1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi\ \hat{\mathbf{r}} \\ u_\phi\ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi\ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned}\partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) = \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ + \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ + \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\}\end{aligned} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{3}$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\mathrm{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{4}$$

$$\begin{aligned}\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi) \\ + \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right)\end{aligned} \tag{5}$$

2 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_z \tag{6}$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$
\begin{gathered}
\frac{1}{r^2}\partial_t\partial_r\Psi-\frac{1}{r}\partial_t\partial_r^2\Psi-\frac{3}{r^4}\partial_z\Psi\partial_r\Psi+\frac{1}{r^3}\partial_r\left(\partial_z\Psi\partial_r\Psi\right)+\frac{2}{r^3}\partial_z\Psi\partial_r^2\Psi-\frac{1}{r^2}\partial_r\left(\partial_z\Psi\partial_r^2\Psi\right) \\
-\frac{2}{r^3}\partial_r\Psi\partial_r\partial_z\Psi+\frac{1}{r^2}\partial_r\left(\partial_r\Psi\partial_r\partial_z\Psi\right)= \\
\partial_r\left(B_\phi\partial_z B_\phi\right)+\frac{2}{r^3}\partial_z^2 A\partial_z A-\frac{1}{r^2}\partial_r\left(\partial_z^2 A\partial_z A\right)+\frac{3}{r^4}\partial_z A\partial_r A-\frac{1}{r^3}\partial_r\left(\partial_z A\partial_r A\right)-\frac{2}{r^3}\partial_z A\partial_r^2 A \\
+\frac{1}{r^2}\partial_r\left(\partial_z A\partial_r^2 A\right)+\frac{1}{\mathrm{Re}}\left[\frac{3}{r^4}\partial_r\Psi-\frac{3}{r^3}\partial_r^2\Psi+\frac{2}{r^2}\partial_r^3\Psi-\frac{1}{r}\partial_r^4\Psi\right] \qquad (7)
\end{gathered}
$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$
\partial_t u_r+[u\cdot\nabla u]_r=[(\nabla\times B)\times B]_r+\frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_r \qquad (8)
$$

and take ∂_z of the expanded equation to obtain

$$
\begin{gathered}
\frac{1}{r}\partial_t\partial_z^2\Psi-\frac{1}{r^3}\partial_z\left(\partial_z\Psi\partial_z\Psi\right)+\frac{1}{r^2}\partial_z\left(\partial_z\Psi\partial_z\partial_r\Psi\right)-\frac{1}{r^2}\partial_z\left(\partial_r\Psi\partial_z^2\Psi\right)-\frac{1}{r}2u_\phi\partial_z u_\phi \\
=-\frac{1}{r^2}\partial_z^3 A\partial_r A-\frac{1}{r^2}\partial_z^2 A\partial_r\partial_z A+\frac{2}{r^3}\partial_r\partial_z A\partial_r A-\frac{1}{r^2}\partial_r^2\partial_z A\partial_r A-\frac{1}{r^2}\partial_r^2 A\partial_r\partial_z A \\
+\frac{1}{\mathrm{Re}}\left[-\frac{1}{r^2}\partial_z^2\partial_r\Psi+\frac{1}{r}\partial_z^2\partial_r^2\Psi+\frac{1}{r}\partial_z^4\Psi\right] \qquad (9)
\end{gathered}
$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$
\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\phi\partial_z u_\phi \qquad (10)
$$

Note that the relevant quantity appears to be $\nabla^2\Psi-\frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$
\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right] \qquad (11)
$$

Full Ψ equation according to Susan:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$
$$= J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi$$
$$+ \frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \quad (12)$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

The derivation of the non-viscous term on the righthand side of the momentum equation ($\mathbf{J} \times \mathbf{B}$) is as follows.

$$\partial_z\left(\left[(\nabla \times B) \times B\right]_r\right) - \partial_r\left(\left[(\nabla \times B) \times B\right]_z\right) \quad (13)$$

$$= \partial_z\left(\left[\left(\partial_z B_r - \partial_r B_z\right)B_z - \left(\frac{1}{r}\partial_r\left(rB_\phi\right)\right)B_\phi\right]\right) - \partial_r\left(\left[\left(-\partial_z B_\phi\right)B_\phi - \left(\partial_z B_r - \partial_r B_z\right)B_r\right]\right) \quad (14)$$

$$= -\frac{1}{r^2}\partial_z^3 A\partial_r A + \frac{1}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{2}{r^3}\partial_z^2 A\partial_z A$$
$$+ \frac{1}{r^2}\partial_z^2\partial_r A\partial_z A + \frac{3}{r^4}\partial_r A\partial_z A - \frac{3}{r^3}\partial_r^2 A\partial_z A + \frac{1}{r^2}\partial_r^3 A\partial_z A - \frac{2}{r}B_\phi\partial_z B_\phi \quad (15)$$

This simplifies to

$$J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \quad (16)$$

Full derivation of viscous term:

$$\partial_z\left(\frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_r\right) - \partial_r\left(\frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_z\right) \quad (17)$$

$$= \frac{1}{\mathrm{Re}}\left[\partial_z\left(\nabla^2 u_r - \frac{1}{r^2}u_r\right) - \partial_r\left(\nabla^2 u_z\right)\right] \quad (18)$$

$$= \frac{1}{\mathrm{Re}}\left[-\frac{2}{r^2}\partial_z^2\partial_r\Psi + \frac{2}{r}\partial_z^2\partial_r^2\Psi + \frac{1}{r}\partial_z^4\Psi - \frac{3}{r^4}\partial_r\Psi + \frac{3}{r^3}\partial_r^2\Psi - \frac{2}{r^2}\partial_r^3\Psi + \frac{1}{r}\partial_r^4\Psi\right] \quad (19)$$

3 Recovery of Narrow Gap Equations

4 Nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$\begin{aligned}\widetilde{u} &= \Omega_0 r_0 \delta u \\ \widetilde{x} &= \delta r_0 x \\ \widetilde{\nabla} &= \frac{\nabla}{\delta r_0} \\ \widetilde{t} &= \frac{t}{\Omega_0} \\ \widetilde{B} &= B_0 B \\ \widetilde{P} &= P_0 P \\ \widetilde{\rho} &= \rho_0 \rho\end{aligned} \tag{20}$$

If we define the velocity scale v_0 as the local sound speed scale $\left(c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}}\right)$, then $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$.

Nondimensionalizing the momentum equation and dividing by a factor of $\Omega_0^2 r_0 \delta$ yields

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\rho_0 \Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - 2\mathbf{\Omega} \times \mathbf{u} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \tag{21}$$

If we define

$$\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \tag{22}$$

where $P_{mag} = \frac{B_0^2}{8\pi}$, then the factor in front of $(\nabla \times \mathbf{B}) \times \mathbf{B}$ should be $\frac{2}{\beta}$. Agreed?

5 Perturbed Equations

We perturb the wide gap equations according to

$$\mathbf{B} = B_0 \hat{z} + \mathbf{B_1}, \tag{23}$$

which is the same perturbation we used in the thin-gap construction. But wait! We perturbed the thin-gap equations according to

$$\mathbf{u} = -q \Omega_0 r \hat{\phi} + \mathbf{u_1} \tag{24}$$

but I don't think this is valid in the wide-gap case. To be sure, I'm going to perturb instead by

$$\mathbf{u} = r\Omega(r)\hat{\phi} + \mathbf{u_1} \tag{25}$$

where $\Omega(r) = \Omega_0 \left(\frac{r}{r_0}\right)^{-q} = \Omega_0 \left(r\right)^{-q}$ in dimensionless coordinates (but keeping Ω_0 to flag a rotational term...) Agreed?

We also add the Coriolis and centrifugal terms $-2\mathbf{\Omega}\times\mathbf{u}-\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ which expand as follows:

$$-2\Omega(r)\hat{\mathbf{z}}\times\left(\mathbf{u_1}+r\Omega(r)\hat{\phi}\right) = -2\Omega(r)u_r\hat{\phi} + 2\Omega(r)u_\phi\hat{\mathbf{r}} + 2r\Omega(r)^2\hat{\mathbf{r}} \tag{26}$$

$$-\Omega(r)\hat{\mathbf{z}}\times(\Omega(r)\hat{\mathbf{z}}\times r\hat{\mathbf{r}}) = +\Omega(r)^2 r\hat{\mathbf{r}} \tag{27}$$

The $\hat{\mathbf{r}}$ terms will have ∂_z applied to them, which will destroy the $3\Omega(r)^2 r\hat{\mathbf{r}}$ term and so the righthand side of the Ψ equation will ultimately gain only the term $2\Omega(r)\partial_z u_\phi\hat{\mathbf{r}}$. The $\hat{\phi}$ equation gains the term $-\frac{2}{r}\Omega(r)\partial_z\Psi$ on the righthand side.

The Ψ equation becomes

$$\begin{aligned}\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\phi\partial_z u_\phi-\partial_z\left(u_\phi\Omega(r)\right)\\ =\frac{2}{\beta}J\left(A,\frac{1}{r^2}\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\right)-\frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi+2\Omega(r)\partial_z u_\phi\\ +\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right]+\frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\end{aligned} \tag{28}$$

Note that one of the terms gained from the base state in the $(\mathbf{u}\cdot\nabla)\mathbf{u}$ term can be combined with the term gained from the Coriolis term, but here I write them separately so we can check the equation. Equation 3 becomes

$$\begin{aligned}\partial_t u_\phi+\frac{J(\psi,u_\phi)}{r}+\frac{u_\phi\partial_z\psi}{r^2}+\partial_z\Psi\left(\frac{2}{r}\Omega(r)+\partial_r\Omega(r)\right)=\\ \frac{2}{\beta}\frac{J(A,B_\phi)}{r}+\frac{2}{\beta}\frac{B_\phi\partial_z A}{r^2}+\frac{1}{\mathrm{Re}}\left(\nabla^2 u_\phi-\frac{u_\phi}{r}\right)+\frac{2}{\beta}B_0\partial_z B_\phi-\frac{2}{r}\Omega(r)\partial_z\Psi\end{aligned} \tag{29}$$

Equation 4 picks up $+\Omega(r)B_\phi$ from the term $-(u_0\hat{\phi}\cdot\nabla)\mathbf{B_1}$ and $-\Omega(r)B_\phi$ from the term $+(\mathbf{B_1}\cdot\nabla)\hat{\phi}$ and, so those cancel and the equation becomes

$$\partial_t A=\frac{1}{r}J(A,\psi)+\frac{1}{\mathrm{Rm}}\left(\nabla^2 A-\frac{2\partial_r A}{r}\right)+B_0\partial_z\Psi \tag{30}$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The $\hat{\phi}$ component of the induction equation, Equation 5, becomes

$$\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi)$$
$$+\frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \frac{1}{\mathrm{Rm}}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) + B_0\partial_z u_\phi + \partial_z A\left(\frac{2}{r}\Omega(r) + \partial_r\Omega(r)\right) \tag{31}$$

6 Matrix Formulation

This is all pending a rigorous check of Equations 28 - 31, which Jeff is working on, but I'll start putting this into a matrix construction.

First, the nonlinear vector, on the lefthand side of the equation, is

$$\mathbf{N} = \begin{bmatrix} J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{2}{\beta}J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}u_\phi\partial_z u_\phi + \frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi \\ \frac{1}{r}J\left(\Psi, u_\phi\right) - \frac{1}{r}\frac{2}{\beta}J\left(A, B_\phi\right) + \frac{1}{r^2}u_\phi\partial_z\Psi - \frac{2}{\beta}\frac{1}{r^2}B_\phi\partial_z A \\ -\frac{1}{r}J\left(A, \Psi\right) \\ -\frac{1}{r}J\left(A, u_\phi\right) - \frac{1}{r}J\left(B_\phi, \Psi\right) - \frac{1}{r^2}B_\phi\partial_z\Psi + \frac{1}{r^2}u_\phi\partial_z A \end{bmatrix} \tag{32}$$

Note that these differ from the thin-gap nonlinear terms not only because of the curvature terms in the Jacobians, but also because of the additional advective terms in all but the **A** equation.

The ∂_t terms are grouped together into

$$\partial_t D = \partial_t \begin{bmatrix} \frac{1}{r}\nabla^2 - \frac{2}{r^2}\partial_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{34}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{35}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{36}$$

We also need the ϕ component of the convective derivative $\mathbf{u} \cdot \nabla \mathbf{u}$,

$$[\mathbf{u} \cdot \nabla \mathbf{u}]_\phi = \mathbf{u} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{37}$$

and finally, the vector Laplacian,

$$(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \tag{38}$$

$$(\nabla^2 \mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \tag{39}$$

$$(\nabla^2 \mathbf{u})_z = \nabla^2 u_z, \tag{40}$$

where ∇ on the vector components is given by equation (34).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$