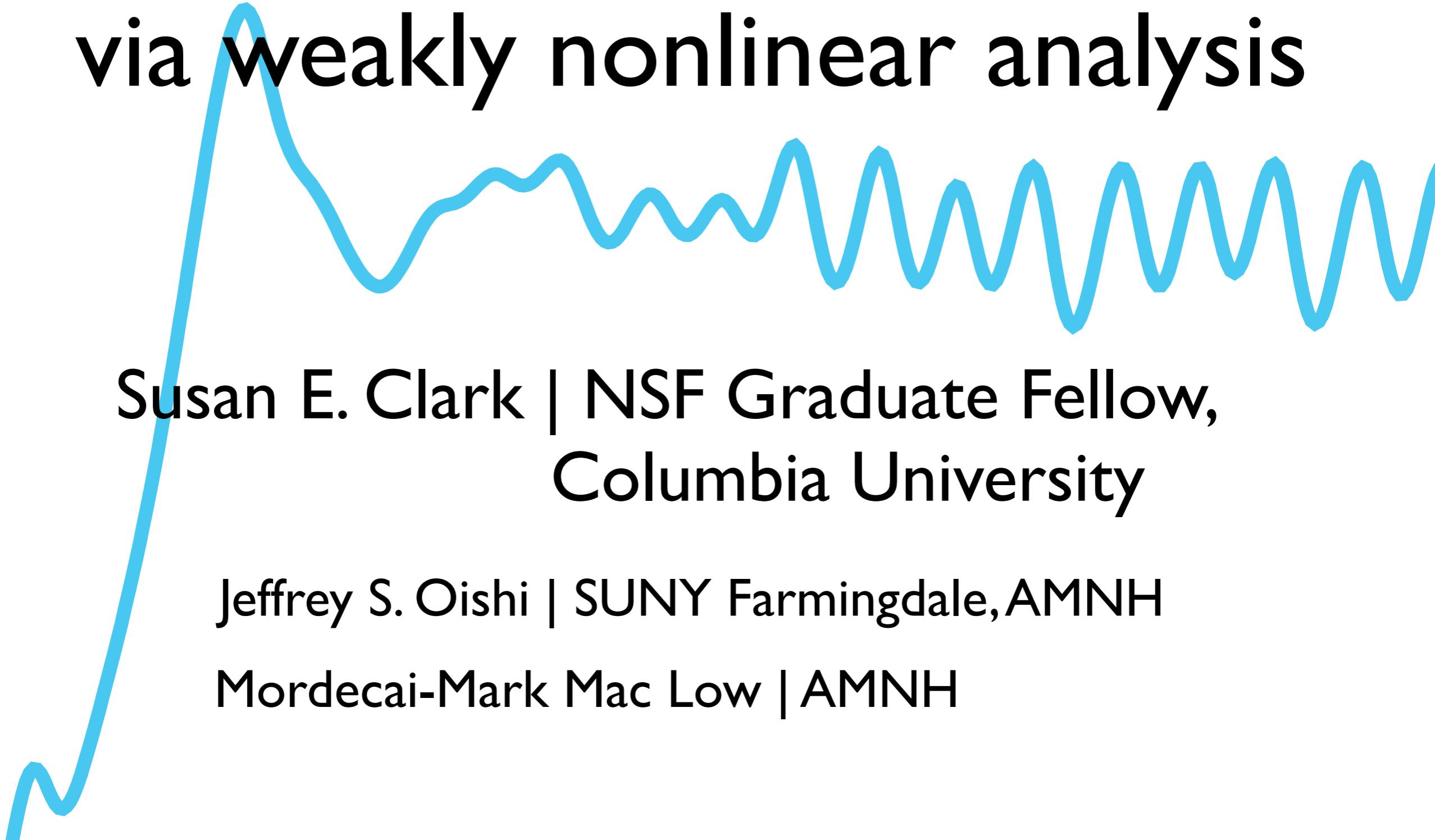


Exploring the saturation of the MRI

via weakly nonlinear analysis



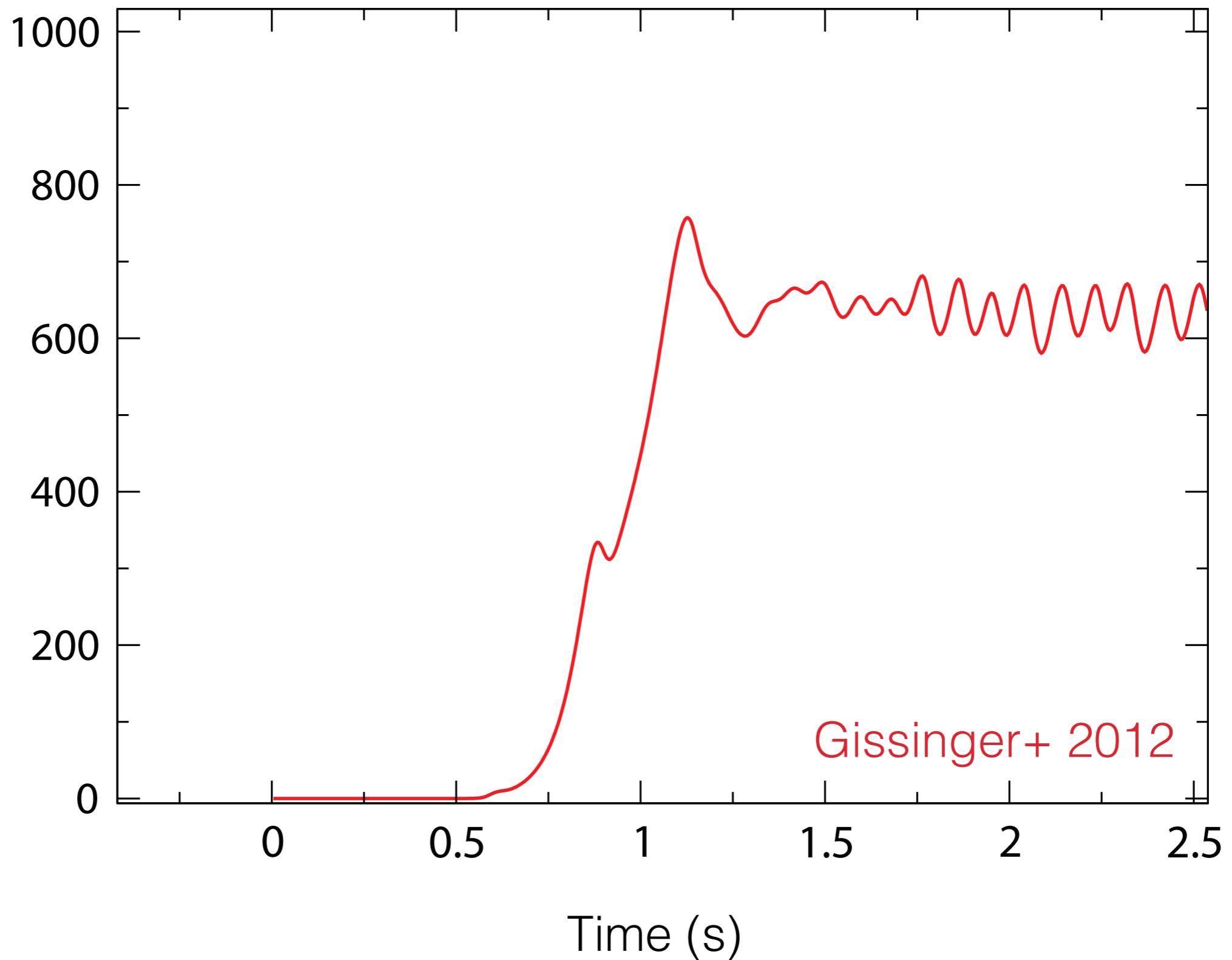
Susan E. Clark | NSF Graduate Fellow,
Columbia University

Jeffrey S. Oishi | SUNY Farmingdale, AMNH

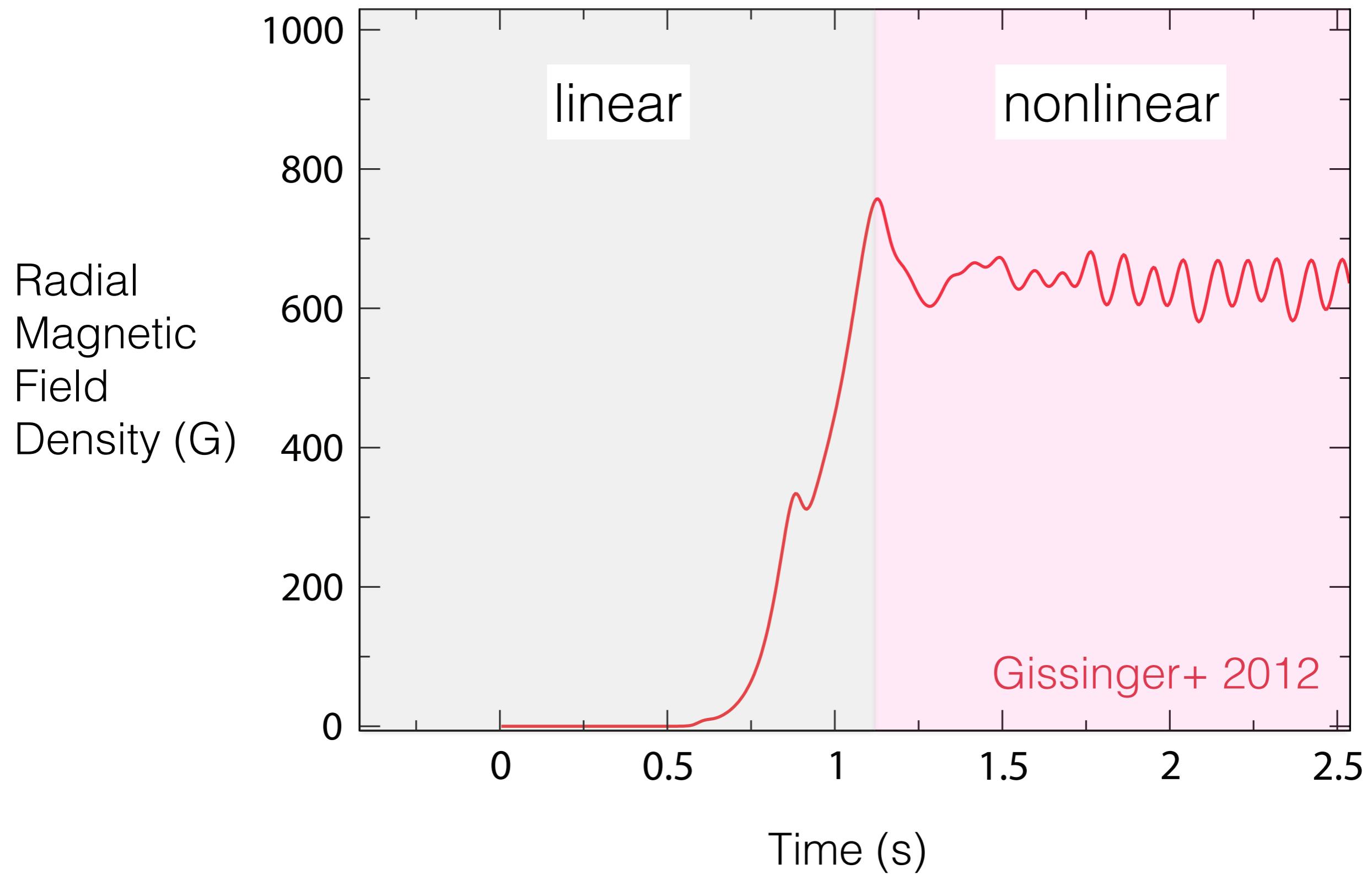
Mordecai-Mark Mac Low | AMNH

The MRI evolves in stages.

Radial
Magnetic
Field
Density (G)

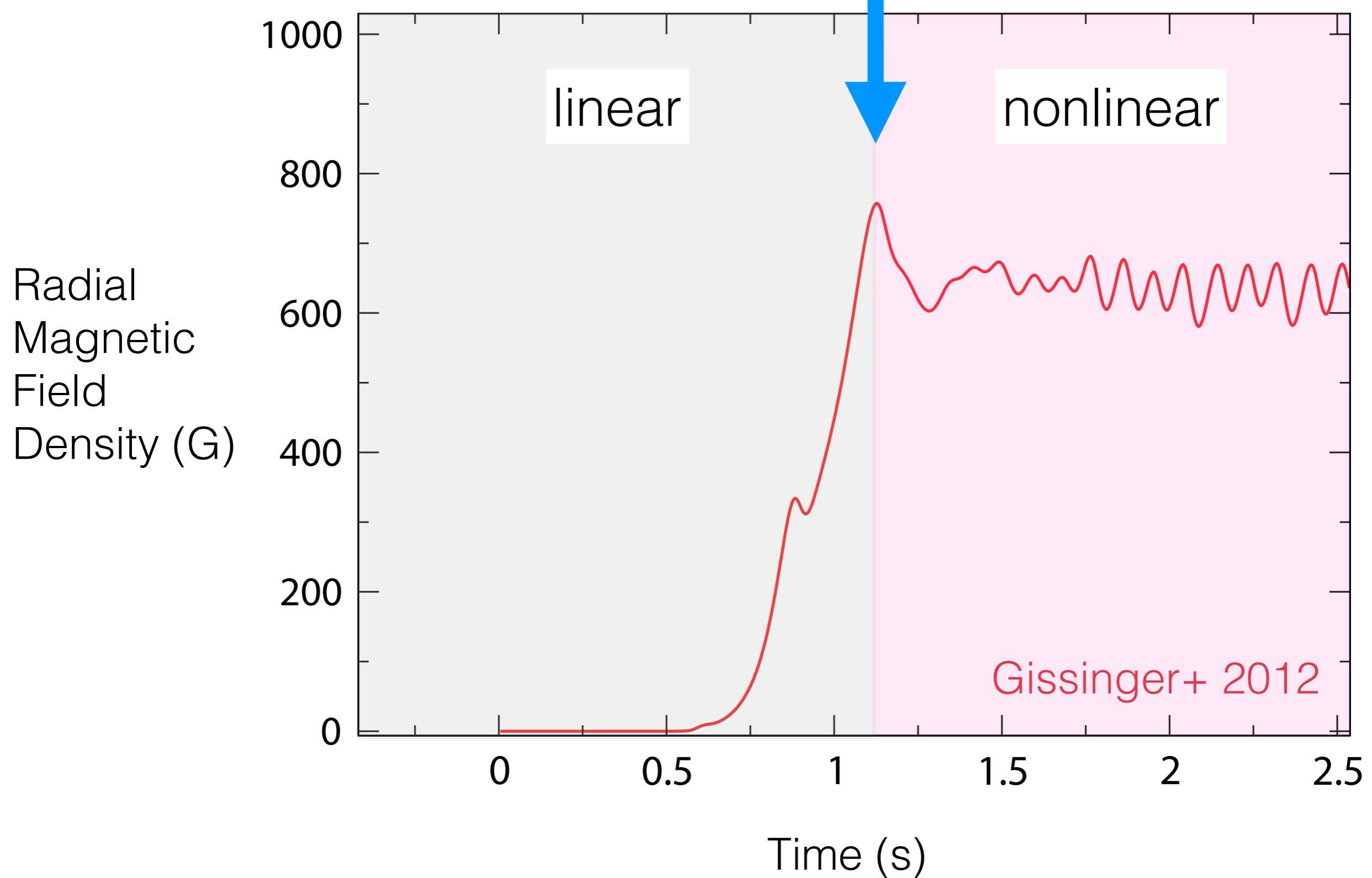


The MRI evolves in stages.



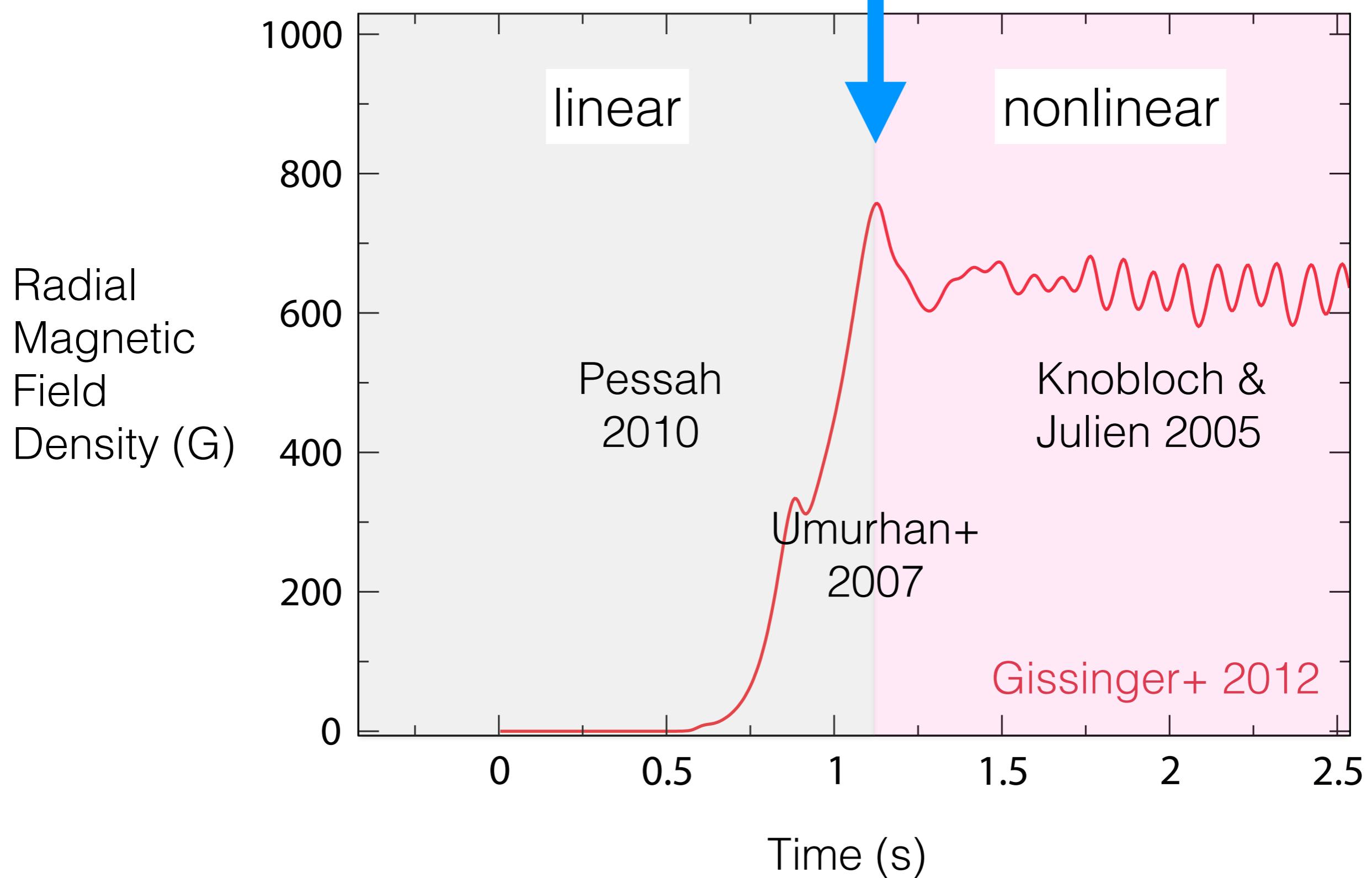
The MRI evolves in stages.

weakly nonlinear

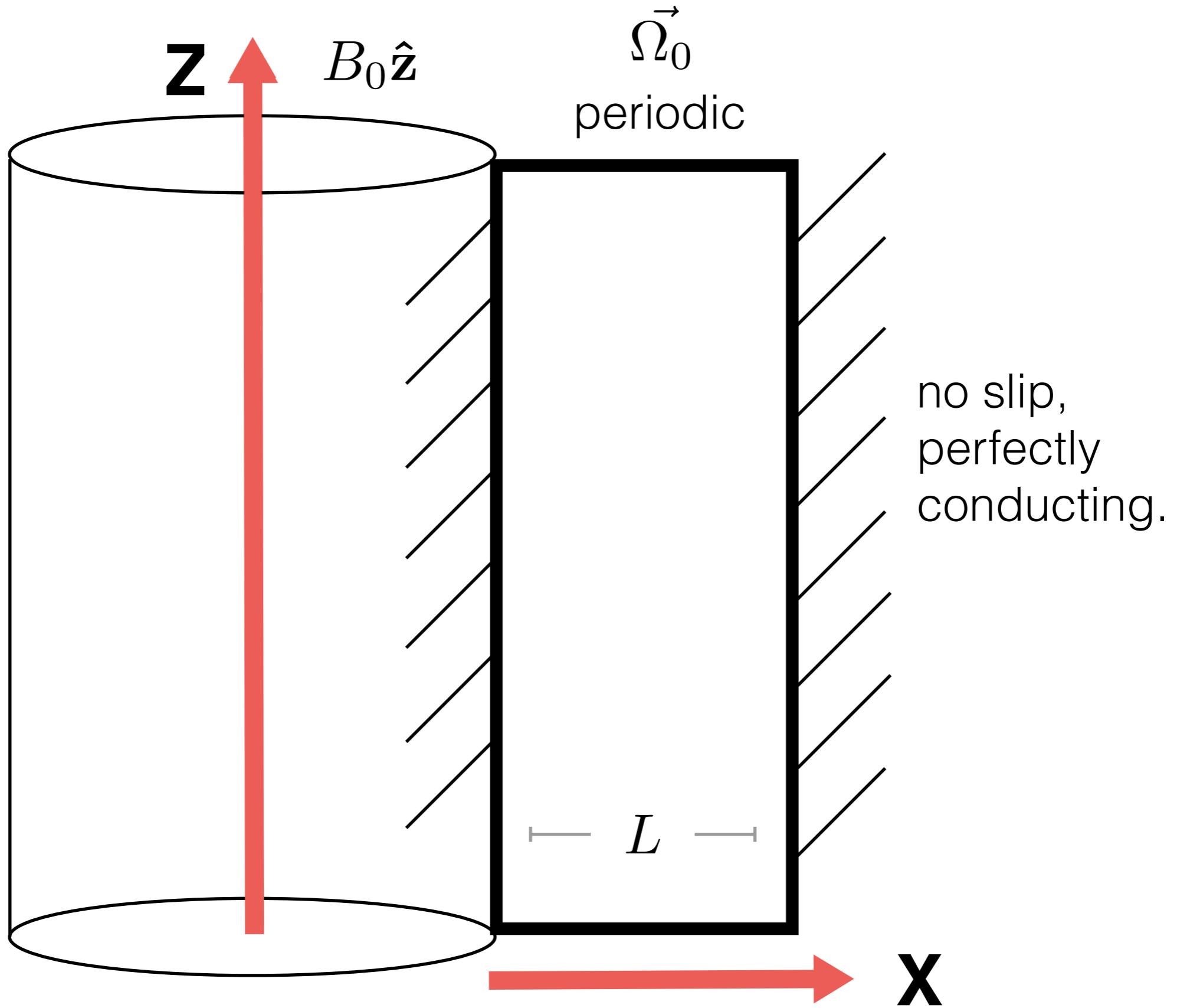


The MRI evolves in stages.

weakly nonlinear



We use a thin-gap Taylor Couette setup.



We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$



kinematic
viscosity

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$



magnetic
resistivity

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0} \right)^{-q}$$

shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$

background field

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$

Reynolds number

$$\nabla \cdot \mathbf{B} = 0$$

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$

magnetic Reynolds number

$$\beta \equiv \frac{8\pi\rho_0\Omega_0^2 L^2}{B_0^2}$$

plasma beta

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \Psi \\ u_y \\ A \\ B_y \end{bmatrix}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2 - q) \Omega_0 \partial_z \Psi + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \frac{1}{Re} \nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2 - q) \Omega_0 \partial_z \Psi + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \boxed{\frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J(A, B_y) - J(\Psi, u_y)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + \boxed{J(A, \Psi)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J(A, u_y) - J(\Psi, B_y)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

nonlinear

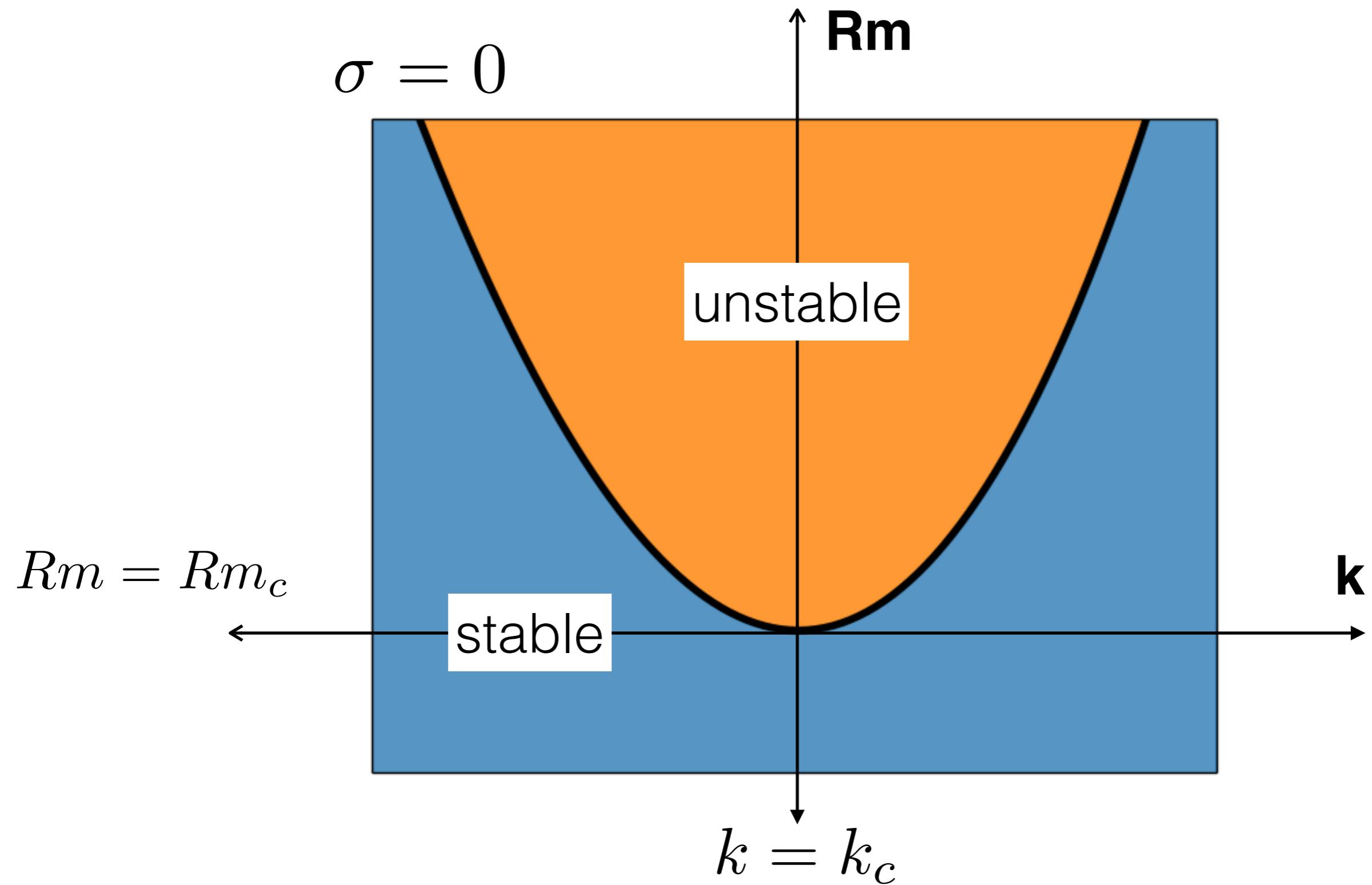
viscous

shear

resistive

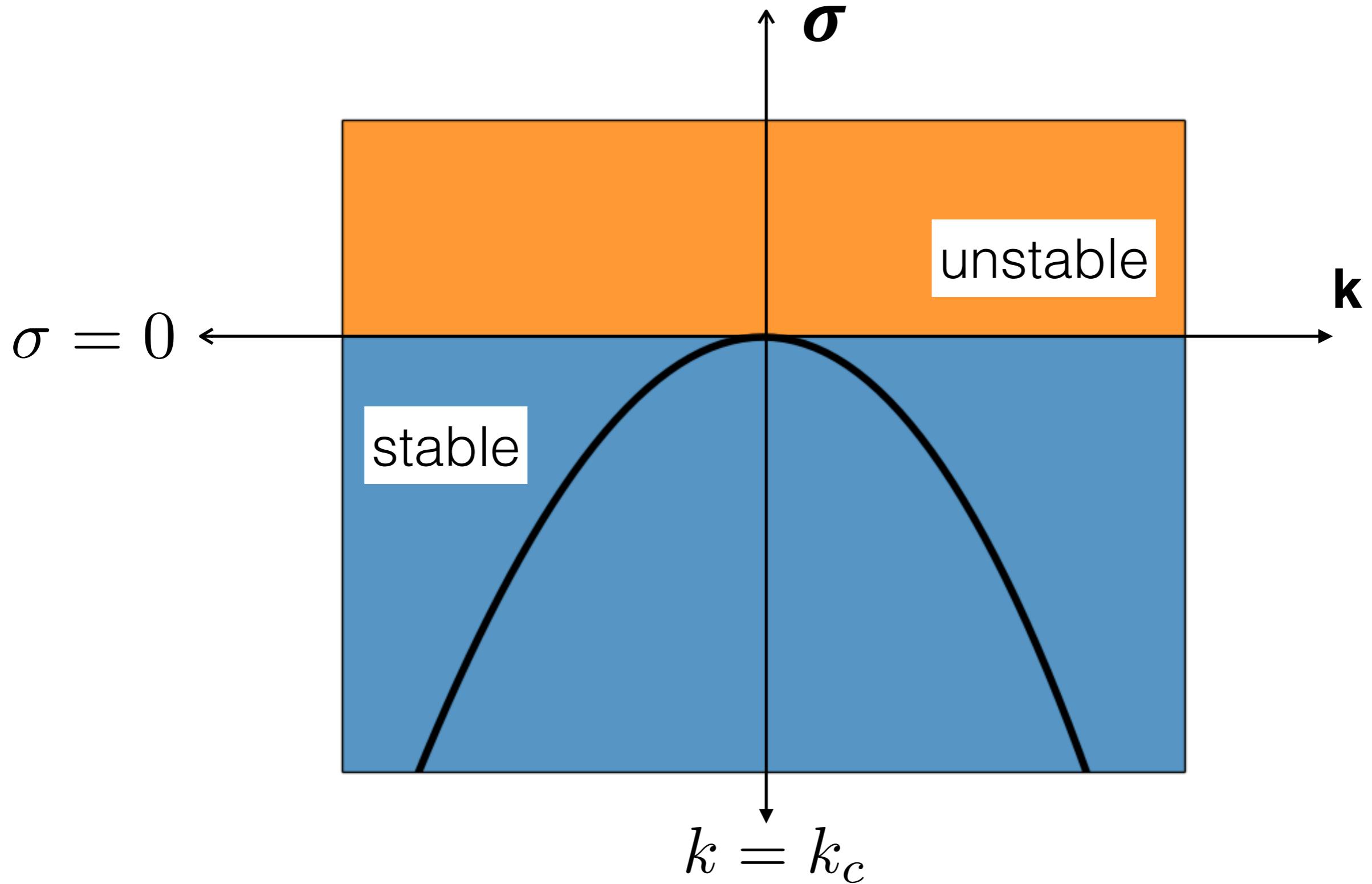
Weakly nonlinear analysis explores behavior at the margin of instability.

$$e^{ikz+\sigma t}$$

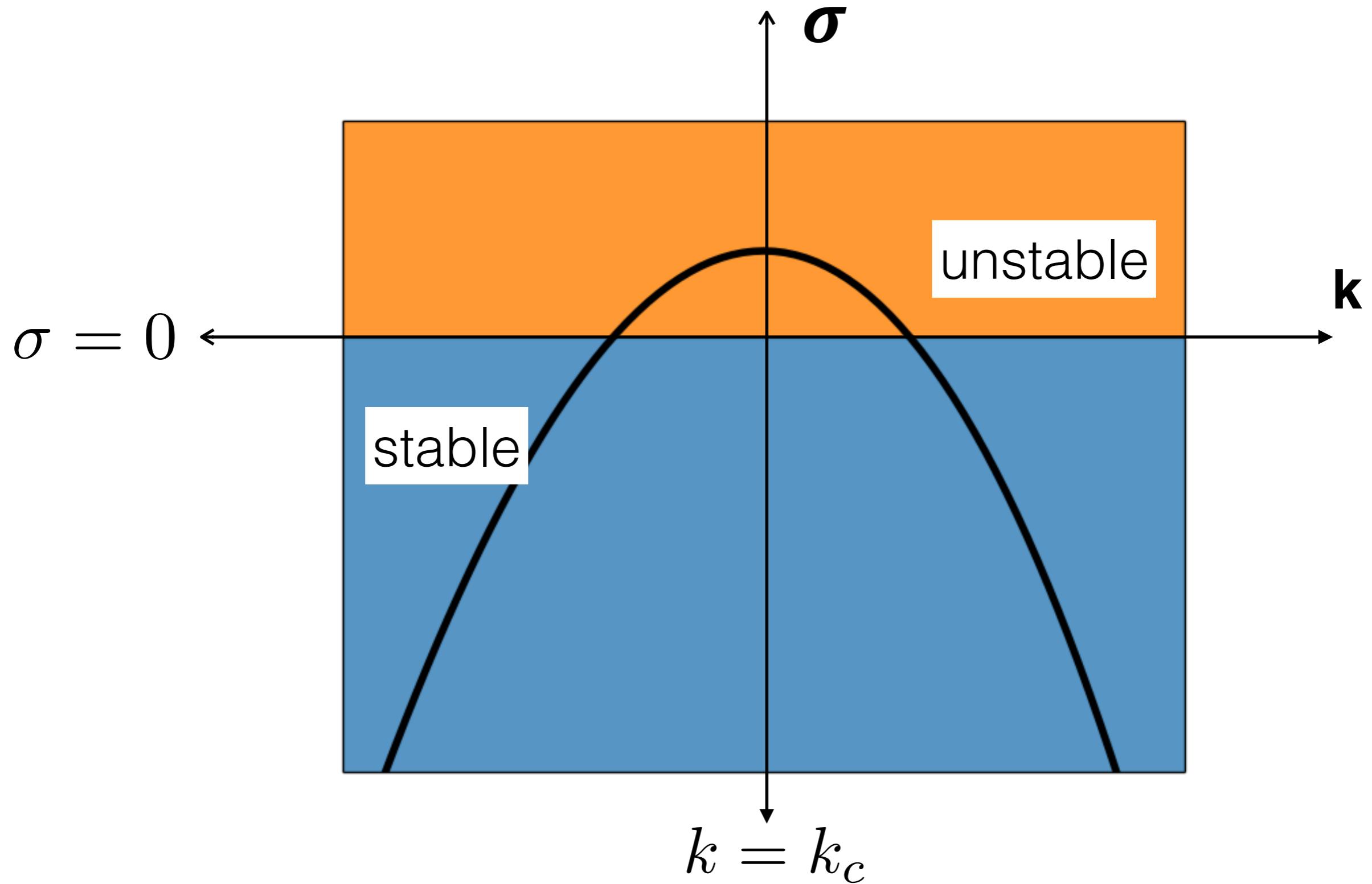


Weakly nonlinear analysis explores behavior at the margin of instability.

Fixed Rm



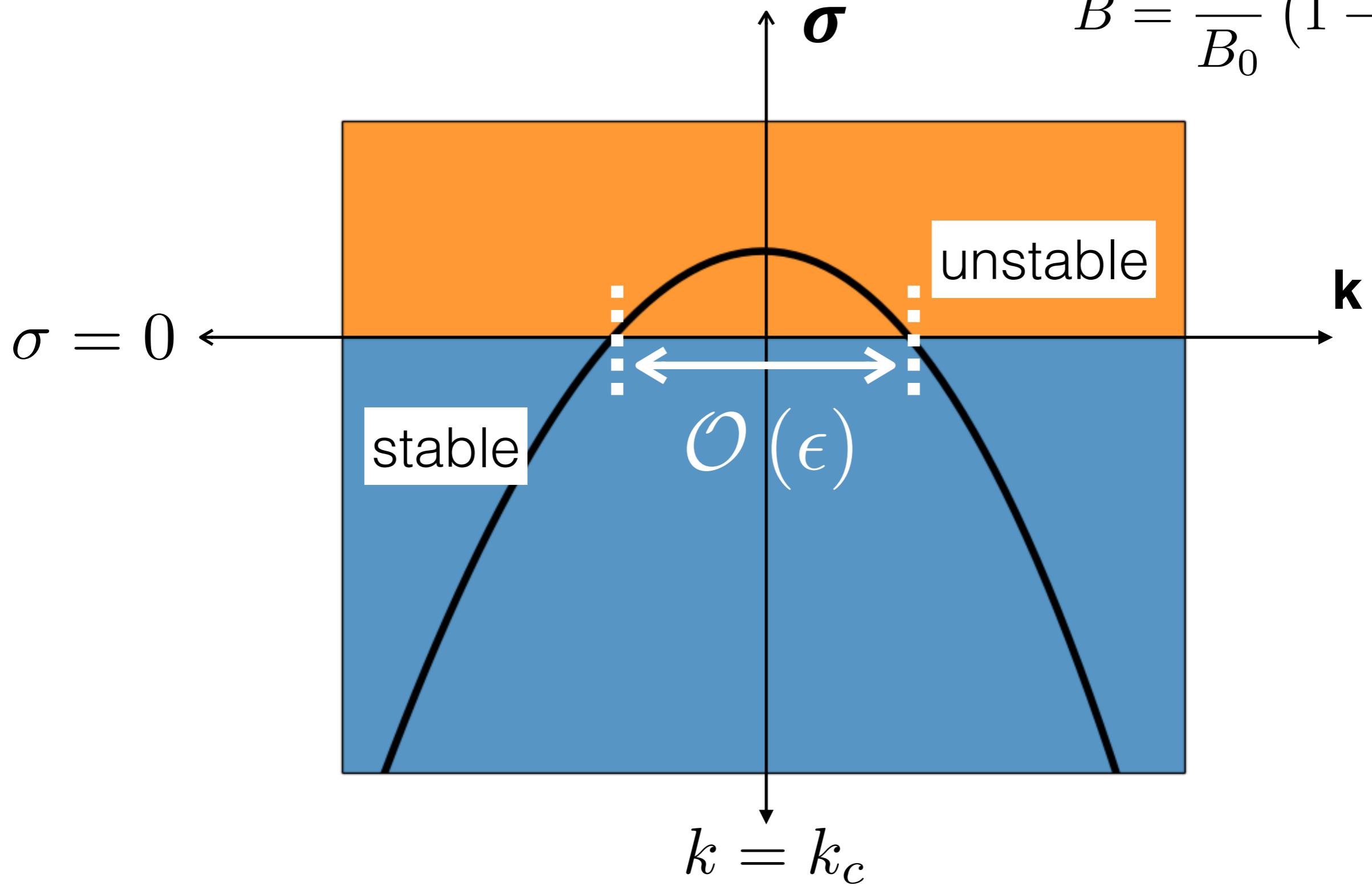
Tune the most unstable mode just over the threshold of instability.



Tune the most unstable mode just over the threshold of instability.

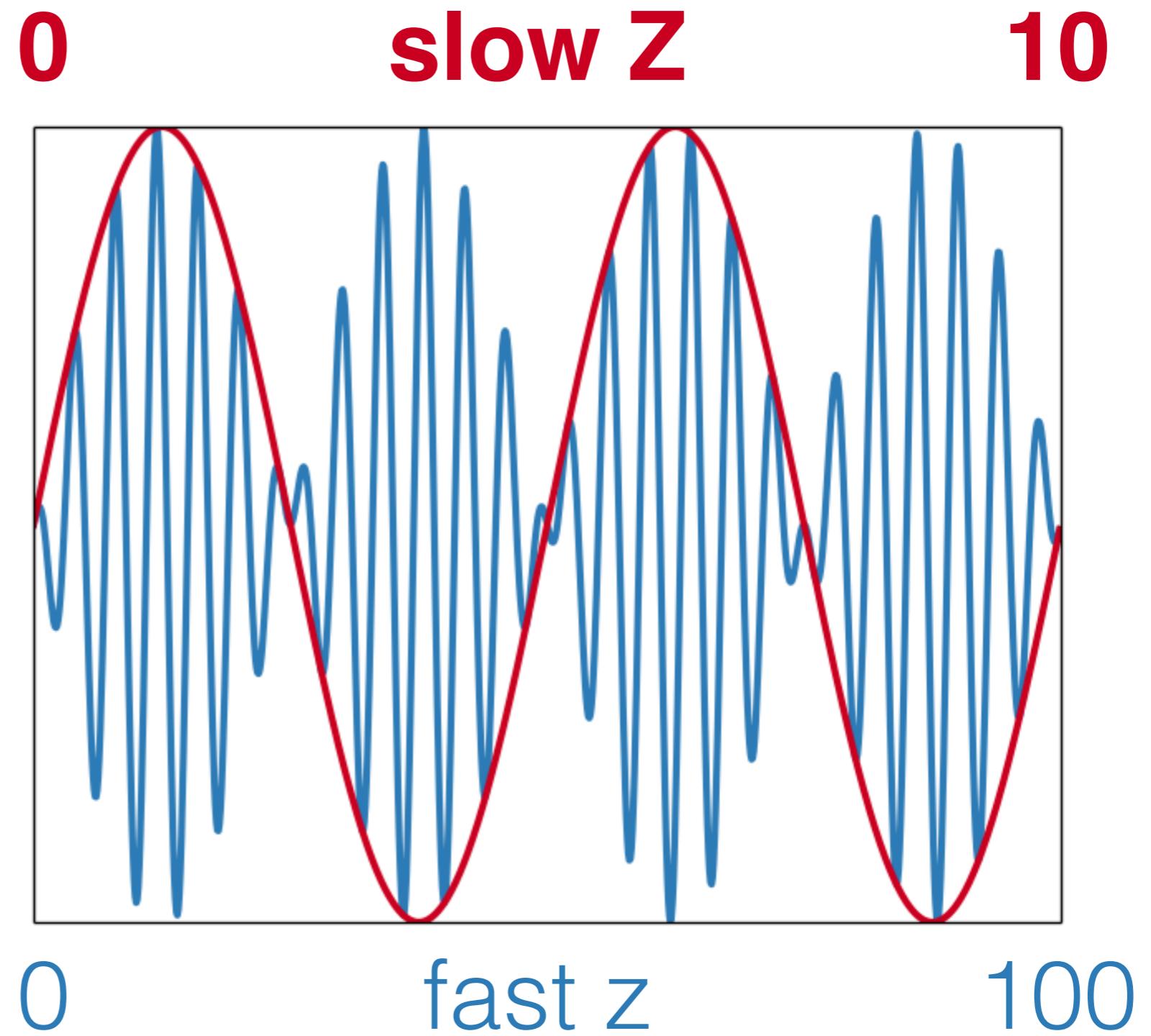
small parameter
↓

$$B = \frac{B}{B_0} (1 - \epsilon^2)$$



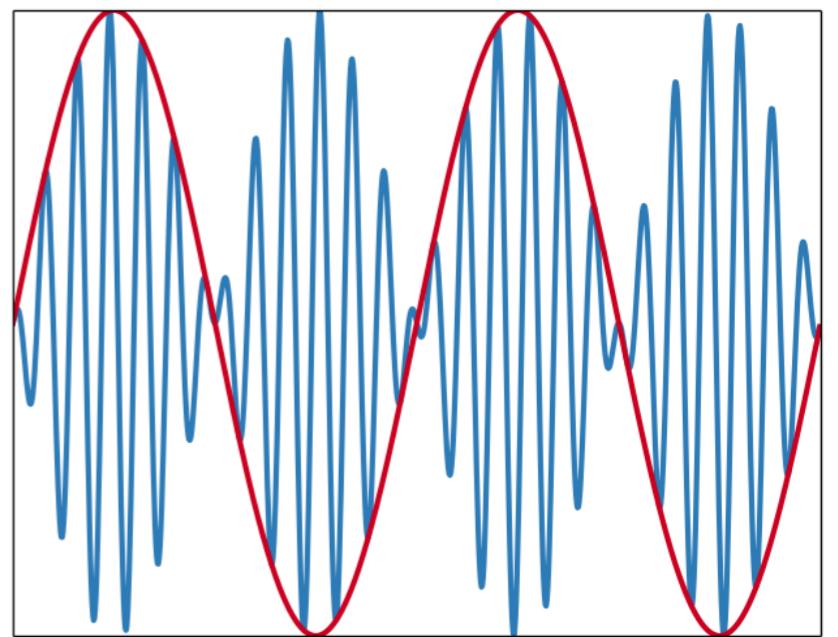
Multiscale analysis tracks the evolution of fast and slow variables.

$$Z \equiv \epsilon z$$



We choose an ansatz state vector form.

$$\mathbf{V} = \alpha(Z, T) V(x) e^{ik_c z}$$



We choose an ansatz state vector form.

$$V = \alpha(Z, T) V(x) e^{ik_c z}$$

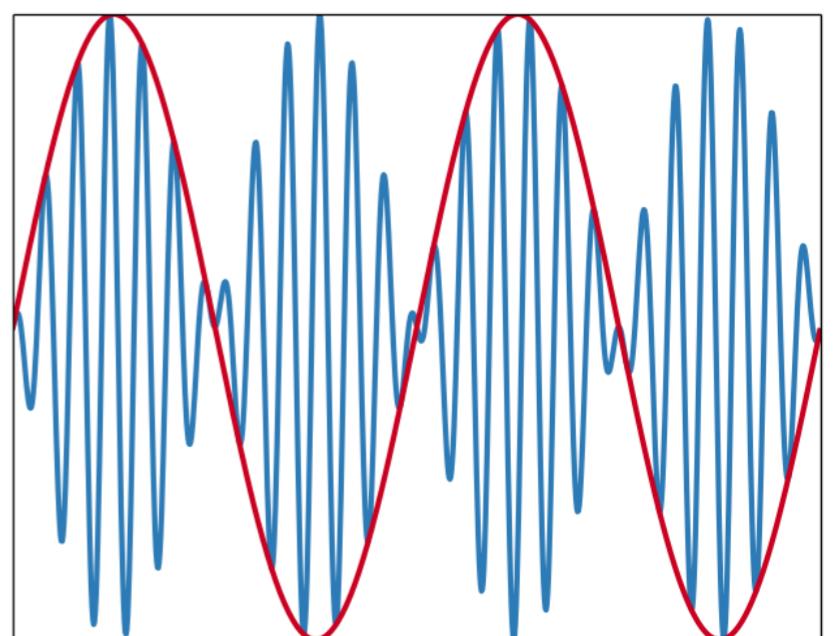
x dependence

↓

↑

amplitude function

vertical periodicity



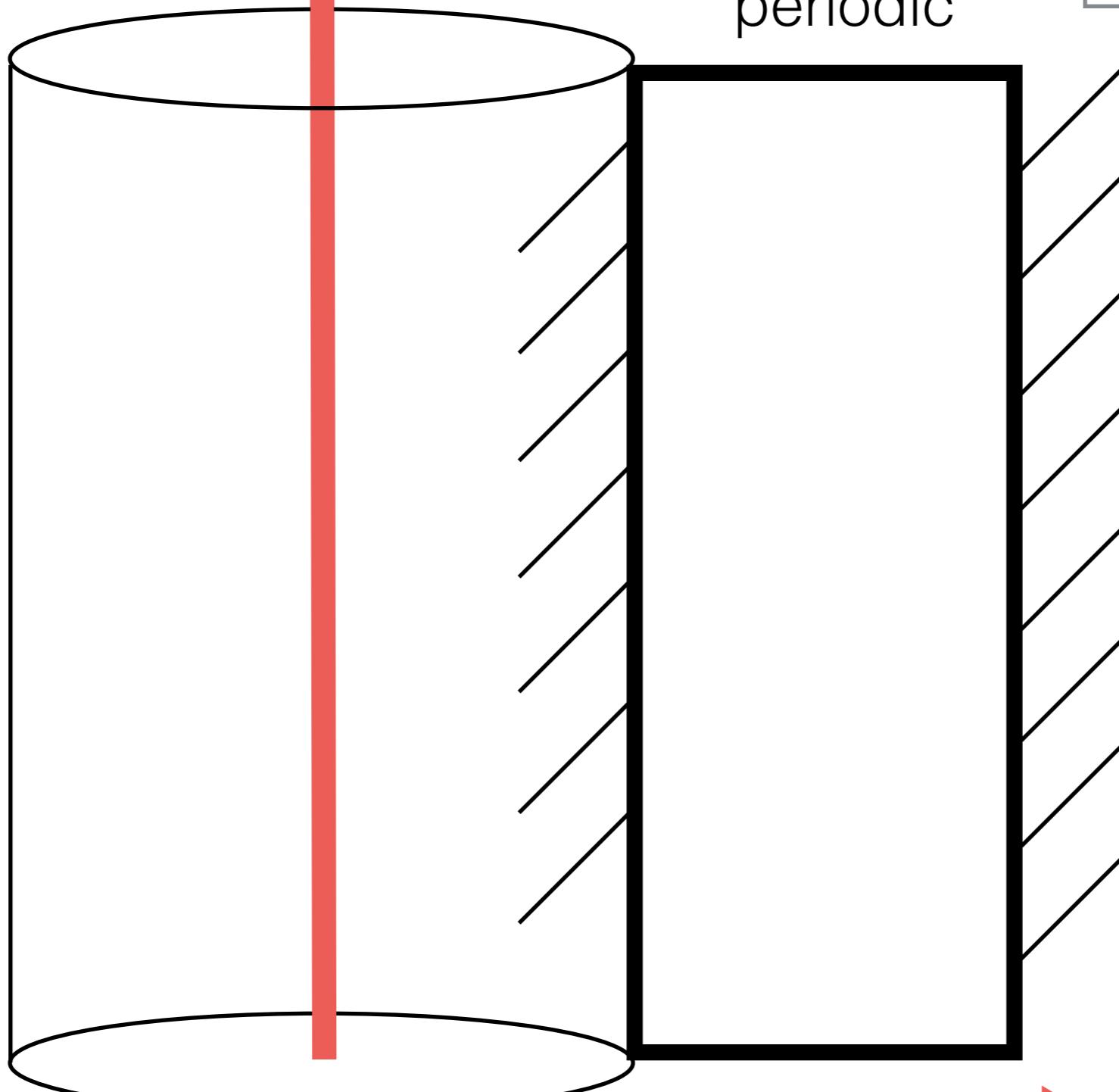
The fluid quantities are expanded
in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + \epsilon^3 \mathbf{V}_3 + \dots$$

Dedalus is a general-purpose spectral code.

Fourier

Z



dedalus-project.org

Burns+, in prep.

no slip,
perfectly
conducting.

X Chebyshev

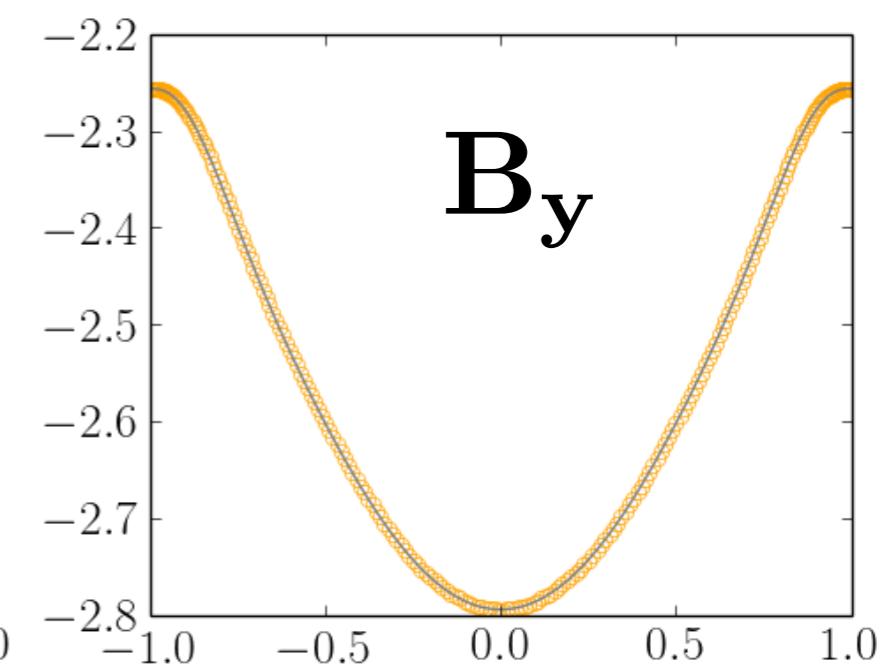
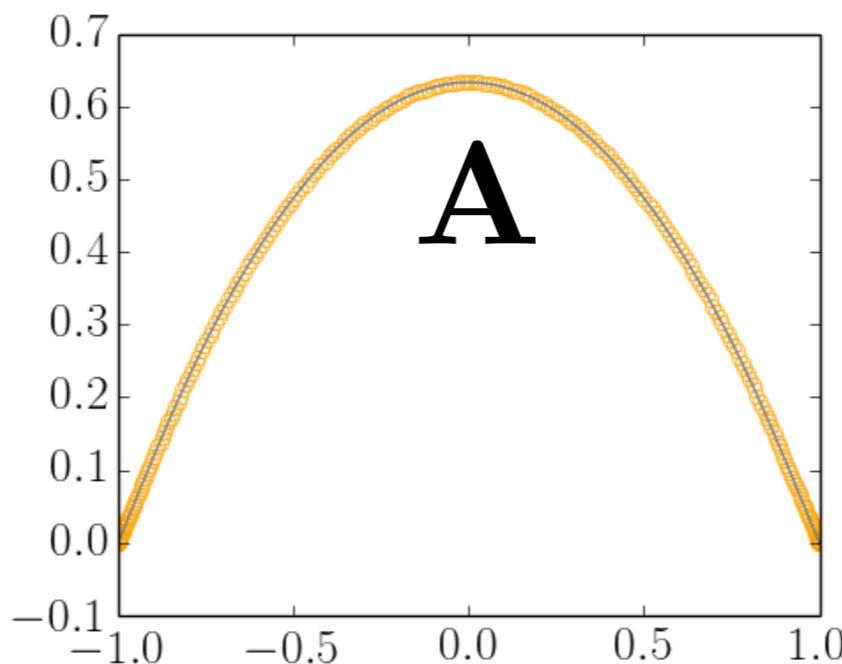
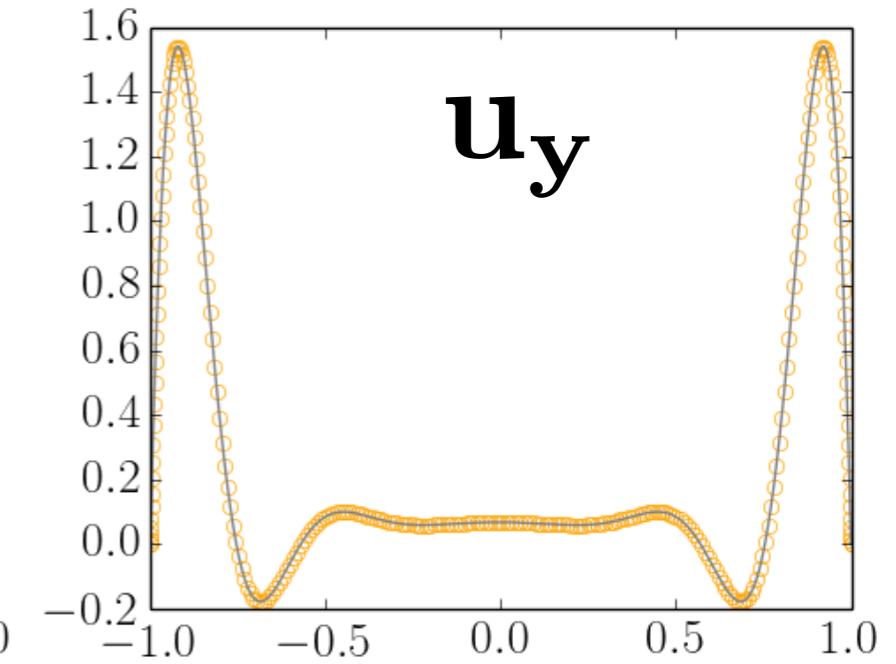
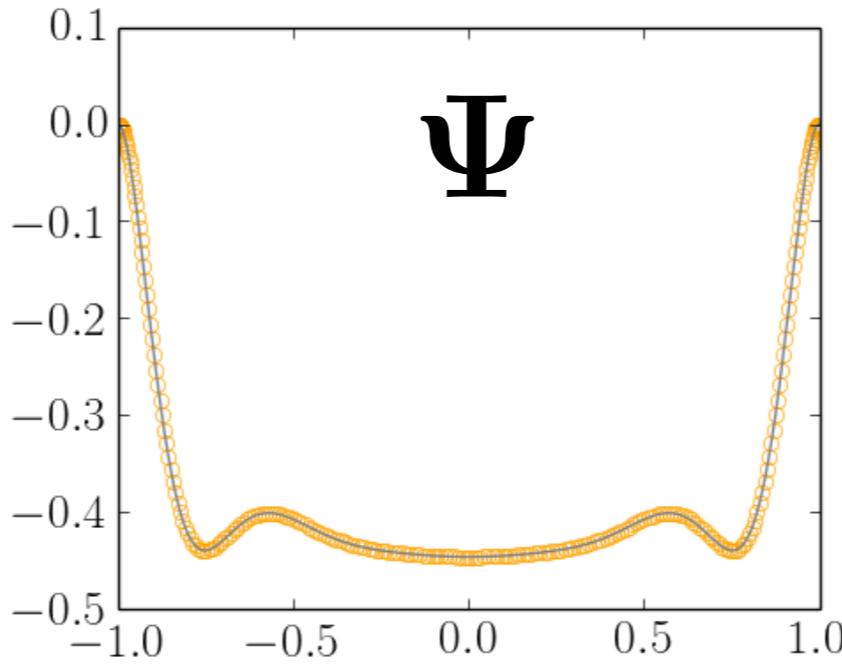
Spectrally solve the most unstable mode
of the linear MRI.

$$V_{11}(x)$$

$$\mathbf{V} = \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + \epsilon^3 \mathbf{V}_3 + \dots$$

$$\mathcal{L}\mathbf{V}_1 = 0$$

$$\mathbf{V}_1 = \alpha(T, Z) V_{11}(x) e^{ik_c z}$$



X

X

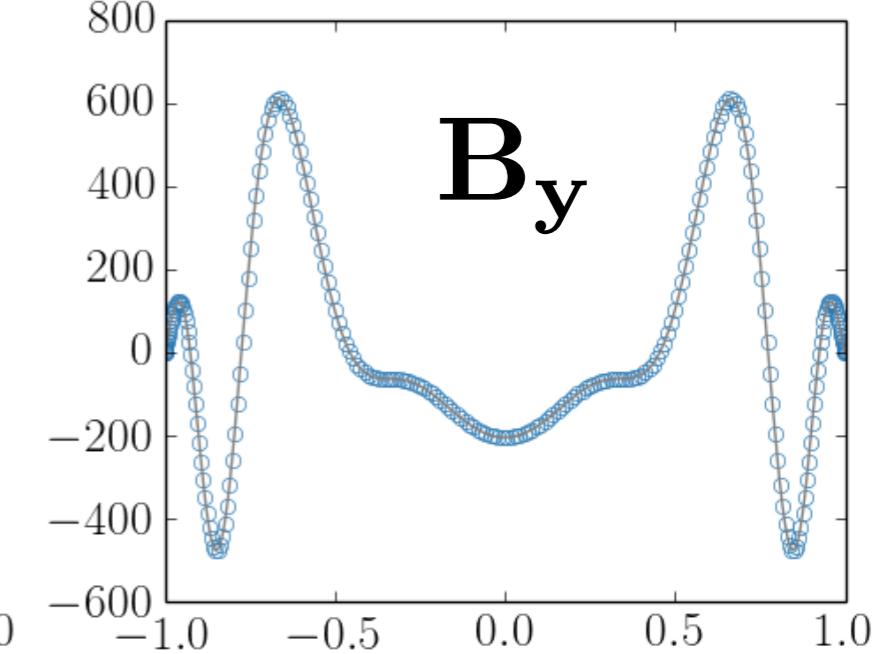
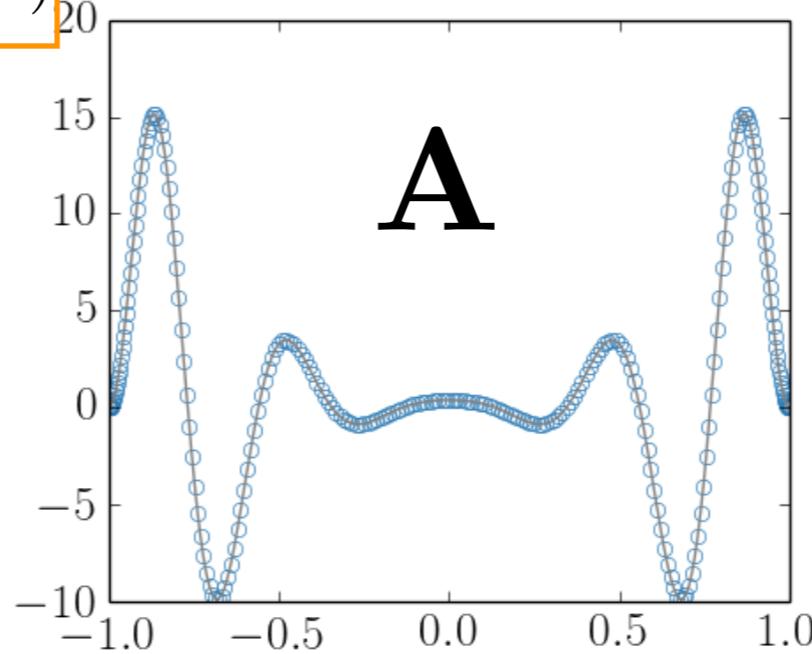
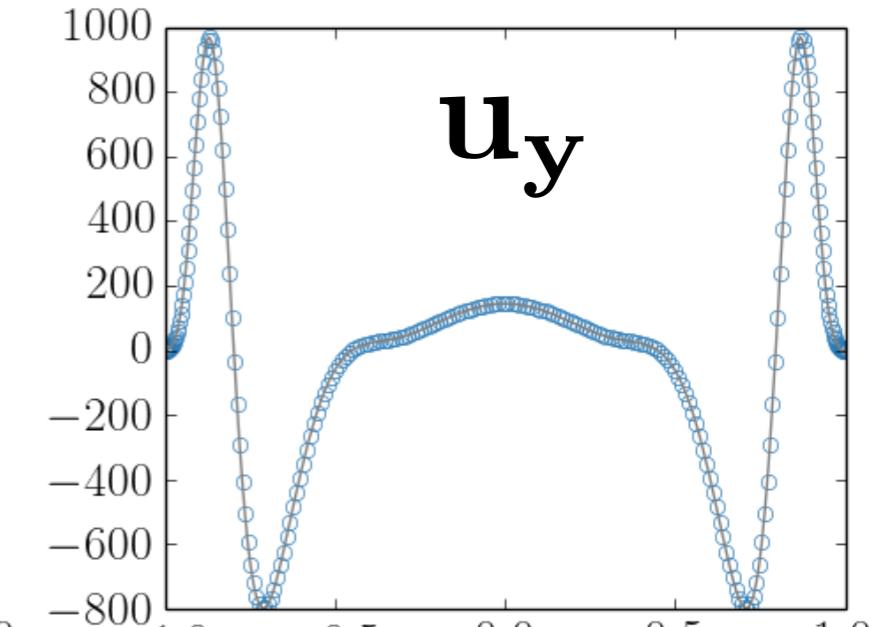
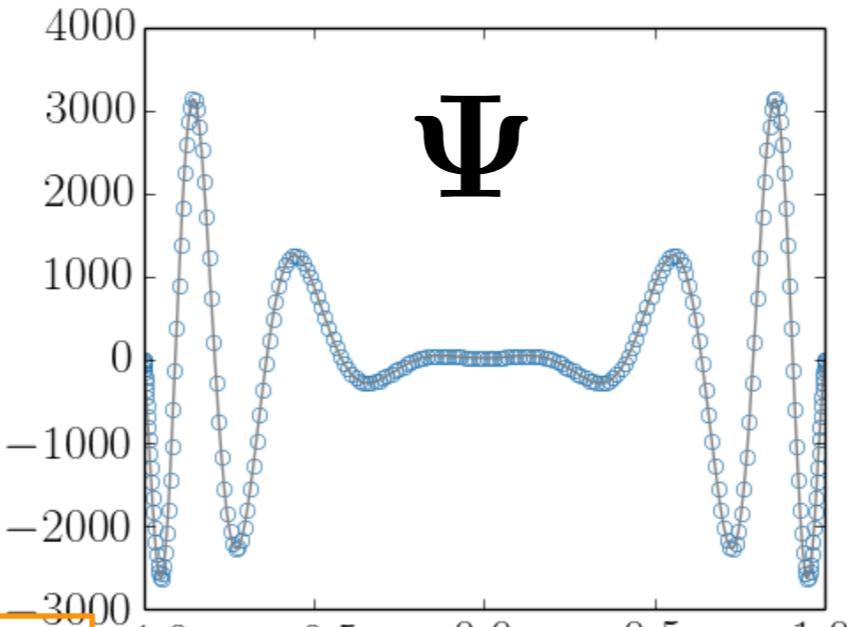
We solve each term in the expanded equations at each order.

$$\mathbf{V} = \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + \epsilon^3 \mathbf{V}_3 + \dots$$

$$\mathcal{L} \mathbf{V}_2 = \mathbf{N}_2 - \tilde{\mathcal{L}}_1 \partial_Z \mathbf{V}_1$$

$$\mathbf{N}_2 = \alpha^2 \mathbf{N}_{22}(x) e^{i2k_c z} + |\alpha|^2 \boxed{\mathbf{N}_{20}(x)}$$

$$\mathbf{N}_{20}(x)$$



X

X

$$\mathcal{D}\partial_t \mathbf{V} + \epsilon^2 \mathcal{D}\partial_T \mathbf{V} + \epsilon^2 \mathcal{D}^*\partial_Z^2 \partial_t \mathbf{V} + 2\epsilon \mathcal{D}^*\partial_z \partial_Z \partial_t \mathbf{V} + \mathbf{N} = \mathcal{L}\mathbf{V} + \epsilon \mathcal{L}_1 \partial_Z \mathbf{V} + \epsilon^2 \mathcal{L}_2 \partial_Z^2 \mathbf{V} - \epsilon^2 \partial_z \mathcal{X} \mathbf{V} - \epsilon^2 \partial_z^3 \mathcal{L}_3 \mathbf{V} + \mathcal{O}(\epsilon^3)$$

$$\mathbf{N} = [N^{(\Psi)}, N^{(u_{1y})}, N^{(A)}, N^{(B_{1y})}]^{\mathbf{T}}. \quad N^{(\Psi)} = J(\Psi, \nabla^2 \Psi) - J(A, \nabla^2 A):$$

The removal of secular terms yields solvability criteria.

$$J(\Psi, \nabla^2 \Psi) = \partial_x \Psi \partial_x \nabla^2 \Psi - \partial_x \Psi \partial_z \nabla^2 \Psi - \partial_z \Psi \partial_x \nabla^2 \Psi - \partial_z \Psi \partial_z \nabla^2 \Psi - \partial_z \Psi \partial_x \nabla^2 \Psi - \partial_x \Psi \partial_z \nabla^2 \Psi + \epsilon \partial_z \Psi \partial_x \partial_z \partial_Z \Psi + 2\epsilon \partial_z \Psi \partial_x \partial_z \partial_Z \Psi - 2\epsilon \partial_x \Psi \partial_z \partial_Z \Psi + \epsilon \partial_Z \Psi \partial_x \nabla^2 \Psi - \epsilon \partial_x \Psi \partial_Z \nabla^2 \Psi + \epsilon^3 \partial_Z \Psi \partial_x \partial_Z^2 \Psi - \epsilon^3 \partial_x \Psi \partial_Z^3 \Psi + 2\epsilon^2 \partial_Z \Psi \partial_x \partial_z \partial_Z \Psi - 2\epsilon^2 \partial_x \Psi \partial_z \partial_Z^2 \Psi = J(\Psi, \nabla^2 \Psi) + \epsilon^2 J(\Psi, \partial_Z^2 \Psi) + 2\epsilon J(\Psi, \partial_z \partial_Z \Psi) + \epsilon \tilde{J}(\Psi, \nabla^2 \Psi) + \epsilon^3 \tilde{J}(\Psi, \partial_Z^2 \Psi) + 2\epsilon^2 \tilde{J}(\Psi, \partial_z \partial_Z \Psi)$$

$$\tilde{J}(f, g) \equiv \partial_Z f \partial_x g - \partial_x f \partial_Z g$$

$$J(a+b, c+d) = J(a, c) + J(a, d) + J(b, c) + J(b, d).$$

$$J(\epsilon \Psi_1 + \epsilon^2 \Psi_2, \epsilon \nabla^2 \Psi_1 + \epsilon^2 \nabla^2 \Psi_2)$$

$$= J(\epsilon \Psi_1, \epsilon \nabla^2 \Psi_1) + J(\epsilon \Psi_1, \epsilon^2 \nabla^2 \Psi_2) + J(\epsilon^2 \Psi_2, \epsilon \nabla^2 \Psi_1) + J(\epsilon^2 \Psi_2, \epsilon^2 \nabla^2 \Psi_2)$$

$$= \epsilon^2 J(\Psi_1, \nabla^2 \Psi_1) + \epsilon^3 J(\Psi_1, \nabla^2 \Psi_2) + \epsilon^3 J(\Psi_2, \nabla^2 \Psi_1) + \epsilon^4 J(\Psi_2, \nabla^2 \Psi_2)$$

$$2\epsilon J(\epsilon \Psi_1 + \epsilon^2 \Psi_2, \epsilon \partial_z \partial_Z \Psi_1 + \epsilon^2 \partial_z \partial_Z \Psi_2)$$

$$= 2\epsilon^3 J(\Psi_1, \partial_z \partial_Z \Psi_1) + 2\epsilon^4 J(\Psi_1, \partial_z \partial_Z \Psi_2) + 2\epsilon^4 J(\Psi_2, \partial_z \partial_Z \Psi_1) + 2\epsilon^5 J(\Psi_2, \partial_z \partial_Z \Psi_2)$$

$$\epsilon \tilde{J}(\epsilon \Psi_1 + \epsilon \Psi_2, \epsilon \nabla^2 \Psi_1 + \epsilon^2 \nabla^2 \Psi_2)$$

$$= \epsilon^3 \tilde{J}(\Psi_1, \nabla^2 \Psi_1) + \epsilon^4 \tilde{J}(\Psi_1, \nabla^2 \Psi_2) + \epsilon^4 \tilde{J}(\Psi_2, \nabla^2 \Psi_1) + \epsilon^5 \tilde{J}(\Psi_2, \nabla^2 \Psi_2)$$

$$N^{(\Psi)} \rightarrow J(\Psi, \nabla^2 \Psi) \rightarrow \epsilon^2 J(\Psi_1, \nabla^2 \Psi_1) + \epsilon^3 J(\Psi_1, \nabla^2 \Psi_2) + \epsilon^3 J(\Psi_2, \nabla^2 \Psi_1) + 2\epsilon^3 J(\Psi_1, \partial_z \partial_Z \Psi_1) + \epsilon^3 \tilde{J}(\Psi_1, \nabla^2 \Psi_1) + \mathcal{O}(\epsilon^4)$$

$$N^{(\Psi)} \rightarrow \epsilon^2 J(\Psi_1, \nabla^2 \Psi_1) - \epsilon^2 \frac{1}{4\pi} J(A_1, \nabla^2 A_1) + \epsilon^3 J(\Psi_1, \nabla^2 \Psi_2) - \epsilon^3 \frac{1}{4\pi} J(A_1, \nabla^2 A_2) + \epsilon^3 J(\Psi_2, \nabla^2 \Psi_1) - \epsilon^3 \frac{1}{4\pi} J(A_2, \nabla^2 A_1) + 2\epsilon^3 J(\Psi_1, \partial_z \partial_Z \Psi_1)$$

$$2\epsilon^3 \frac{1}{4\pi} J(A_1, \partial_z \partial_Z A_1) + \epsilon^3 \tilde{J}(\Psi_1, \nabla^2 \Psi_1) - \epsilon^3 \frac{1}{4\pi} \tilde{J}(A_1, \nabla^2 A_1) + \mathcal{O}(\epsilon^4)$$

$$N^{(u_{1y})} = J(\Psi, u_{1y}) - \frac{1}{4\pi} J(A, B_{1y})$$

$$= J(\Psi, u_{1y}) + \epsilon \tilde{J}(\Psi, u_{1y}) - \frac{1}{4\pi} J(A, B_{1y}) - \frac{1}{4\pi} \epsilon \tilde{J}(A, B_{1y})$$

$$N^{(u_{1y})} \rightarrow \epsilon^2 J(\Psi_1, u_1) - \frac{1}{4\pi} \epsilon^2 J(A_1, B_1) + \epsilon^3 J(\Psi_1, u_2) + \epsilon^3 J(\Psi_2, u_1) + \epsilon^3 \tilde{J}(\Psi_1, u_1) - \frac{1}{4\pi} \epsilon^3 J(A_1, B_2) - \frac{1}{4\pi} \epsilon^3 J(A_2, B_1) - \frac{1}{4\pi} \epsilon^3 \tilde{J}(A_1, B_1) + \mathcal{O}(\epsilon^4)$$

$$N^{(A)} = -J(A, \Psi) = -J(A, \Psi) - \epsilon \tilde{J}(A, \Psi)$$

$$N^{(A)} \rightarrow -\epsilon^2 J(A_1, \Psi_1) - \epsilon^3 J(A_1, \Psi_2) - \epsilon^3 J(A_2, \Psi_1) - \epsilon^4 J(A_2, \Psi_2) - \epsilon^3 \tilde{J}(A_1, \Psi_1) - \epsilon^4 \tilde{J}(A_1, \Psi_2) - \epsilon^4 \tilde{J}(A_2, \Psi_1) - \epsilon^5 \tilde{J}(A_2, \Psi_2)$$

$$= J(\Psi, B_{1y}) + \epsilon \tilde{J}(\Psi, B_{1y}) - J(A, u_{1y}) - \epsilon \tilde{J}(A, u_{1y})$$

$$= \epsilon^2 J(\Psi_1, B_1) + \epsilon^3 J(\Psi_1, B_2) + \epsilon^3 J(\Psi_2, B_1) + \epsilon^4 J(\Psi_2, B_2) + \epsilon^3 \tilde{J}(\Psi_1, B_1) + \epsilon^4 \tilde{J}(\Psi_1, B_2) + \epsilon^4 \tilde{J}(\Psi_2, B_1) + \epsilon^5 \tilde{J}(\Psi_2, B_2) - \epsilon^2 J(A_1, u_1) -$$

$$\epsilon^3 J(A_1, u_2) - \epsilon^3 J(A_2, u_1) - \epsilon^4 J(A_2, u_2) - \epsilon^3 \tilde{J}(A_1, u_1) - \epsilon^4 \tilde{J}(A_1, u_2) - \epsilon^4 \tilde{J}(\Psi_2, B_1) - \epsilon^5 \tilde{J}(\Psi_2, B_2)$$

$$\mathbf{N} = \epsilon^2 \mathbf{N}_1 + \epsilon^3 \mathbf{N}_2 + \mathcal{O}(\epsilon^4)$$

$$N^{(\Psi)} = \epsilon^2 N_2^{(\Psi)} + \epsilon^3 N_3^{(\Psi)} + \mathcal{O}(\epsilon^4) \quad N^{(u)} = \epsilon^2 N_2^{(u)} + \epsilon^3 N_3^{(u)} + \mathcal{O}(\epsilon^4) \quad N^{(A)} = \epsilon^2 N_2^{(A)} + \epsilon^3 N_3^{(A)} + \mathcal{O}(\epsilon^4) \quad N^{(B)} = \epsilon^2 N_2^{(B)} + \epsilon^3 N_3^{(B)} + \mathcal{O}(\epsilon^4)$$

$$N_2^{(\Psi)} = J(\Psi_1, \nabla^2 \Psi_1) - \frac{1}{4\pi} J(A_1, \nabla^2 A_1)$$

$$N_2^{(u)} = J(\Psi_1, u_1) - \frac{1}{4\pi} J(A_1, B_1)$$

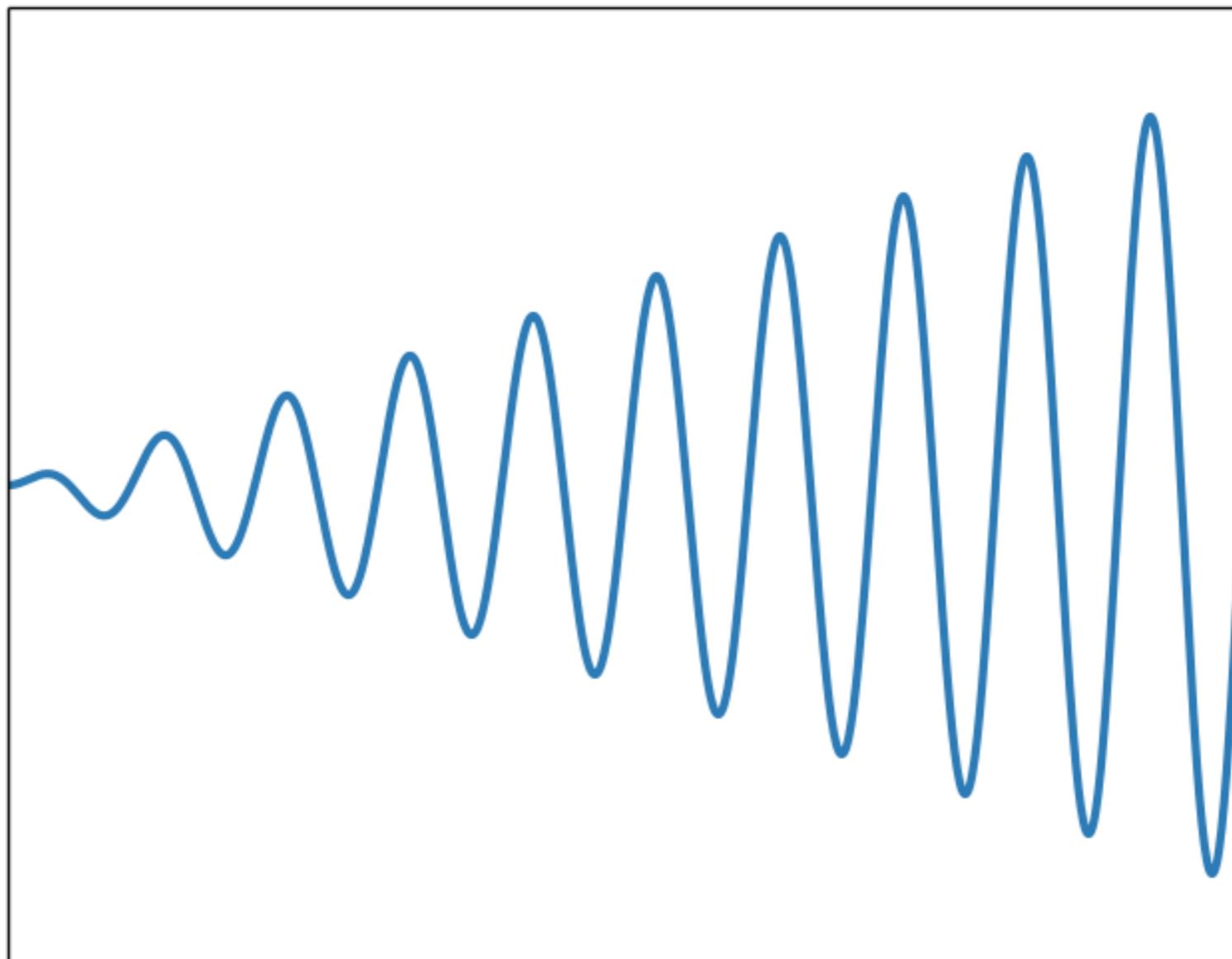
$$N_2^{(A)} = -J(A_1, \Psi_1)$$

$$N_2^{(B)} = J(\Psi_1, B_1) - J(A_1, u_1)$$

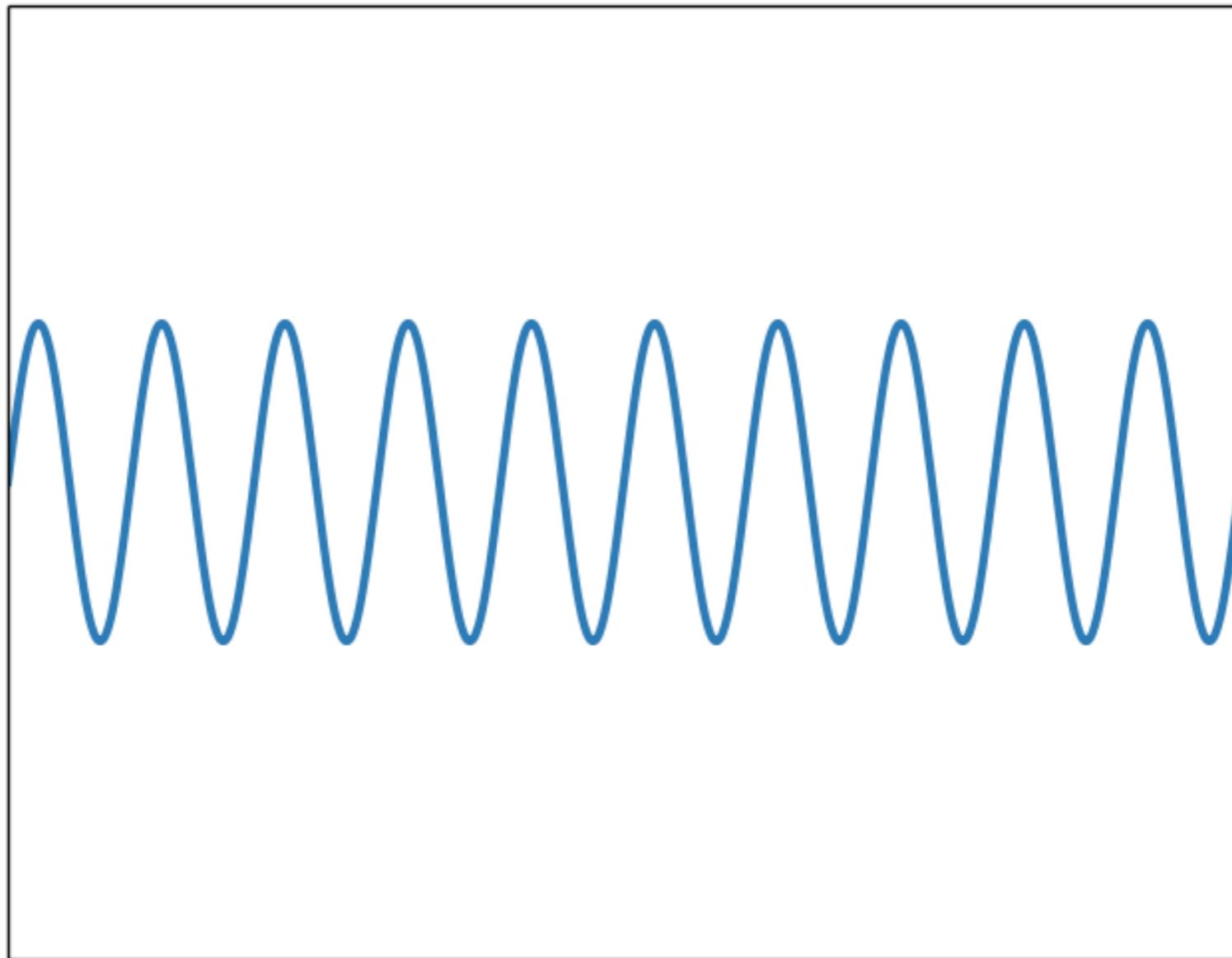
$$N_3^{(\Psi)} = J(\Psi_1, \nabla^2 \Psi_2) - \frac{1}{4\pi} J(A_1, \nabla^2 A_2) + J(\Psi_2, \nabla^2 \Psi_1) - \frac{1}{4\pi} J(A_2, \nabla^2 A_1) + 2J(\Psi_1, \partial_z \partial_Z \Psi_1) - 2\frac{1}{4\pi} J(A_1, \partial_z \partial_Z A_1) + \tilde{J}(\Psi_1, \nabla^2 \Psi_1) -$$

$$-\frac{1}{4\pi} \tilde{J}(A_1, \nabla^2 A_2)$$

The removal of secular terms yields solvability criteria.



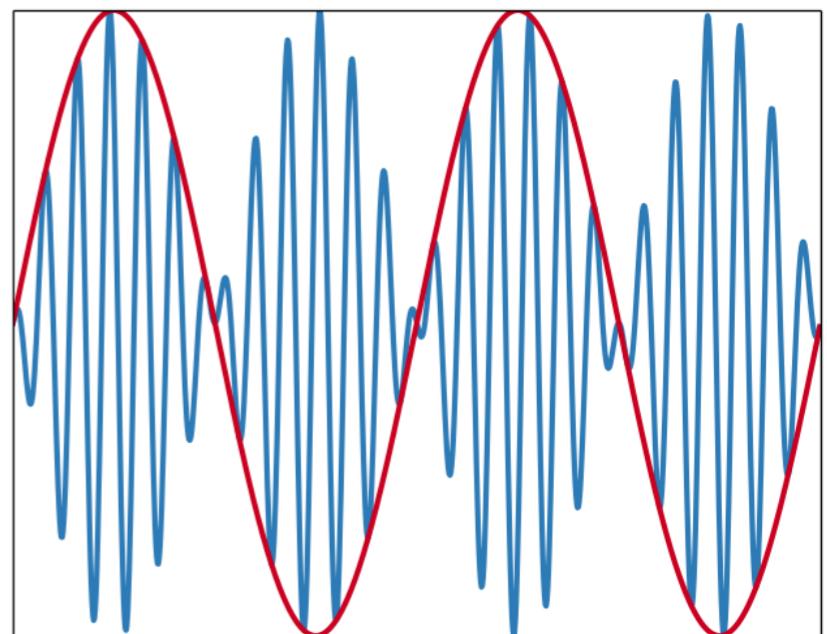
The removal of secular terms yields solvability criteria.



The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha^2| + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

α



The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha^2| + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

diffusion term 
nonlinear term 
linear growth 

The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b \partial_Z \alpha - c \alpha |\alpha^2| + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

?

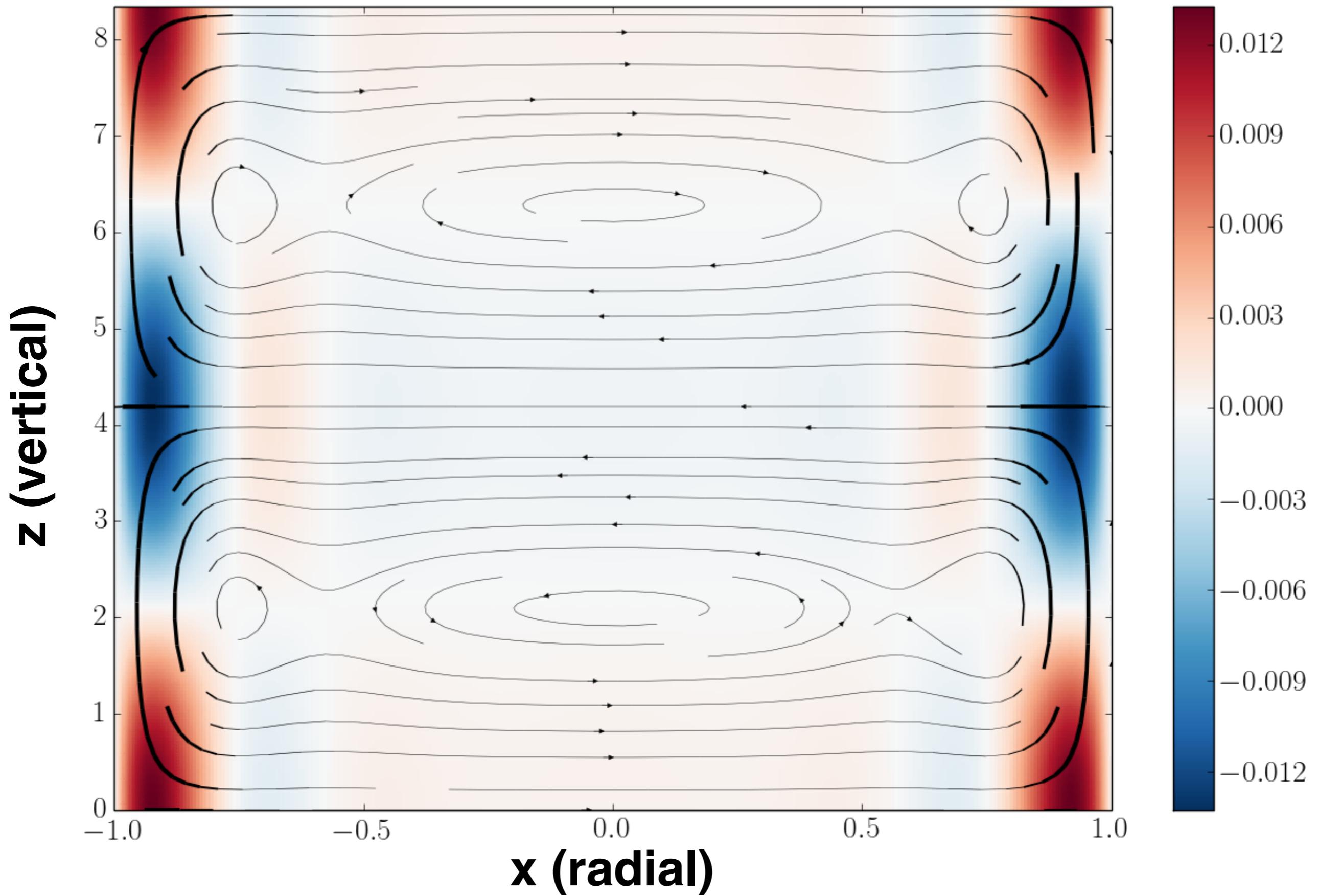
diffusion term 

nonlinear term 

linear growth 

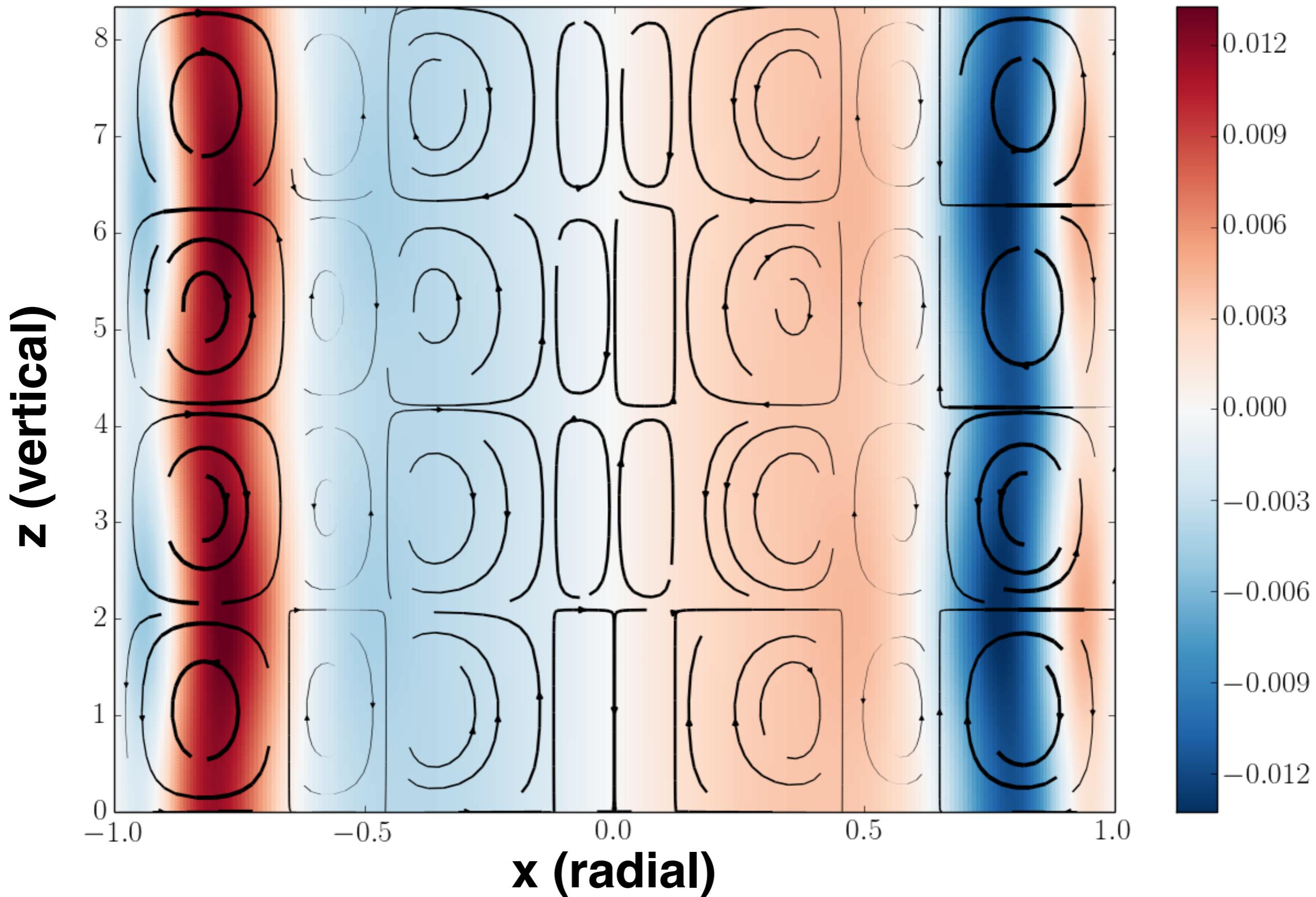
First order velocity perturbations

u_y



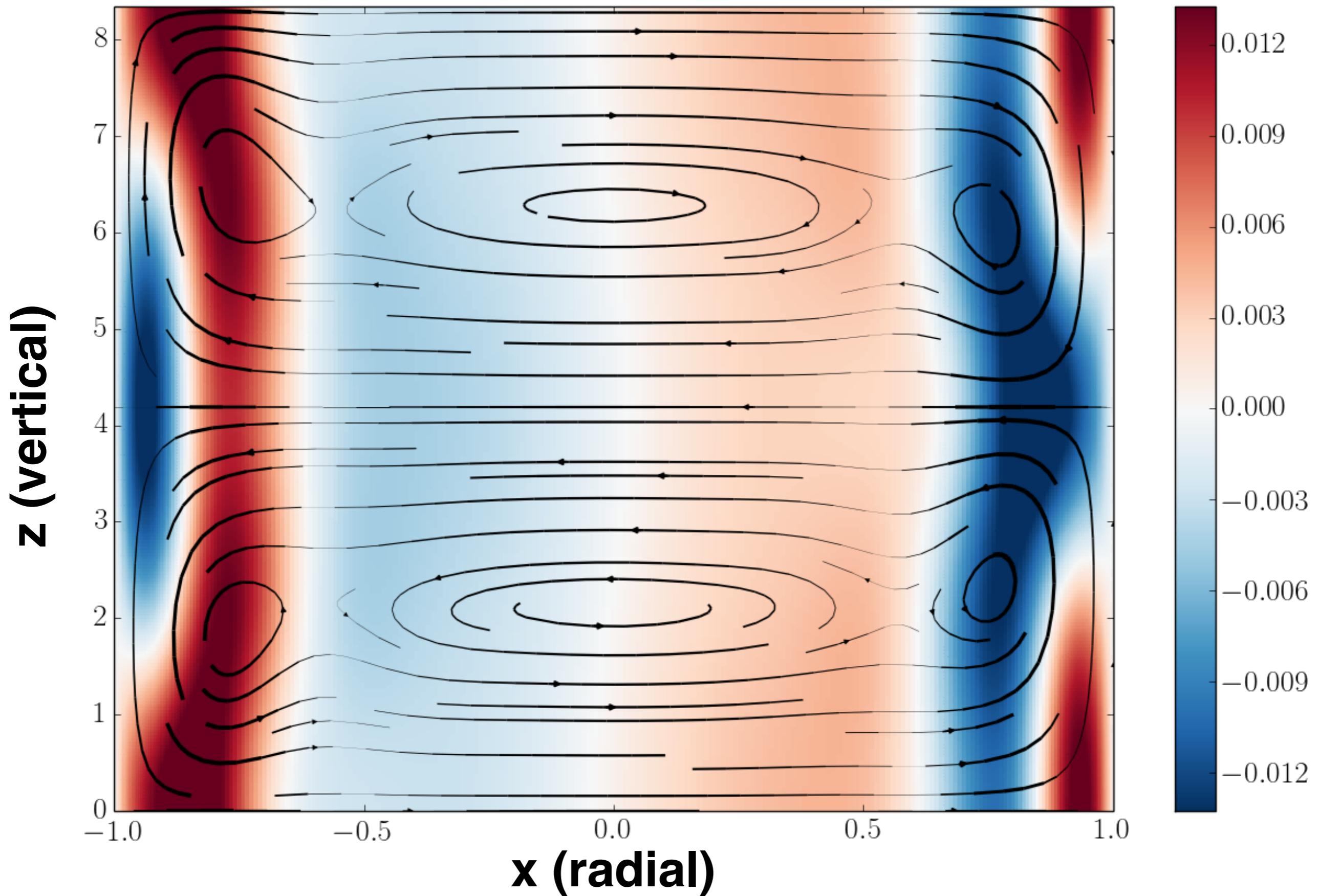
Second order velocity perturbations

u_y

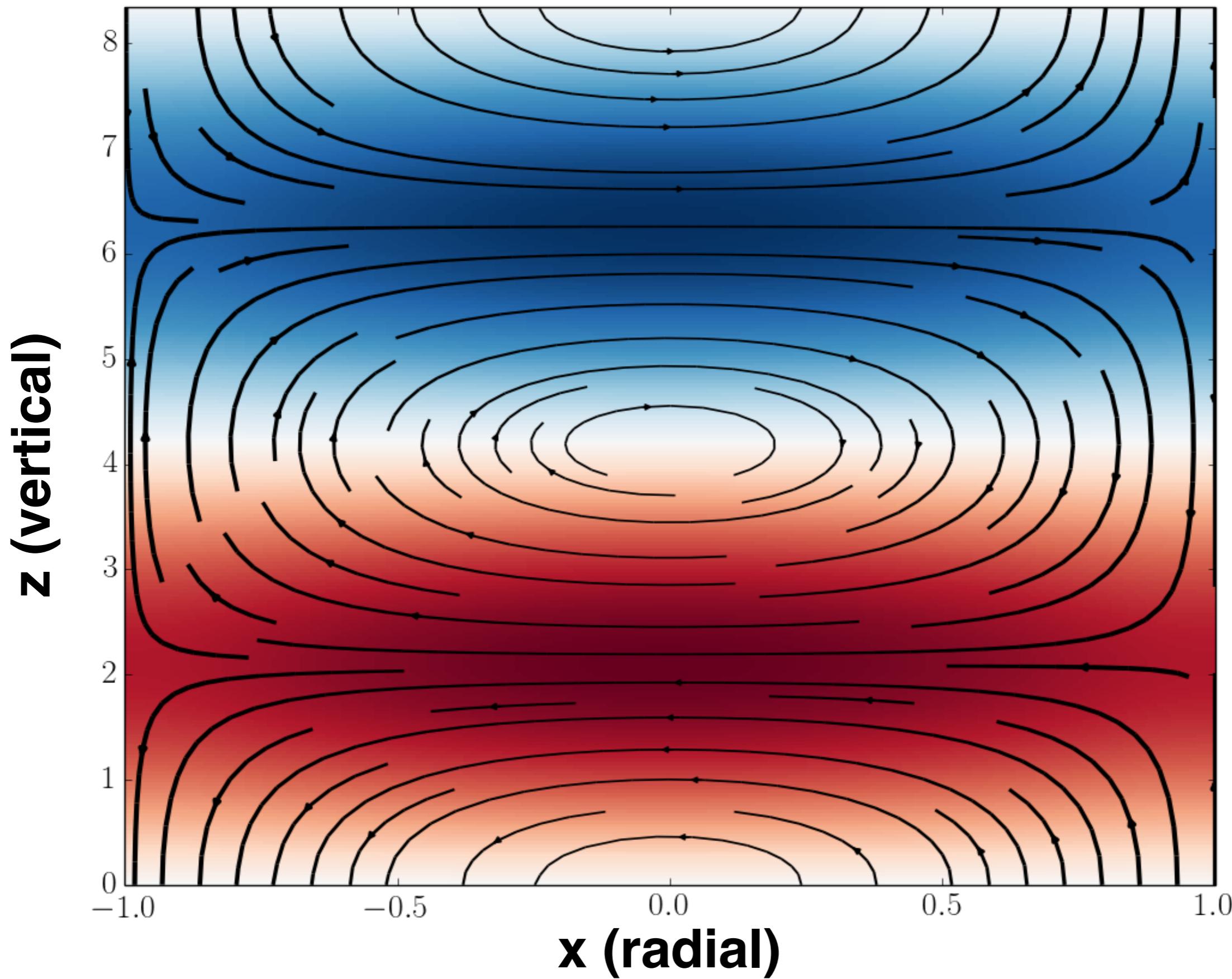
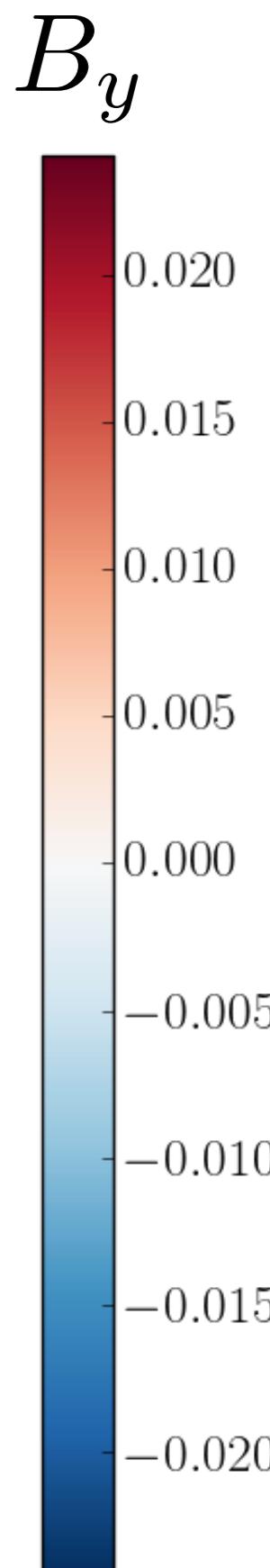


First and second order velocity perturbations

u_y

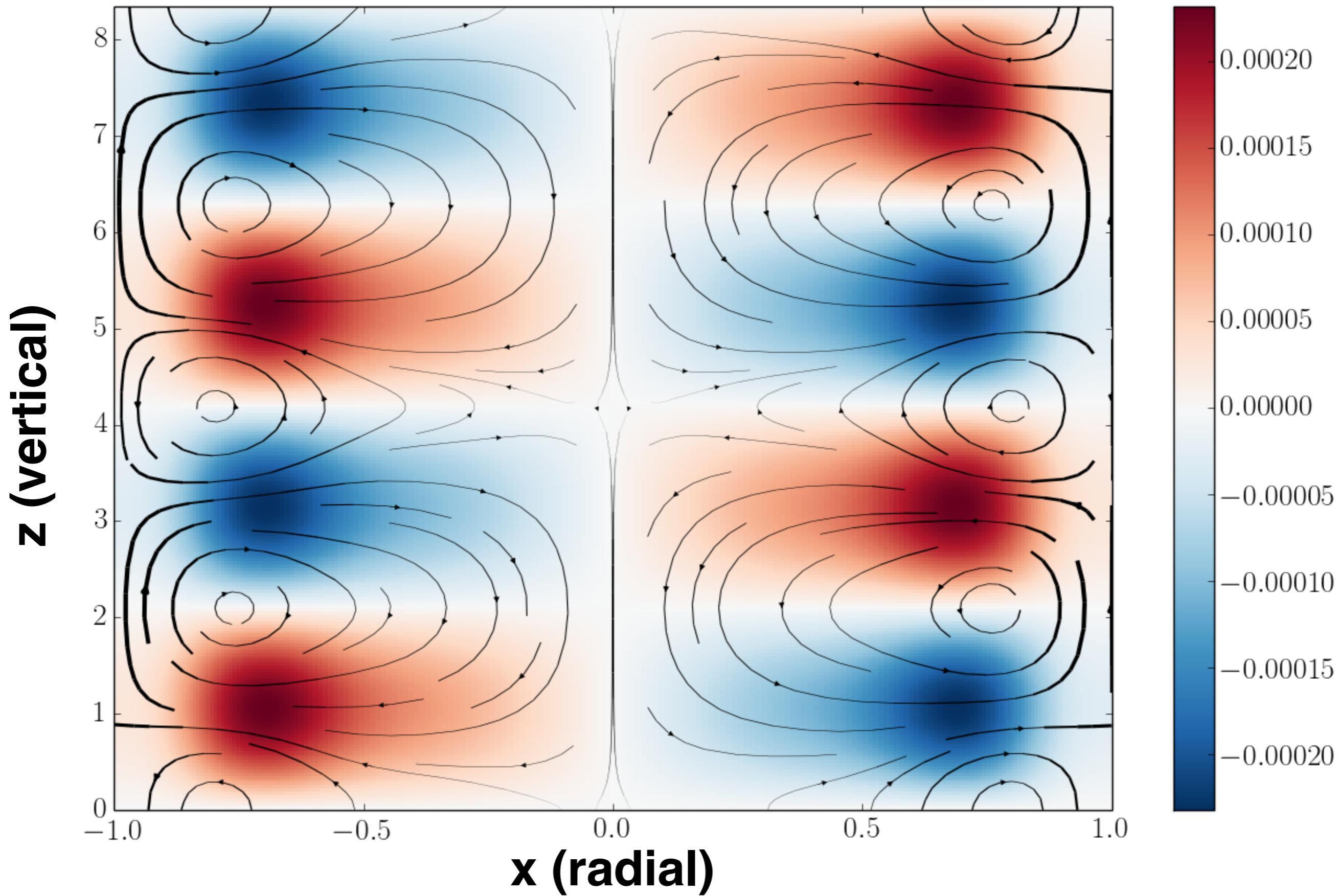


First order magnetic field perturbations

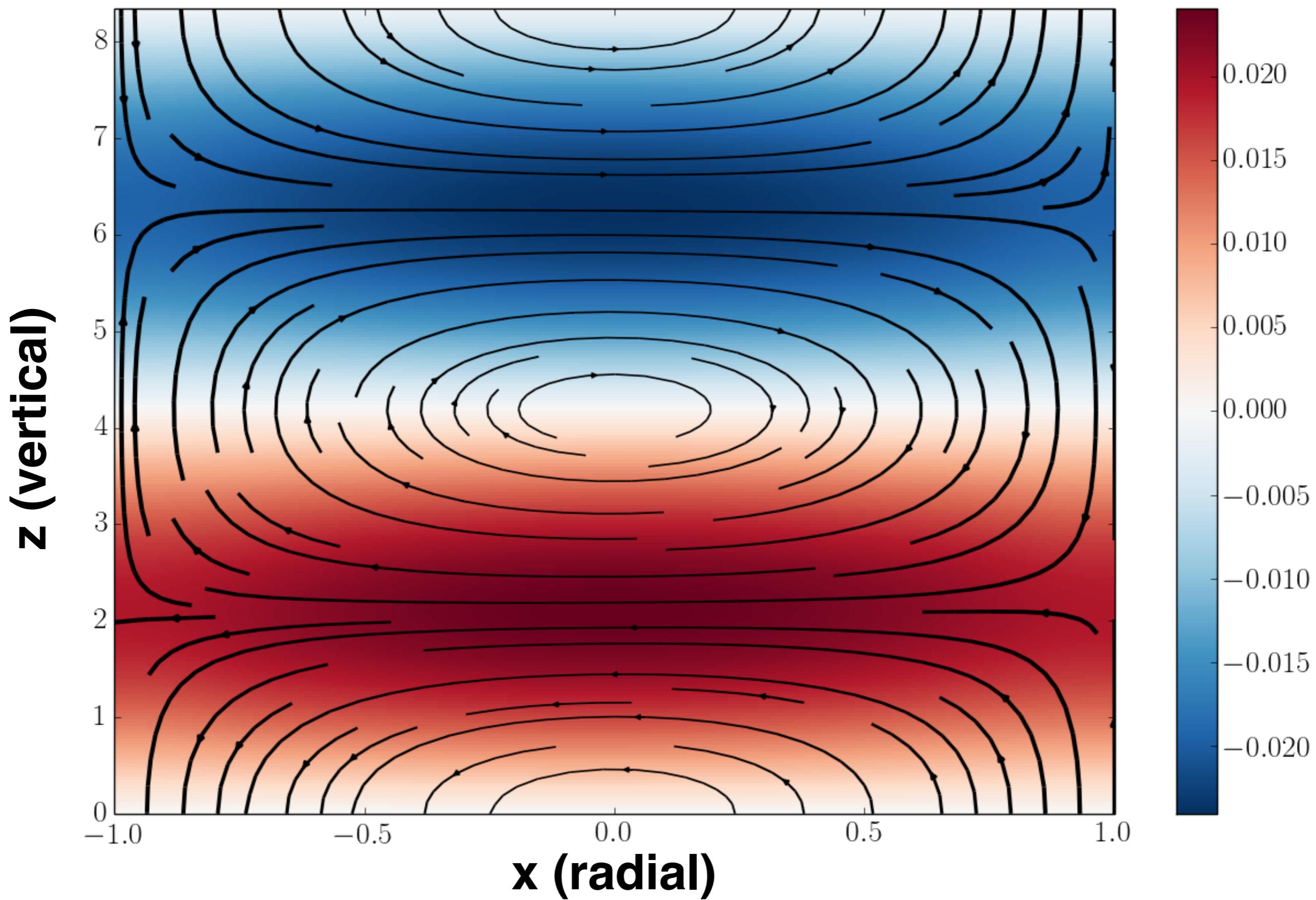


two OOM smaller!

Second order magnetic field perturbations



First and second order magnetic field perturbations B_y



Future work

- explore parameter space
- relax thin gap approximation
- comparison to experiment
- helical MRI

PPPL MRI experiment



Conclusions

- We derive a robust analytical framework for solving