1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi \ \hat{\mathbf{r}} \\ u_\phi \ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi \ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Susan's question about the stream function definition:
This is defined as

$$\mathbf{u} = -\nabla \times \Psi(r,z)\hat{\phi} \tag{2}$$

Which expands to become

$$\mathbf{u} = \hat{\phi} \times \nabla\Psi = \hat{\phi} \times \left(\partial_r \hat{\mathbf{r}} + \frac{1}{r}\partial_\phi\hat{\phi} + \partial_z\hat{\mathbf{z}}\right) \tag{3}$$

According to the definitions below. Wouldn't that yield

$$\mathbf{u} = \begin{bmatrix} \partial_z\psi \ \hat{\mathbf{r}} \\ u_\phi \ \hat{\phi} \\ -\partial_r\psi \ \hat{\mathbf{z}} \end{bmatrix}; \tag{4}$$

i.e. the same thing as above but without the $\frac{1}{r}$ terms? I must be missing something...

Using the definitions in

$$\begin{aligned}\partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) = \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ + \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ + \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\}\end{aligned} \tag{5}$$

$$\partial_t u_\phi + J(\psi, u_\phi) + \frac{u_\phi\partial_r\psi}{r^2} = J(B_\phi, A) - \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{6}$$

$$\partial_t A = \frac{1}{r^2}J(\psi, A) - \eta\left[\frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right] \tag{7}$$

$$\begin{aligned}\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi) \\ + \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right)\end{aligned} \tag{8}$$

2 Recovery of Narrow Gap Equations

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{9}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{10}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{11}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{12}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \tag{13}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \tag{14}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{15}$$

where ∇ on the vector components is given by equation (9).