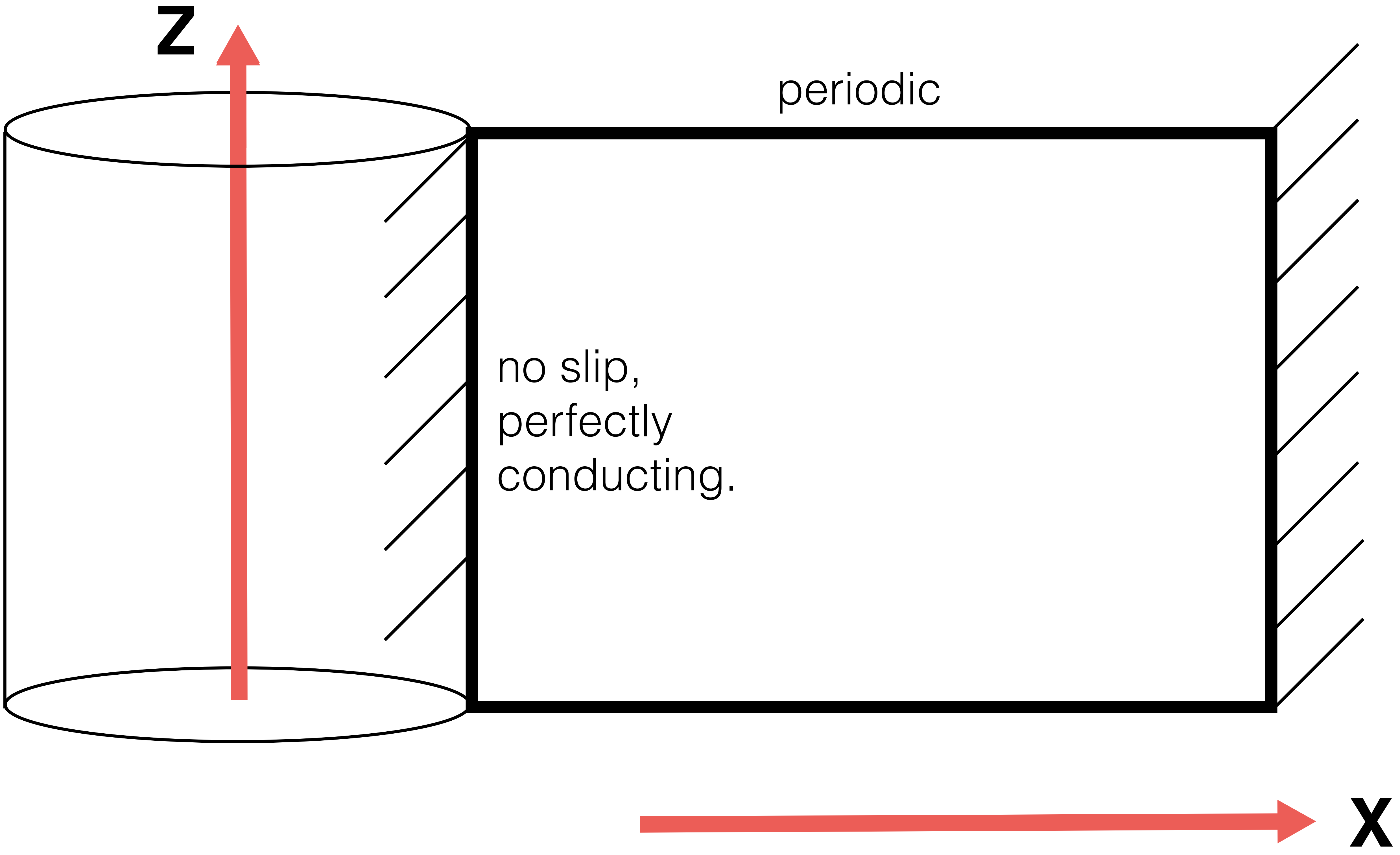
**induction**

$$\partial_t A = B_0 \partial_z \Psi + \boxed{J\left(A, \Psi\right)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

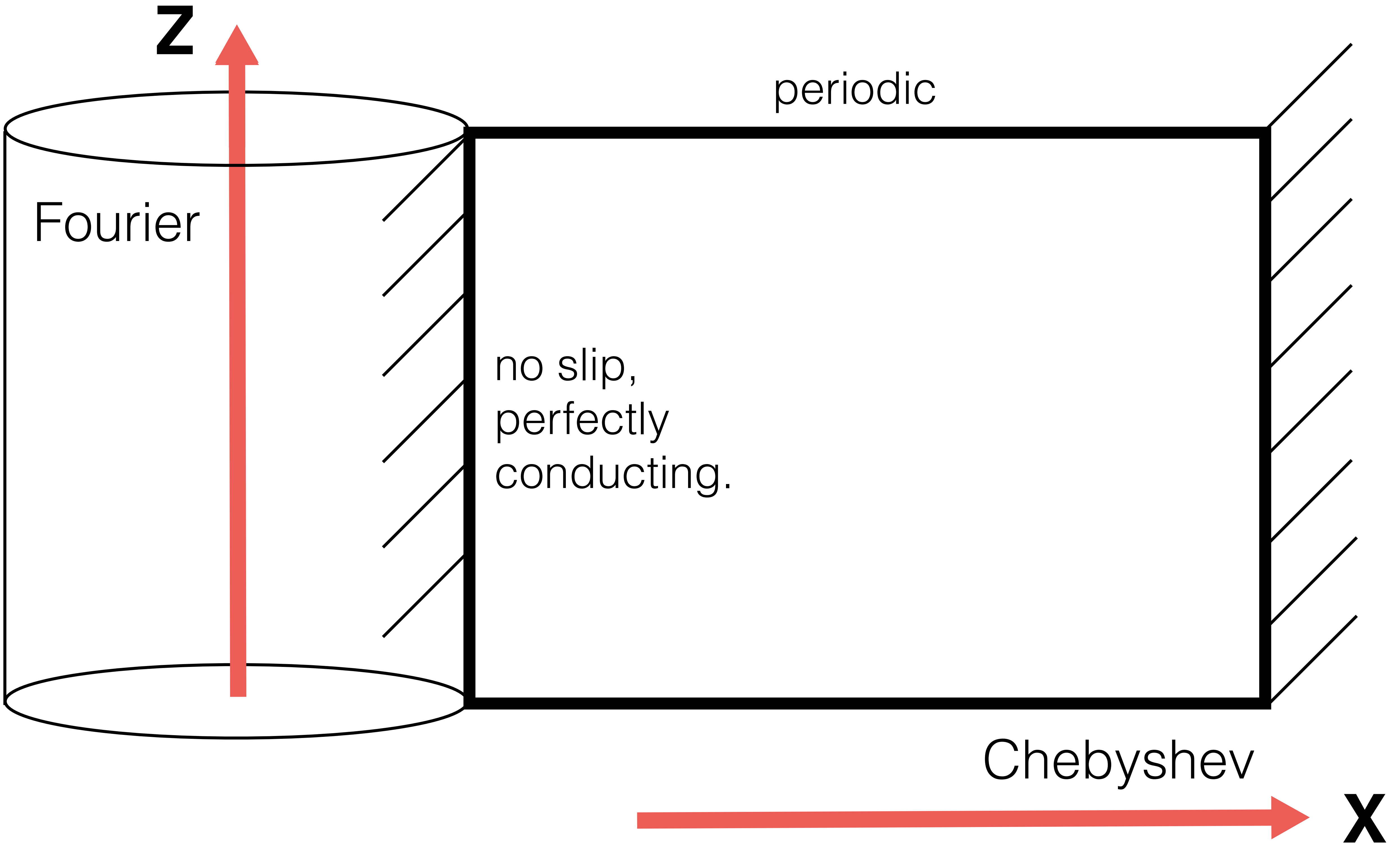
resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J\left(A, u_y\right) - J\left(\Psi, B_y\right)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

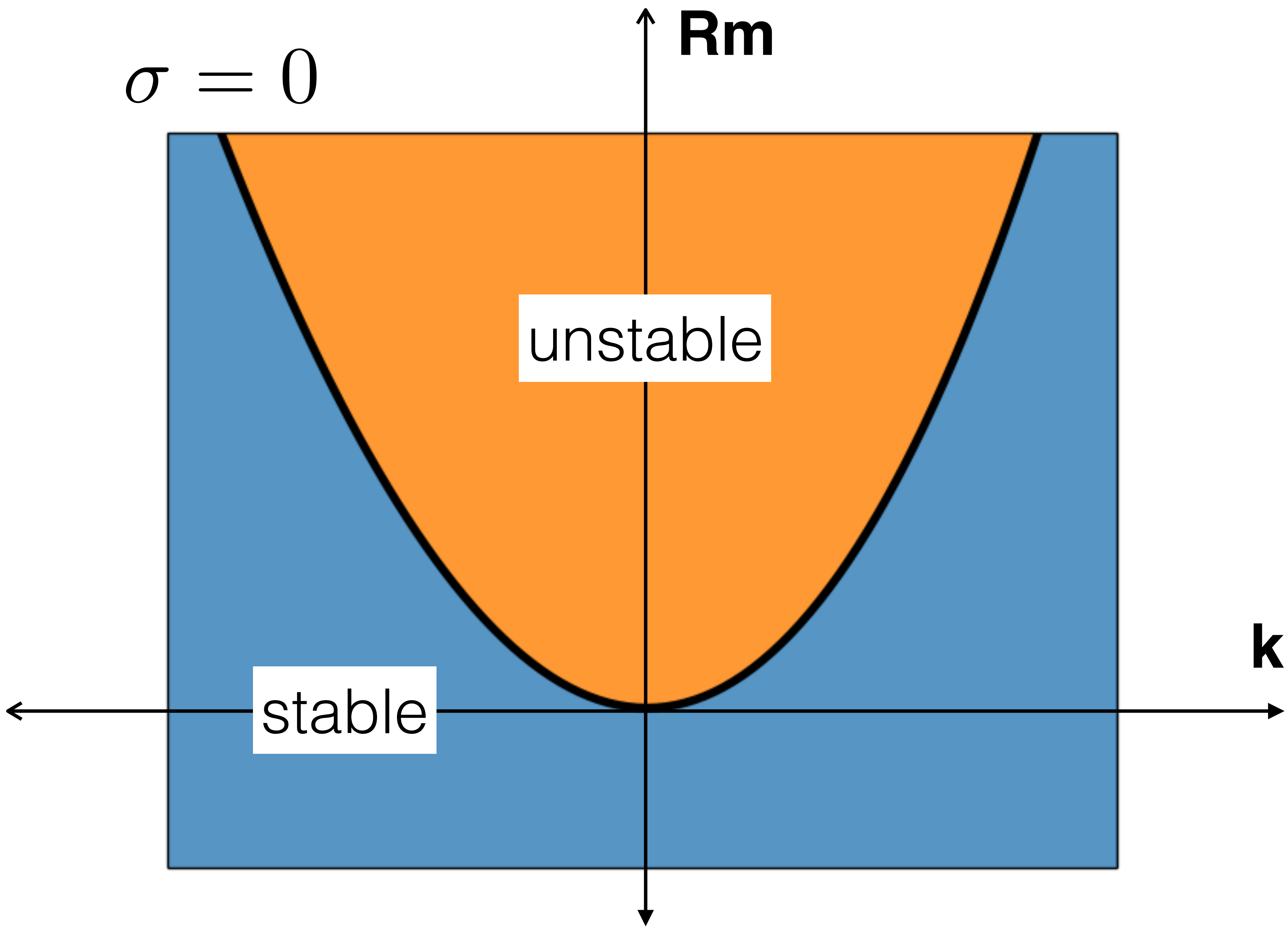
We use experimentally relevant boundary conditions.



Dedalus is a general-purpose spectral code.



Weakly nonlinear analysis explores behavior at the margin of instability.



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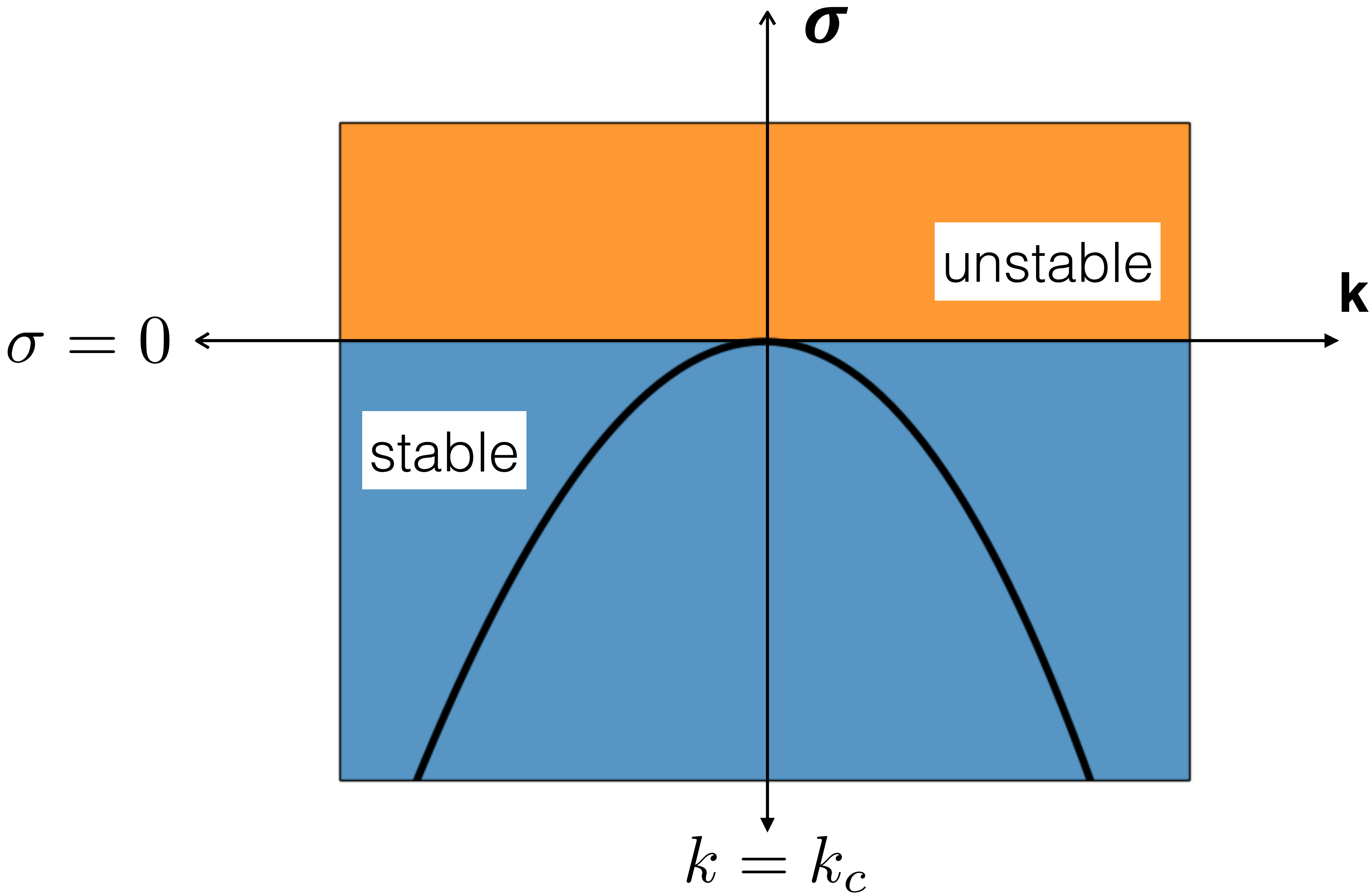

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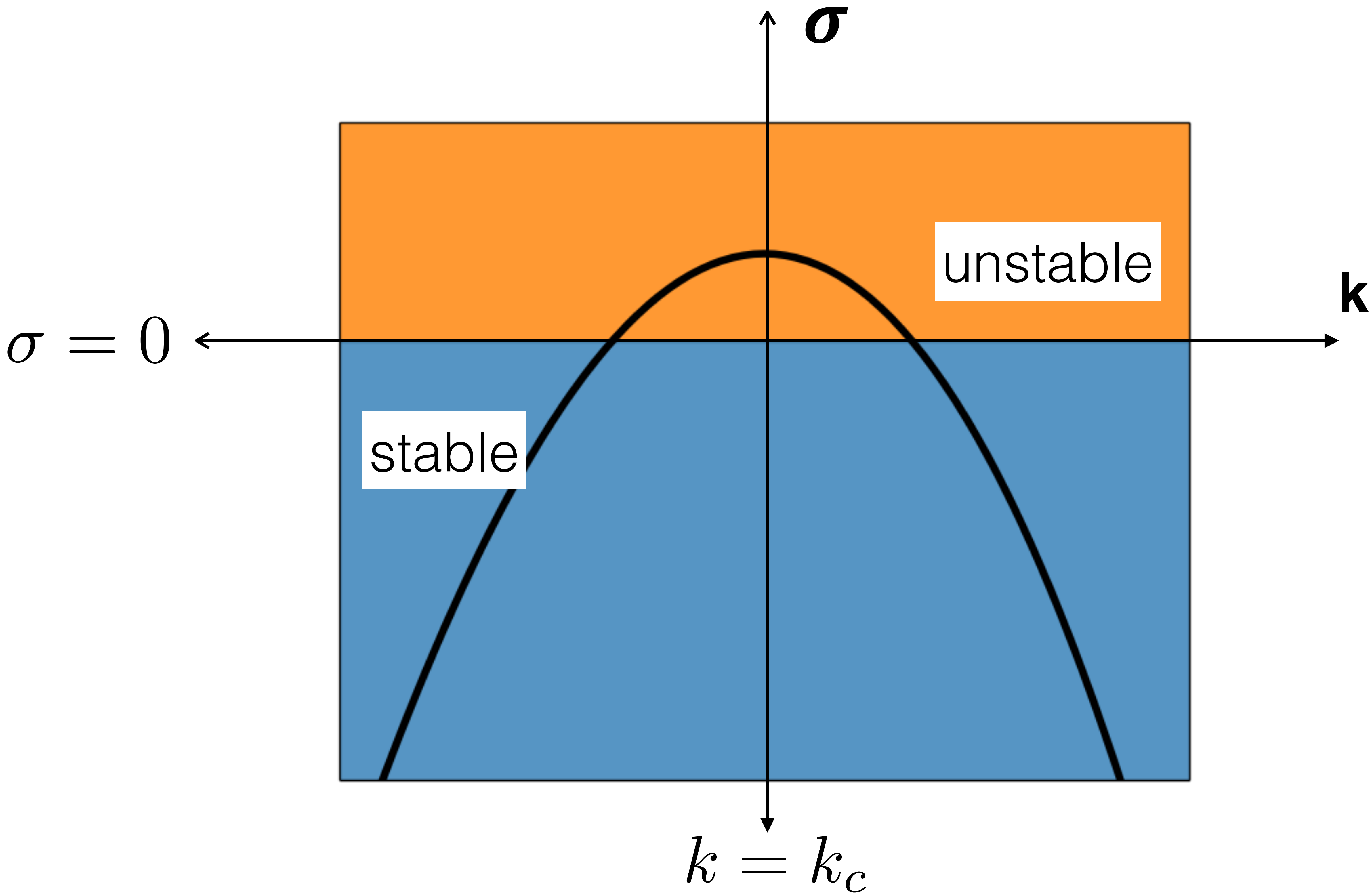

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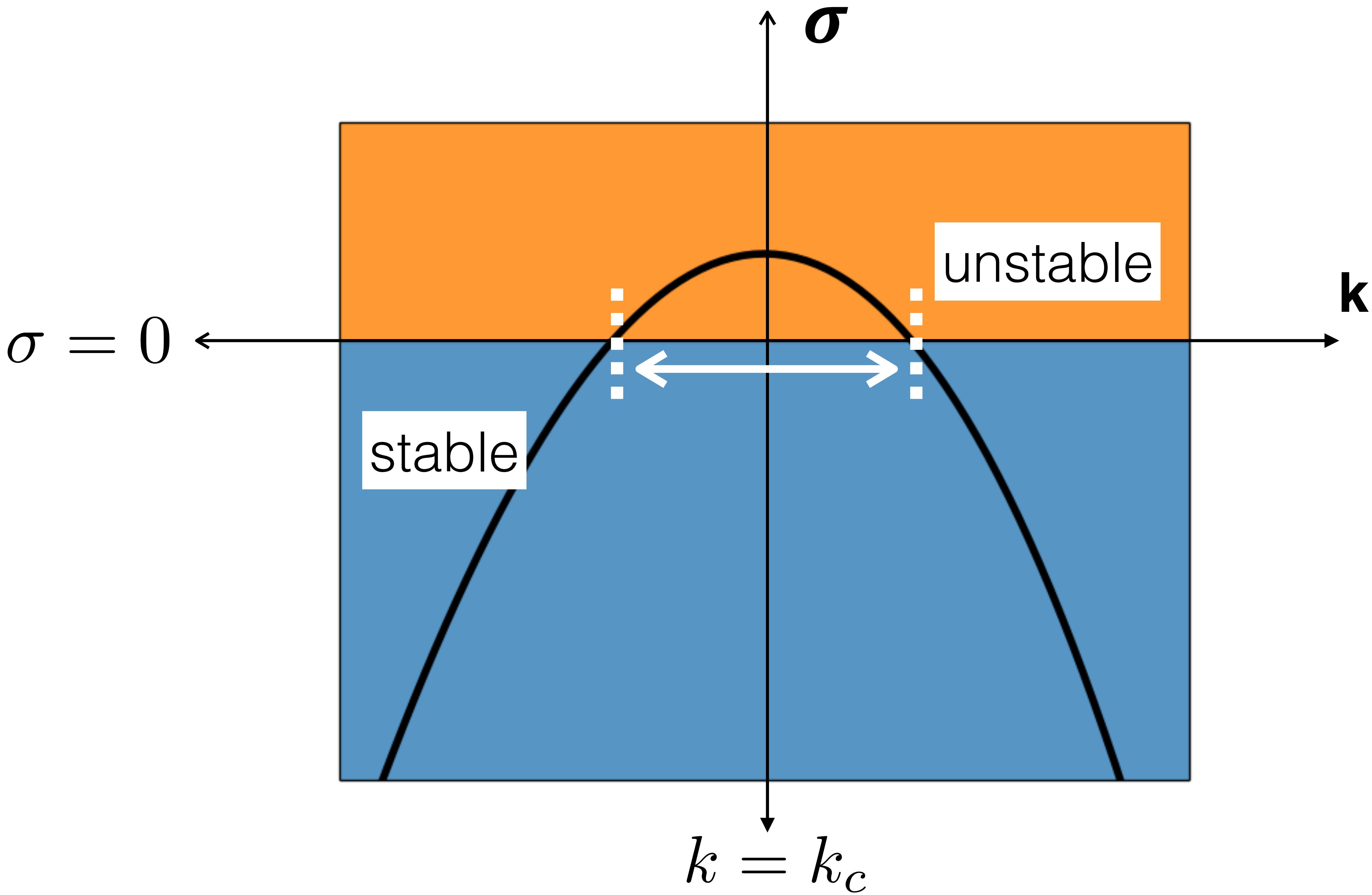
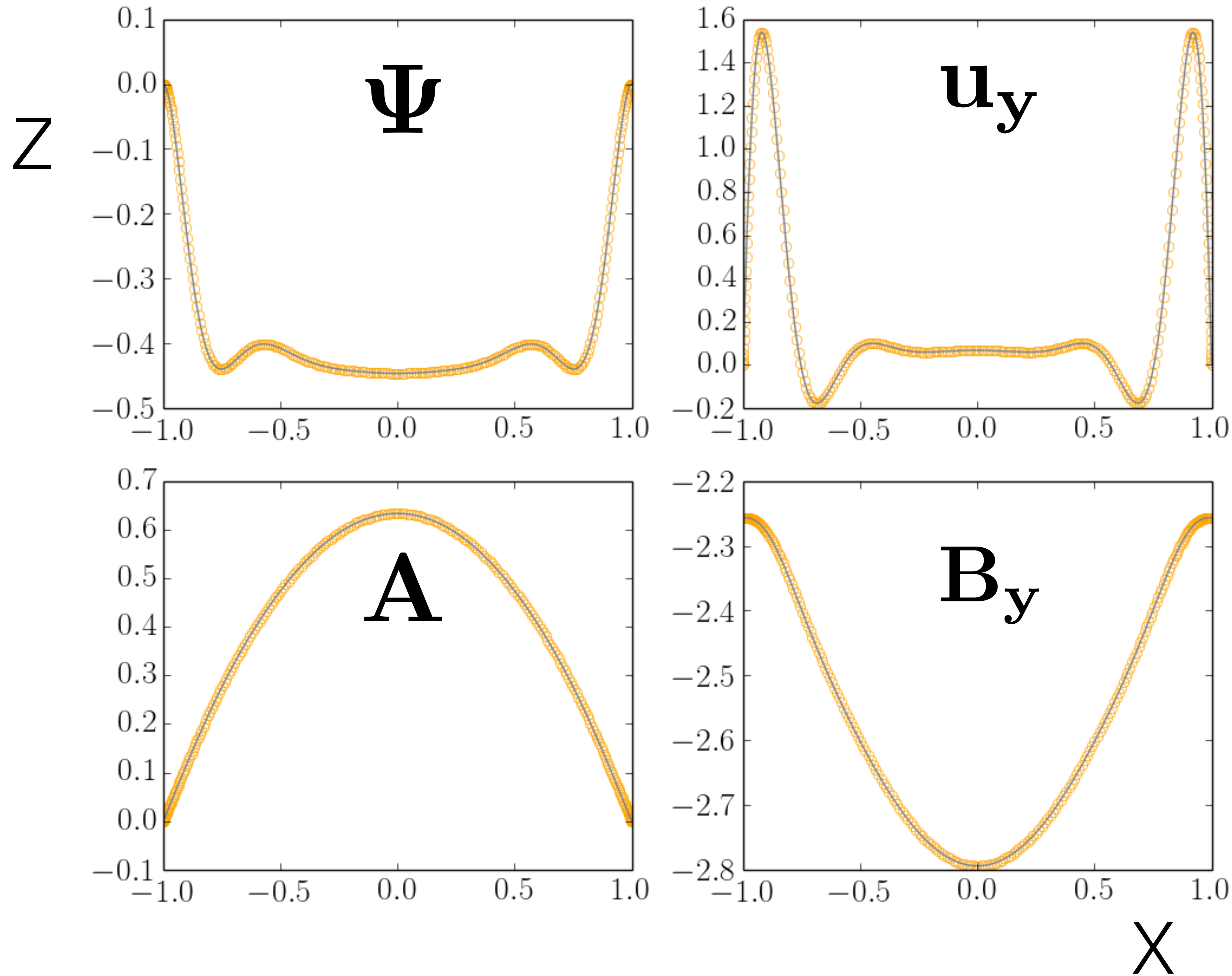


Tune the most unstable mode just over the threshold of instability.

$$\epsilon^2 \equiv 1 - B_0$$

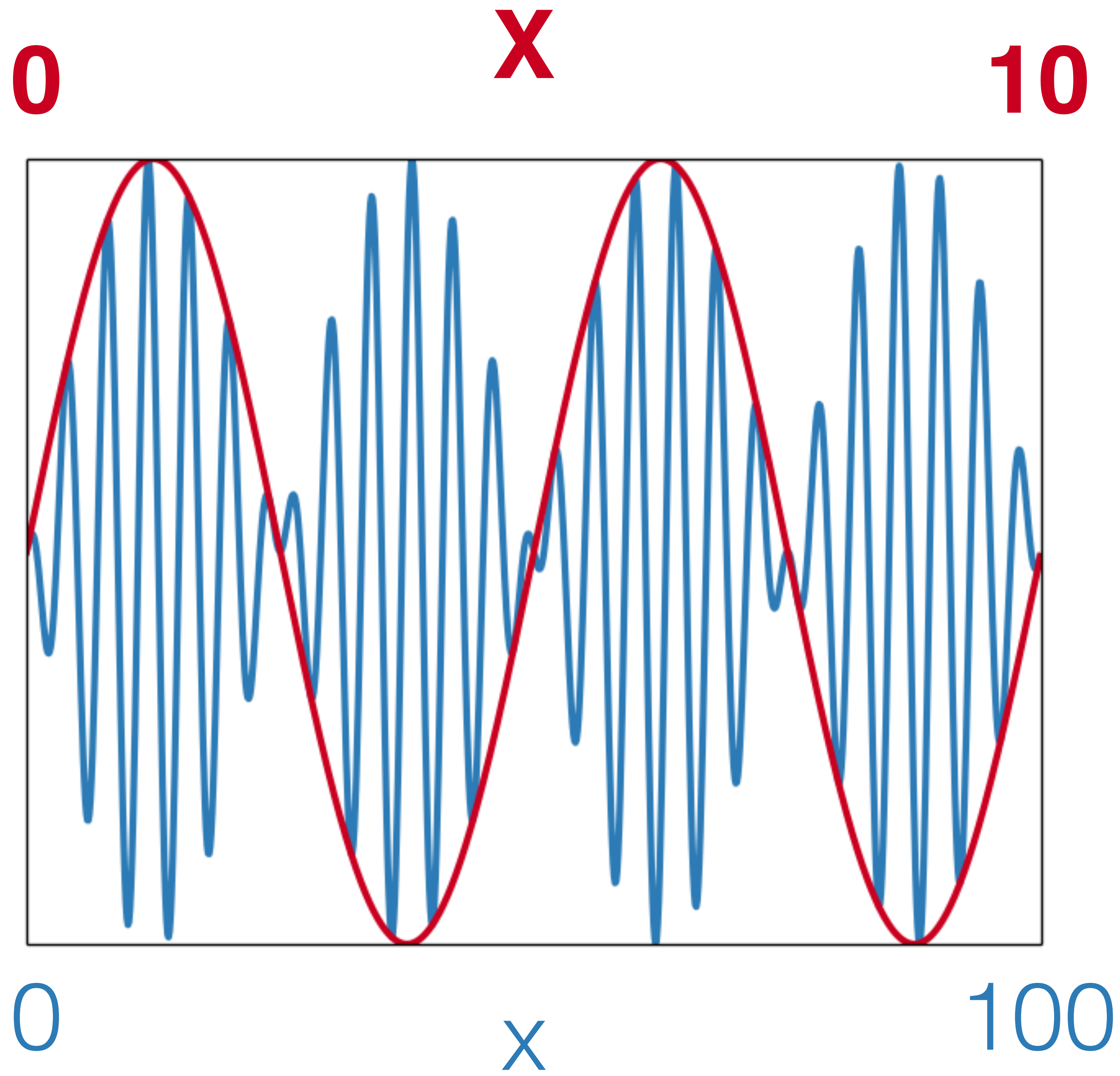
**small parameter**

Identify the most unstable mode of the linear MRI.



Multiscale analysis tracks the evolution of fast and slow variables.

$$X \equiv \epsilon x$$

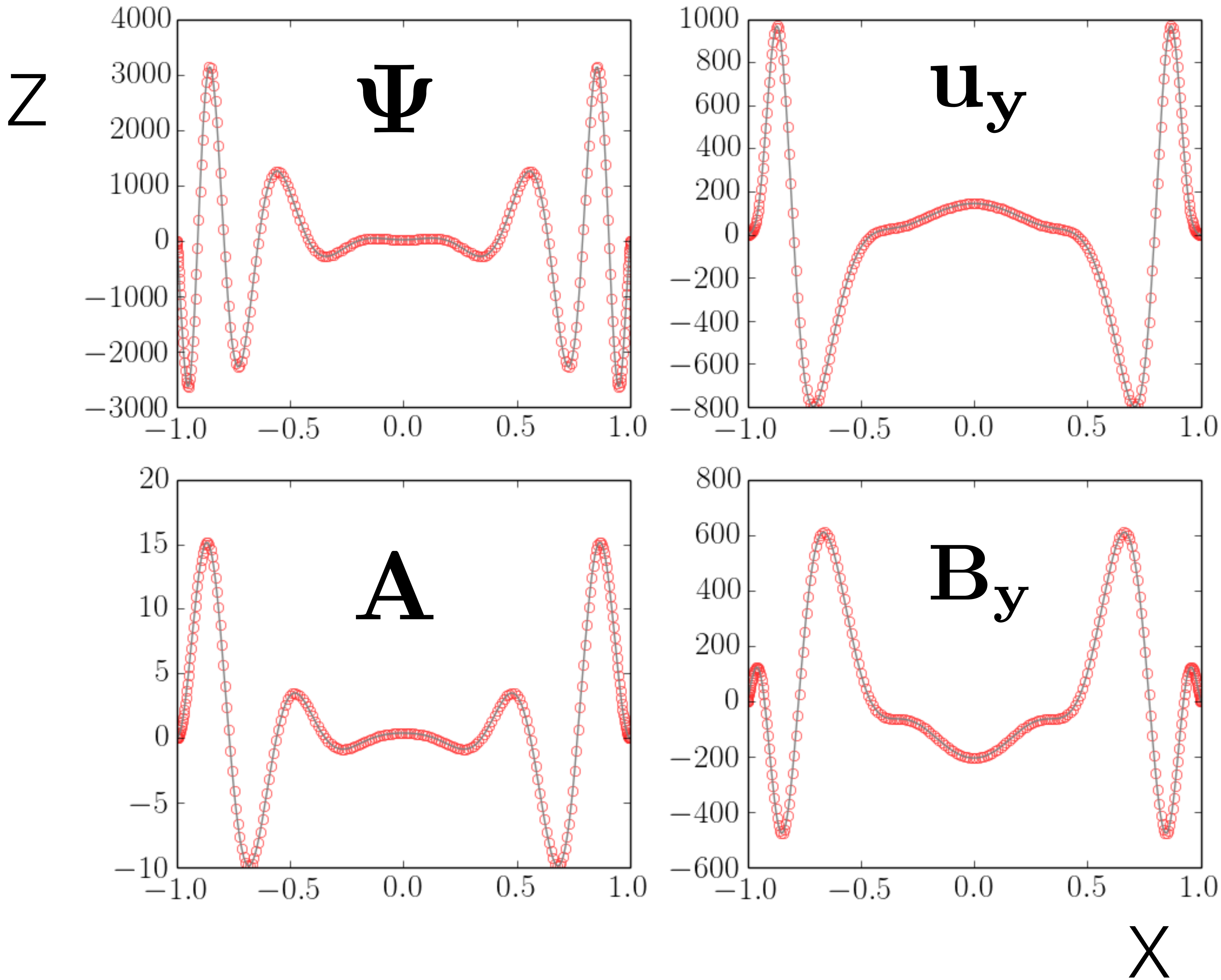


Equations are solved in a matrix formulation.

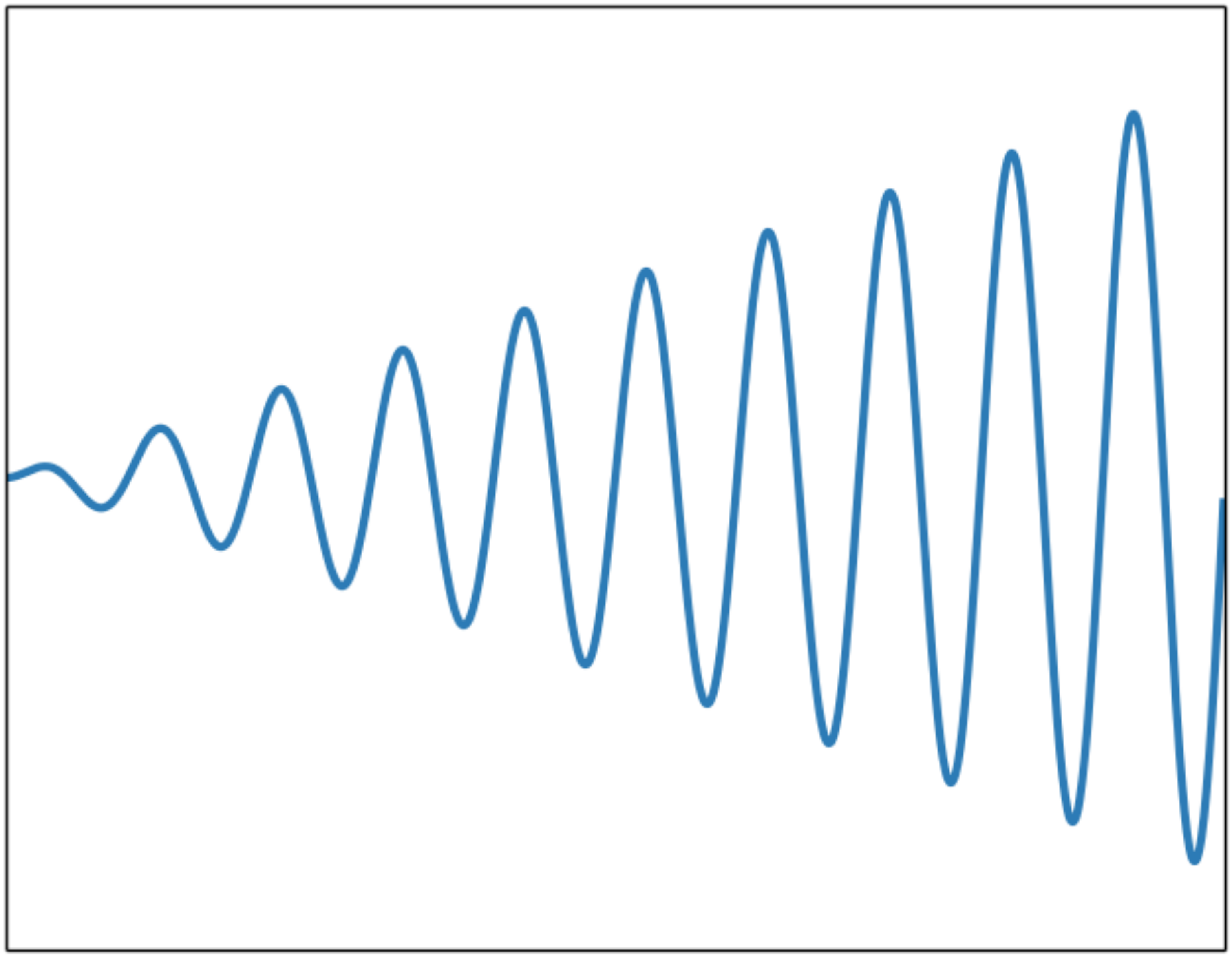
The fluid quantities are expanded
in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

something about boundary layers?



The removal of secular terms yields solvability criteria.



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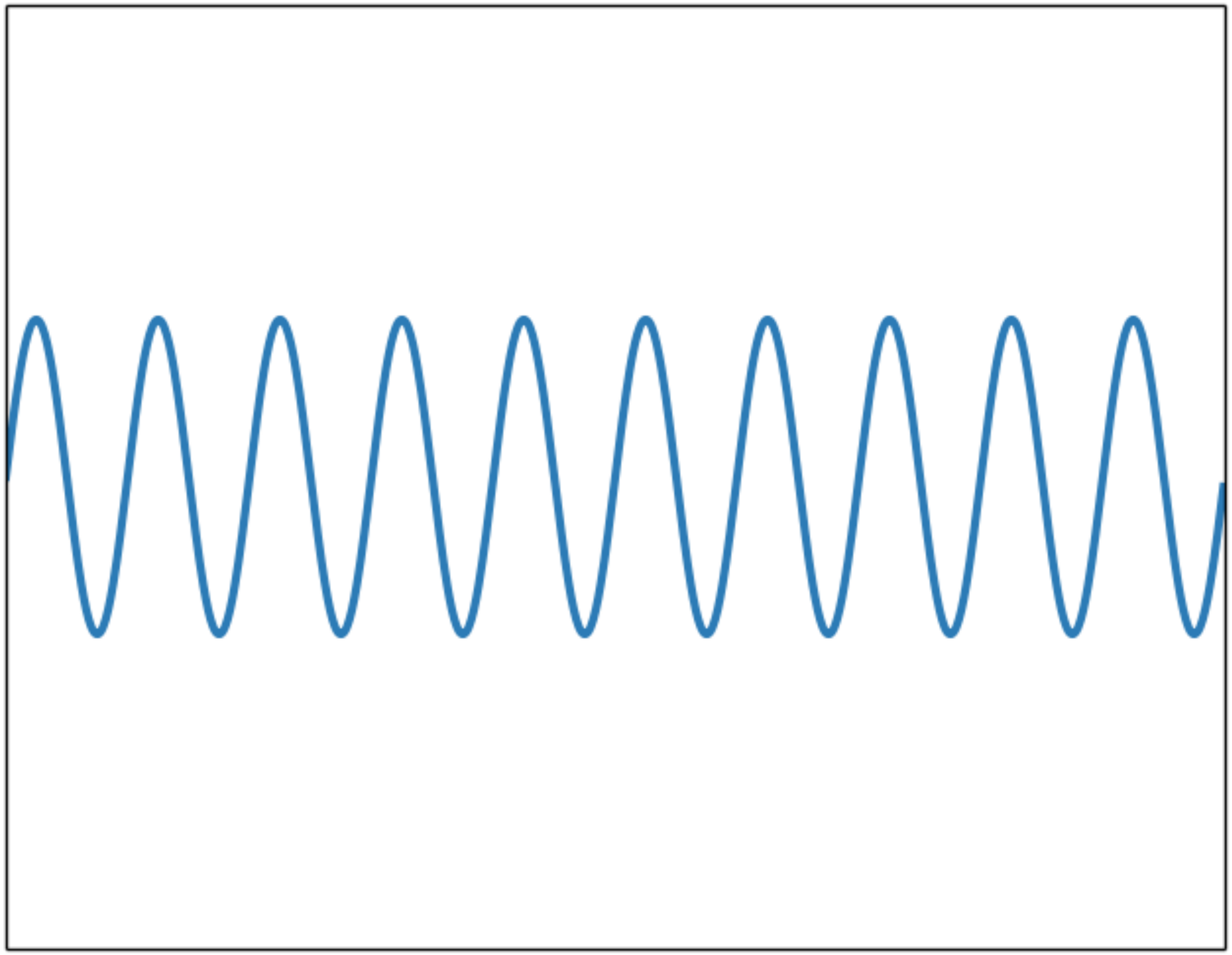


The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + giQ^3 \alpha$$

The result is an amplitude equation for the most unstable mode.

diffusion term

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + giQ^3 \alpha$$

nonlinear term

linear growth

The result is an amplitude equation for the most unstable mode.

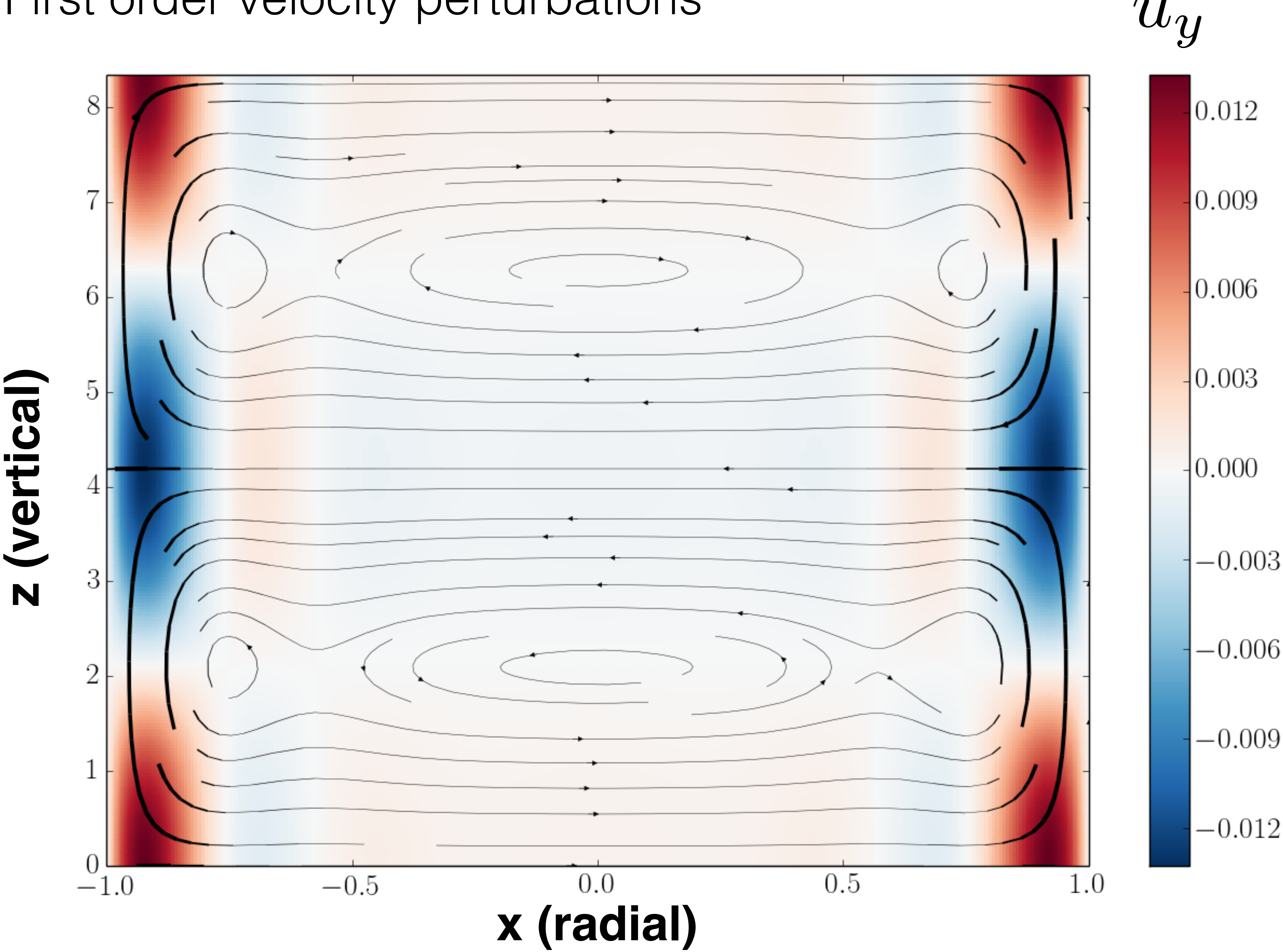
**diffusion term**

**?**

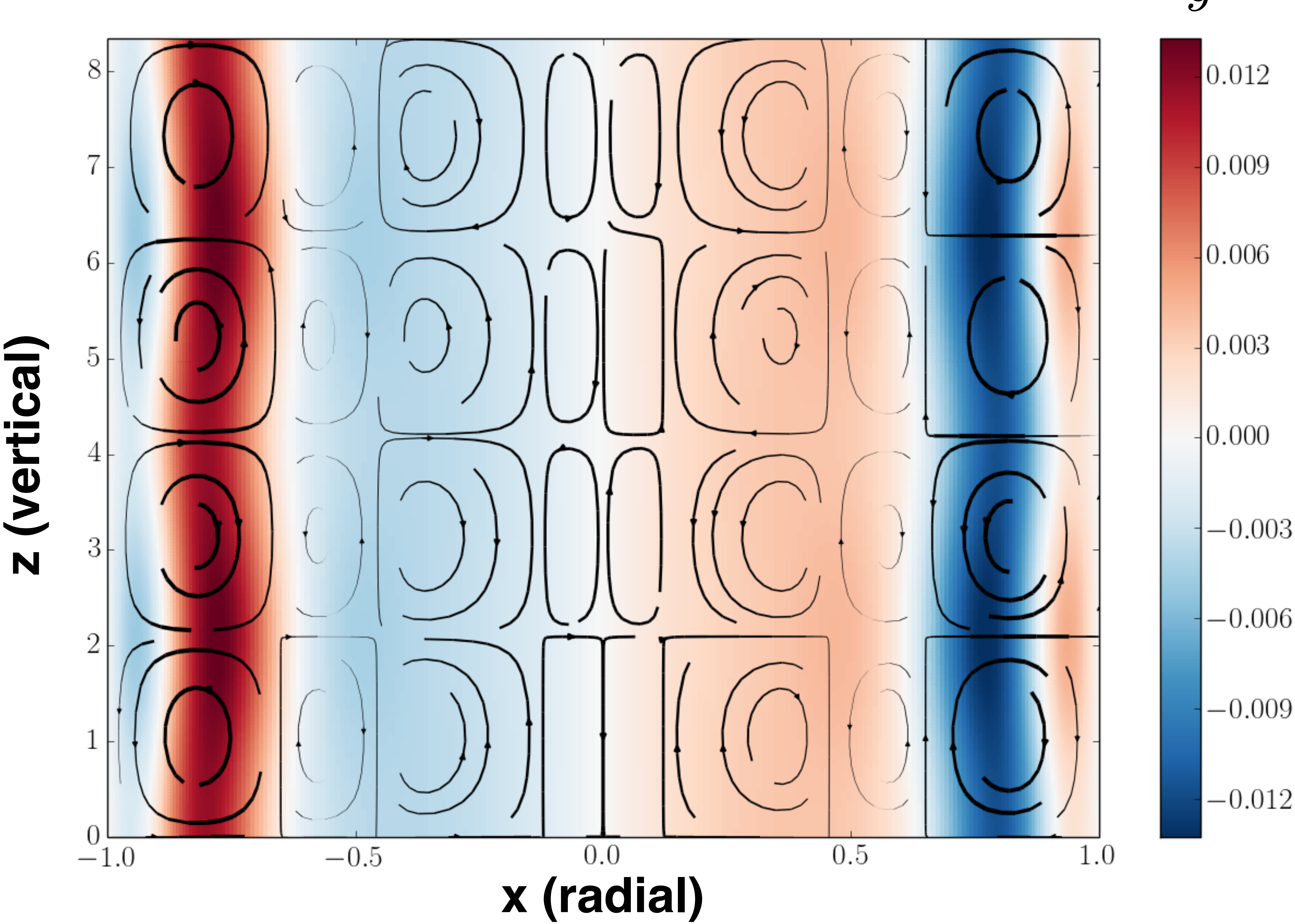
$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha\left|\alpha^2\right| + h\partial_Z^2 \alpha + giQ^3 \alpha$$

**nonlinear term** **linear growth**

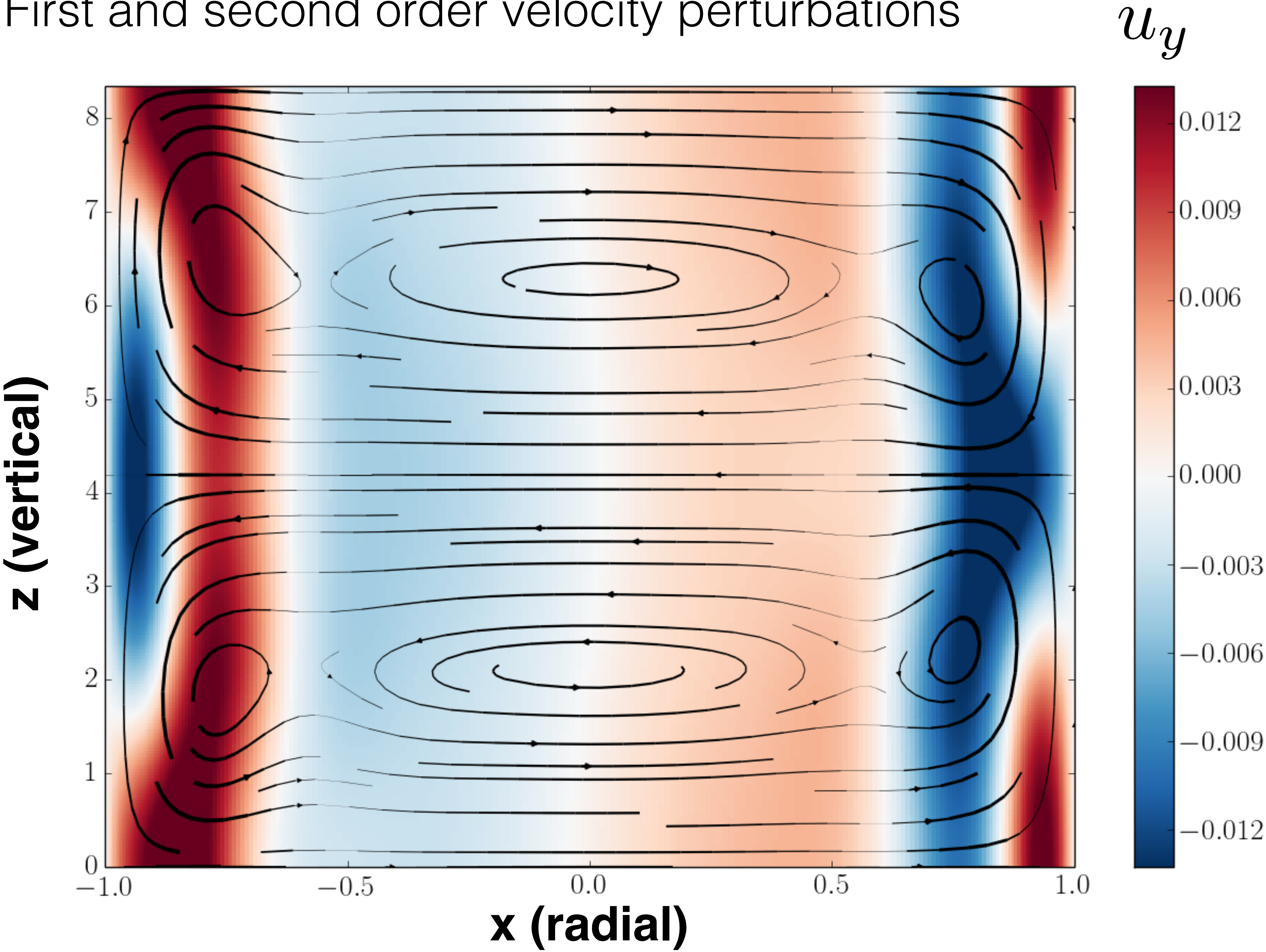
# First order velocity perturbations



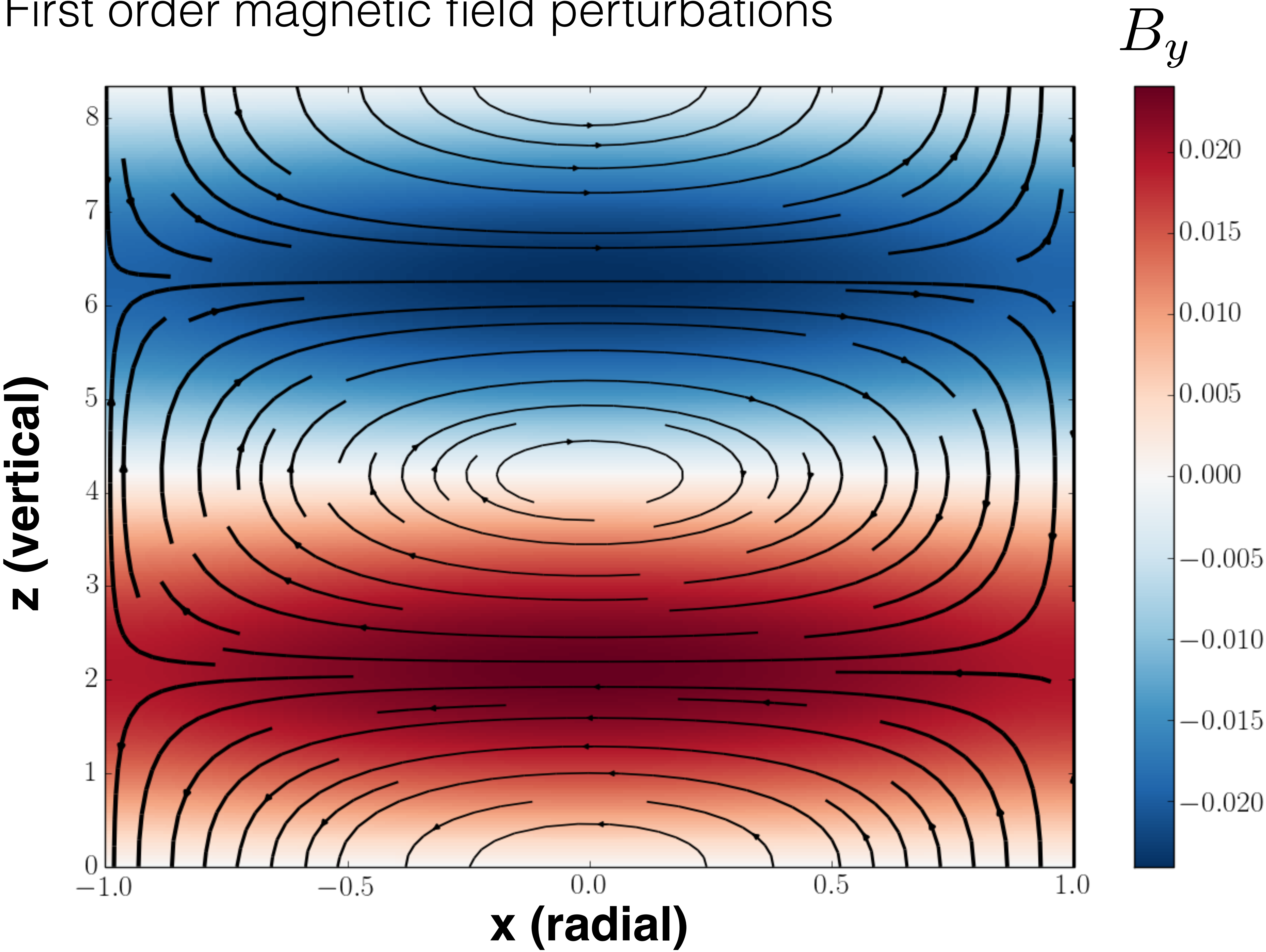
Second order velocity perturbations



# First and second order velocity perturbations

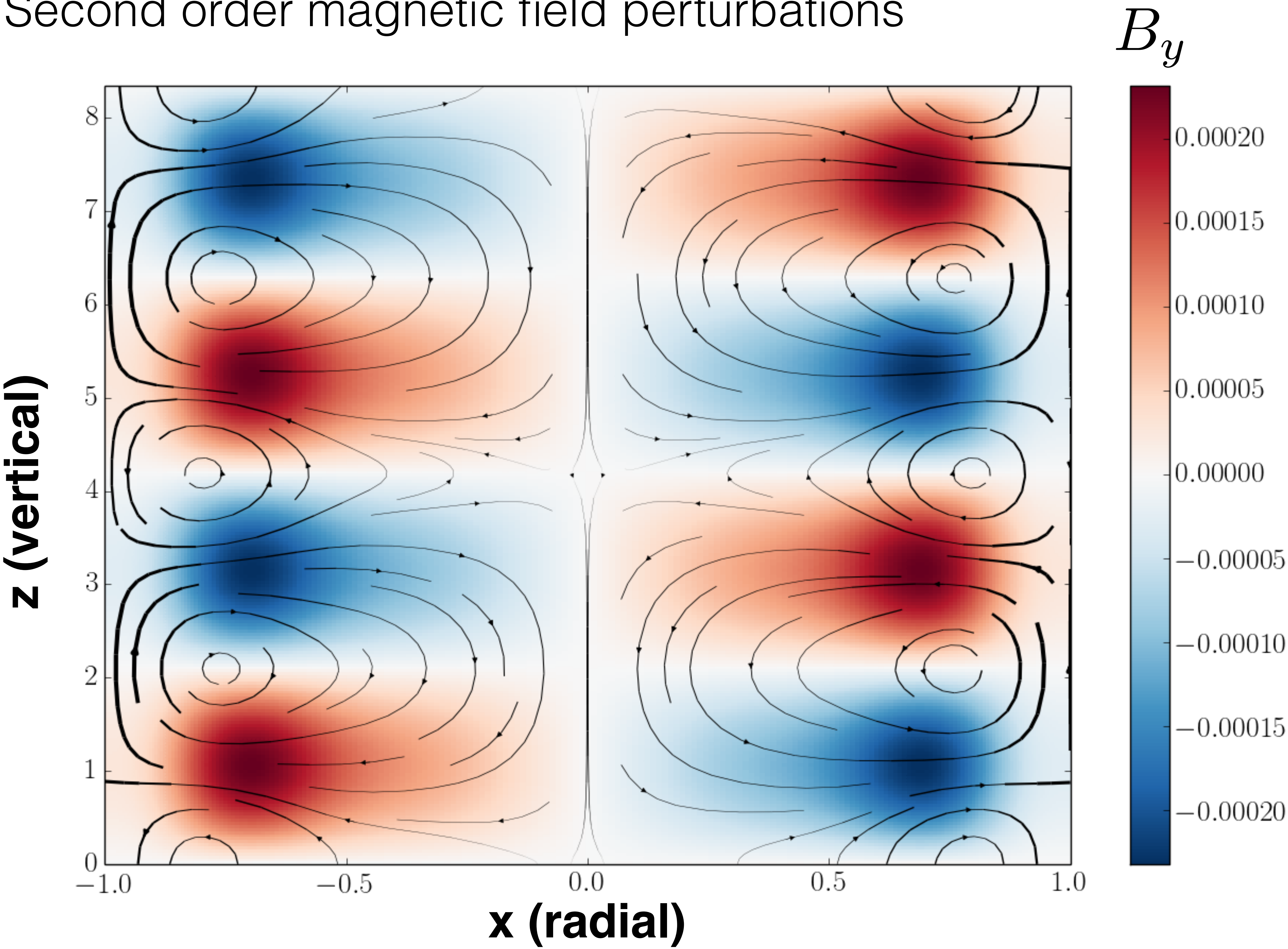


# First order magnetic field perturbations

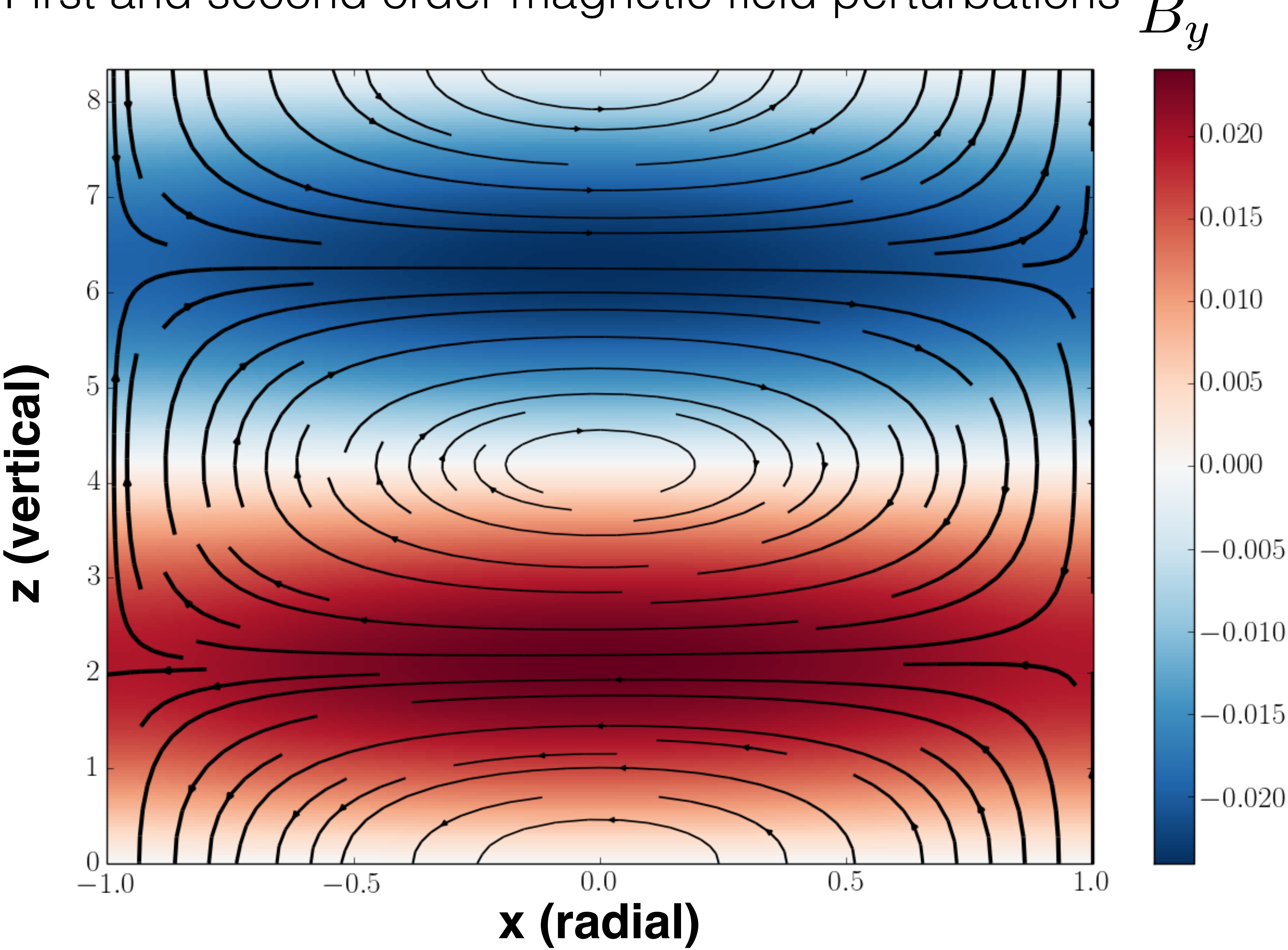


two OOM smaller!

# Second order magnetic field perturbations



First and second order magnetic field perturbations



Future work:

non-thin gap approximation
helical MRI
explore parameter space
comparison to experiment