Exploring the saturation of the MRI via weakly nonlinear analysis

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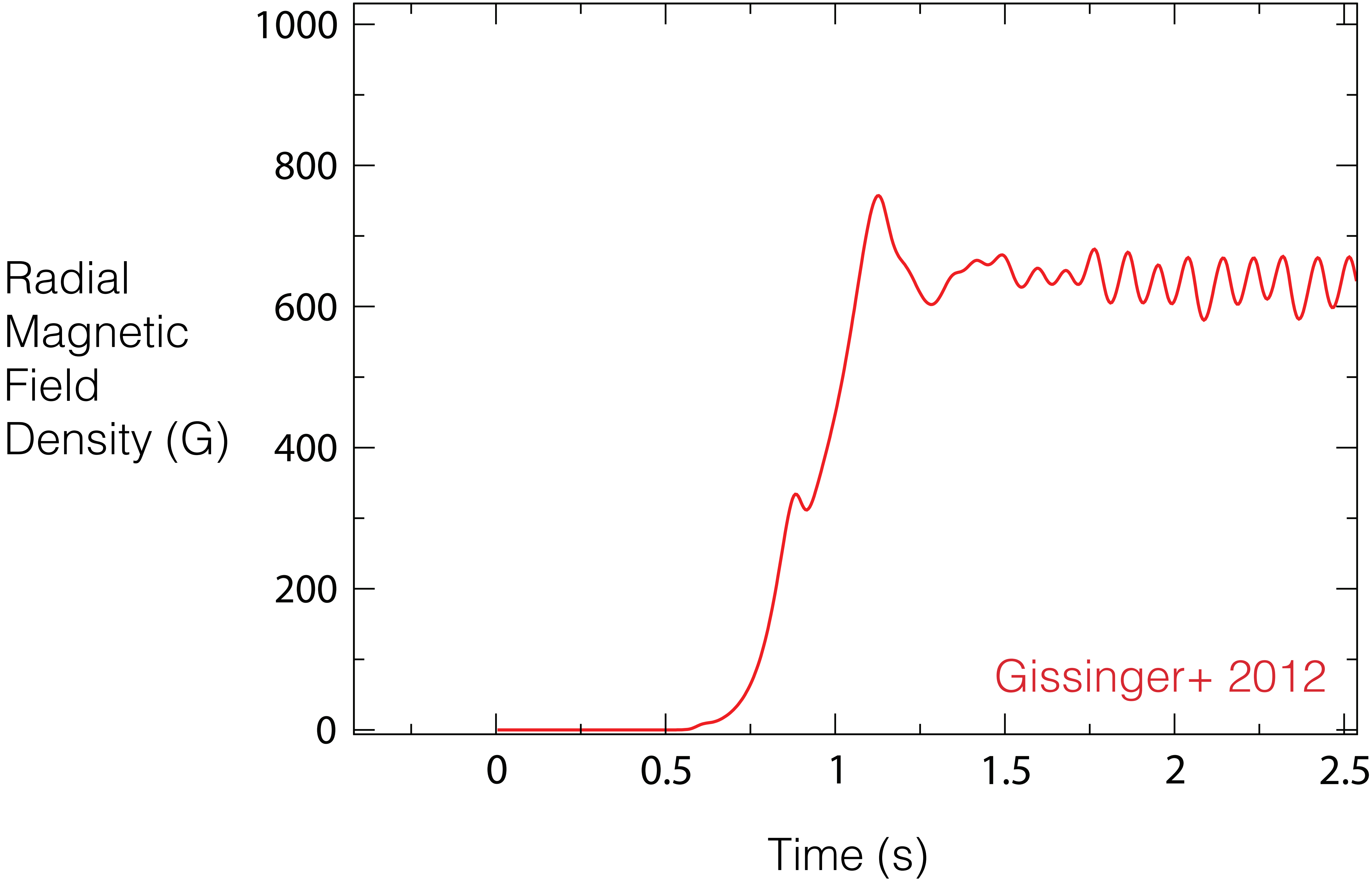
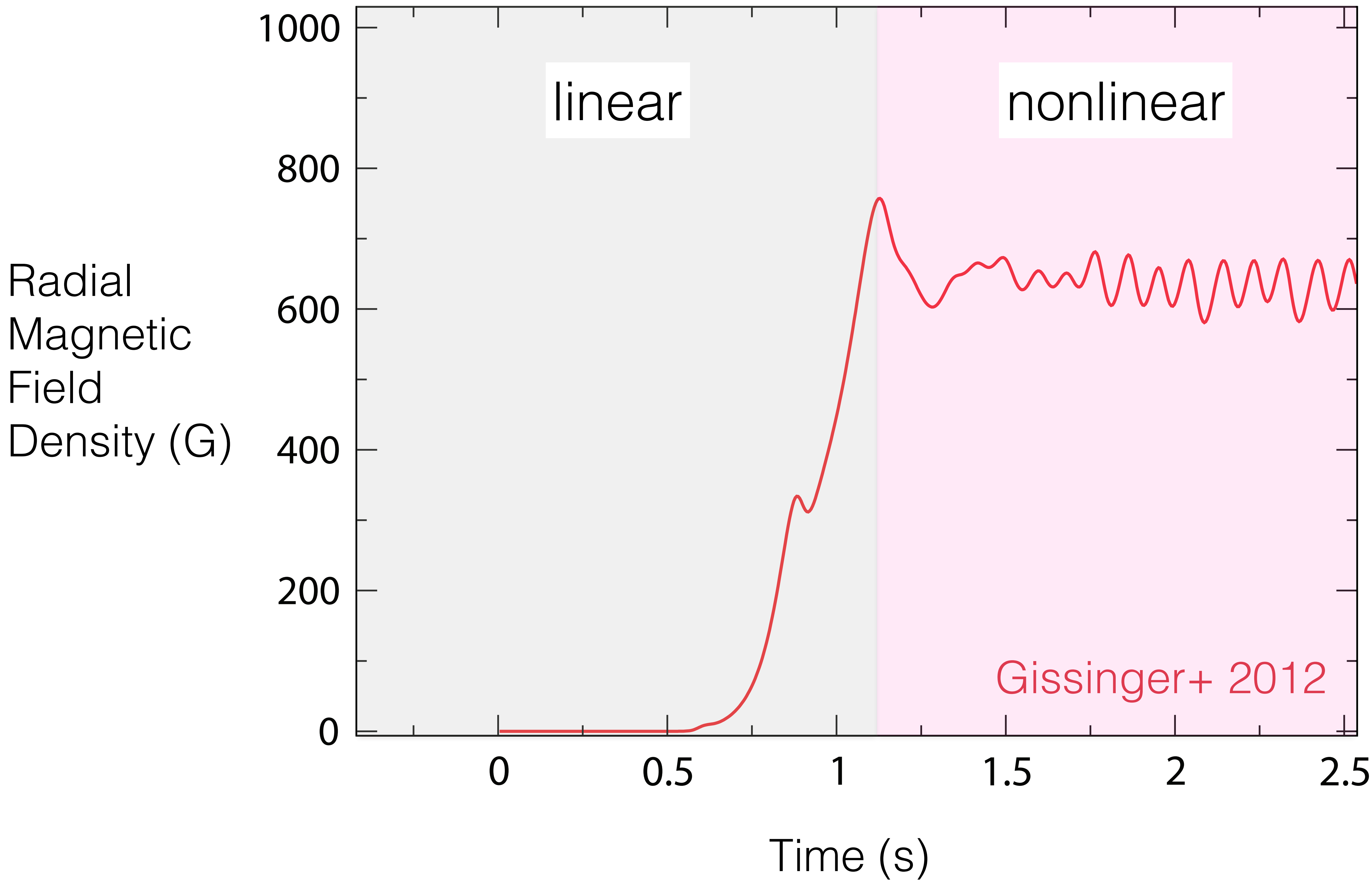
MRI saturation is well-studied in simulation.



The MRI evolves in stages.



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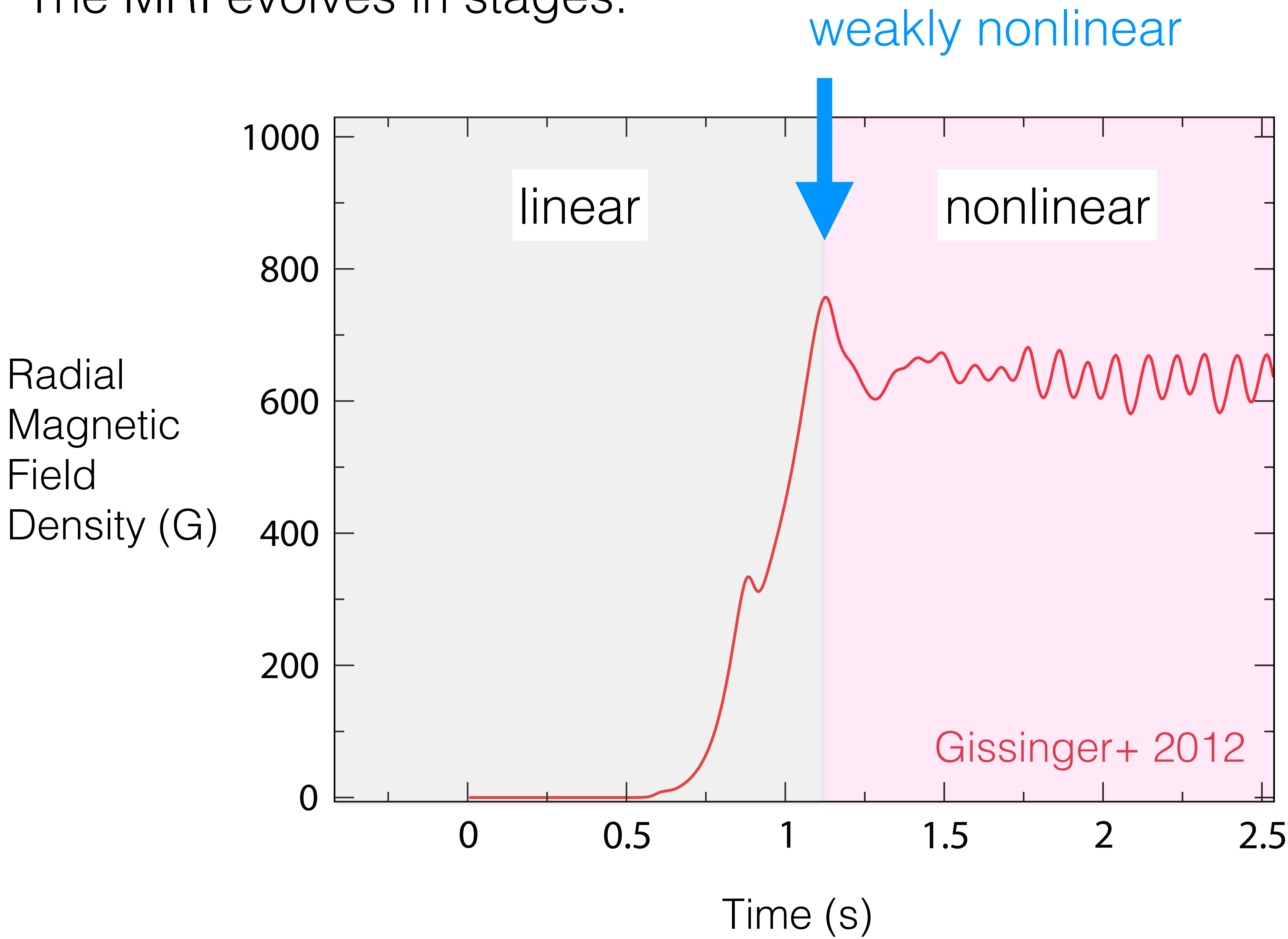

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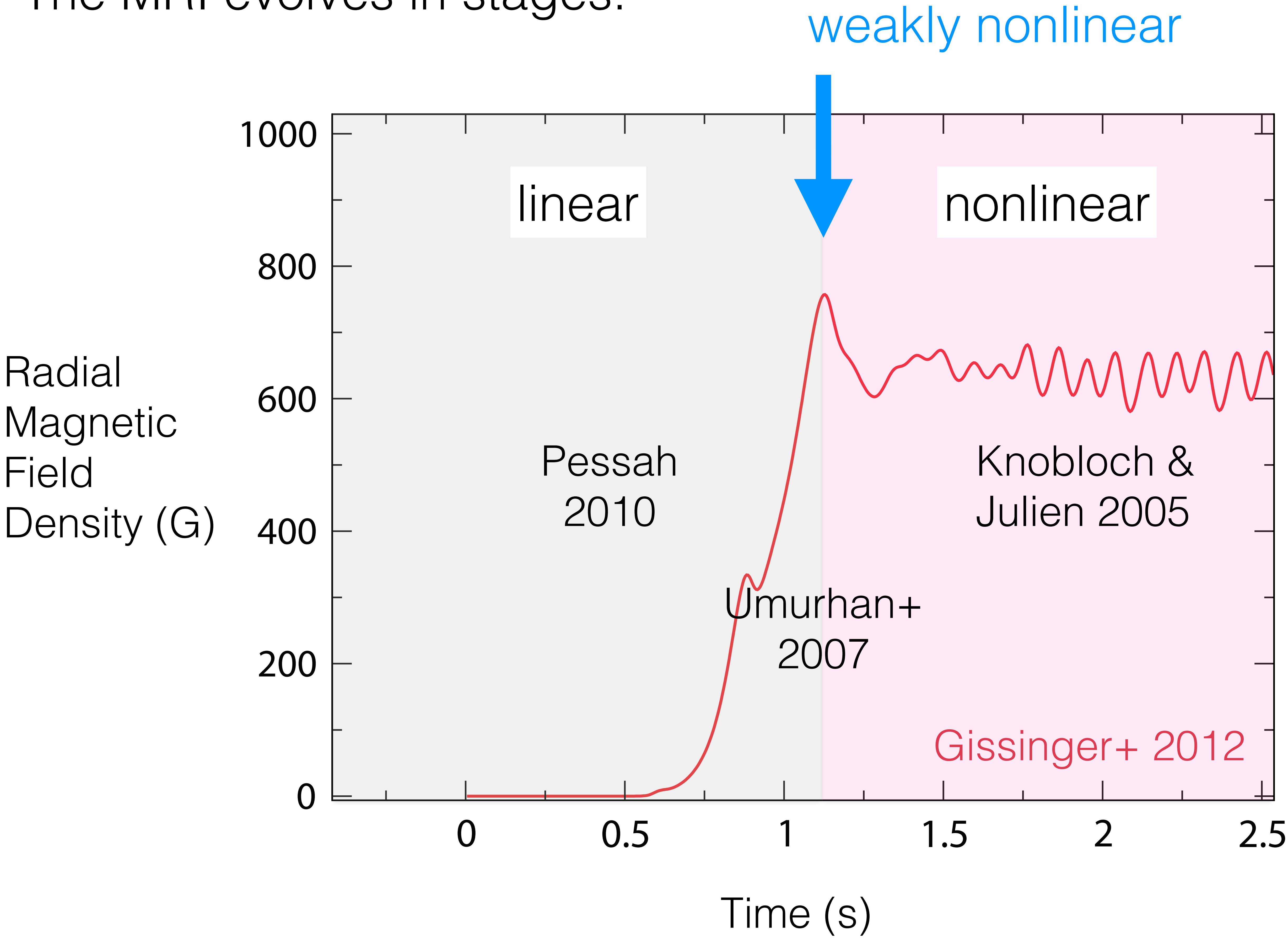
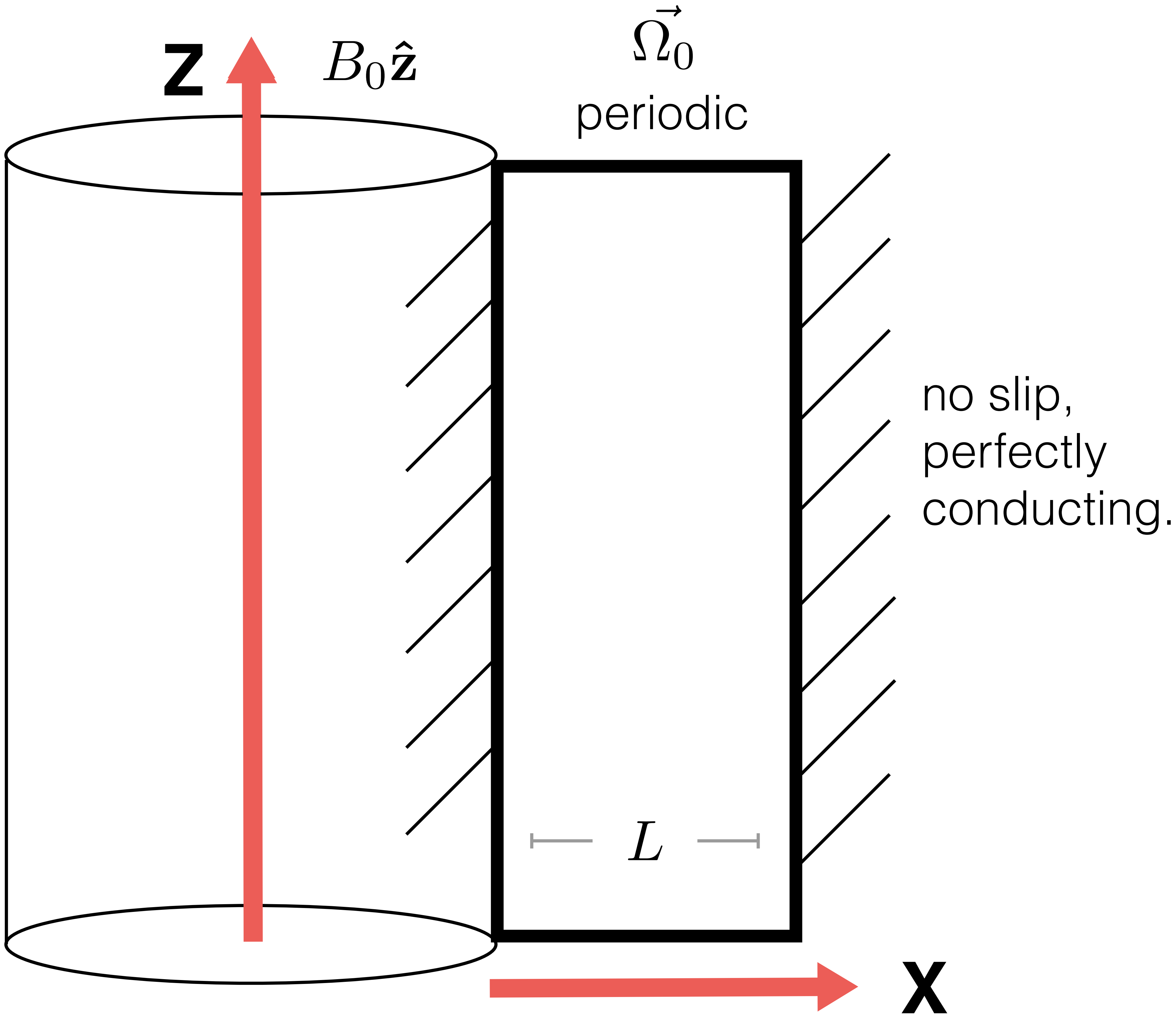


We use a thin-gap Taylor Couette setup.



We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

kinematic viscosity

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

magnetic resistivity

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0}\right)^{-q}$$ shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$ background field

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$ Reynolds number

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$ magnetic Reynolds number

$$\beta \equiv \frac{8\pi \rho_0 \Omega_0^2 L^2}{B_0^2}$$ plasma beta

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{u_y} \\ \mathbf{A} \\ \mathbf{B_y} \end{bmatrix}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \frac{1}{Re} \nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re}\nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re}\nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm}\nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm}\nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re}\nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re}\nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm}\nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm}\nabla^2 B_y}$$

We work in terms of flux and stream functions.

$$J\left(f,g\right) = \partial_z f \partial_x g - \partial_x f \partial_z g$$

momentum

nonlinear

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \boxed{\frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

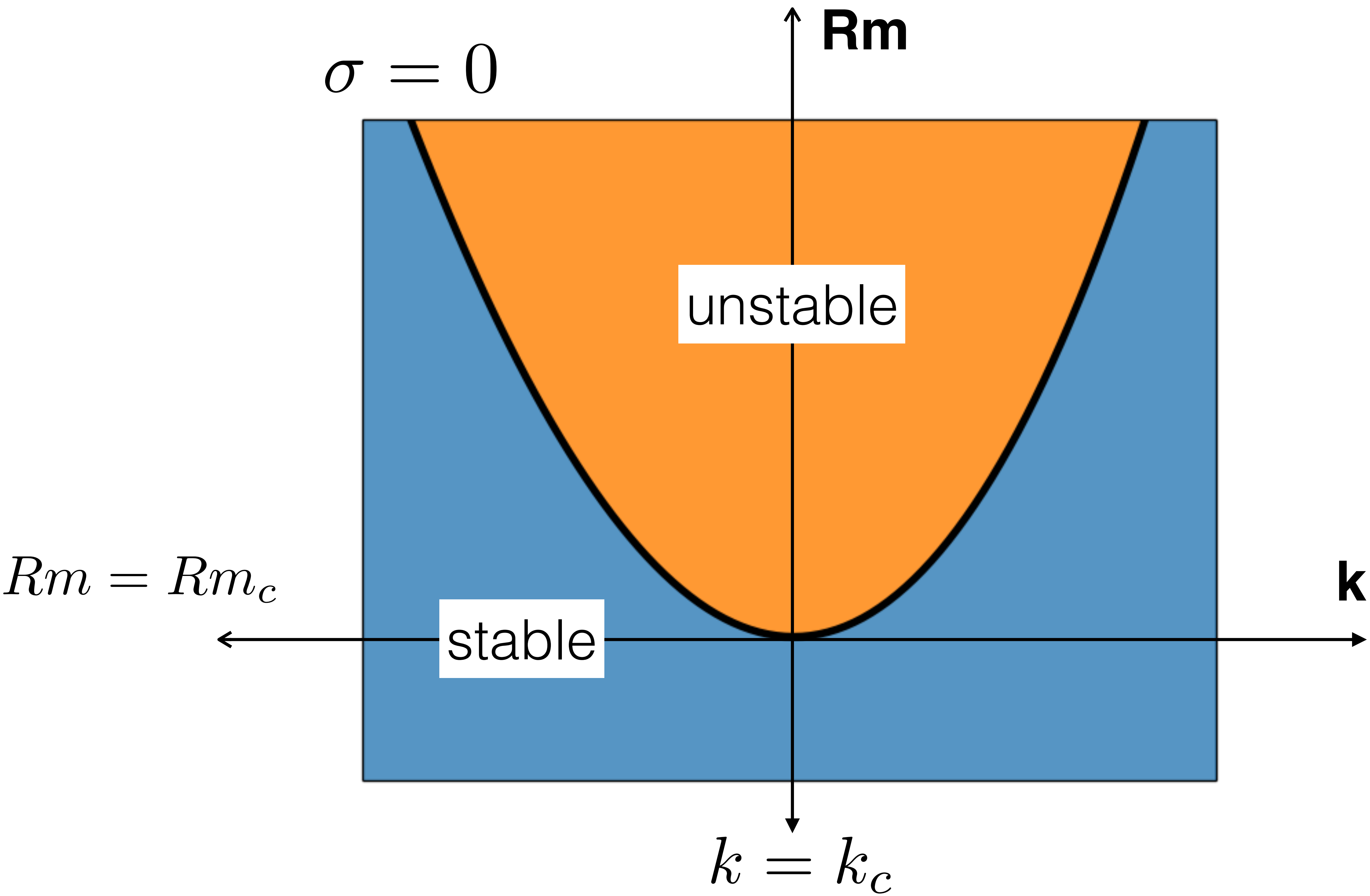
$$\partial_t A = B_0 \partial_z \Psi + \boxed{J\left(A, \Psi\right)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J\left(A, u_y\right) - J\left(\Psi, B_y\right)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

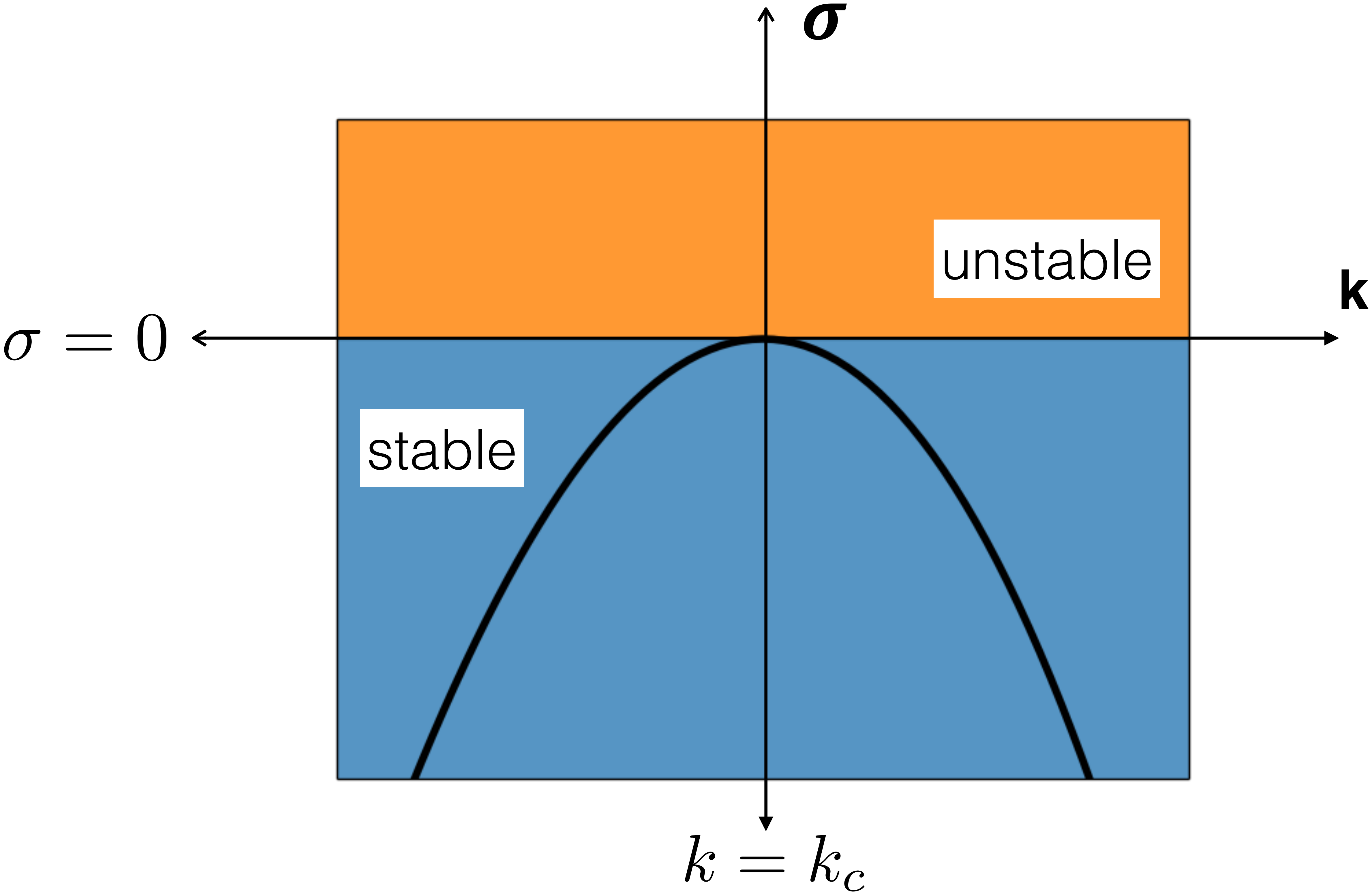
Weakly nonlinear analysis explores behavior at the margin of instability.

$$\boxed{e^{ikz+\sigma t}}$$

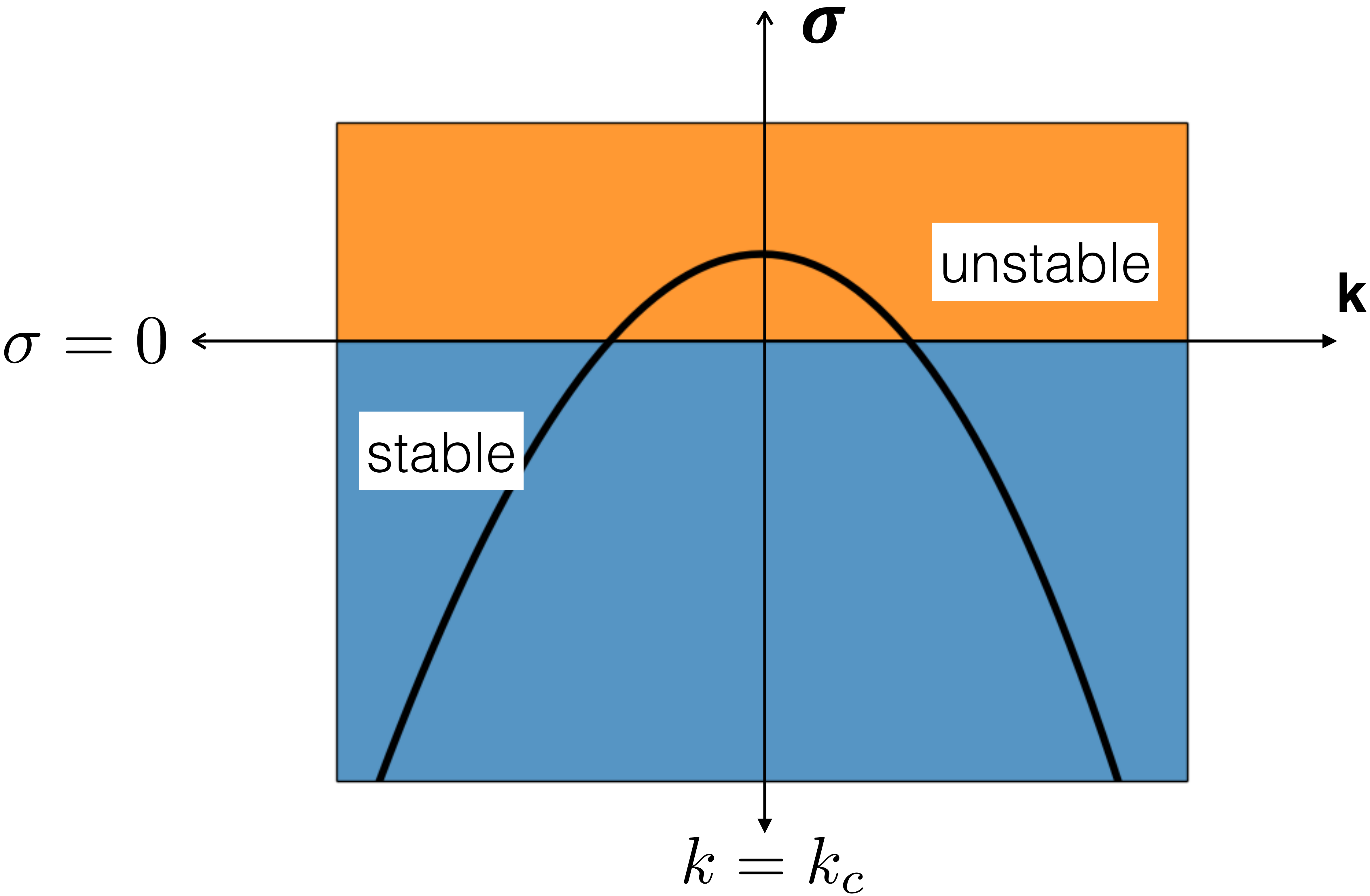


Weakly nonlinear analysis explores behavior at the margin of instability.

$Fixed\ Rm$



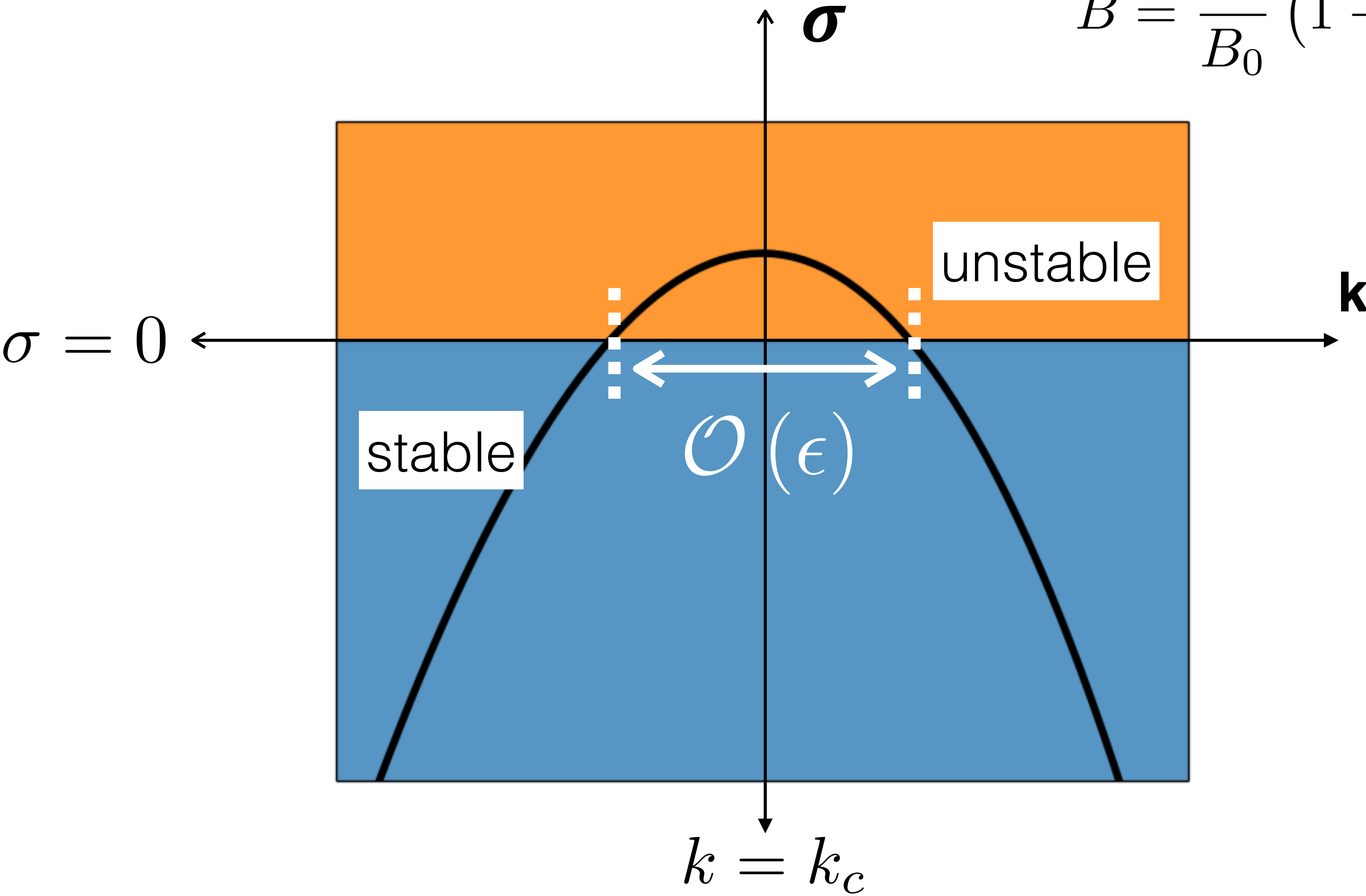
Tune the most unstable mode just over the threshold of instability.



Tune the most unstable mode just over the threshold of instability.

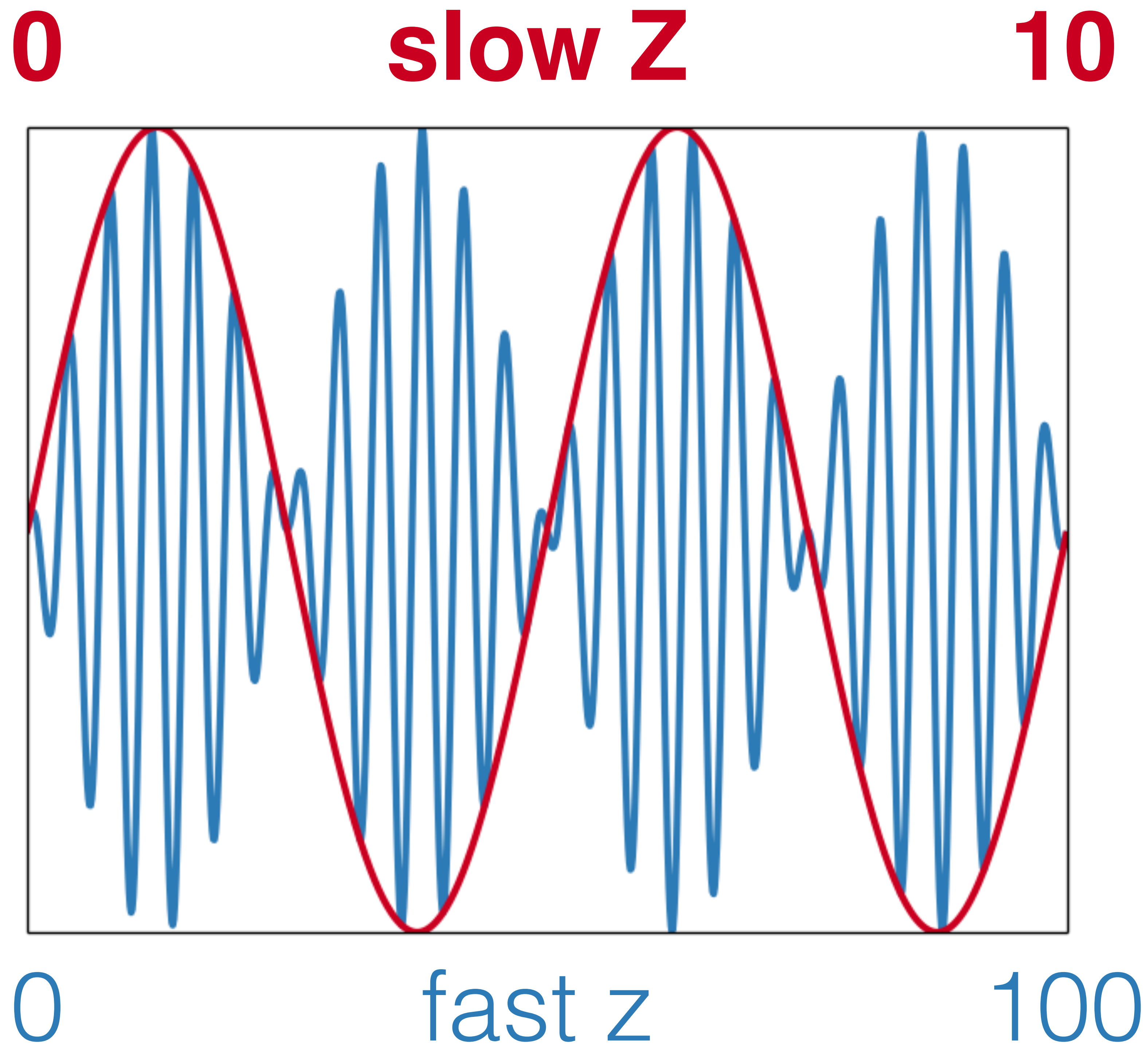
small parameter

$$B = \frac{B}{B_0}\left(1 - \epsilon^2\right)$$



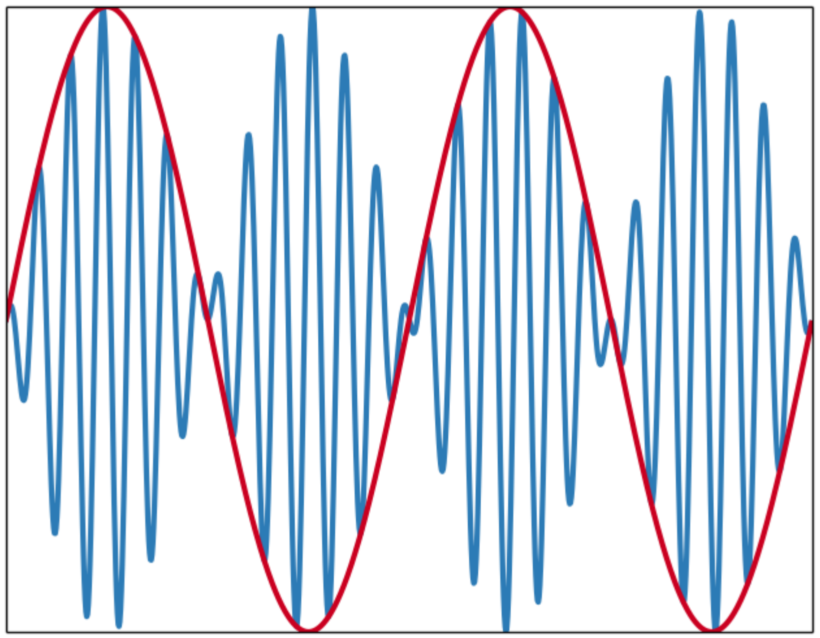
Multiscale analysis tracks the evolution of fast and slow variables.

$$Z \equiv \epsilon z$$



We choose an ansatz state vector form.

$$\mathbf{V} = \alpha\left(Z, T\right) \mathrm{V}\left(x\right) e^{ik_c z}$$



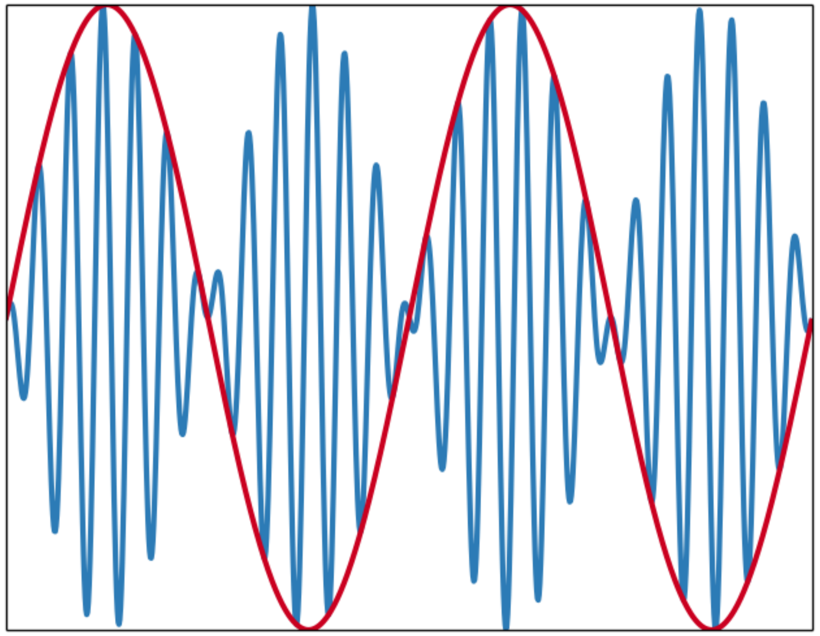
We choose an ansatz state vector form.

x dependence

$$\mathbf{V} = \alpha\left(Z, T\right)\mathrm{V}\left(x\right)e^{ik_c z}$$

vertical periodicity

amplitude function

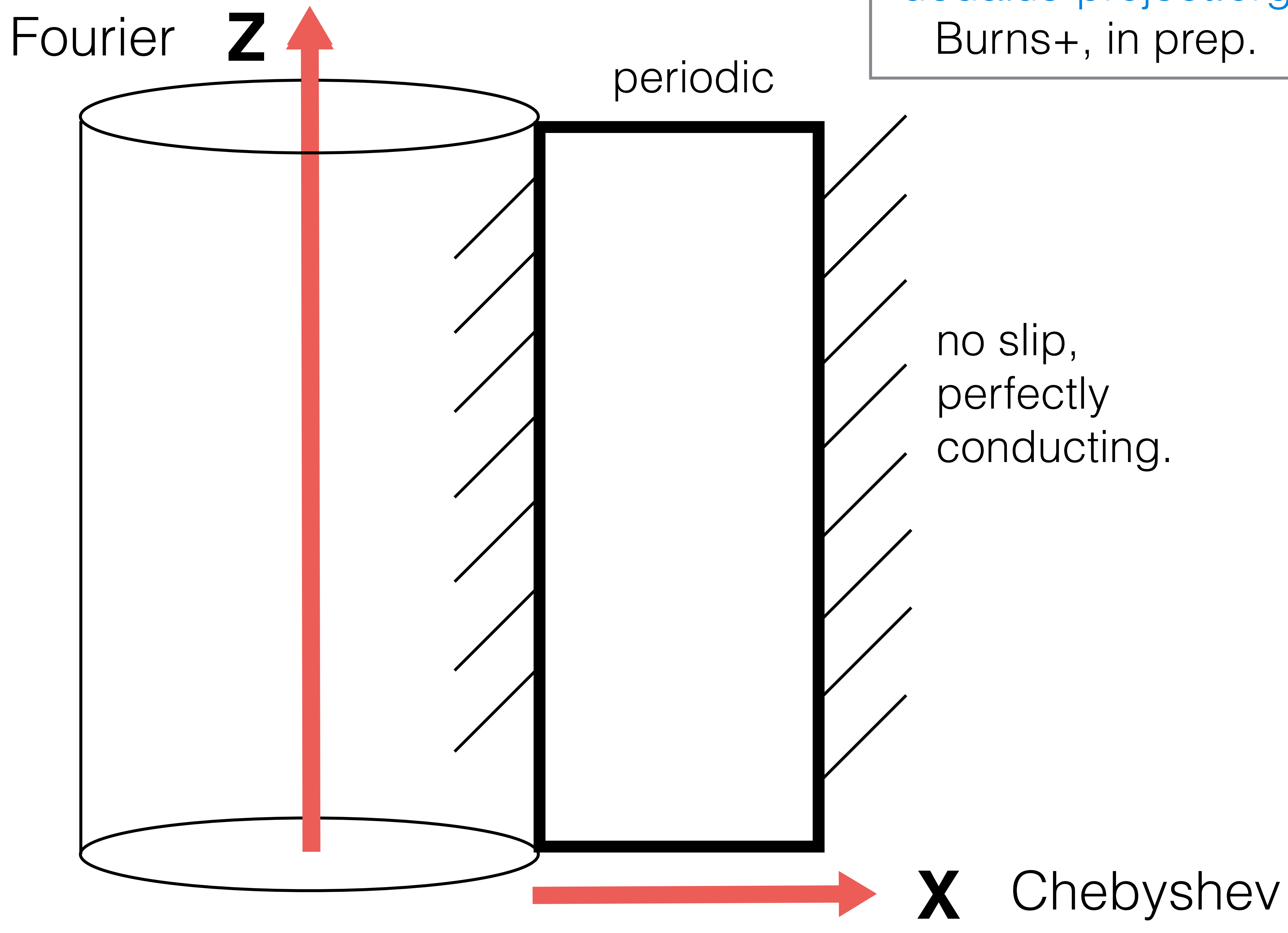


The fluid quantities are expanded in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

Dedalus is a general-purpose spectral code.

dedalus-project.org
Burns+, in prep.



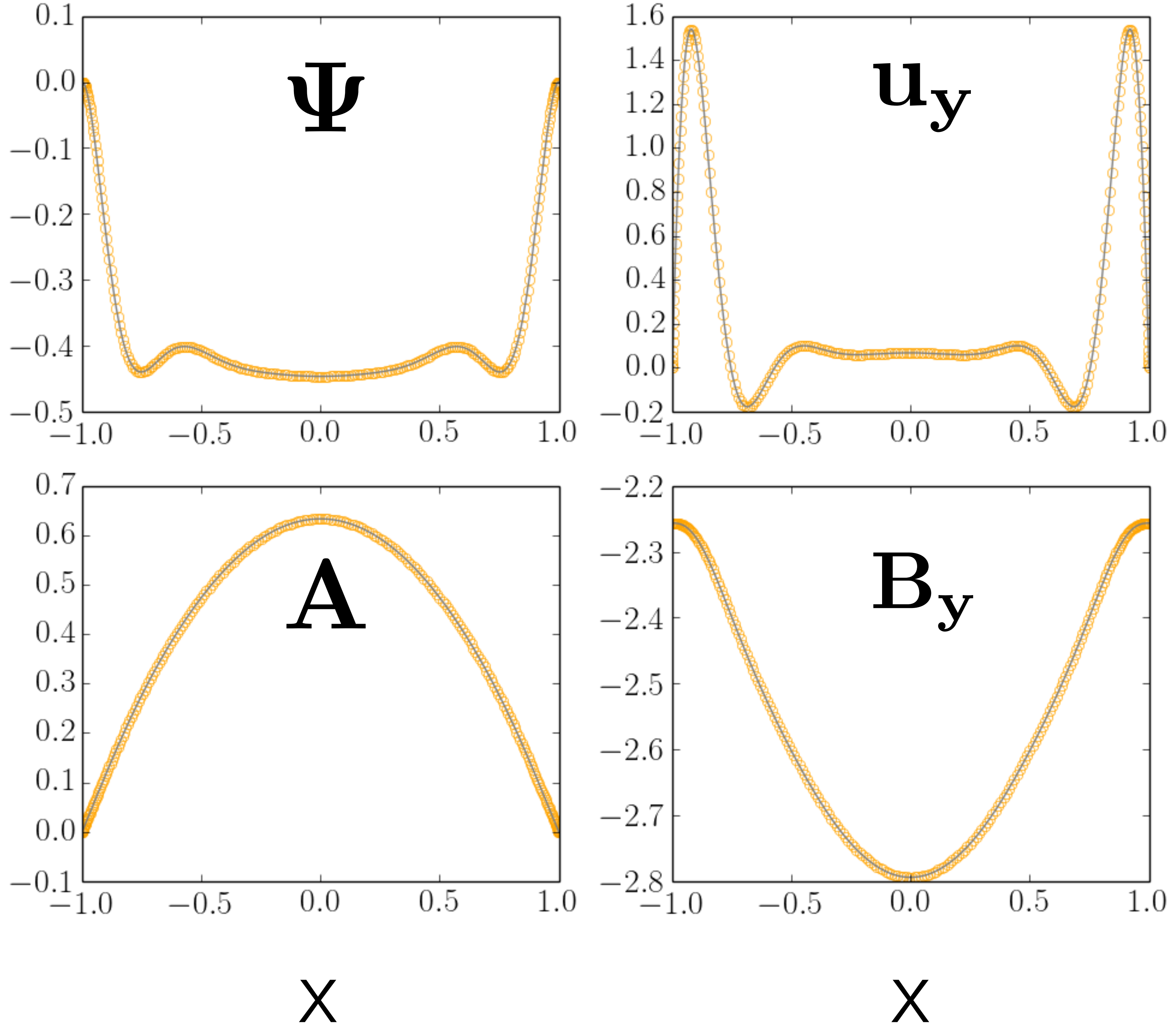
Spectrally solve the most unstable mode of the linear MRI.

$\boxed{V_{11}(x)}$

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + \ldots$$

$$\mathcal{L}\mathbf{V}_1 = 0$$

$$\mathbf{V}_1 = \alpha\left(T, Z\right) \boxed{V_{11}(x)} e^{ik_c z}$$



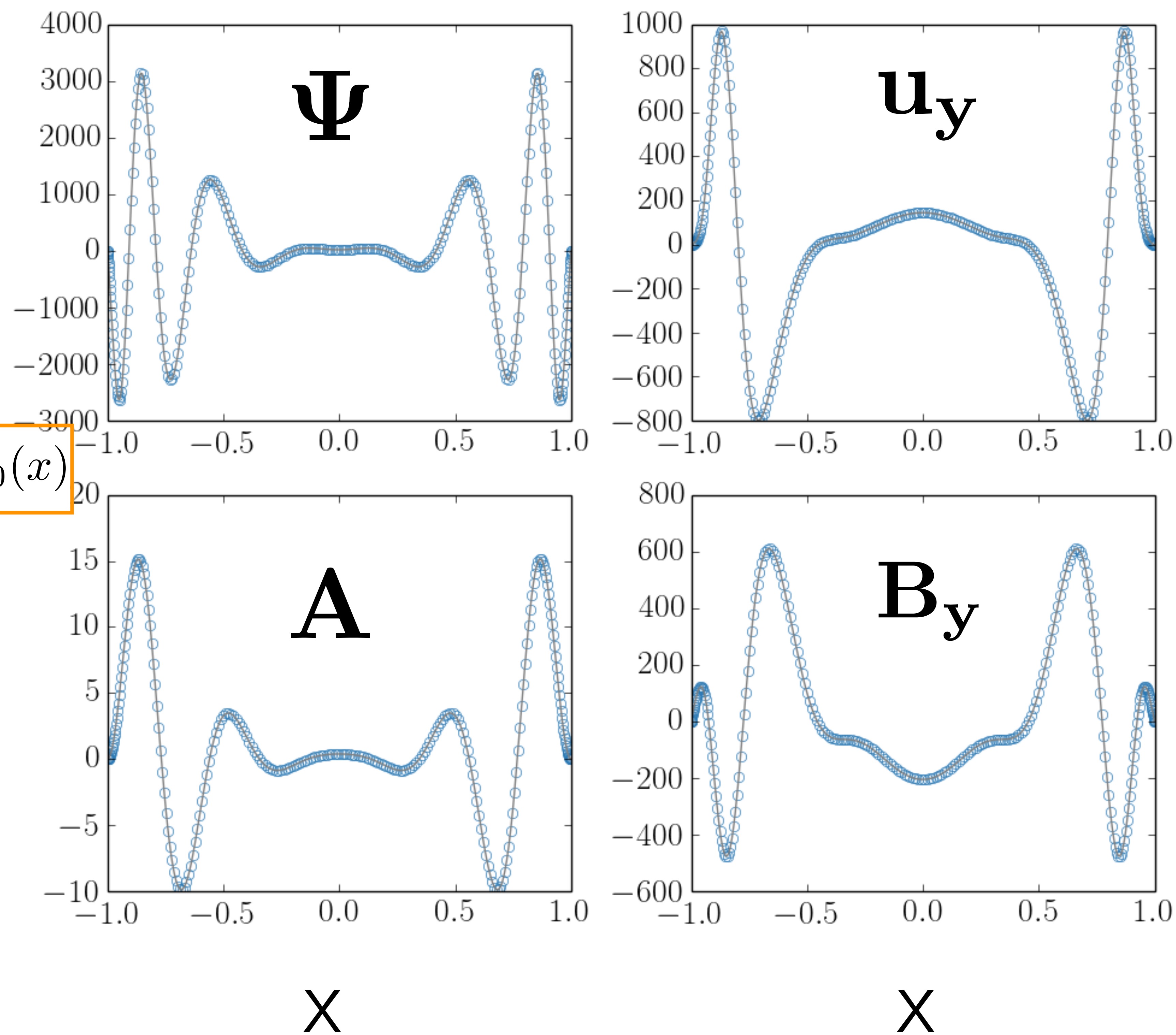
We solve each term in the expanded equations at each order.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + \ldots$$

$$\mathcal{L}\mathbf{V}_2 = \mathbf{N}_2 - \widetilde{\mathcal{L}}_1 \partial_Z \mathbf{V}_1$$

$$\mathbf{N}_2 = \alpha^2 \mathbf{N}_{22}(x) e^{i2k_c z} + |\alpha|^2 \boxed{\mathbf{N}_{20}(x)}$$

$$\boxed{\mathbf{N}_{20}(x)}$$



The removal of secular terms yields solvability criteria.

$\mathcal{D}\partial_t\mathbf{V} + \epsilon^2\mathcal{D}\partial_T\mathbf{V} + \epsilon^2\mathcal{D}^*\partial_Z^2\partial_t\mathbf{V} + 2\epsilon\mathcal{D}^*\partial_z\partial_Z\partial_t\mathbf{V} + \mathbf{N} = \mathcal{L}\mathbf{V} + \epsilon\mathcal{L}_1\partial_Z\mathbf{V} + \epsilon^2\mathcal{L}_2\partial_Z^2\mathbf{V} - \epsilon^2\partial_z\mathcal{X}\mathbf{V} - \epsilon^2\partial_z^3\mathcal{L}_3\mathbf{V} + \mathcal{O}(\epsilon^3)$

$\mathbf{N} = \left[N^{(\Psi)}, N^{(u_{1y})}, N^{(A)}, N^{(B_{1y})}\right]^{\mathbf{T}}$. $N^{(\Psi)} = J\left(\Psi,\nabla^2\Psi\right) - J\left(A,\nabla^2 A\right)$:

$J\left(\Psi,\nabla^2\Psi\right) = \partial_z\Psi\partial_x\nabla^2\Psi - \partial_x\Psi\partial_z\nabla^2\Psi =$

$\partial_z\Psi\partial_x\nabla^2\Psi - \partial_x\Psi\partial_z\nabla^2\Psi + \epsilon^2\partial_z\Psi\partial_x\partial_Z^2\Psi - \epsilon^2\partial_x\Psi\partial_z\partial_Z^2\Psi + 2\epsilon\partial_z\Psi\partial_x\partial_z\partial_Z\Psi - 2\epsilon\partial_x\Psi\partial_z^2\partial_Z\Psi + \epsilon\partial_Z\Psi\partial_x\nabla^2\Psi - \epsilon\partial_x\Psi\partial_Z\nabla^2\Psi + \epsilon^3\partial_Z\Psi\partial_x\partial_Z^2\Psi -$
$\epsilon^3\partial_x\Psi\partial_Z^3\Psi + 2\epsilon^2\partial_Z\Psi\partial_x\partial_z\partial_Z\Psi - 2\epsilon^2\partial_x\Psi\partial_z\partial_Z^2\Psi$

$= J\left(\Psi,\nabla^2\Psi\right) + \epsilon^2 J\left(\Psi,\partial_Z^2\Psi\right) + 2\epsilon J\left(\Psi,\partial_z\partial_Z\Psi\right) + \epsilon\widetilde{J}\left(\Psi,\nabla^2\Psi\right) + \epsilon^3\widetilde{J}\left(\Psi,\partial_Z^2\Psi\right) + 2\epsilon^2\widetilde{J}\left(\Psi,\partial_z\partial_Z\Psi\right)$

$\widetilde{J}(f,g) \equiv \partial_Z f\partial_x g - \partial_x f\partial_Z g$

$J(a+b,c+d) = J(a,c) + J(a,d) + J(b,c) + J(b,d).$

$J(\epsilon\Psi_1 + \epsilon^2\Psi_2, \epsilon\nabla^2\Psi_1 + \epsilon^2\nabla^2\Psi_2)$
$= J(\epsilon\Psi_1,\epsilon\nabla^2\Psi_1) + J(\epsilon\Psi_1,\epsilon^2\nabla^2\Psi_2) + J(\epsilon^2\Psi_2,\epsilon\nabla^2\Psi_1) + J(\epsilon^2\Psi_2,\epsilon^2\nabla^2\Psi_2)$
$= \epsilon^2 J(\Psi_1,\nabla^2\Psi_1) + \epsilon^3 J(\Psi_1,\nabla^2\Psi_2) + \epsilon^3 J(\Psi_2,\nabla^2\Psi_1) + \epsilon^4 J(\Psi_2,\nabla^2\Psi_2)$

$2\epsilon J(\epsilon\Psi_1 + \epsilon^2\Psi_2, \epsilon\partial_z\partial_Z\Psi_1 + \epsilon^2\partial_z\partial_Z\Psi_2)$
$= 2\epsilon^3 J(\Psi_1,\partial_z\partial_Z\Psi_1) + 2\epsilon^4 J(\Psi_1,\partial_z\partial_Z\Psi_2) + 2\epsilon^4 J(\Psi_2,\partial_z\partial_Z\Psi_1) + 2\epsilon^5 J(\Psi_2,\partial_z\partial_Z\Psi_2)$

$\epsilon\widetilde{J}(\epsilon\Psi_1 + \epsilon\Psi_2, \epsilon\nabla^2\Psi_1 + \epsilon^2\nabla^2\Psi_2)$
$= \epsilon^3\widetilde{J}(\Psi_1,\nabla^2\Psi_1) + \epsilon^4\widetilde{J}(\Psi_1,\nabla^2\Psi_2) + \epsilon^4\widetilde{J}(\Psi_2,\nabla^2\Psi_1) + \epsilon^5\widetilde{J}(\Psi_2,\nabla^2\Psi_2)$

$N^{(\Psi)} \to J(\Psi,\nabla^2\Psi) \to \epsilon^2 J(\Psi_1,\nabla^2\Psi_1) + \epsilon^3 J(\Psi_1,\nabla^2\Psi_2) + \epsilon^3 J(\Psi_2,\nabla^2\Psi_1) + 2\epsilon^3 J(\Psi_1,\partial_z\partial_Z\Psi_1) + \epsilon^3\widetilde{J}(\Psi_1,\nabla^2\Psi_1) + \mathcal{O}(\epsilon^4)$

$N^{(\Psi)} \to \epsilon^2 J(\Psi_1,\nabla^2\Psi_1) - \epsilon^2\frac{1}{4\pi}J(A_1,\nabla^2 A_1) + \epsilon^3 J(\Psi_1,\nabla^2\Psi_2) - \epsilon^3\frac{1}{4\pi}J(A_1,\nabla^2 A_2) + \epsilon^3 J(\Psi_2,\nabla^2\Psi_1) - \epsilon^3\frac{1}{4\pi}J(A_2,\nabla^2 A_1) + 2\epsilon^3 J(\Psi_1,\partial_z\partial_Z\Psi_1)$
$2\epsilon^3\frac{1}{4\pi}J(A_1,\partial_z\partial_Z A_1) + \epsilon^3\widetilde{J}(\Psi_1,\nabla^2\Psi_1) - \epsilon^3\frac{1}{4\pi}\widetilde{J}(A_1,\nabla^2 A_1) + \mathcal{O}(\epsilon^4)$

$N^{(u_{1y})} = J(\Psi,u_{1y}) - \frac{1}{4\pi}J(A,B_{1y})$
$= J(\Psi,u_{1y}) + \epsilon\widetilde{J}(\Psi,u_{1y}) - \frac{1}{4\pi}J(A,B_{1y}) - \frac{1}{4\pi}\epsilon\widetilde{J}(A,B_{1y})$

$N^{(u_{1y})} \to \epsilon^2 J(\Psi_1,u_1) - \frac{1}{4\pi}\epsilon^2 J(A_1,B_1) + \epsilon^3 J(\Psi_1,u_2) + \epsilon^3 J(\Psi_2,u_1) + \epsilon^3\widetilde{J}(\Psi_1,u_1) - \frac{1}{4\pi}\epsilon^3 J(A_1,B_2) - \frac{1}{4\pi}\epsilon^3 J(A_2,B_1) - \frac{1}{4\pi}\epsilon^3\widetilde{J}(A_1,B_1) +$
$\mathcal{O}(\epsilon^4)$

$N^{(A)} = -J(A,\Psi) = -J(A,\Psi) - \epsilon\widetilde{J}(A,\Psi)$

$N^{(A)} \to -\epsilon^2 J(A_1,\Psi_1) - \epsilon^3 J(A_1,\Psi_2) - \epsilon^3 J(A_2,\Psi_1) - \epsilon^4 J(A_2,\Psi_2) - \epsilon^3\widetilde{J}(A_1,\Psi_1) - \epsilon^4\widetilde{J}(A_1,\Psi_2) - \epsilon^4\widetilde{J}(A_2,\Psi_1) - \epsilon^5\widetilde{J}(A_2,\Psi_2)$

$= J(\Psi,B_{1y}) + \epsilon\widetilde{J}(\Psi,B_{1y}) - J(A,u_{1y}) - \epsilon\widetilde{J}(A,u_{1y})$
$= \epsilon^2 J(\Psi_1,B_1) + \epsilon^3 J(\Psi_1,B_2) + \epsilon^3 J(\Psi_2,B_1) + \epsilon^4 J(\Psi_2,B_2) + \epsilon^3\widetilde{J}(\Psi_1,B_1) + \epsilon^4\widetilde{J}(\Psi_1,B_2) + \epsilon^4\widetilde{J}(\Psi_2,B_1) + \epsilon^5\widetilde{J}(\Psi_2,B_2) - \epsilon^2 J(A_1,u_1) -$
$\epsilon^3 J(A_1,u_2) - \epsilon^3 J(A_2,u_1) - \epsilon^4 J(A_2,u_2) - \epsilon^3\widetilde{J}(A_1,u_1) - \epsilon^4\widetilde{J}(A_1,u_2) - \epsilon^4\widetilde{J}(\Psi_2,B_1) - \epsilon^5\widetilde{J}(\Psi_2,B_2)$

$\mathbf{N} = \epsilon^2\mathbf{N_1} + \epsilon^3\mathbf{N_2} + \mathcal{O}(\epsilon^4)$

$N^{(\Psi)} = \epsilon^2 N_2^{(\Psi)} + \epsilon^3 N_3^{(\Psi)} + \mathcal{O}(\epsilon^4)$ $N^{(u)} = \epsilon^2 N_2^{(u)} + \epsilon^3 N_3^{(u)} + \mathcal{O}(\epsilon^4)$ $N^{(A)} = \epsilon^2 N_2^{(A)} + \epsilon^3 N_3^{(A)} + \mathcal{O}(\epsilon^4)$ $N^{(B)} = \epsilon^2 N_2^{(B)} + \epsilon^3 N_3^{(B)} + \mathcal{O}(\epsilon^4)$

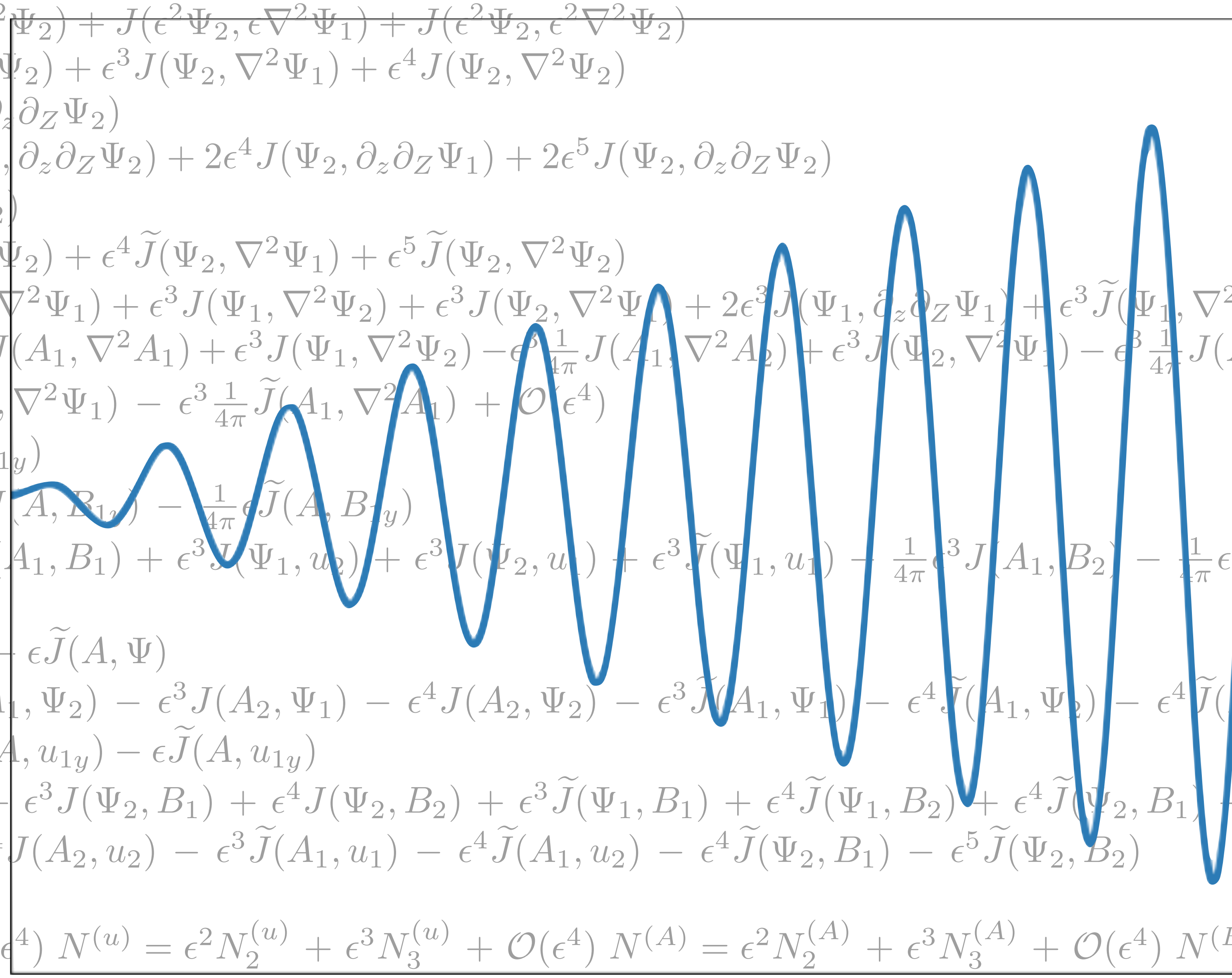
$N_2^{(\Psi)} = J(\Psi_1,\nabla^2\Psi_1) - \frac{1}{4\pi}J(A_1,\nabla^2 A_1)$

$N_2^{(u)} = J(\Psi_1,u_1) - \frac{1}{4\pi}J(A_1,B_1)$

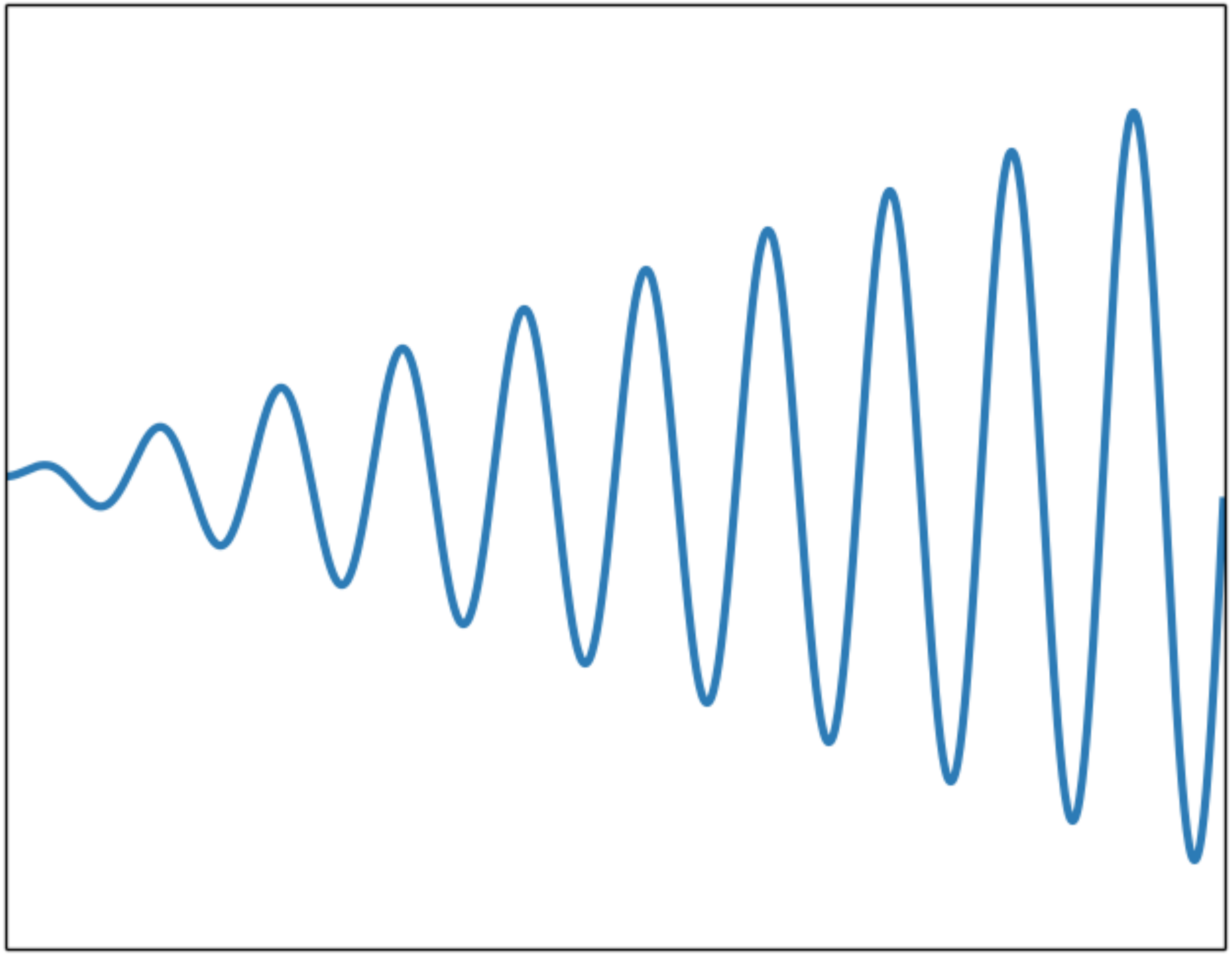
$N_2^{(A)} = -J(A_1,\Psi_1)$

$N_2^{(B)} = J(\Psi_1,B_1) - J(A_1,u_1)$

$N_3^{(\Psi)} = J(\Psi_1,\nabla^2\Psi_2) - \frac{1}{4\pi}J(A_1,\nabla^2 A_2) + J(\Psi_2,\nabla^2\Psi_1) - \frac{1}{4\pi}J(A_2,\nabla^2 A_1) + 2J(\Psi_1,\partial_z\partial_Z\Psi_1) - 2\frac{1}{4\pi}J(A_1,\partial_z\partial_Z A_1) + \widetilde{J}(\Psi_1,\nabla^2\Psi_1) -$
$\frac{1}{4\pi}\widetilde{J}(A_1,\nabla^2 A_1)$



The removal of secular terms yields solvability criteria.



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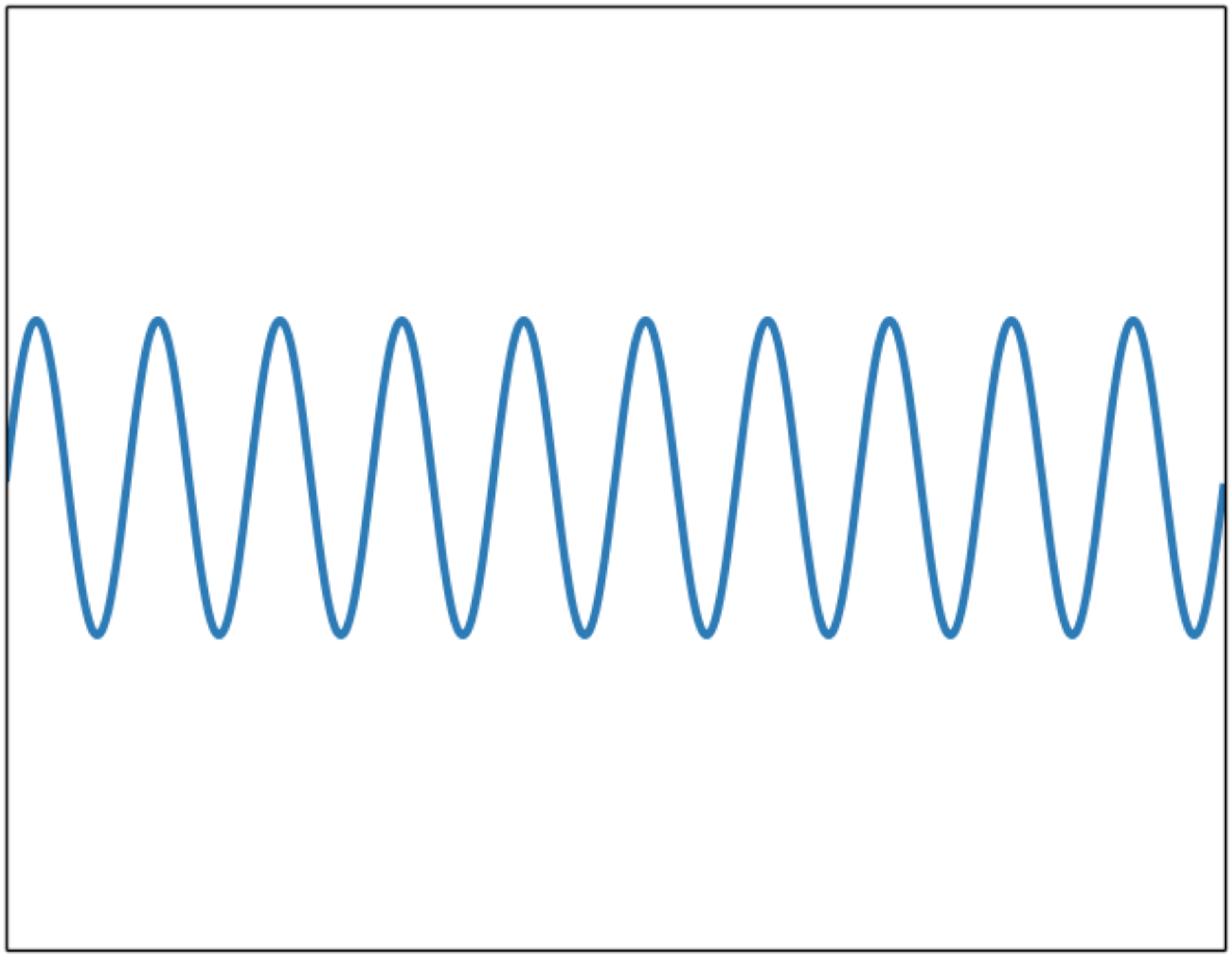
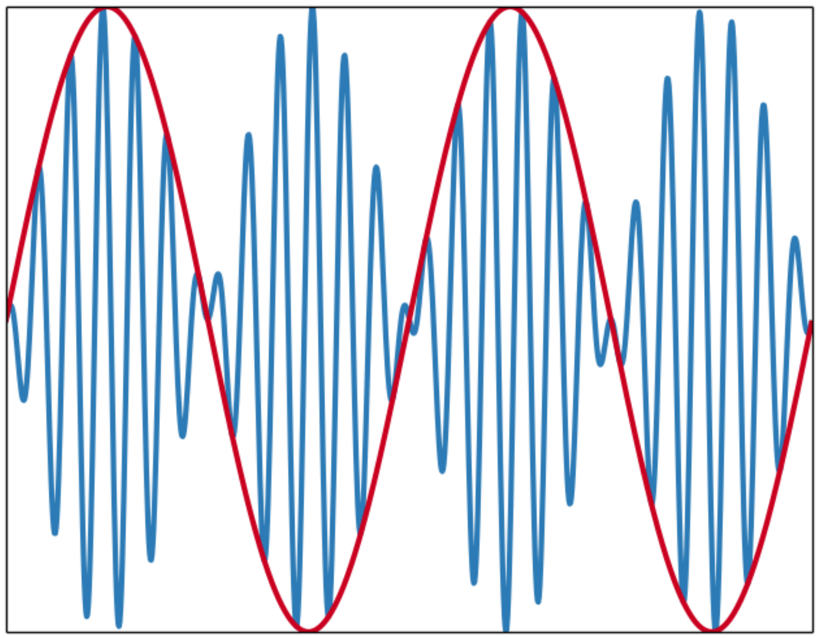


The result is an amplitude equation for the most unstable mode.

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

α



The result is an amplitude equation for the most unstable mode.

diffusion term

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha\left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

nonlinear term

linear growth

The result is an amplitude equation
for the most unstable mode.

diffusion term

?

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha\left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

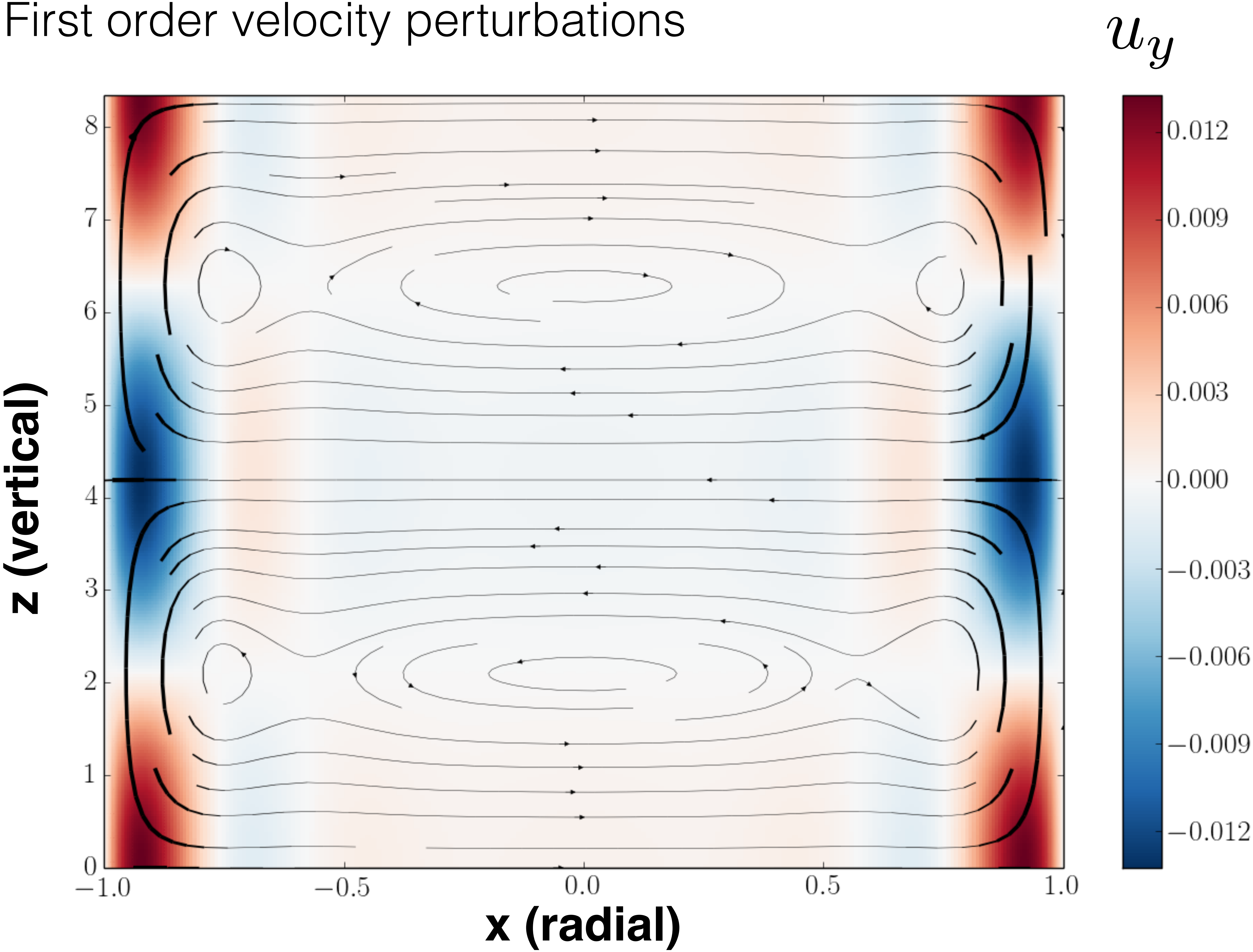
nonlinear term

linear growth

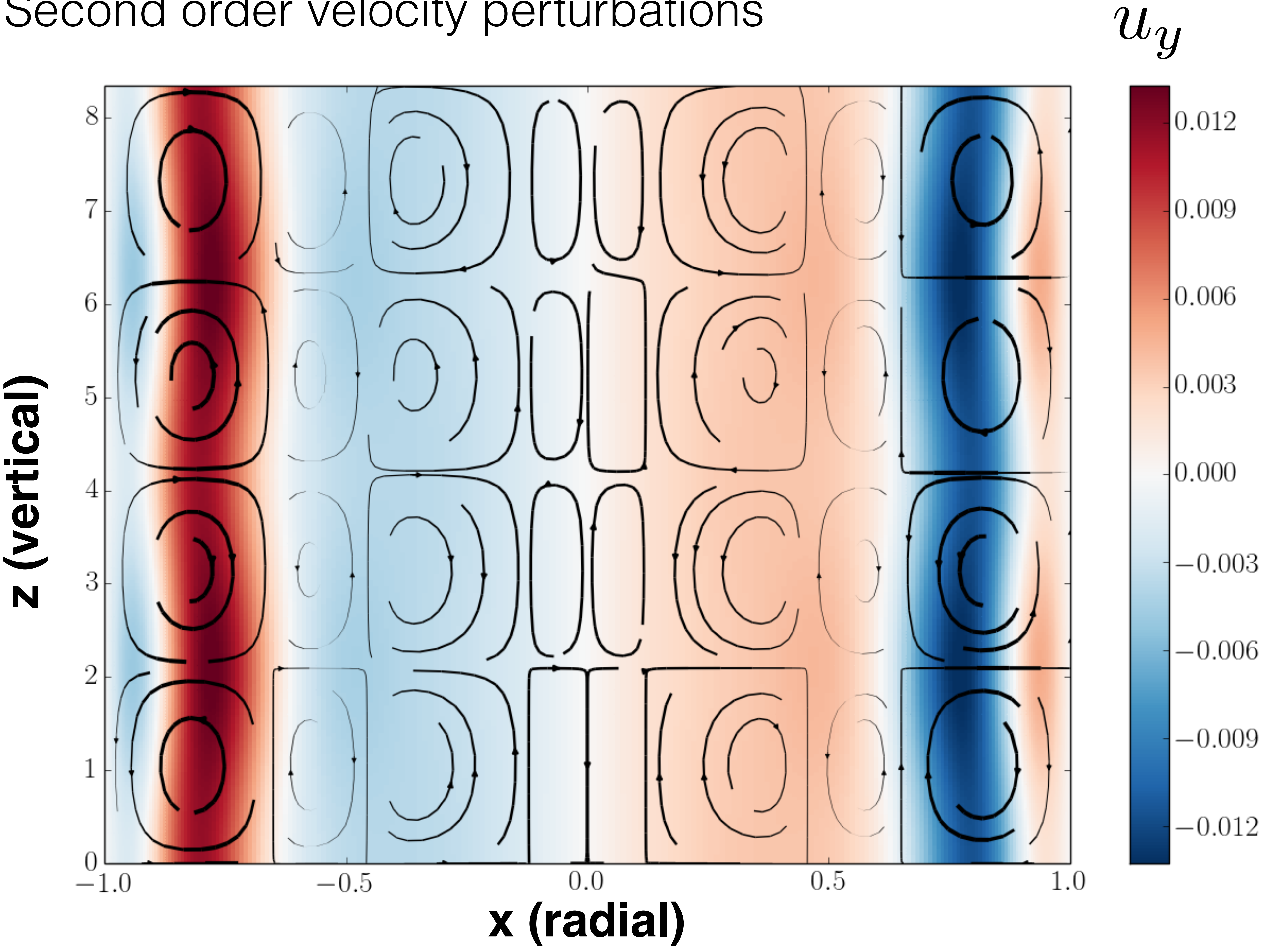
Finally, we obtain the saturation structure of each fluid quantity at each order.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

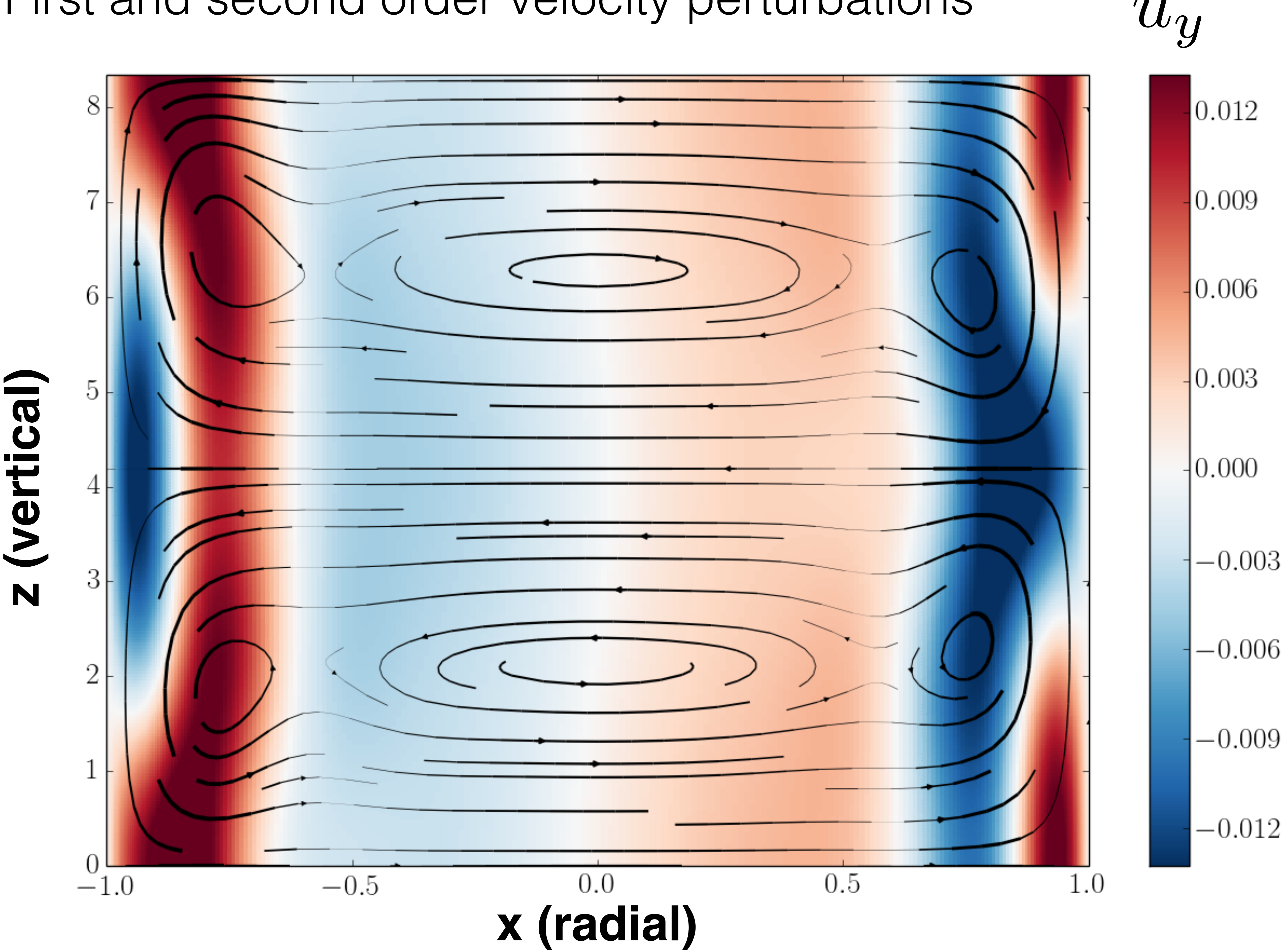
First order velocity perturbations



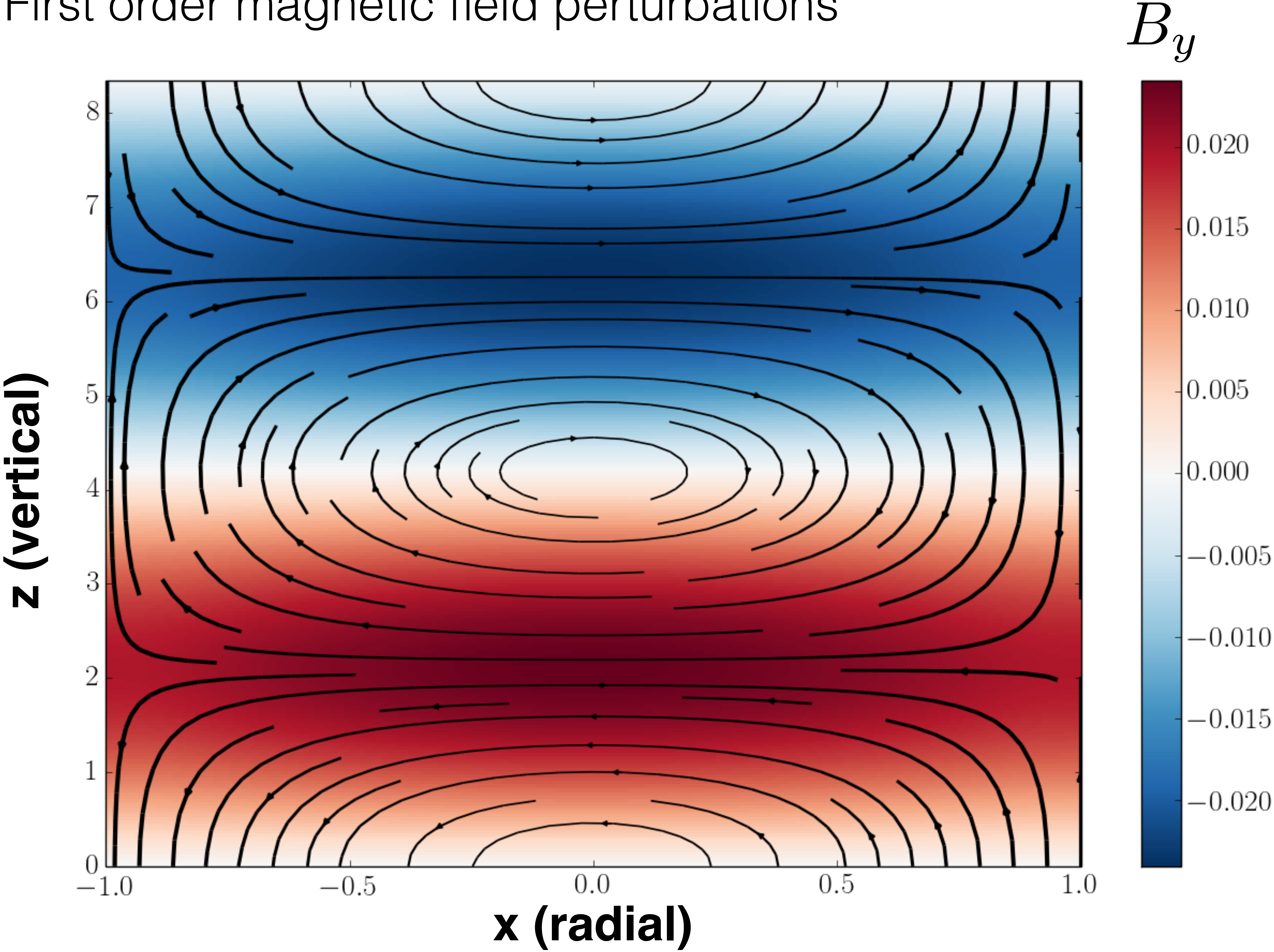
Second order velocity perturbations



First and second order velocity perturbations

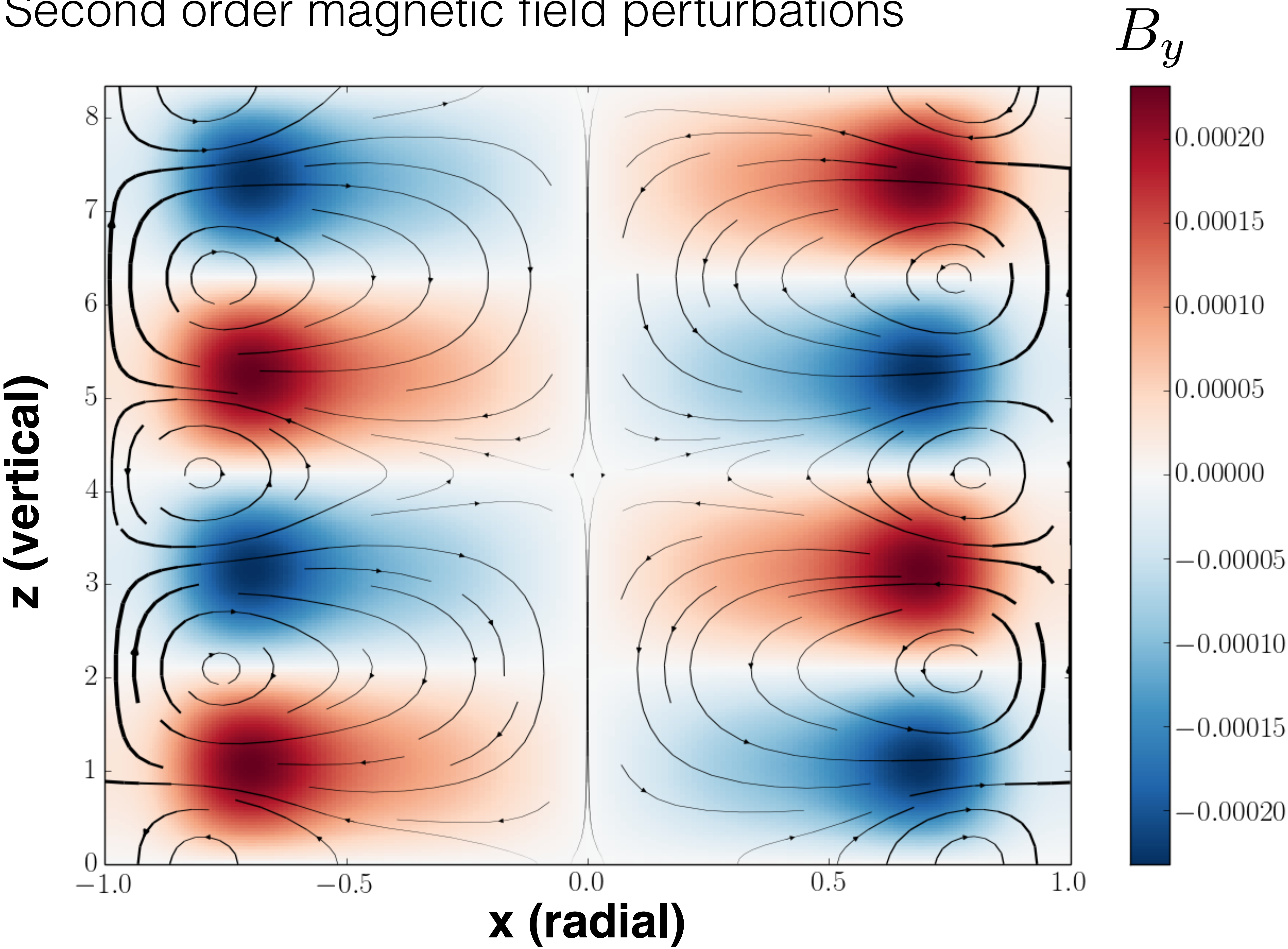


First order magnetic field perturbations

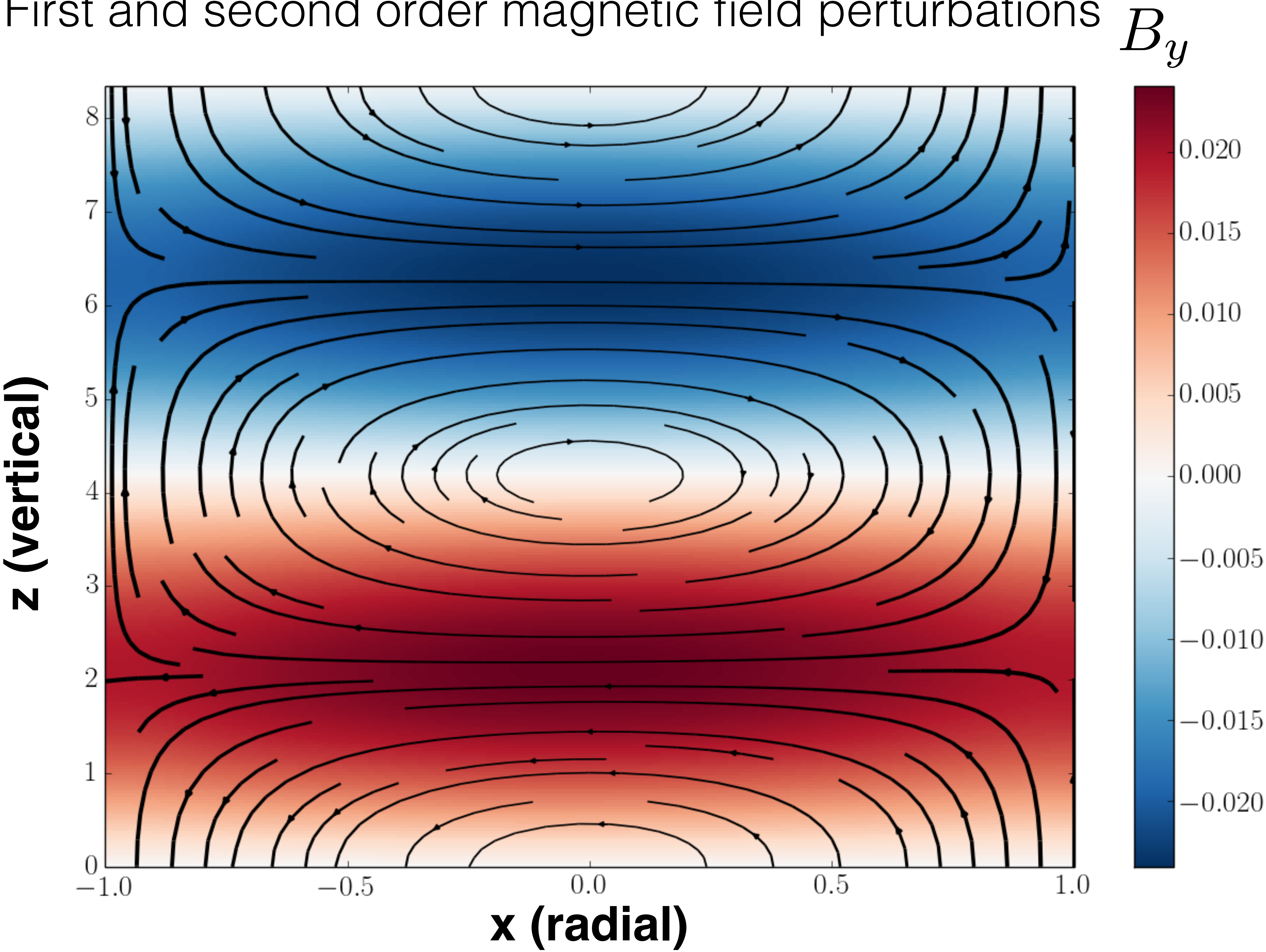


Second order magnetic field perturbations

two OOM smaller!

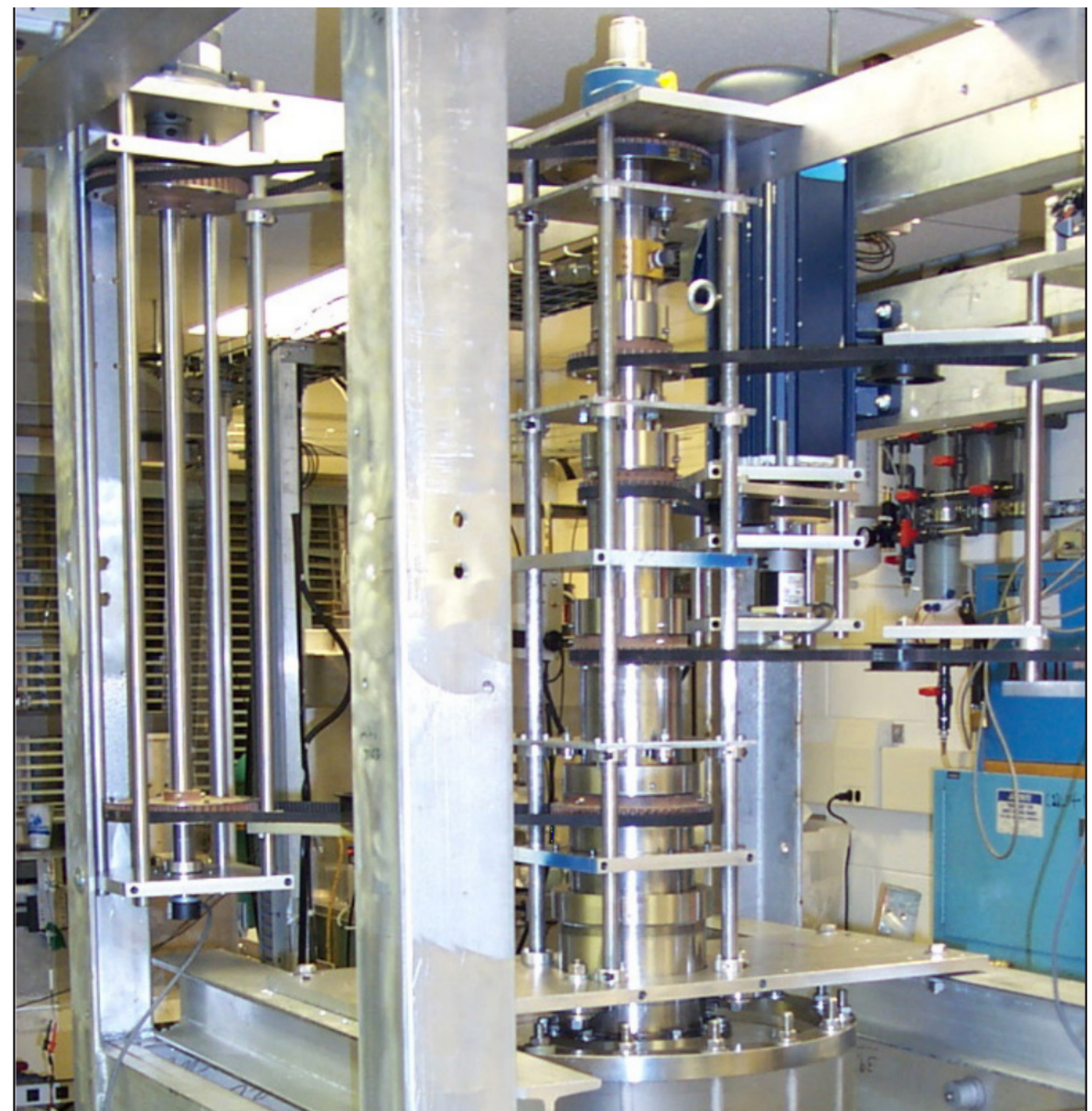


First and second order magnetic field perturbations



Future work

- explore parameter space
- relax thin gap approximation
- comparison to experiment
- helical MRI



PPPL MRI experiment

Conclusions

- We derive a robust analytical framework for solving MRI systems up to second order perturbations.
- We use the spectral code Dedalus to solve the radial components of our equations.
- Preliminary results suggest a shear-related saturation mechanism.