1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi \; \hat{\mathbf{r}} \\ u_\phi \; \hat{\phi} \\ -\frac{1}{r}\partial_r\psi \; \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) &= \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ &+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ &+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \end{aligned} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{3}$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\mathrm{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{4}$$

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r}J(A, u_\phi) &+ \frac{1}{r}J(B_\phi, \psi) \\ &+ \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \end{aligned} \tag{5}$$

2 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_z \tag{6}$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\frac{1}{r^2}\partial_t\partial_r\Psi-\frac{1}{r}\partial_t\partial_r^2\Psi-\frac{3}{r^4}\partial_z\Psi\partial_r\Psi+\frac{1}{r^3}\partial_r\left(\partial_z\Psi\partial_r\Psi\right)+\frac{2}{r^3}\partial_z\Psi\partial_r^2\Psi-\frac{1}{r^2}\partial_r\left(\partial_z\Psi\partial_r^2\Psi\right)$$
$$-\frac{2}{r^3}\partial_r\Psi\partial_r\partial_z\Psi+\frac{1}{r^2}\partial_r\left(\partial_r\Psi\partial_r\partial_z\Psi\right)=$$
$$\partial_r\left(B_\theta\partial_z B_\theta\right)+\frac{2}{r^3}\partial_z^2A\partial_zA-\frac{1}{r^2}\partial_r\left(\partial_z^2A\partial_zA\right)+\frac{3}{r^4}\partial_zA\partial_rA-\frac{1}{r^3}\partial_r\left(\partial_zA\partial_rA\right)-\frac{2}{r^3}\partial_zA\partial_r^2A$$
$$+\frac{1}{r^2}\partial_r\left(\partial_zA\partial_r^2A\right)+\frac{1}{\mathrm{Re}}\left[\frac{3}{r^4}\partial_r\Psi-\frac{3}{r^3}\partial_r^2\Psi+\frac{2}{r^2}\partial_r^3\Psi-\frac{1}{r}\partial_r^4\Psi\right]\quad(7)$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r+\left[u\cdot\nabla u\right]_r=\left[(\nabla\times B)\times B\right]_r+\frac{1}{\mathrm{Re}}\left[\nabla^2u\right]_r\quad(8)$$

and take ∂_z of the expanded equation to obtain

$$\frac{1}{r}\partial_t\partial_z^2\Psi-\frac{1}{r^3}\partial_z\left(\partial_z\Psi\partial_z\Psi\right)+\frac{1}{r^2}\partial_z\left(\partial_z\Psi\partial_z\partial_r\Psi\right)-\frac{1}{r^2}\partial_z\left(\partial_r\Psi\partial_z^2\Psi\right)-\frac{1}{r}2u_\theta\partial_zu_\theta$$
$$=-\frac{1}{r^2}\partial_z^3A\partial_rA-\frac{1}{r^2}\partial_z^2A\partial_r\partial_zA+\frac{2}{r^3}\partial_r\partial_zA\partial_rA-\frac{1}{r^2}\partial_r^2\partial_zA\partial_rA-\frac{1}{r^2}\partial_r^2A\partial_r\partial_zA$$
$$+\frac{1}{\mathrm{Re}}\left[-\frac{1}{r^2}\partial_z^2\partial_r\Psi+\frac{1}{r}\partial_z^2\partial_r^2\Psi+\frac{1}{r}\partial_z^4\Psi\right]\quad(9)$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\theta\partial_zu_\theta\quad(10)$$

Note that the relevant quantity appears to be $\nabla^2\Psi-\frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_θ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right]\quad(11)$$

Full Ψ equation according to Susan:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\theta\partial_z u_\theta$$
$$=J\left(A,\frac{1}{r^2}\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\right)-\frac{2}{r}B_\theta\partial_z B_\theta$$
$$+\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right] \quad (12)$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

3 Recovery of Narrow Gap Equations

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi=\frac{\partial\psi}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi}+\frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \quad (13)$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u}=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{\partial u_z}{\partial z} \quad (14)$$

and

$$\nabla\times\mathbf{u}=\left(\frac{1}{r}\frac{\partial u_z}{\partial\phi}-\frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}}+\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\hat{\phi}+\left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \quad (15)$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi=\mathbf{u}\cdot\nabla u_\phi+\frac{u_r u_\phi}{r}, \quad (16)$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r=\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \quad (17)$$

$$(\nabla^2\mathbf{u})_\phi=\nabla^2 u_\phi+\frac{2}{r^2}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r^2} \quad (18)$$

$$(\nabla^2\mathbf{u})_z=\nabla^2 u_z, \quad (19)$$

where ∇ on the vector components is given by equation (13).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$