Exploring the saturation of the MRI via weakly nonlinear analysis

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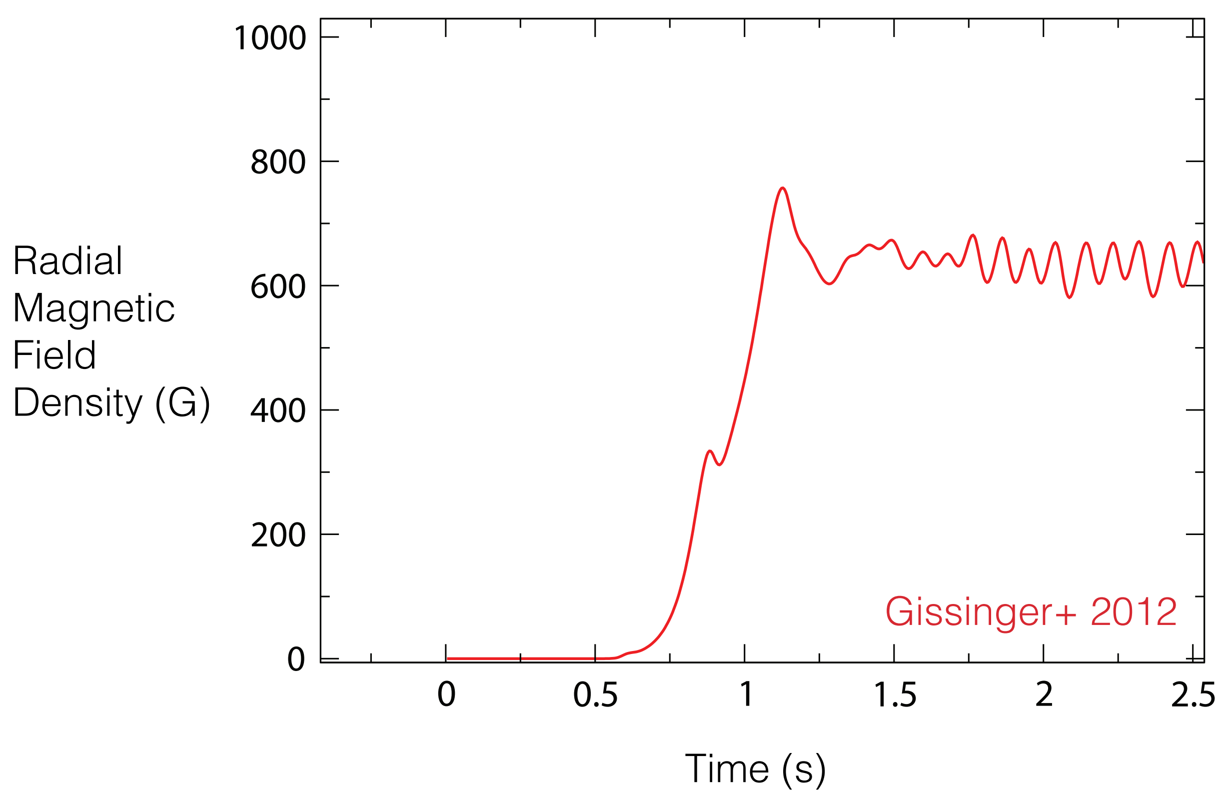
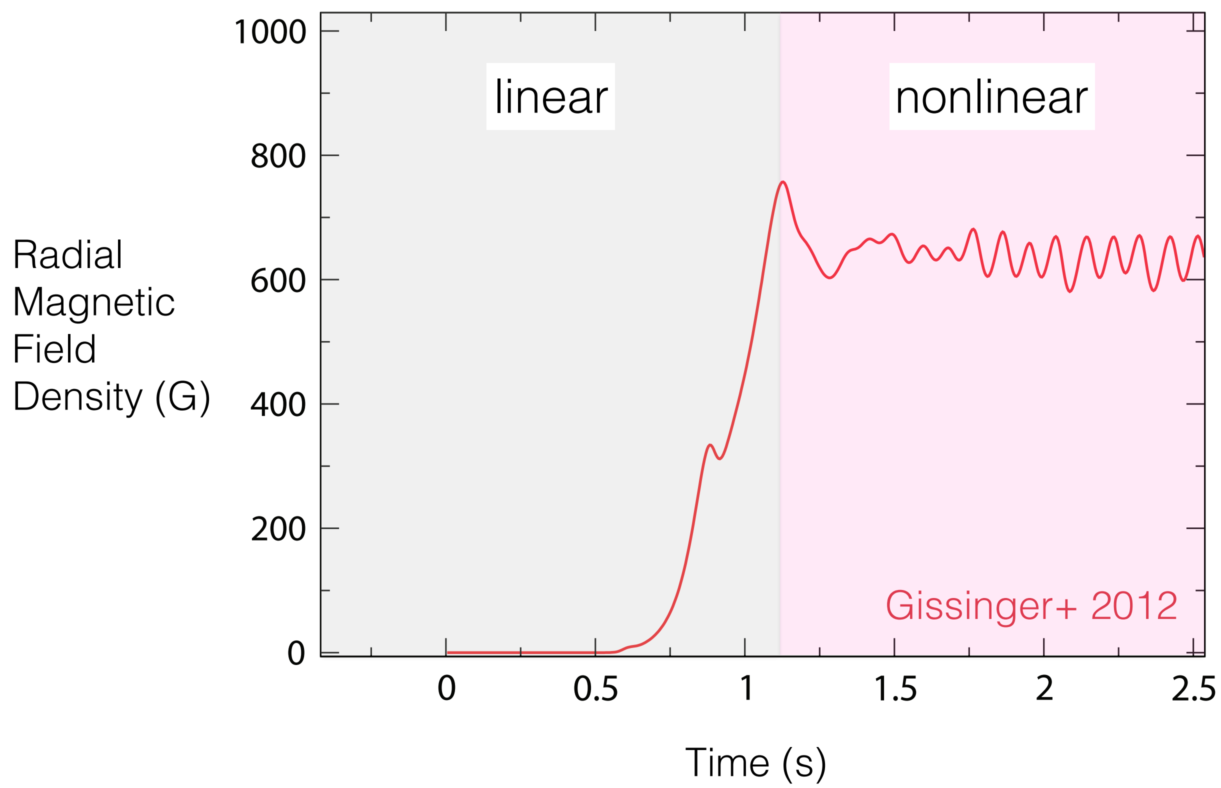
MRI saturation is well-studied in simulation.

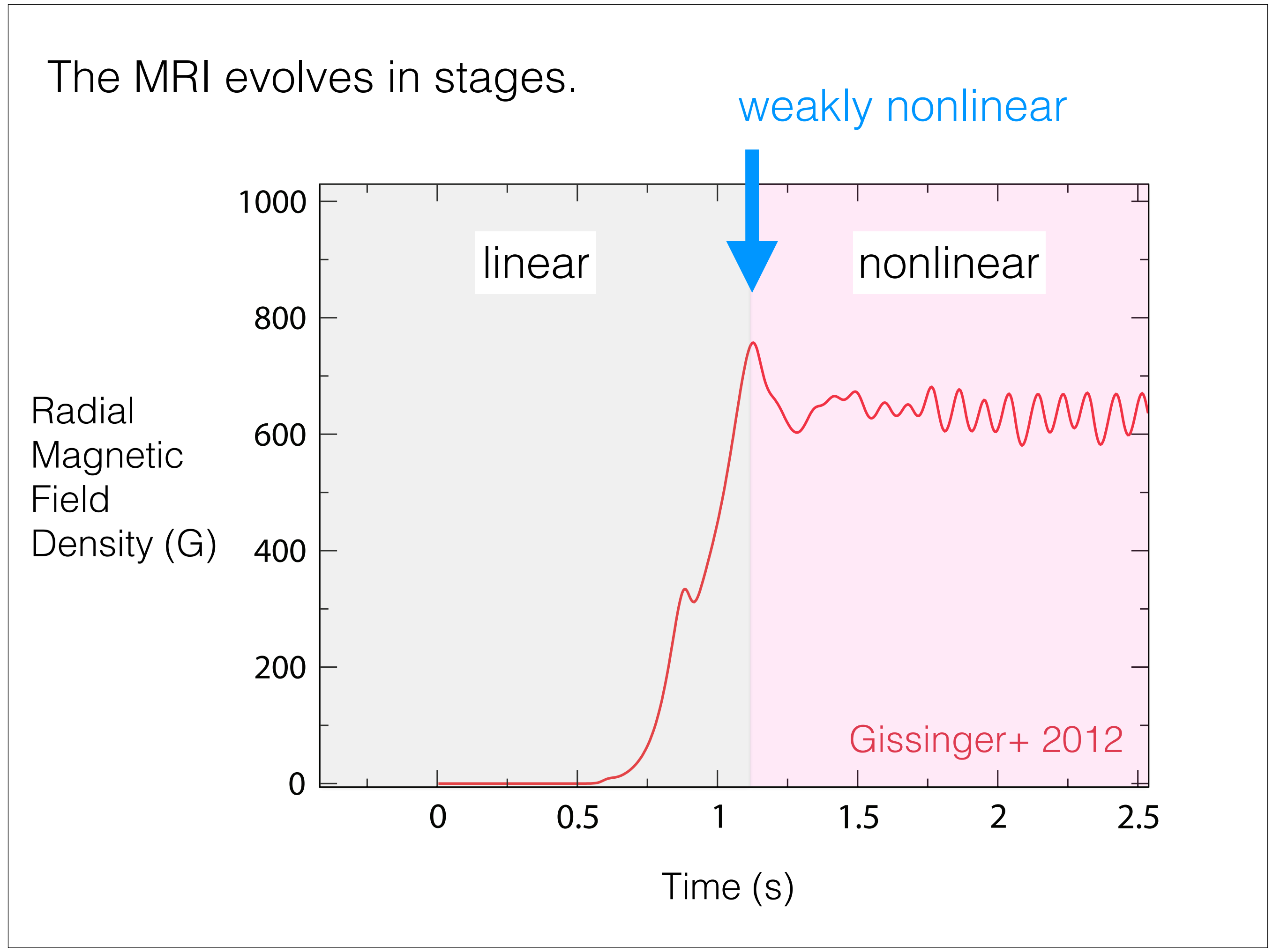


- much of what we know about the saturation of the MRI comes from simulation
- this is simulation data from Gissinger et al 2012, showing radial magnetic field density vs. time.
- I chose this of all simulation papers because this simulation is designed for comparison with the PPPL MRI experiment. We are interested in what our approach can tell us about these experimental results, something I'll touch on later.

The MRI evolves in stages.



- we see that the MRI has a period of linear growth, followed by nonlinear saturation.
-



- this work focuses right at the boundary, where linear growth begins to give way to nonlinear effects.
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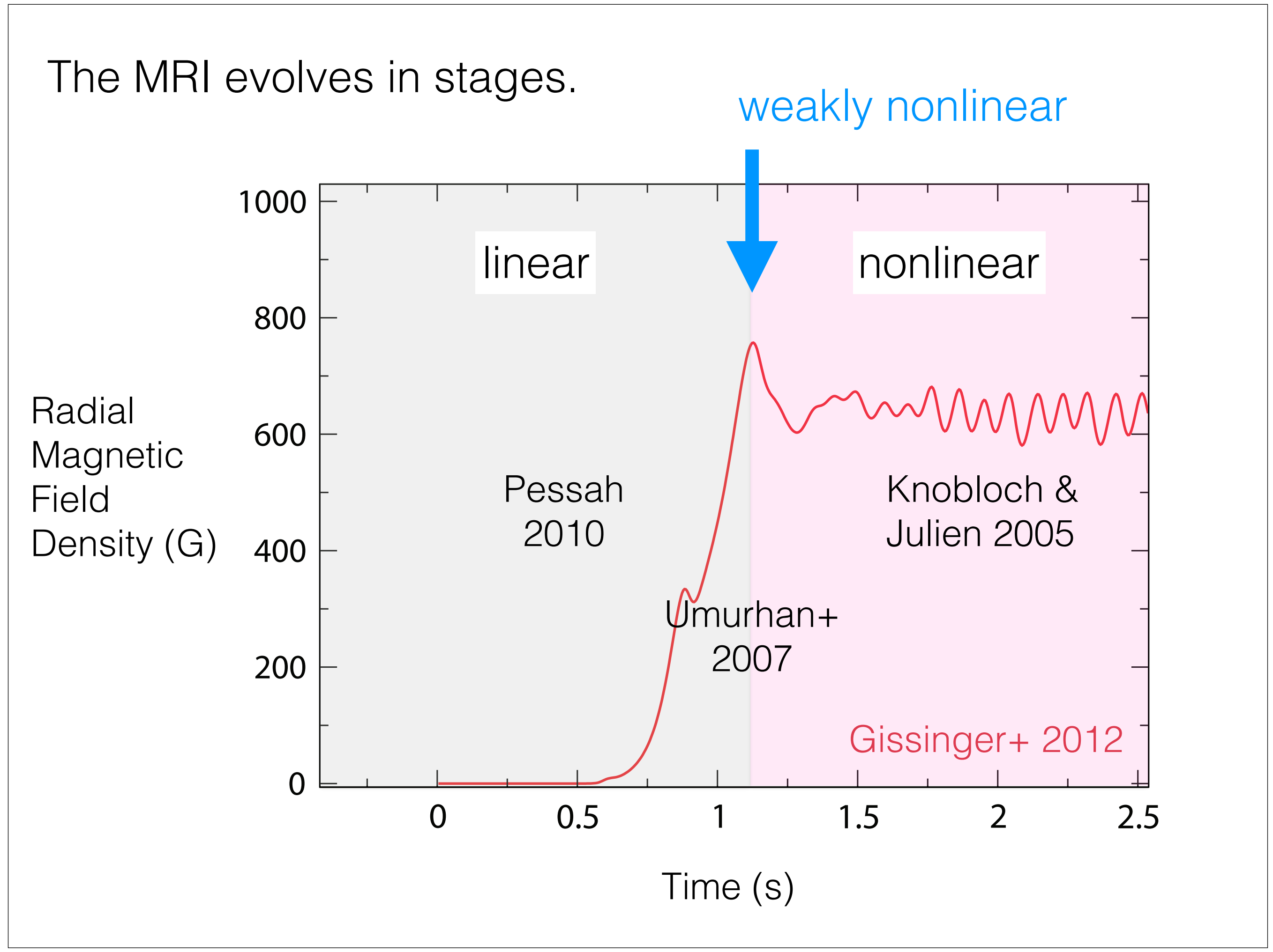
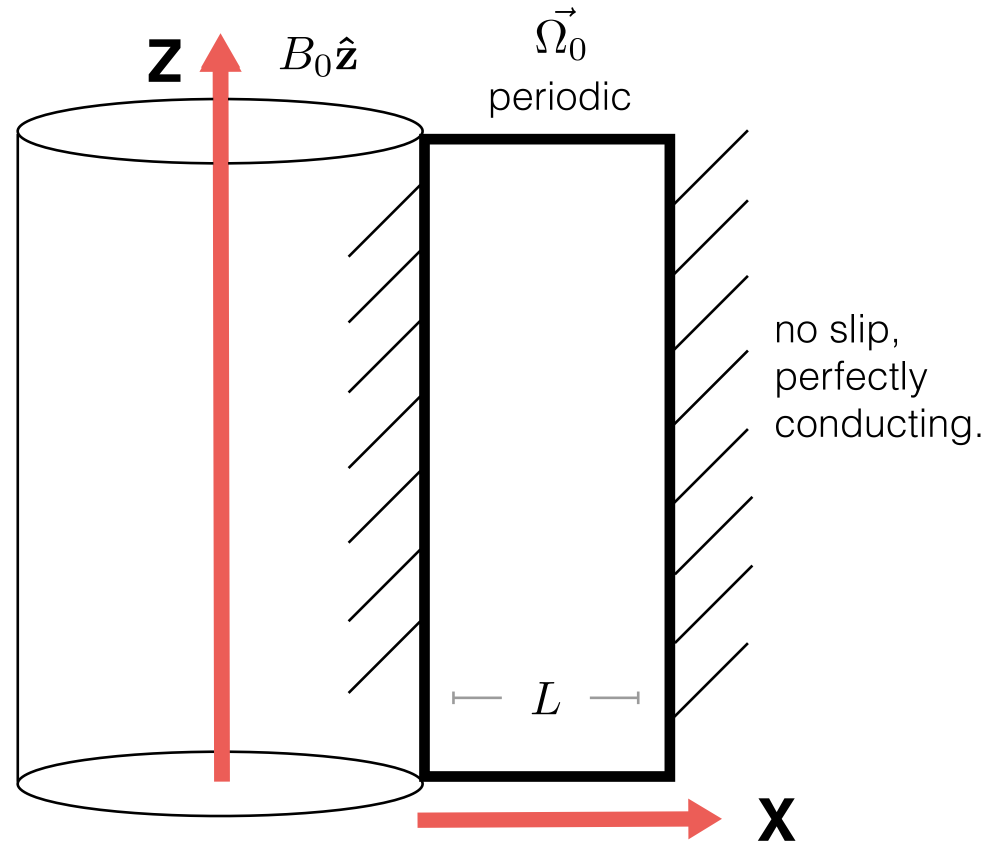


- I'll mention a few of the analytic works that precede this one, notably ….
- … and Umurhan, Regev & Menou 2007, who carried out a weakly nonlinear analysis of the MRI that this work builds on.
- Our investigation is relevant to experimental MRI set-ups as well

We use a thin-gap Taylor Couette setup.



- Our basic set-up is an axisymmetric incompressible flow with a vertical background magnetic field.
- we work in the x-z plane, where x is the radial direction and z is the vertical direction.
- In this cartoon, our domain is the bolded black box.
- We work in the thin-gap limit, where the channel is radially narrow.
- we choose our boundary conditions to be applicable to experimental set-up. We choose periodic boundary conditions in the vertical, and no-slip, perfectly conducting boundary conditions in x.

We solve the non-ideal, incompressible
MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

- We solve the fluid equations for a magnetized Taylor-Couette flow
- in the incompressible limit and subject to the magnetic solenoid constraint.

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}\left(\mathbf{J}\times\mathbf{B}\right) - 2\Omega\times\mathbf{u} - \Omega\times\left(\Omega\times\mathbf{r}\right) + \nu\nabla^2\mathbf{u}$$

kinematic viscosity

induction

$$\partial_t \mathbf{B} = \nabla\times\left(\mathbf{u}\times\mathbf{B}\right) + \eta\nabla^2\mathbf{B}$$

magnetic resistivity

constraints

$$\nabla\cdot\mathbf{u} = 0$$

$$\nabla\cdot\mathbf{B} = 0$$

- We do not work in the ideal limit — we include the effects of kinematic viscosity and magnetic resistivity.

We solve the non-ideal, incompressible MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0}\right)^{-q}$$ shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$ background field

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$ Reynolds number

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$ magnetic Reynolds number

$$\beta \equiv \frac{8\pi \rho_0 \Omega_0^2 L^2}{B_0^2}$$ plasma beta

- We nondimensionalize our equations in terms of several dimensionless parameters
- the shear parameter q, which is 3/2 for Keplerian flow

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{u_y} \\ \mathbf{A} \\ \mathbf{B_y} \end{bmatrix}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

- Rather than work with a system of six fluid quantities (3 components each of the velocity and the magnetic field), we use the incompressibility and magnetic solenoid constraints *and axisymmetry* to define the streamfunction, Psi, and flux function, A, for the x and z components of velocity and magnetic field.
- We will show final results in terms of the more familiar fluid quantities.

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \frac{1}{Re}\nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \frac{1}{Re}\nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \frac{1}{Rm}\nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \frac{1}{Rm}\nabla^2 B_y$$

- Thus we are solving two momentum and two induction equations, where the constraints have been rolled into the definition of the fluid variables.
- I'll point out some key aspects of these equations

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

- you have the viscous and resistive terms

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q \Omega_0 \partial_z A} + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

- shear terms

We work in terms of flux and stream functions.

$$J\left(f,g\right) = \partial_z f \partial_x g - \partial_x f \partial_z g$$

momentum

nonlinear viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \boxed{\frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

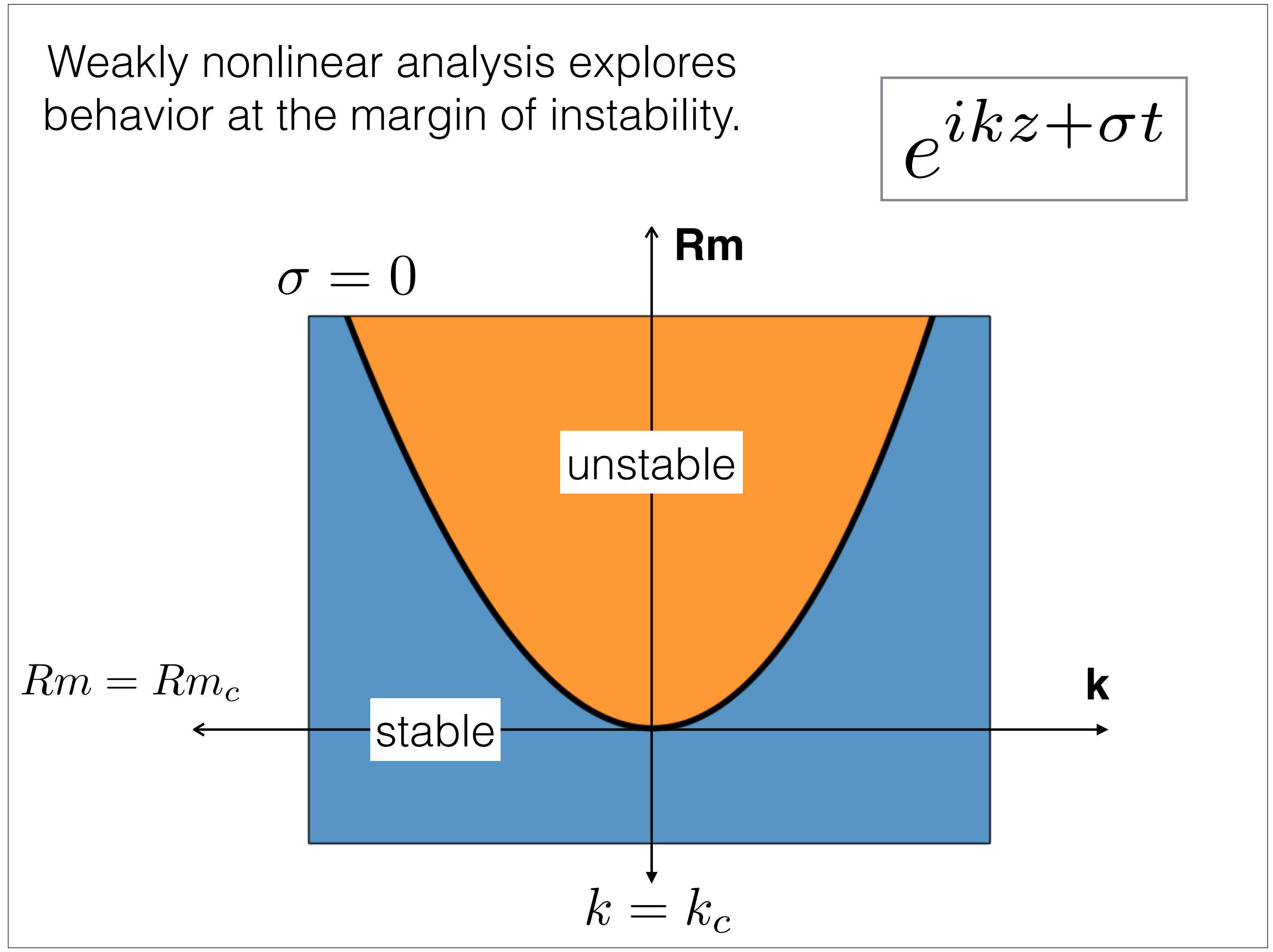
induction

$$\partial_t A = B_0 \partial_z \Psi + \boxed{J\left(A, \Psi\right)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J\left(A, u_y\right) - J\left(\Psi, B_y\right)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

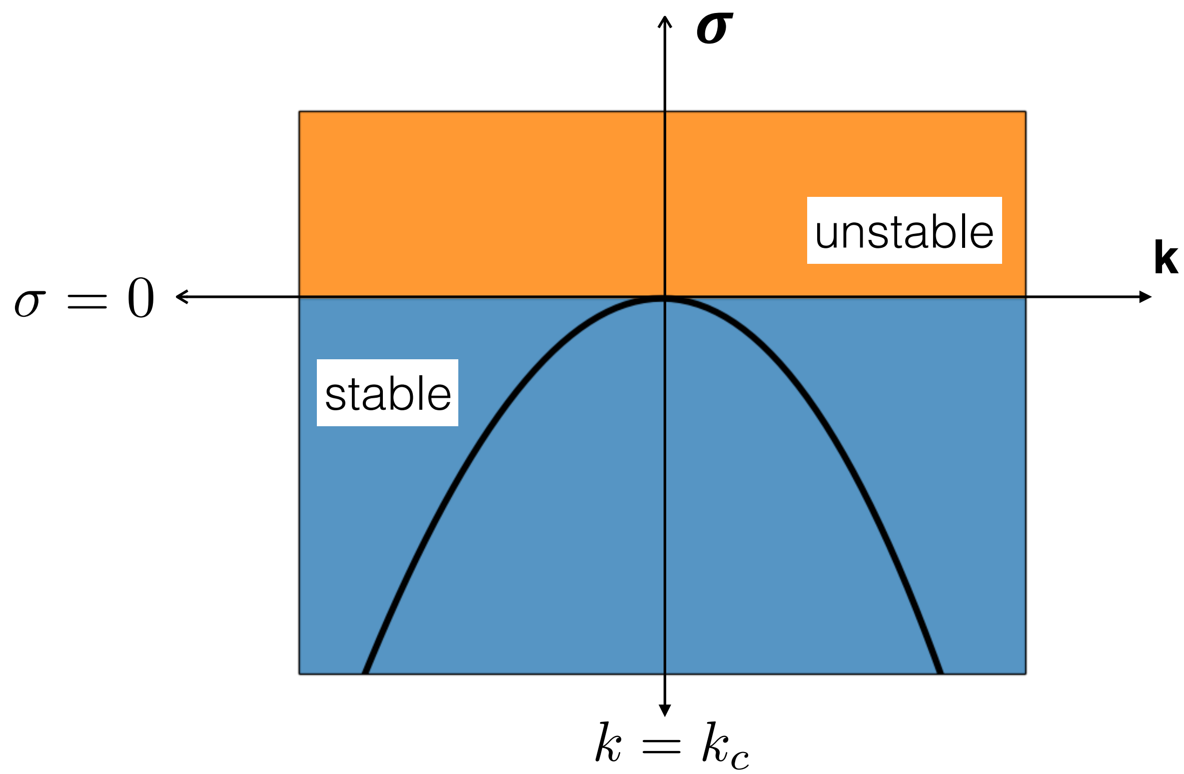
- the nonlinear terms appear in this formulation as Jacobians (J(f, g) = partial_z f partial_x g - partial_x f partial_z g)



- Now we are ready to begin the weakly nonlinear analysis.
- The essence of this analysis is that the MRI is nonlinearly unstable to only the most unstable mode of the linear solution. We identify this mode as a function of the non-dimensional parameters of the system: in this case the magnetic Reynolds number, where all others are held fixed.
- Consider a small perturbation of the form in the upper righthand corner, where sigma is the perturbation growth rate.
- This cartoon shows the curve of marginal stability in the Rm-wavenumber plane: where the growth rate sigma is equal to zero. Above this curve, in the orange region, sigma is greater than zero and the perturbation grows. Below the curve, in the blue, sigma is less than zero and the perturbation decays.
- We identify the critical magnetic Reynolds number and critical wavenumber of the linear MRI.

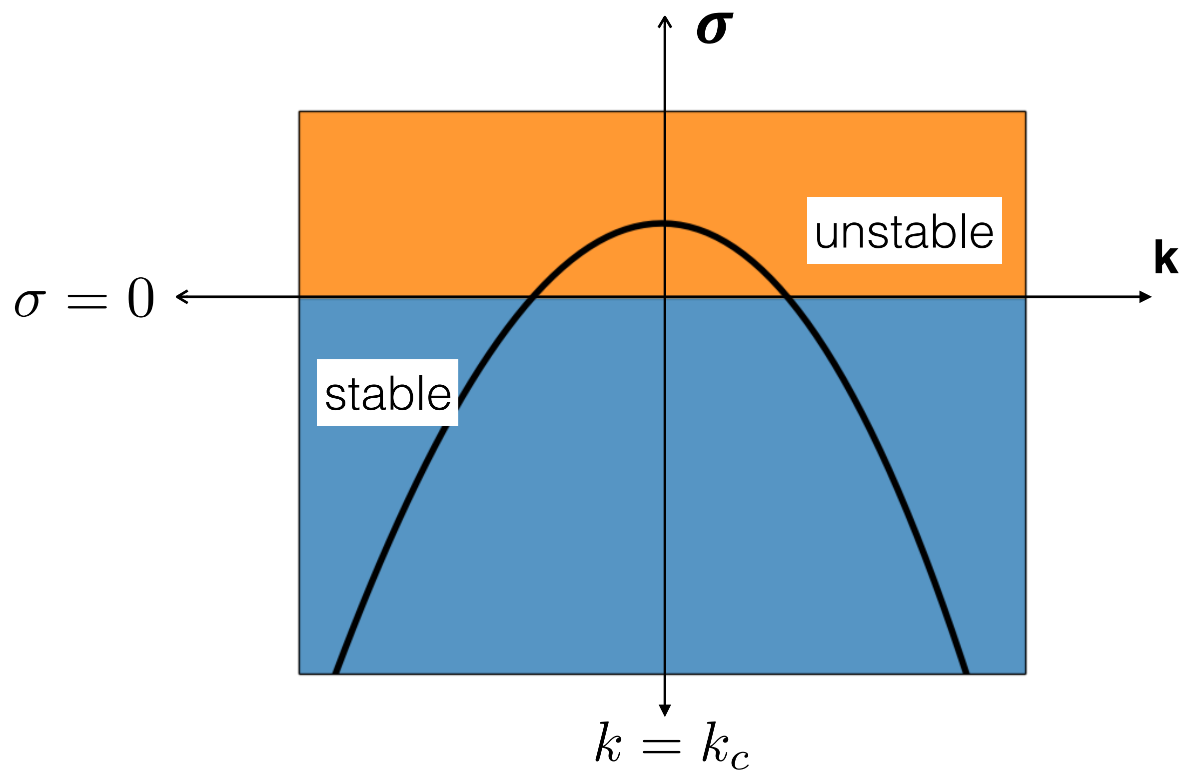
Weakly nonlinear analysis explores behavior at the margin of instability.

$$Fixed\ Rm$$



- Now that we've defined a fixed magnetic reynolds number, we can examine the growth rate as a function of wavenumber. At the critical wavenumber, the growth rate of the system is exactly zero.
- This is the most unstable mode at marginality.

Tune the most unstable mode just over the threshold of instability.

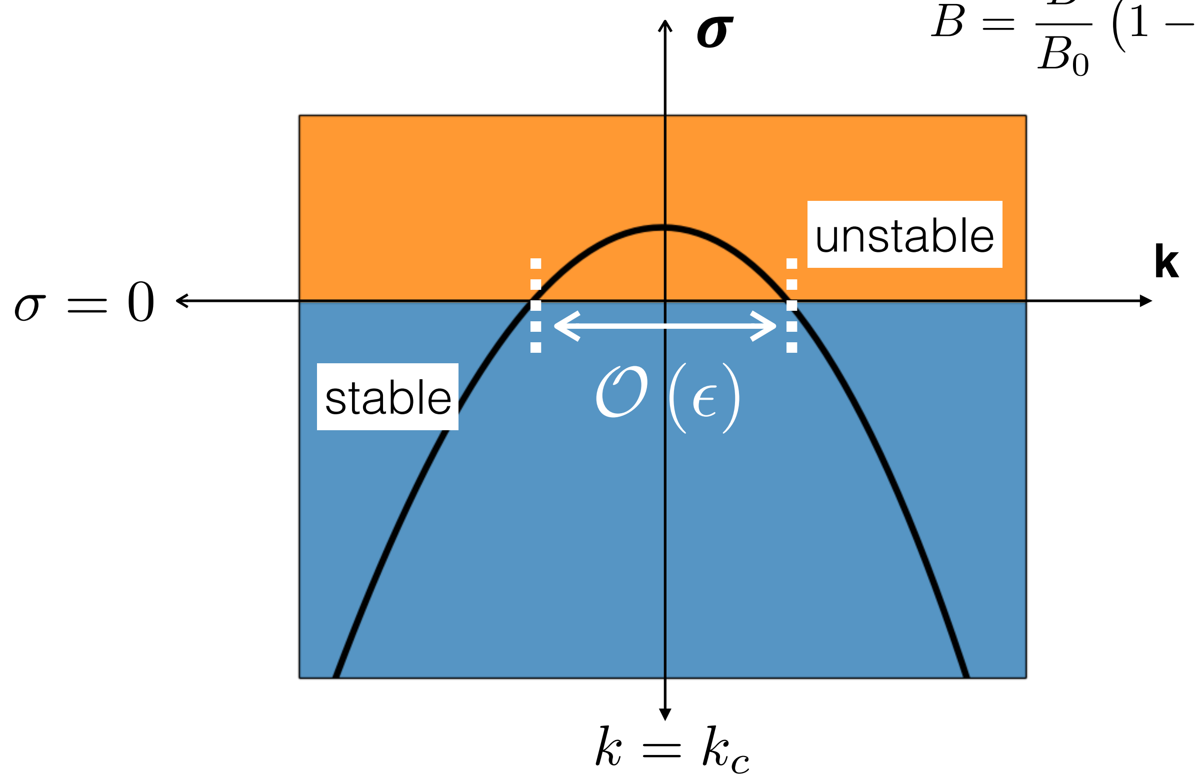


- We tune this most unstable mode just over the threshold of instability.

Tune the most unstable mode just over the threshold of instability.

small parameter

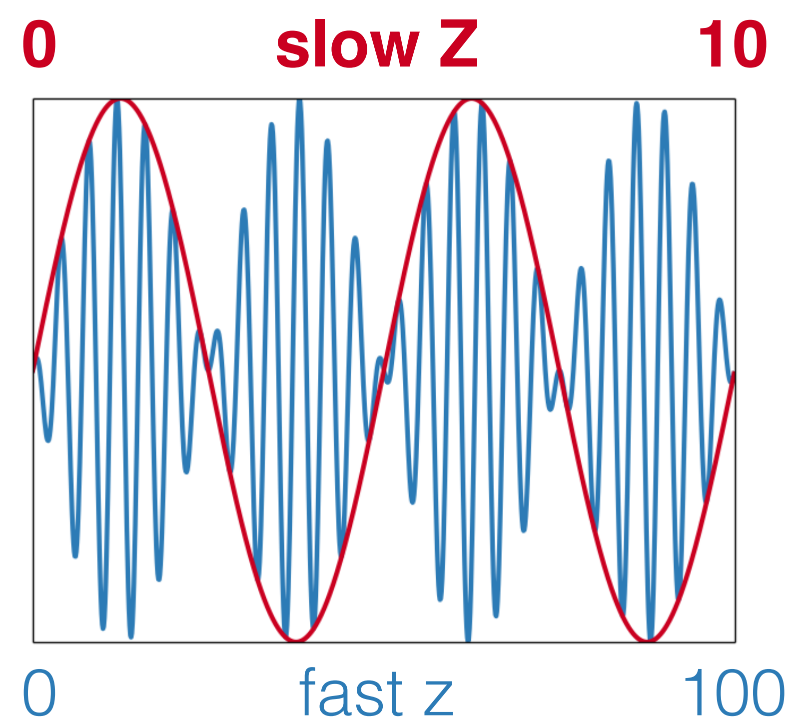
$$B = \frac{B}{B_0}\left(1 - \epsilon^2\right)$$



- We do so by turning the magnetic field down from the marginal state an amount parameterized by the small parameter epsilon.
- This destabilizes a band of wave modes with width of order epsilon, which will now interact nonlinearly.

Multiscale analysis tracks the evolution of fast and slow variables.

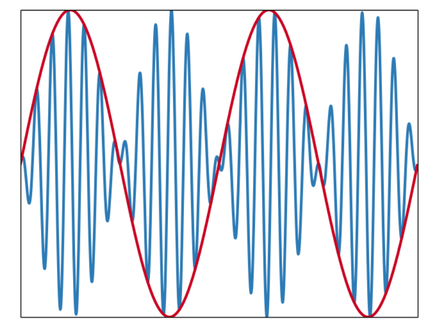
$$Z \equiv \epsilon z$$



- We conduct a formal multiscale analysis, where we track the evolution of our fluid variables on both fast and slow scales.
- In analogy to Rayleigh-Benard convection and hydrodynamic T-C flow, assume slow-scale variation in Z & T only.

We choose an ansatz state vector form.

$$\mathbf{V} = \alpha\left(Z, T\right) \mathrm{V}\left(x\right) e^{i k_c z}$$



- In practice, that means that our ansatz for the state vector containing all of our fluid variables contains severable separable parts.

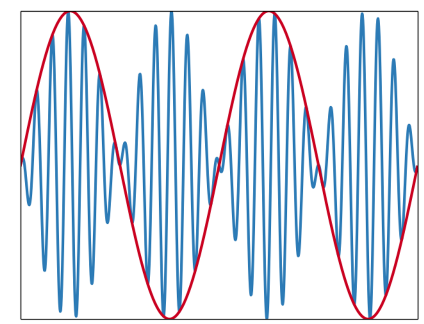
We choose an ansatz state vector form.

x dependence

$$\mathbf{V} = \alpha\left(Z, T\right)\mathrm{V}\left(x\right)e^{ik_c z}$$

vertical periodicity

amplitude function



- we have the amplitude function, the slowly-varying function of Z and time.
- We have the x-dependence, the radial form of the fluid variables, which must be solved on our grid, subject to the no-slip and conducting boundary conditions.
- And finally we have periodicity in the z direction, which our periodic vertical boundary conditions allow us to posit.

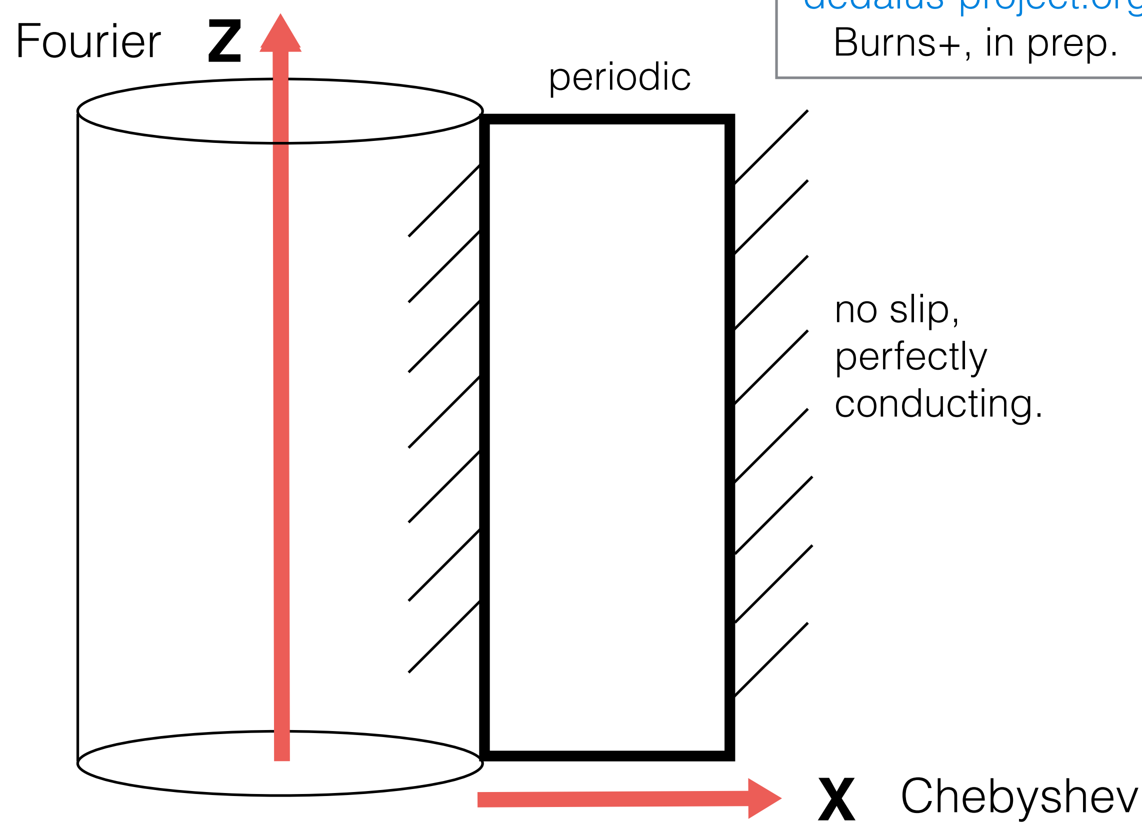
The fluid quantities are expanded in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

- Once we have our ansatz, we expand our state vector in a perturbation series in orders of epsilon.
- We then solve these successively in increasing orders of epsilon.

Dedalus is a general-purpose spectral code.

dedalus-project.org
Burns+, in prep.



- We solve these equations using Dedalus, a general-purpose pseudo spectral code.
- Periodicity in z means that we solve the z dimension on a Fourier basis, and the bounded radial dimension means that we solve the x dimension with a basis of Chebyshev polynomials.

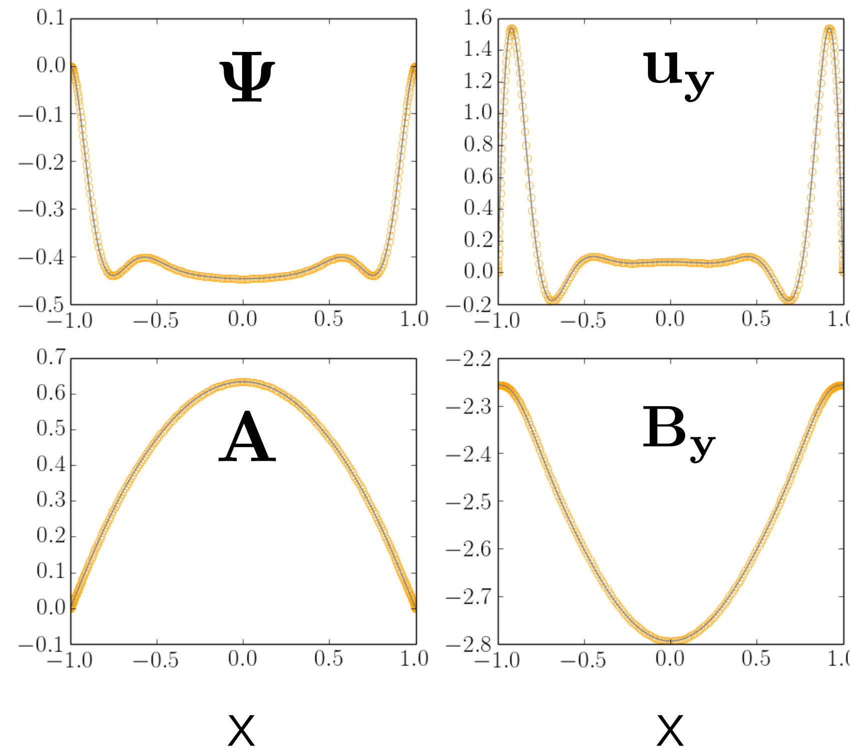
Spectrally solve the most unstable mode of the linear MRI.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

$$\mathcal{L}\mathbf{V}_1 = 0$$

$$\mathbf{V}_1 = \alpha\,(T, Z)\boxed{V_{11}(x)}e^{ik_c z}$$

$\boxed{V_{11}(x)}$



- We first solve the linear eigenfunction problem for the marginal MRI mode, shown here.

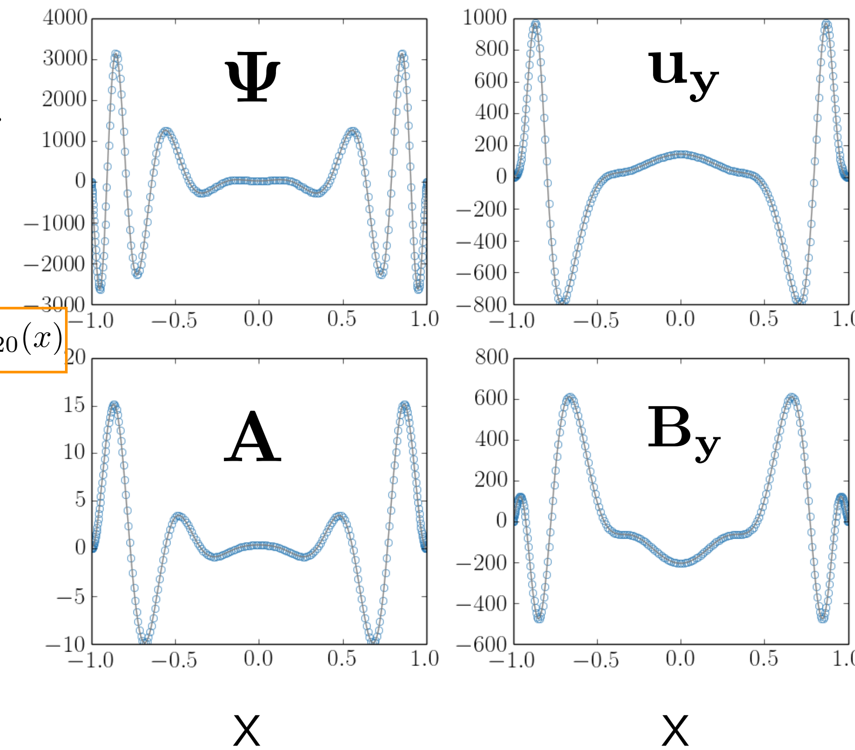
We solve each term in the expanded equations at each order.

$$\boxed{\mathbf{N}_{20}(x)}$$

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

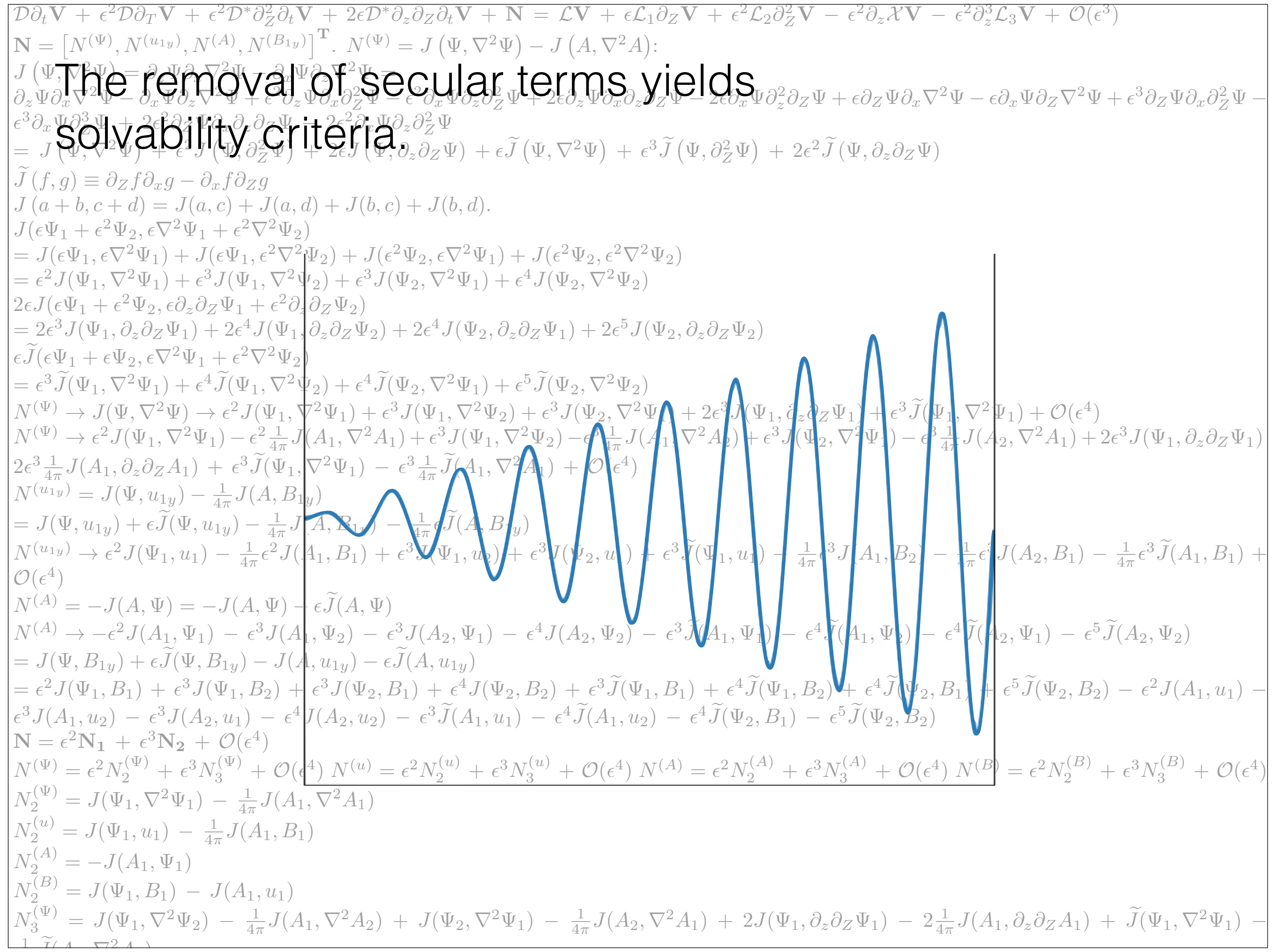
$$\mathcal{L}\mathbf{V}_2 = \mathbf{N}_2 - \widetilde{\mathcal{L}}_1 \partial_Z \mathbf{V}_1$$

$$\mathbf{N}_2 = \alpha^2 \mathbf{N}_{22}(x) e^{i2k_c z} + |\alpha|^2 \boxed{\mathbf{N}_{20}(x)}$$



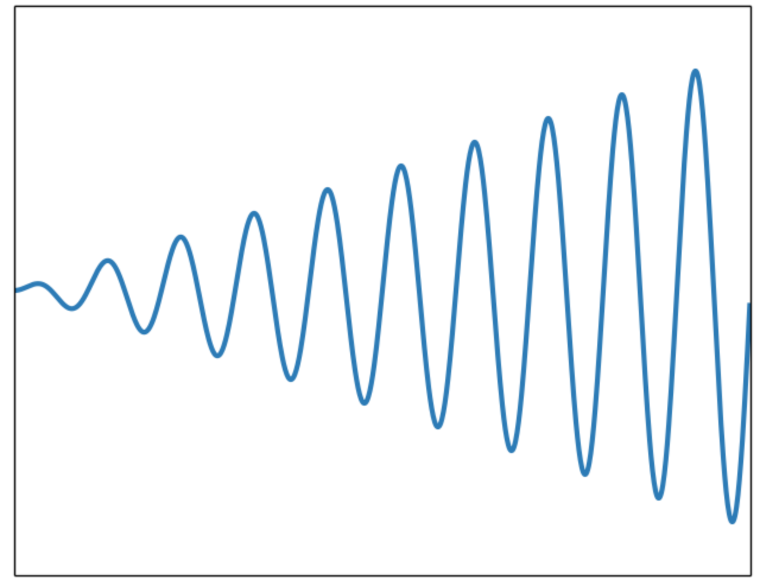
- We then solve up each order of epsilon. Nonlinear terms are functions of lower-epsilon-order linear solutions. Shown here is a third-order nonlinear term. (*change to 2nd order*)
- We solve to third order to close the system at second order.

The removal of secular terms yields solvability criteria.



- We remove secular terms to come up with solvability criteria for the system at each order.
- A secular term is a term on the righthand side of an evolution equation which is the exact solution to the homogenous equation. It causes the system to grow without bound — think of a driven oscillator.

The removal of secular terms yields solvability criteria.



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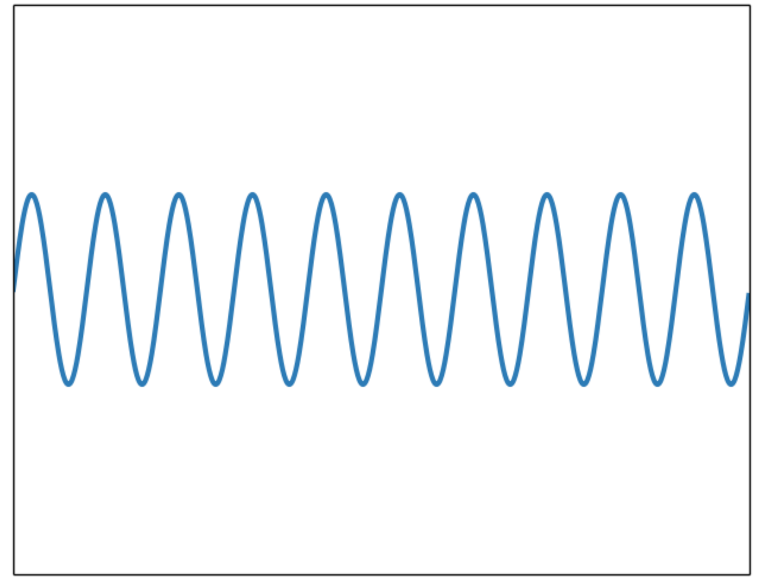
The removal of secular terms yields solvability criteria.

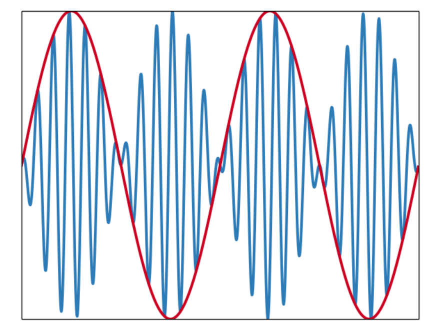


- We remove these terms by taking the inner product of the system with its adjoint homogenous solution. (**do i need to say this?**)

The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b \partial_Z \alpha - c\alpha \left|\alpha^2\right| + h \partial_Z^2 \alpha + g i k_c^3 \alpha$$

α



- The end result of this process is an equation for the evolution of the amplitude function alpha.

The result is an amplitude equation
for the most unstable mode.

diffusion term

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

nonlinear term **linear growth**

- terms

The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

?

diffusion term

nonlinear term

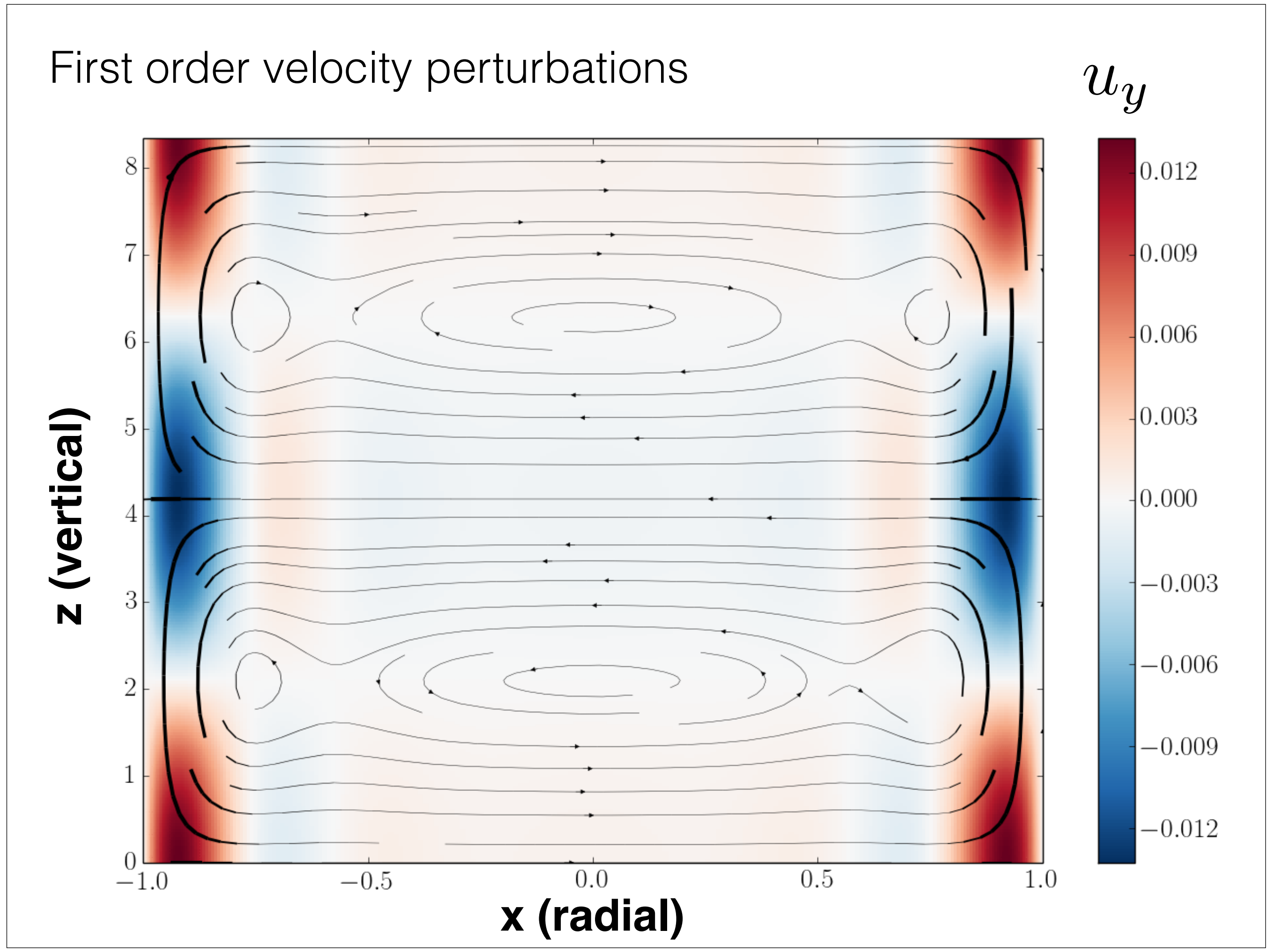
linear growth

- extra term that makes this not a GLE (as has been previously claimed)
- solve this IVP for alpha and take its asymptotic value — the saturation amplitude

Finally, we obtain the saturation structure of each fluid quantity at each order.

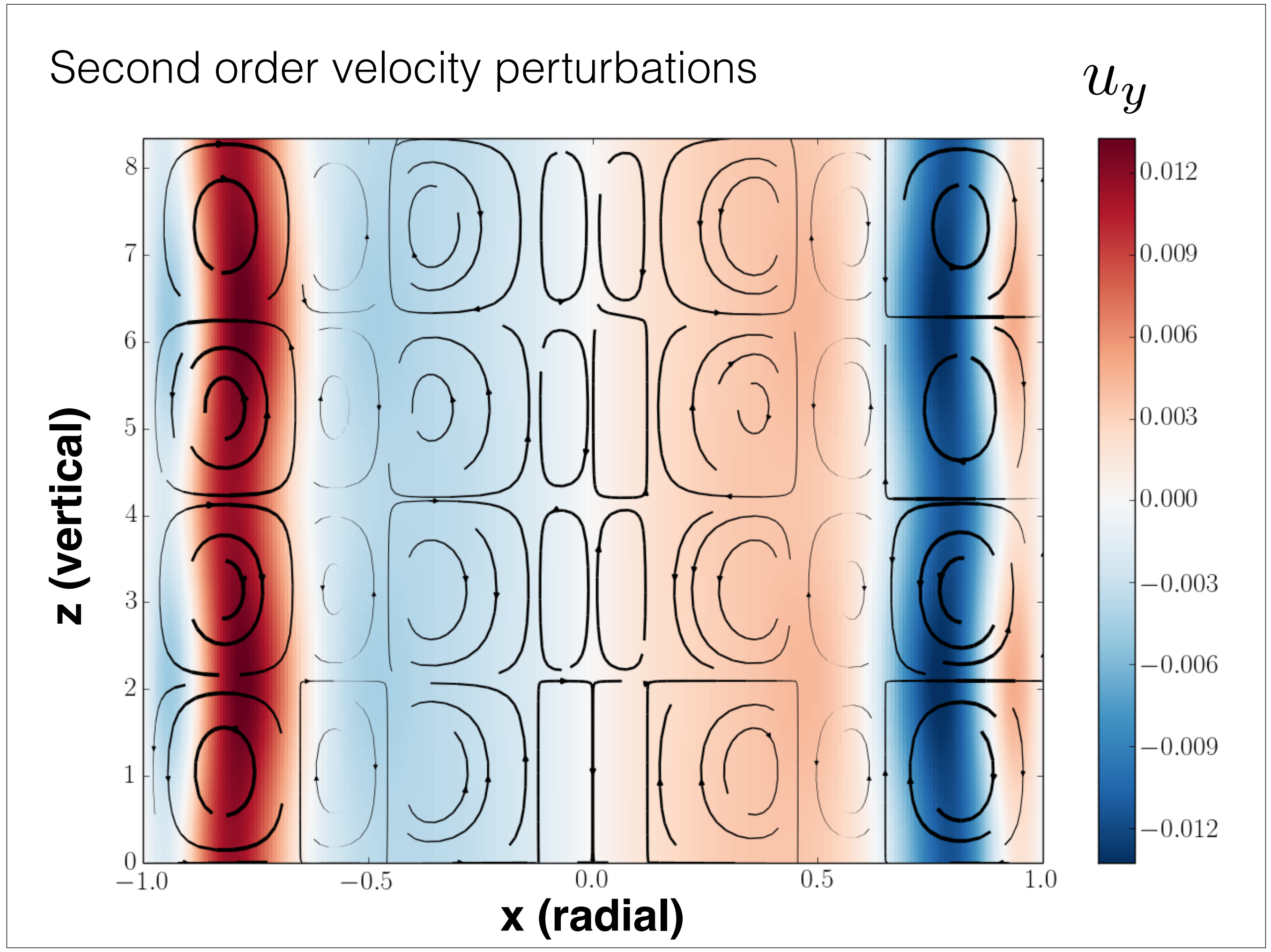
$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + \ldots$$

- We can now put all the pieces together

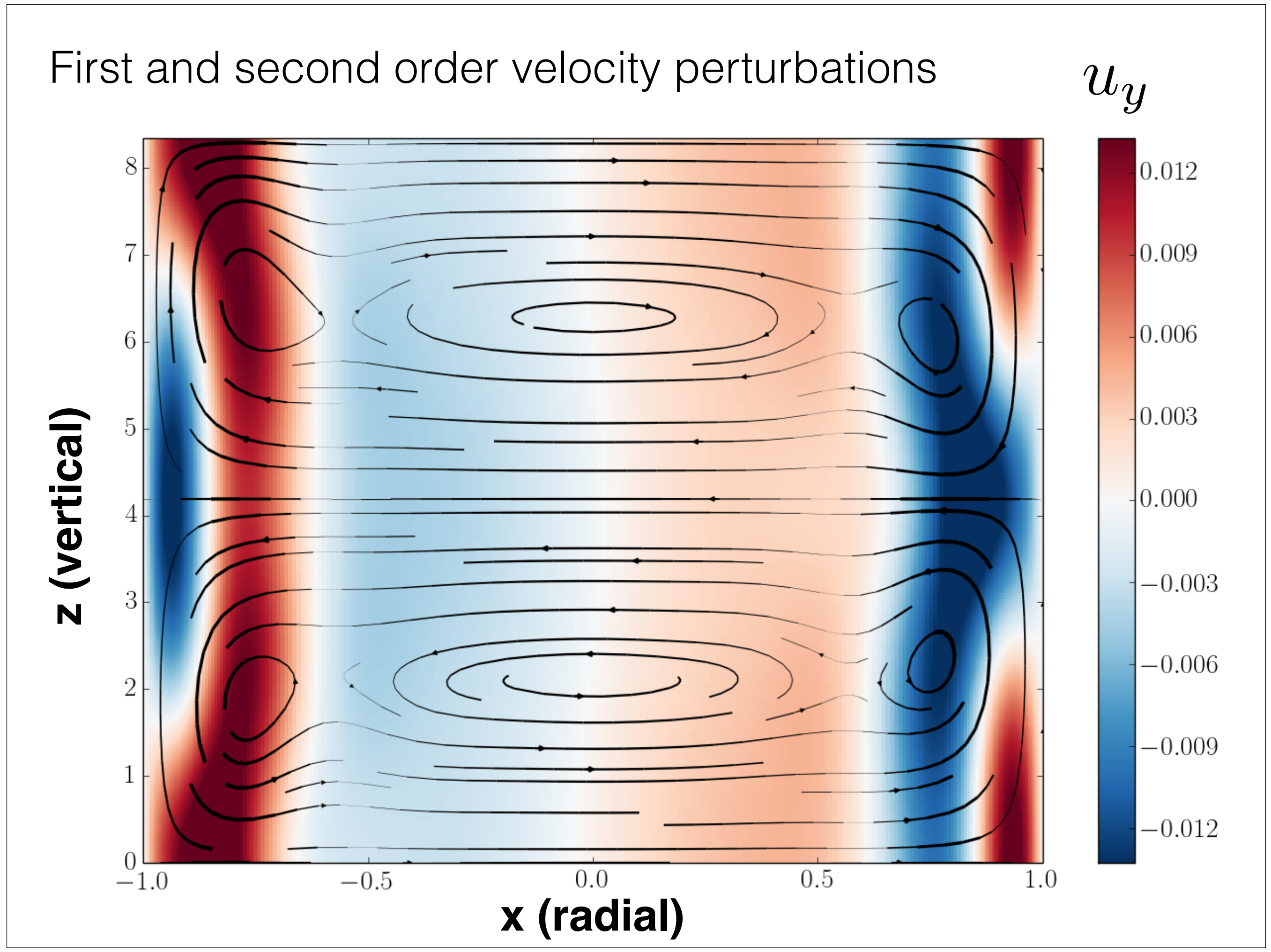


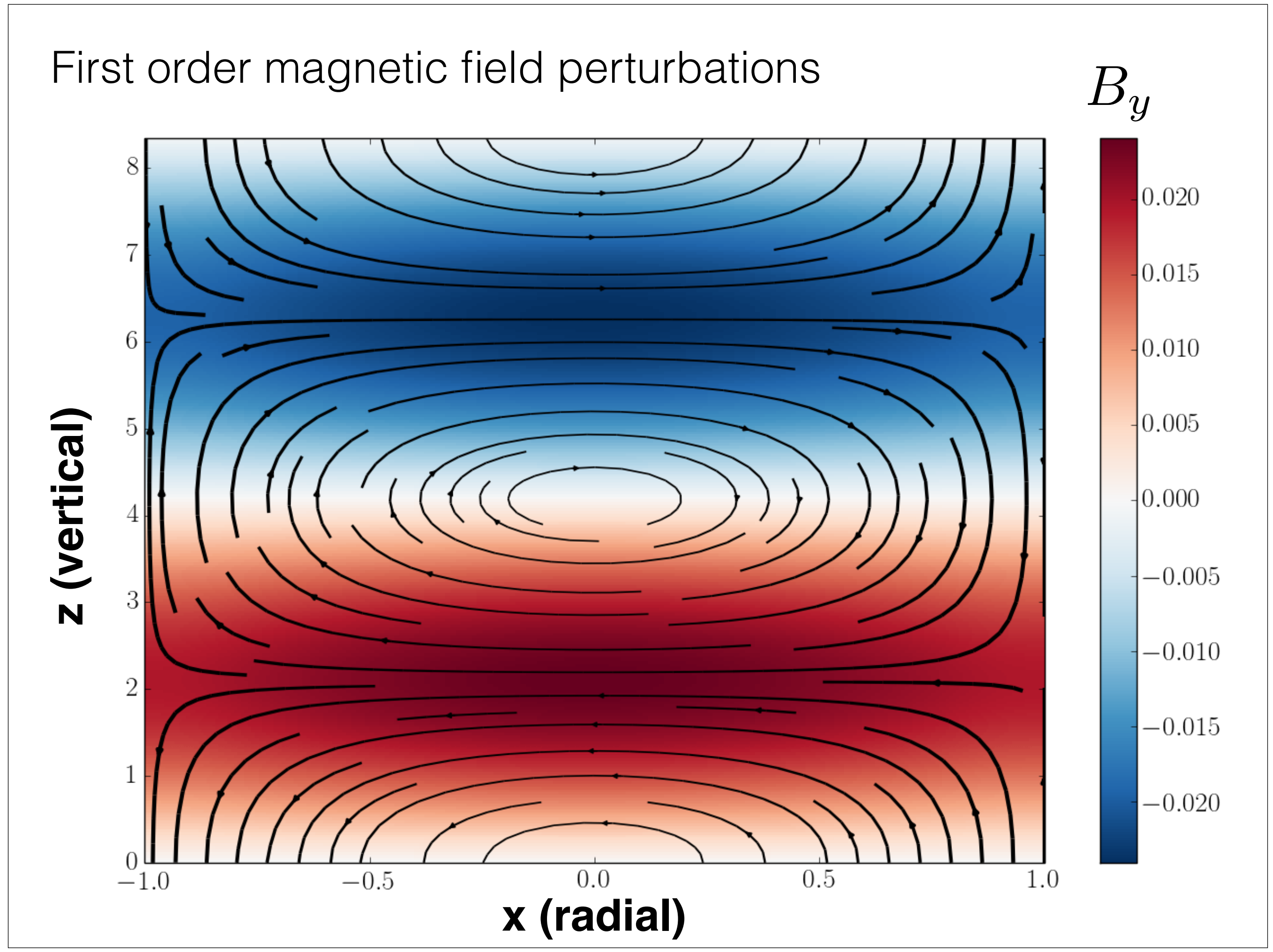
-Here are the first order velocity perturbations.

- colormap, streamfunctions — width of stream functions is proportional to speed in x-z plane



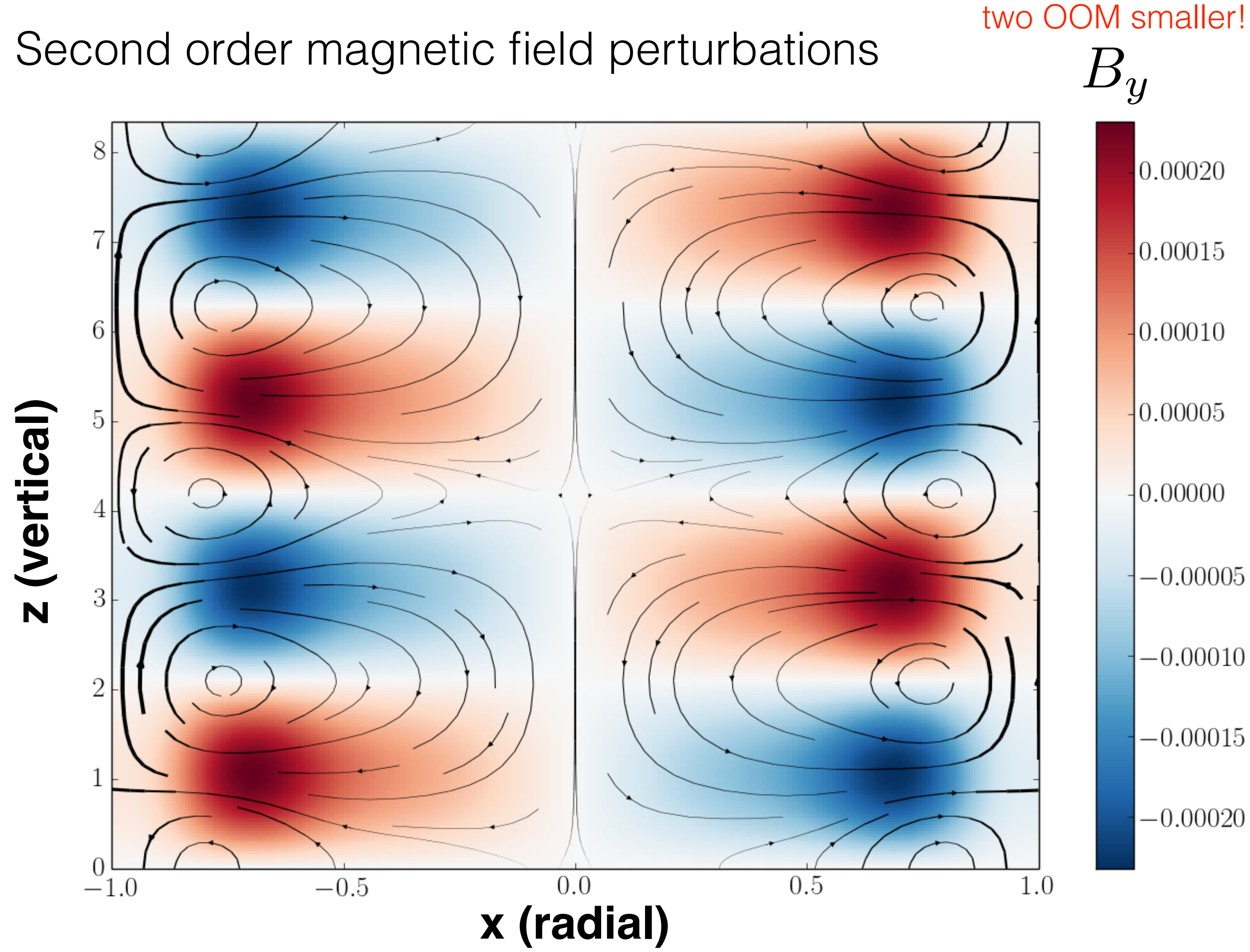
- At second order we see significant structure, particularly in the middle of our domain.
- this suggests that the second order velocity perturbations are instrumental in the saturation of the MRI

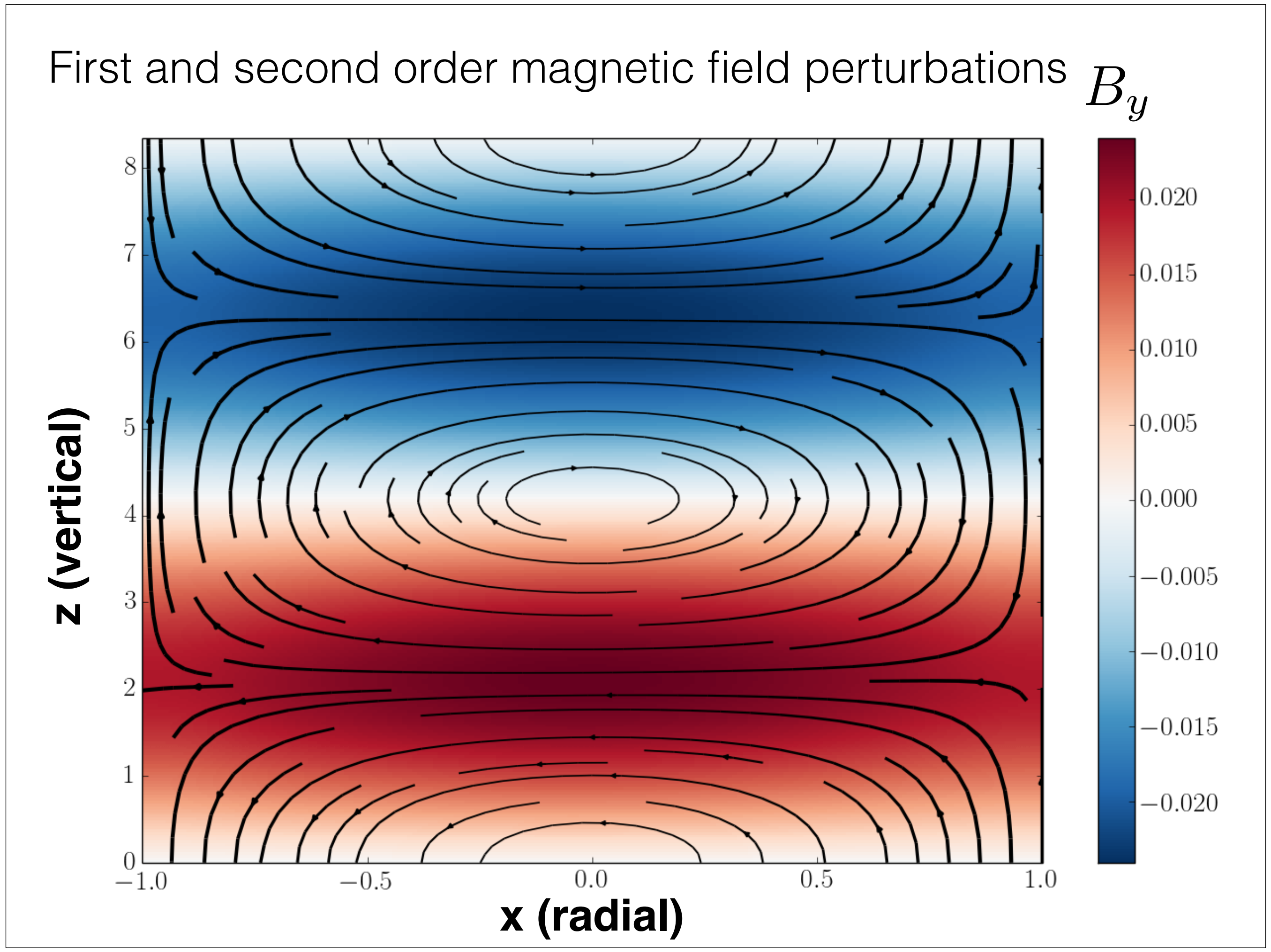

First and second order velocity perturbations
u_y
z (vertical)
x (radial)
0
1
2
3
4
5
6
7
8
−1.0
−0.5
0.0
0.5
1.0
0.012
0.009
0.006
0.003
0.000
−0.003
−0.006
−0.009
−0.012



First order magnetic field perturbations
B_y
z (vertical)
x (radial)
0
1
2
3
4
5
6
7
8
−1.0
−0.5
0.0
0.5
1.0
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020

Second order magnetic field perturbations




First and second order magnetic field perturbations
B_y
z (vertical)
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0
1
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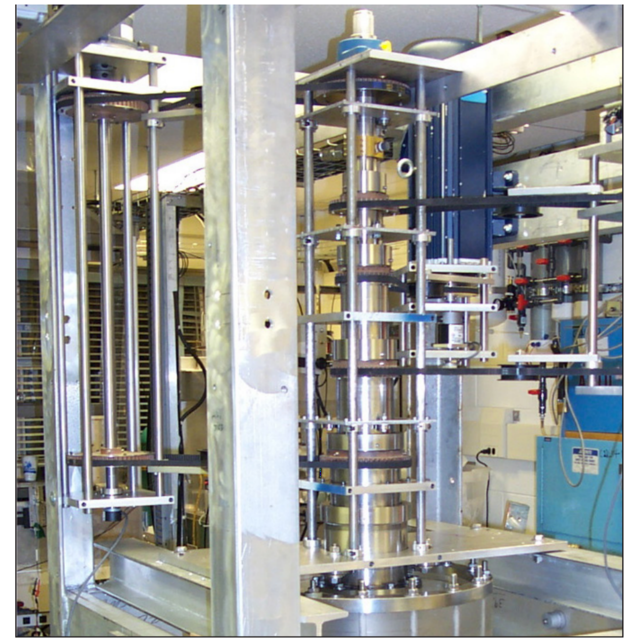
8
−1.0
−0.5
0.0
0.5
1.0
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020

Future work

- explore parameter space
- relax thin gap approximation
- comparison to experiment
- helical MRI



PPPL MRI experiment

Conclusions

- We derive a robust analytical framework for solving MRI systems up to second order perturbations.
- We use the spectral code Dedalus to solve the radial components of our equations.
- Preliminary results suggest a shear-related saturation mechanism.