1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi\ \hat{\mathbf{r}} \\ u_\phi\ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi\ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) &= \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ &+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ &+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \end{aligned} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{3}$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \eta\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{4}$$

Susan gets:

$$\partial_t A = \frac{1}{r}J(A, \Psi) + \frac{1}{Rm}\left[-\frac{1}{r}\partial_r A + \partial_r^2 A + \partial_z^2 A\right] \tag{5}$$

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi) \\ + \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \end{aligned} \tag{6}$$

2 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find x and z components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\text{Re}}\left[\nabla^2 u\right]_z \tag{7}$$

Which becomes

$$\frac{1}{r^2}\partial_t\partial_r\Psi \tag{8}$$

3 Recovery of Narrow Gap Equations

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{9}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{10}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{11}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{12}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \tag{13}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \tag{14}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{15}$$

where ∇ on the vector components is given by equation (8).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$