Exploring the saturation of the MRI via weakly nonlinear analysis

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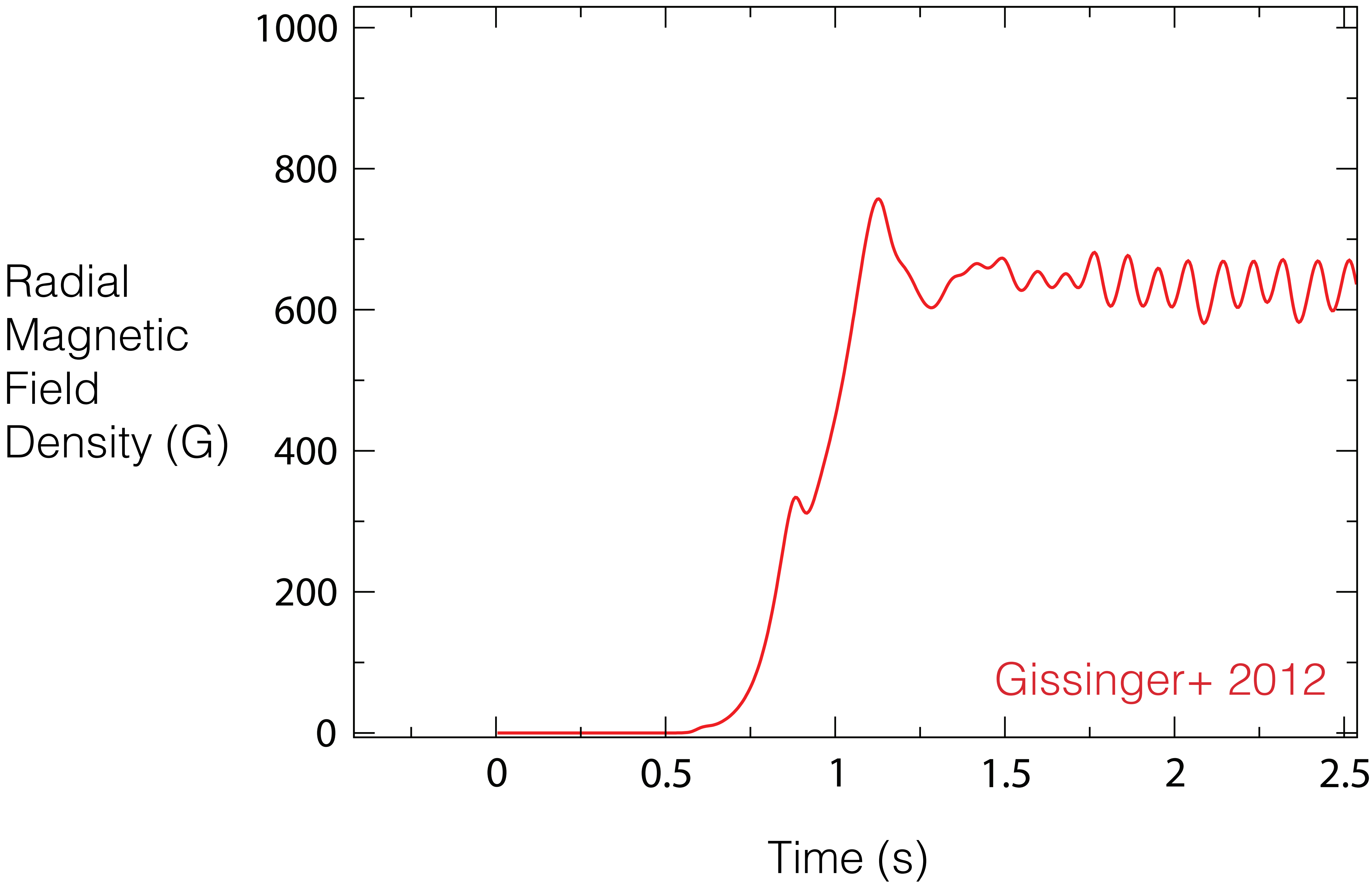
The MRI evolves in stages.



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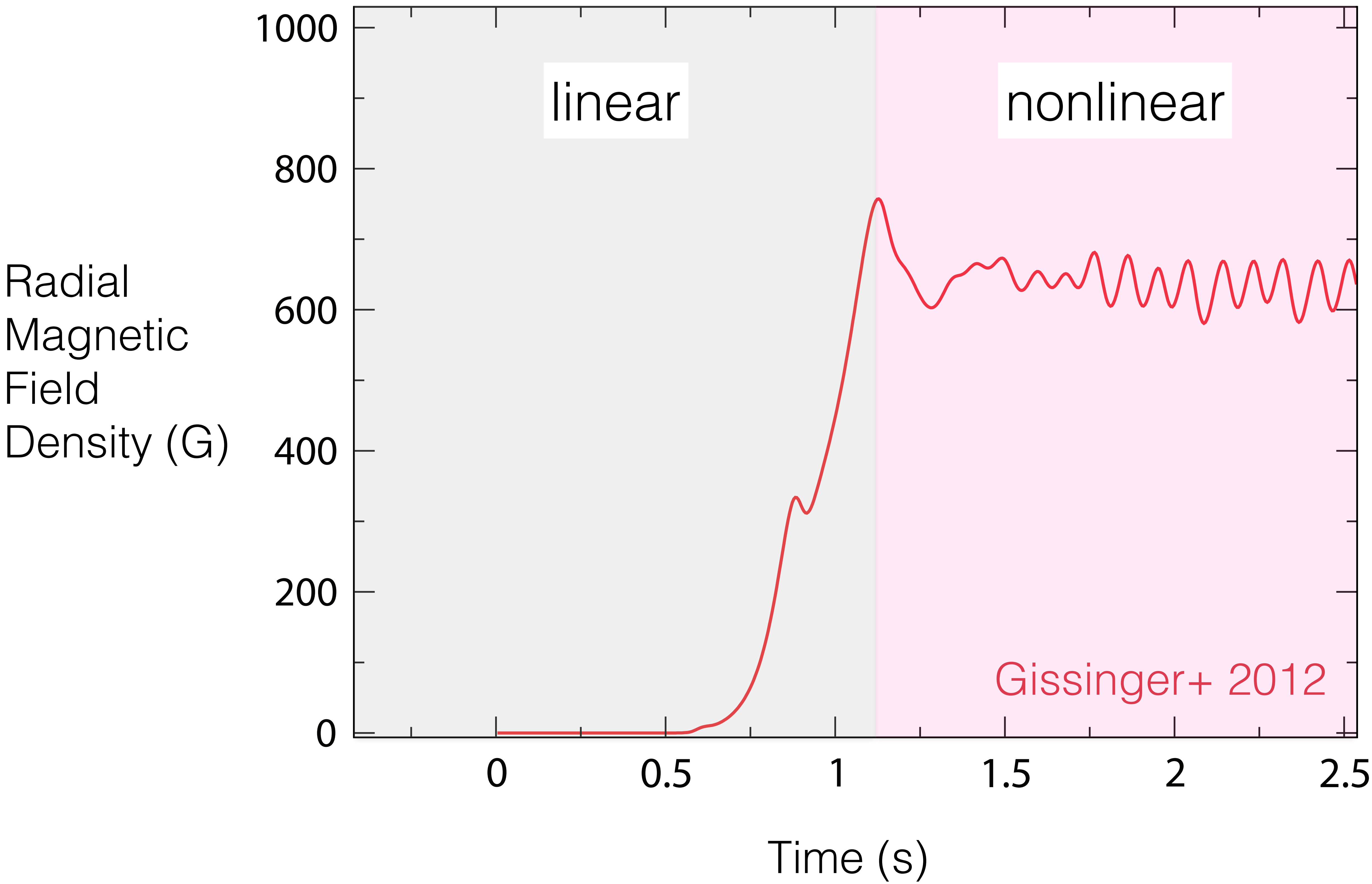

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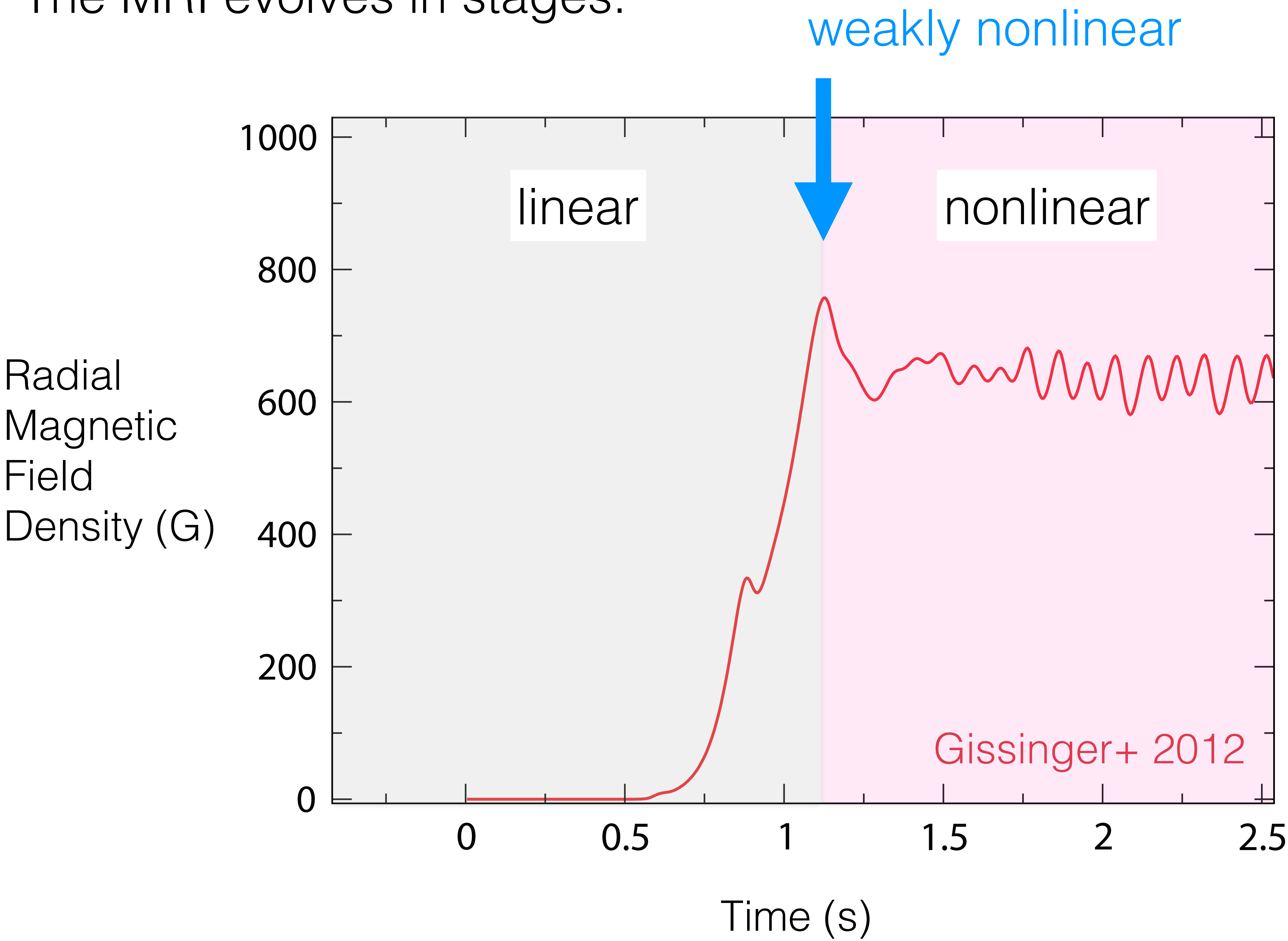

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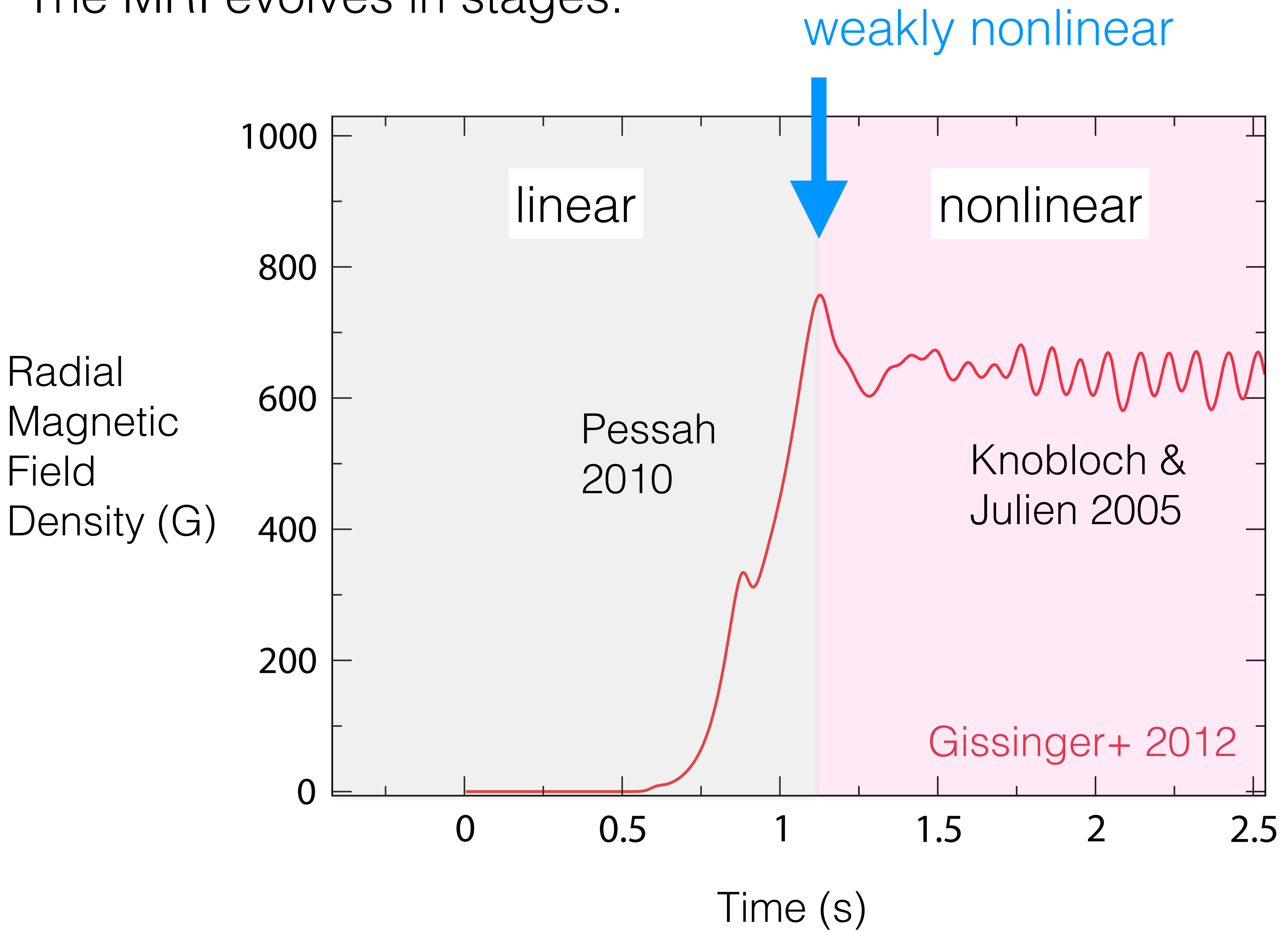

We solve the non-ideal MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi + \frac{1}{\rho}\left(\mathbf{J} \times \mathbf{B}\right) - 2\Omega \times \mathbf{u} - \Omega \times \left(\Omega \times \mathbf{r}\right) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times \left(\mathbf{u} \times \mathbf{B}\right) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu\nabla^2\mathbf{u}$$

microscopic viscosity

induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta\nabla^2\mathbf{B}$$

magnetic resistivity

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We nondimensionalize and perturb the nonlinear MRI equations.

$$\Omega(r) \propto \Omega_0 \left(\frac{r}{r_0}\right)^{-q}$$ shear parameter

$$\mathbf{B} = B_0 \hat{\mathbf{z}}$$ background field

$$Re \equiv \frac{\Omega_0 L^2}{\nu}$$ Reynolds number

$$Rm \equiv \frac{\Omega_0 L^2}{\eta}$$ magnetic Reynolds number

$$\beta \equiv \frac{8\pi\rho_0\Omega_0^2 L^2}{B_0^2}$$ plasma beta

We work in terms of flux and stream functions.

$$\mathbf{V} = \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{u_y} \\ \mathbf{A} \\ \mathbf{B_y} \end{bmatrix}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \frac{1}{Re} \nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

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induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \underbrace{\boxed{\frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right)}}_{\text{nonlinear}} + \underbrace{\boxed{\frac{1}{Re} \nabla^4 \Psi}}_{\text{viscous}}$$

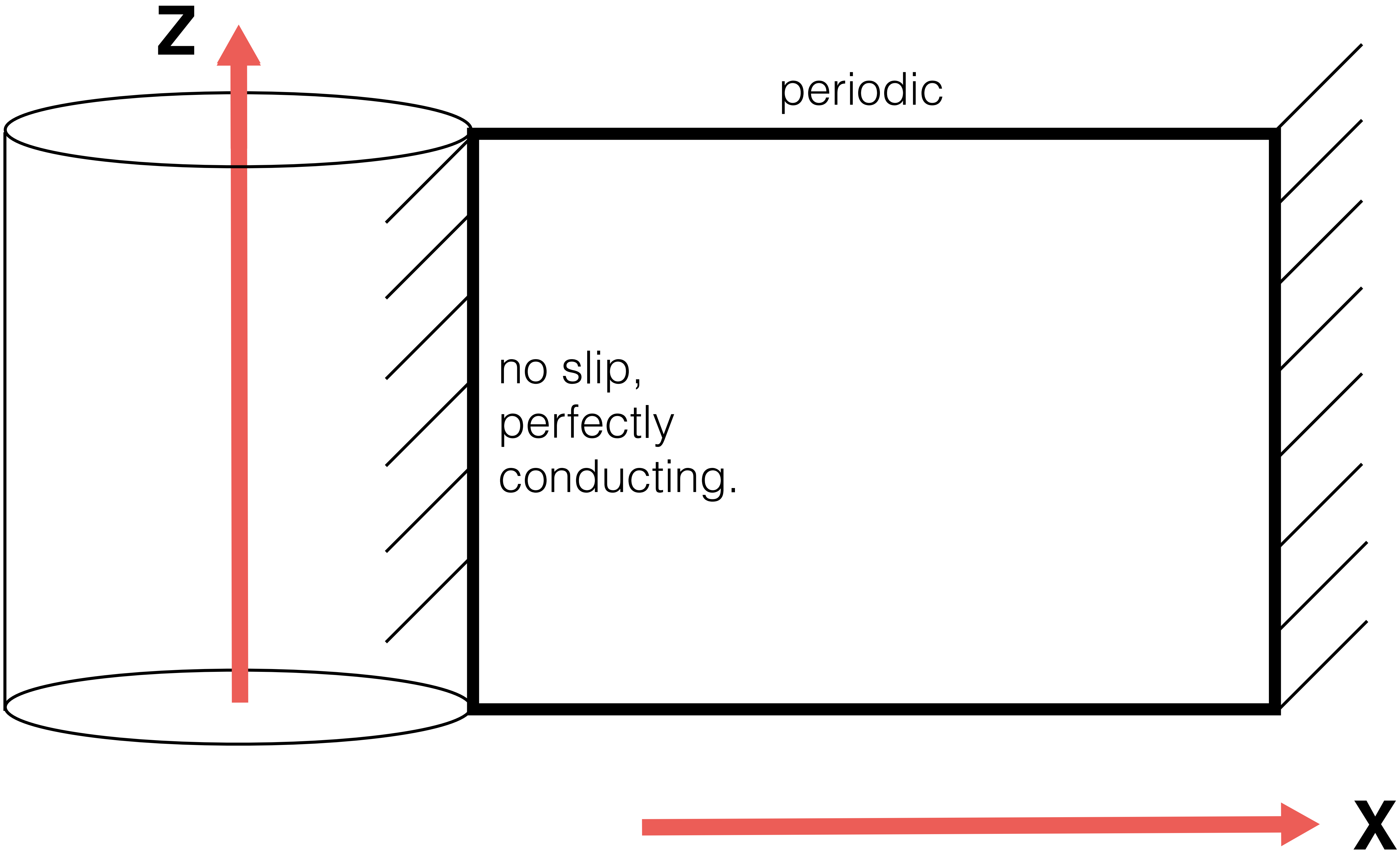
$$\partial_t u_y = \frac{2}{\beta} B_0 \partial_z B_y - \underbrace{\boxed{(2-q)\Omega_0 \partial_z \Psi}}_{\text{shear}} + \boxed{\frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

induction

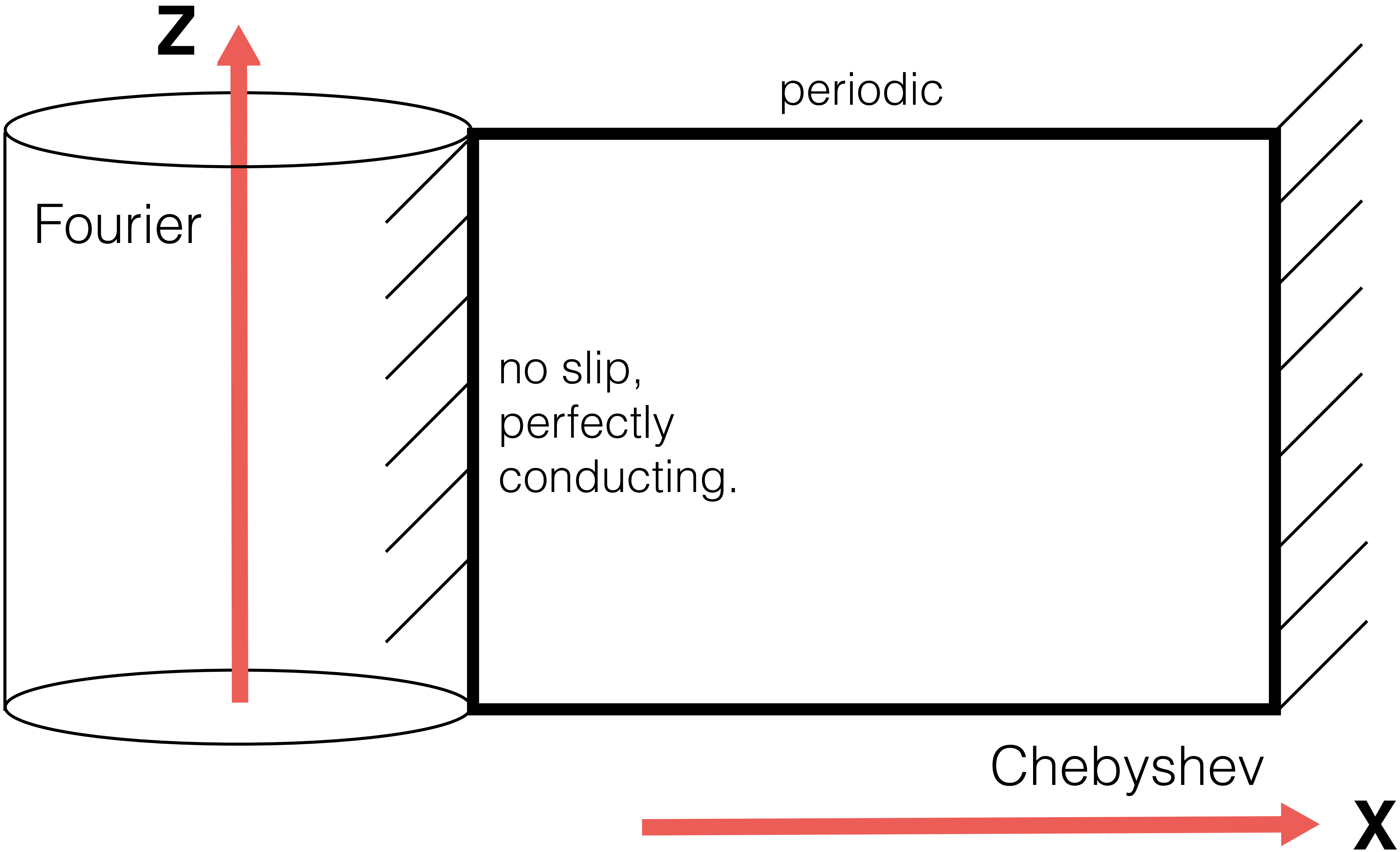
$$\partial_t A = B_0 \partial_z \Psi + \boxed{J\left(A, \Psi\right)} + \underbrace{\boxed{\frac{1}{Rm} \nabla^2 A}}_{\text{resistive}}$$

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J\left(A, u_y\right) - J\left(\Psi, B_y\right)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

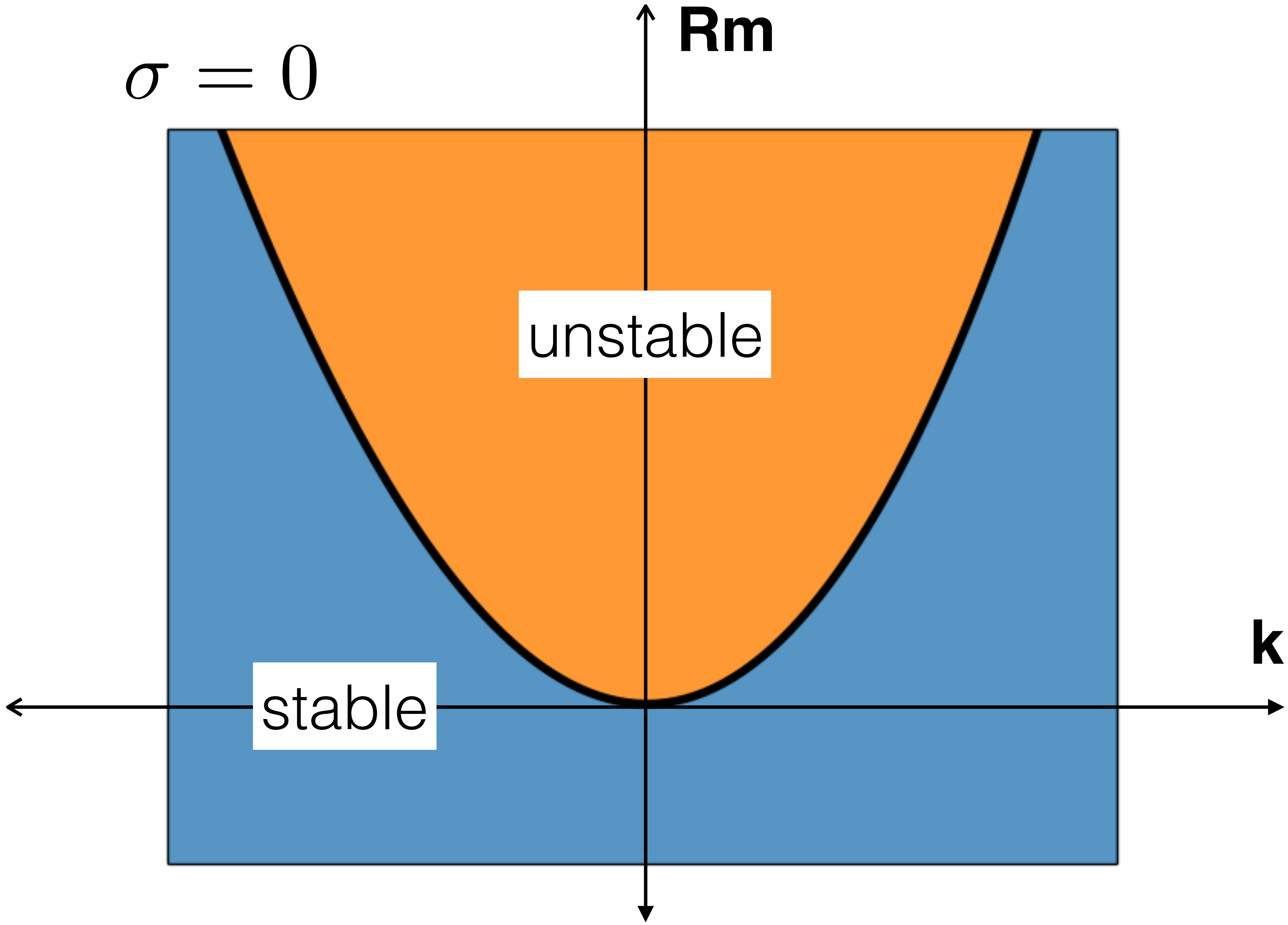
We use experimentally relevant boundary conditions.



Dedalus is a general-purpose spectral code.



Weakly nonlinear analysis explores behavior at the margin of instability.



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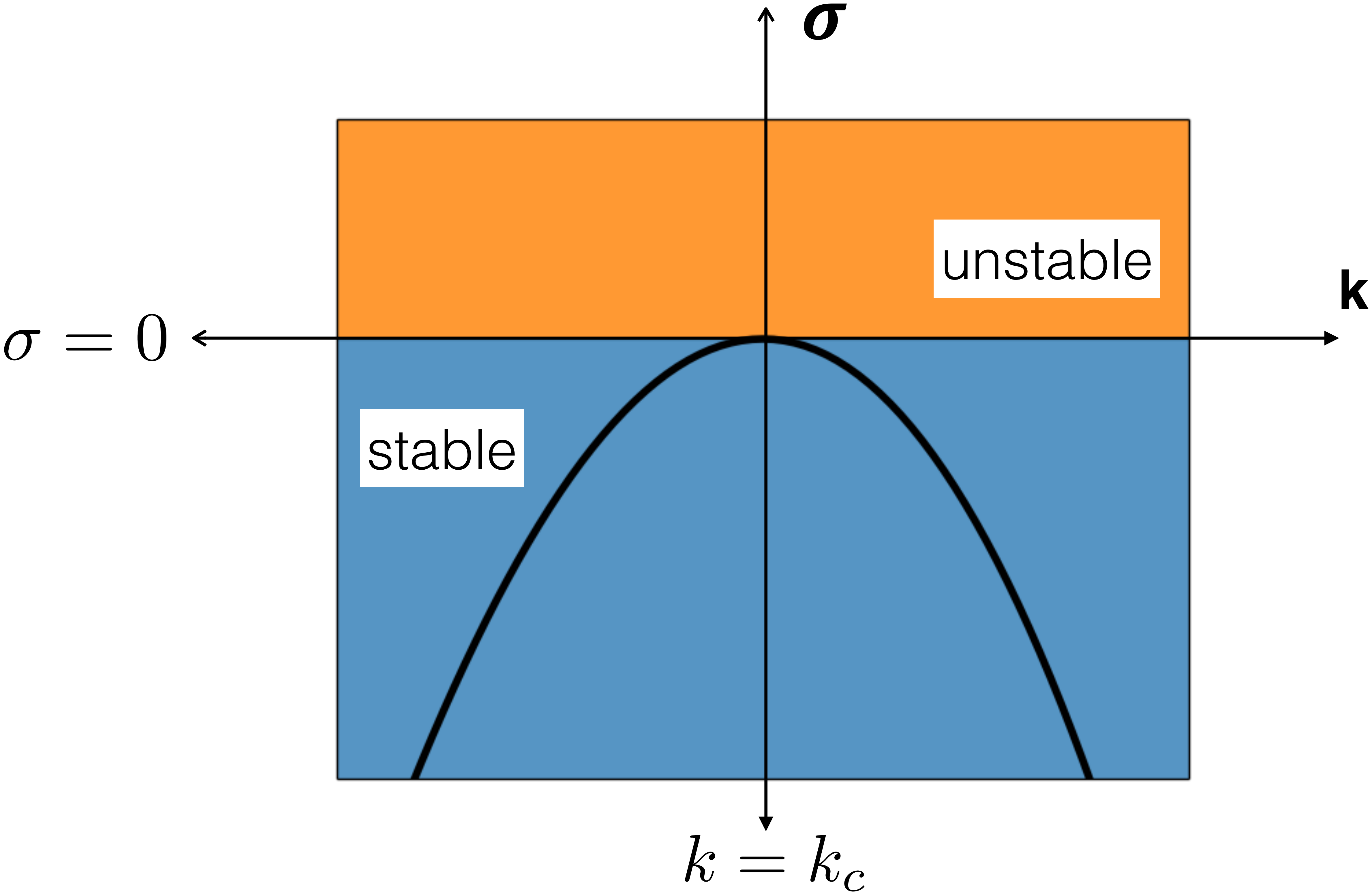

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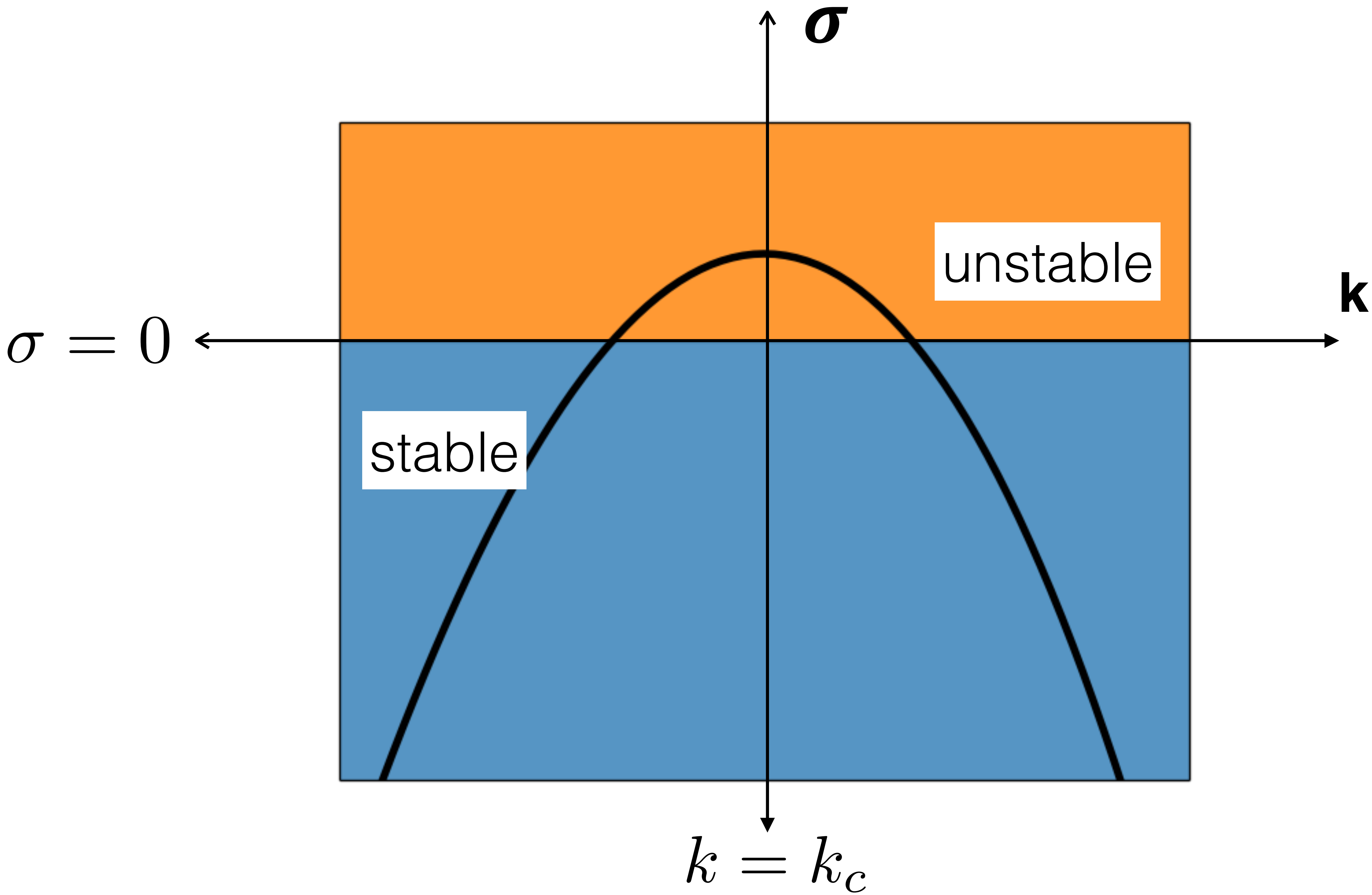

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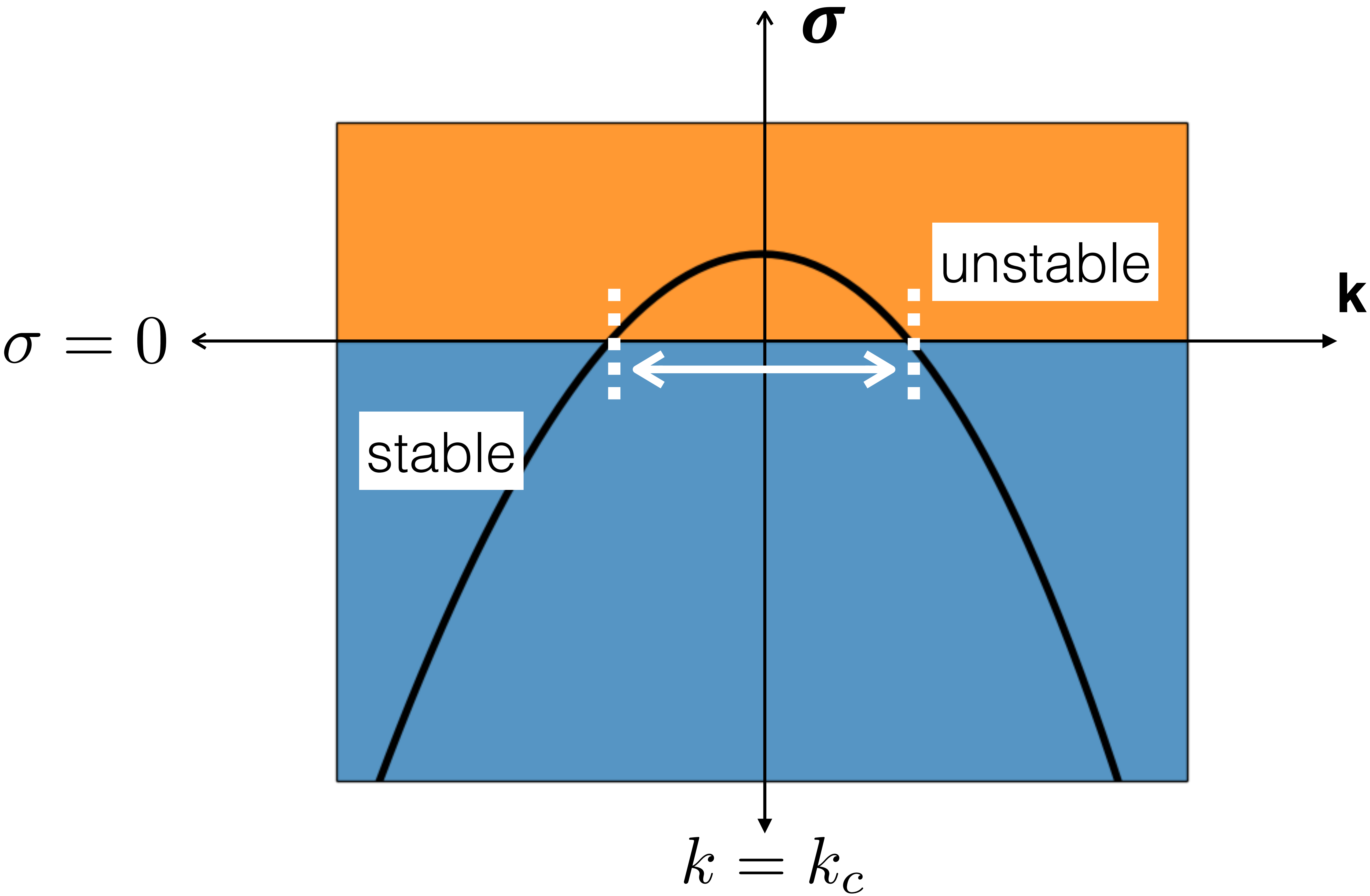
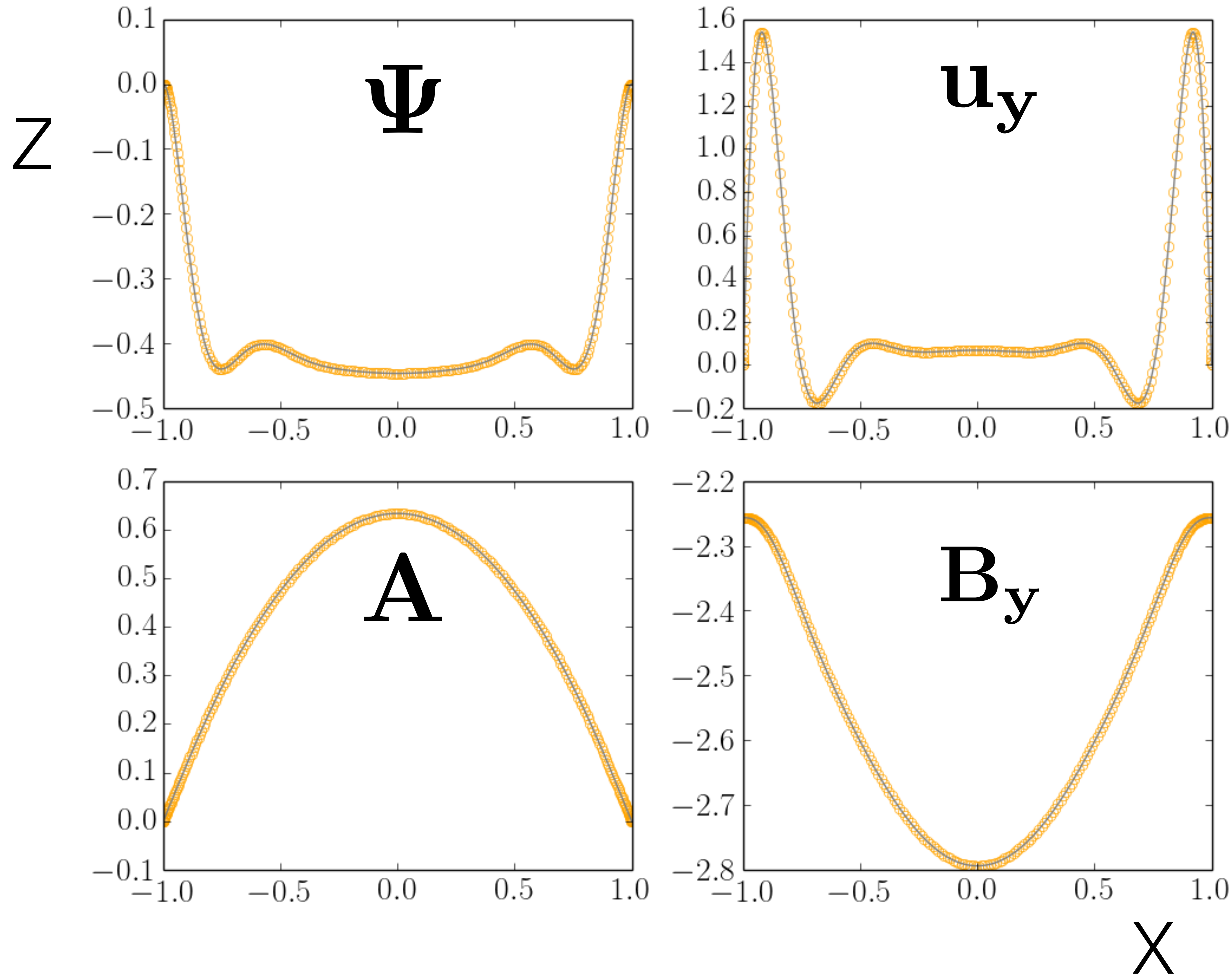


Tune the most unstable mode just over the threshold of instability.

$$\epsilon^2 \equiv 1 - B_0$$

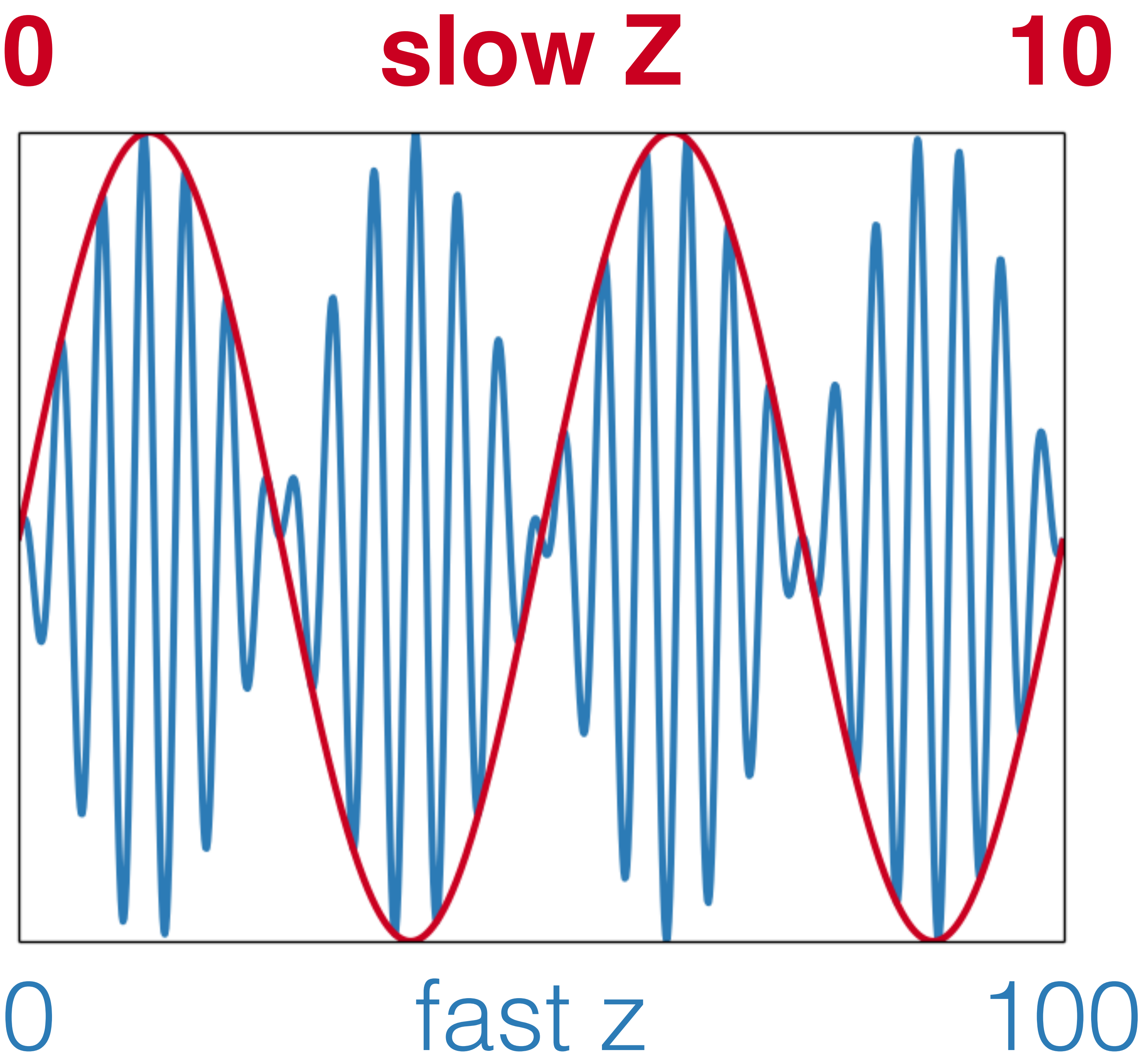
small parameter

Identify the most unstable mode of the linear MRI.



Multiscale analysis tracks the evolution of fast and slow variables.

$$Z \equiv \epsilon z$$



We choose an ansatz state vector form.

$$\mathbf{V} = \alpha\left(Z, T\right) \mathrm{V}\left(x\right) e^{ik_c z}$$

We choose an ansatz state vector form.

x dependence

$$\mathbf{V} = \alpha\left(Z, T\right) \mathrm{V}\left(x\right) e^{ik_c z}$$

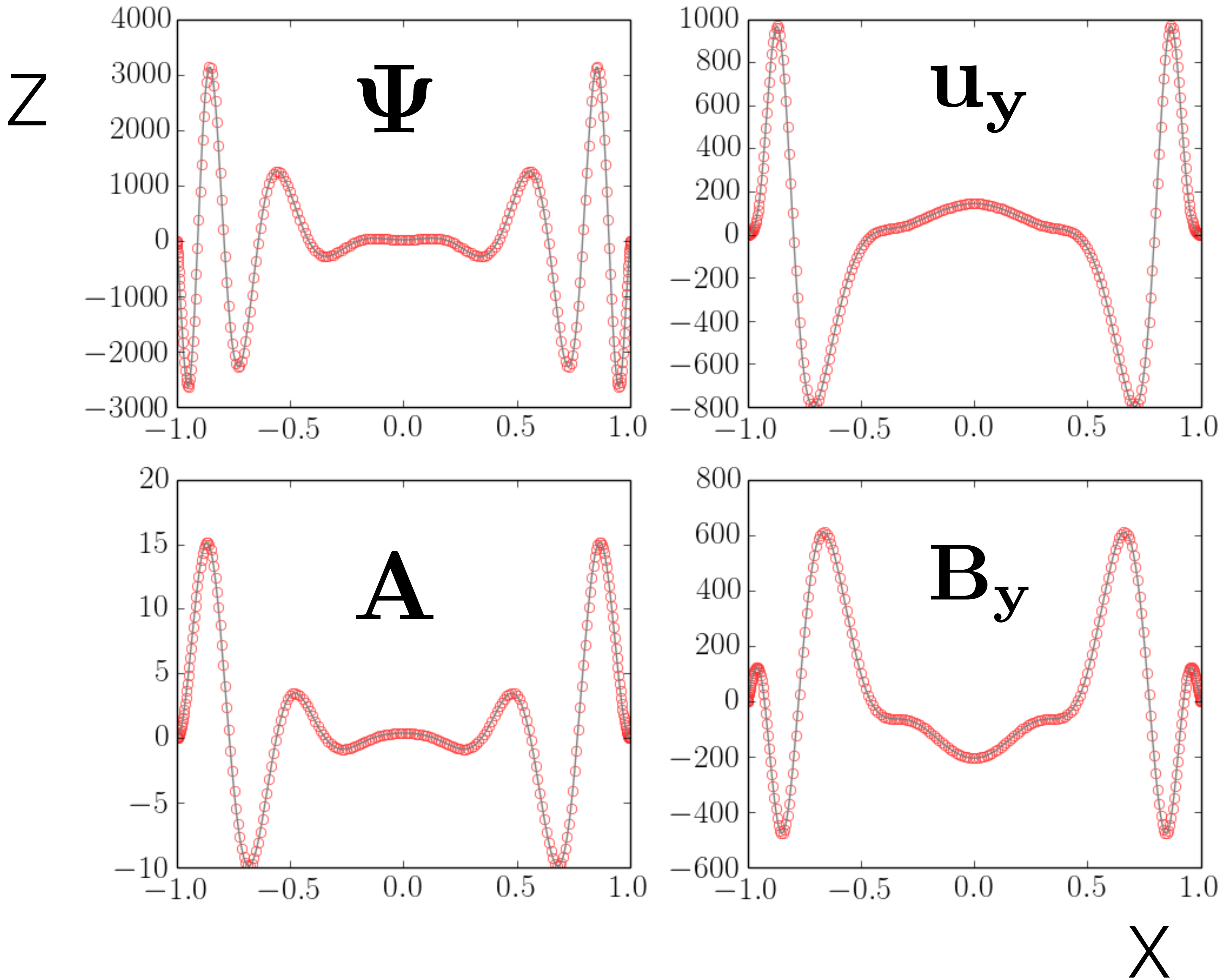
amplitude function

vertical periodicity

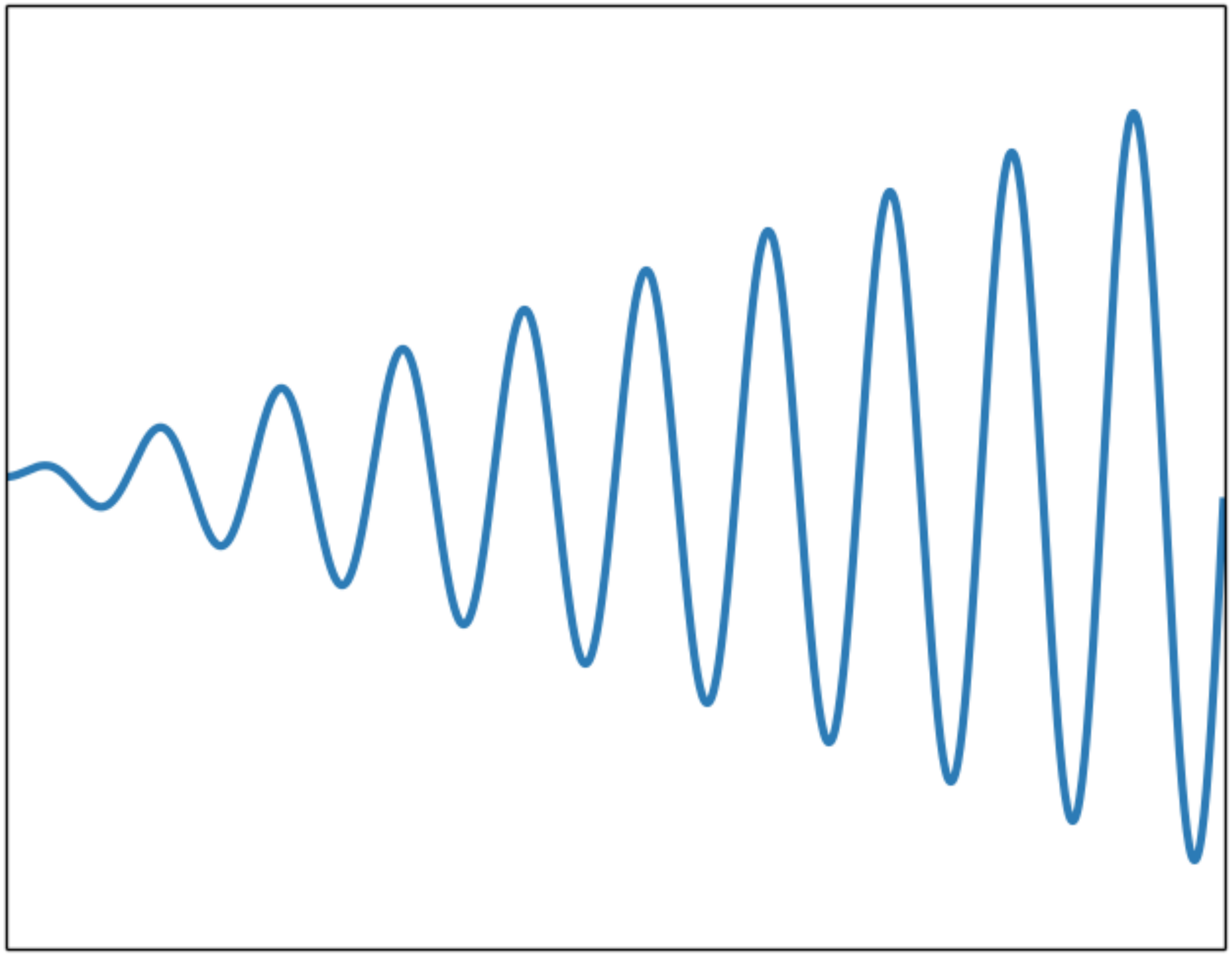
The fluid quantities are expanded in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + \ldots$$

We solve each term in the expanded equations at each order.



The removal of secular terms yields solvability criteria.



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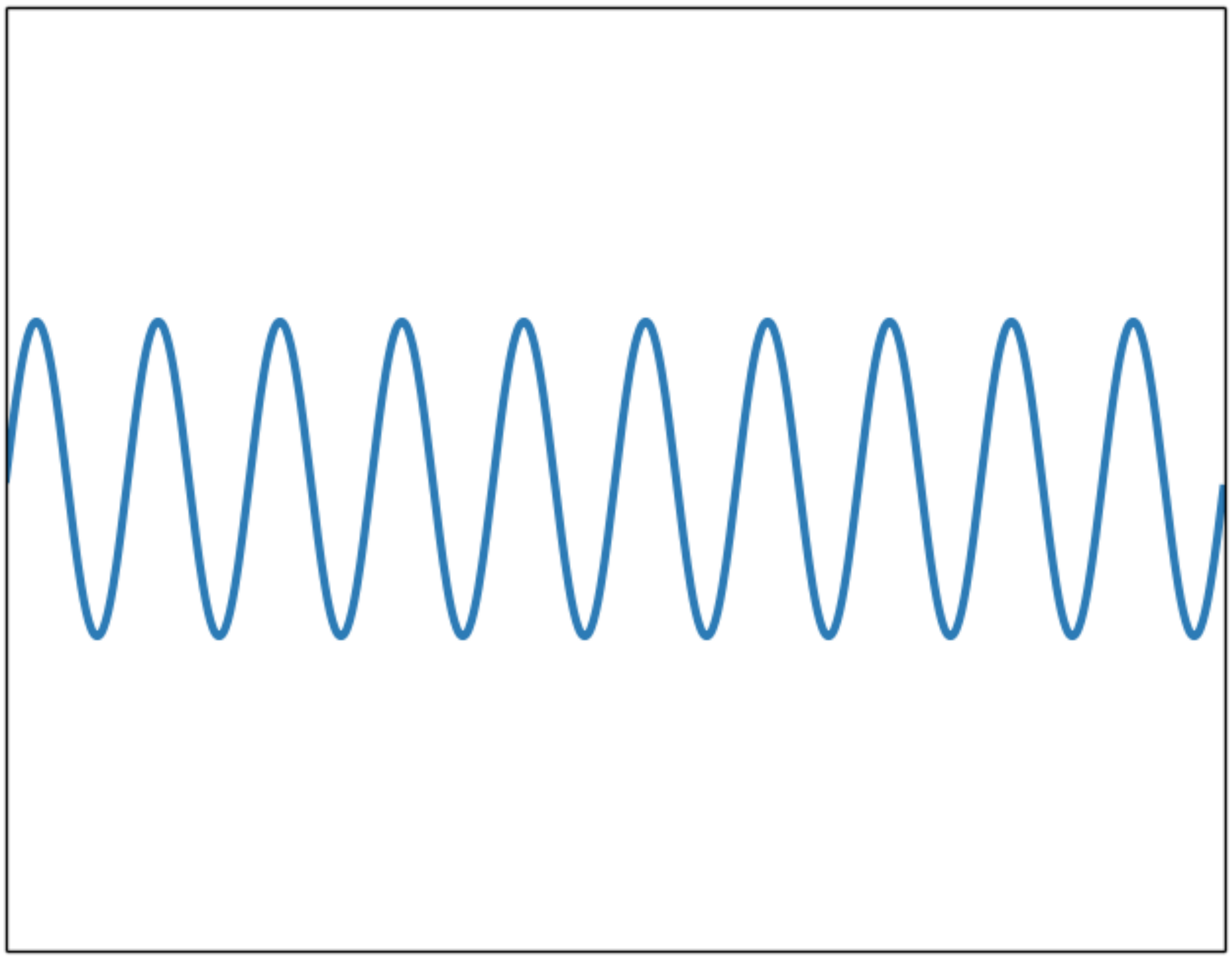


The result is an amplitude equation
for the most unstable mode.

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

The result is an amplitude equation for the most unstable mode.

diffusion term

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + gik_c^3 \alpha$$

nonlinear term

linear growth

The result is an amplitude equation for the most unstable mode.

diffusion term

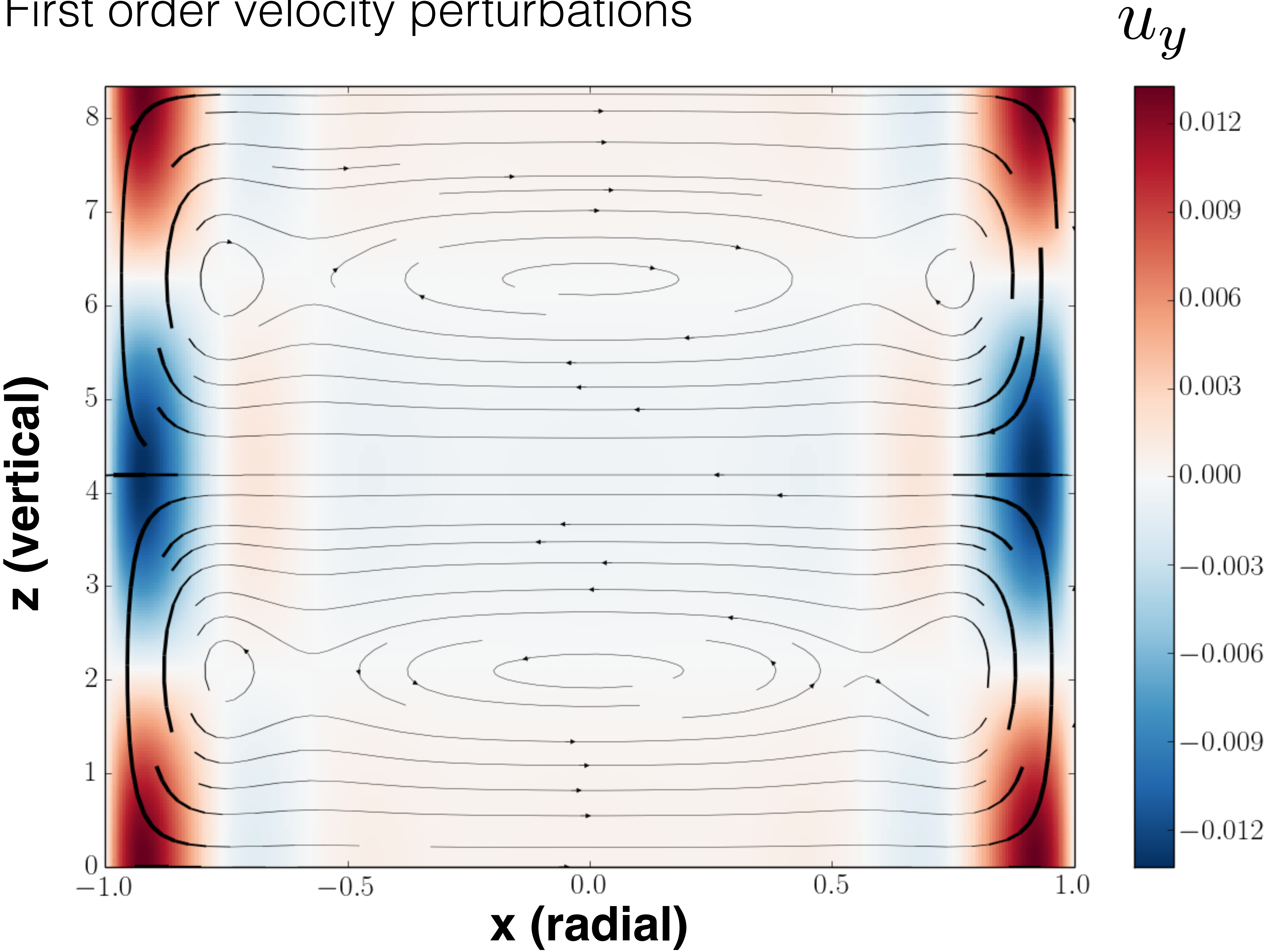
?

$$\partial_T \alpha = -b\partial_Z \alpha - c\alpha\left|\alpha^2\right| + h\partial_Z^2 \alpha + giQ^3\alpha$$

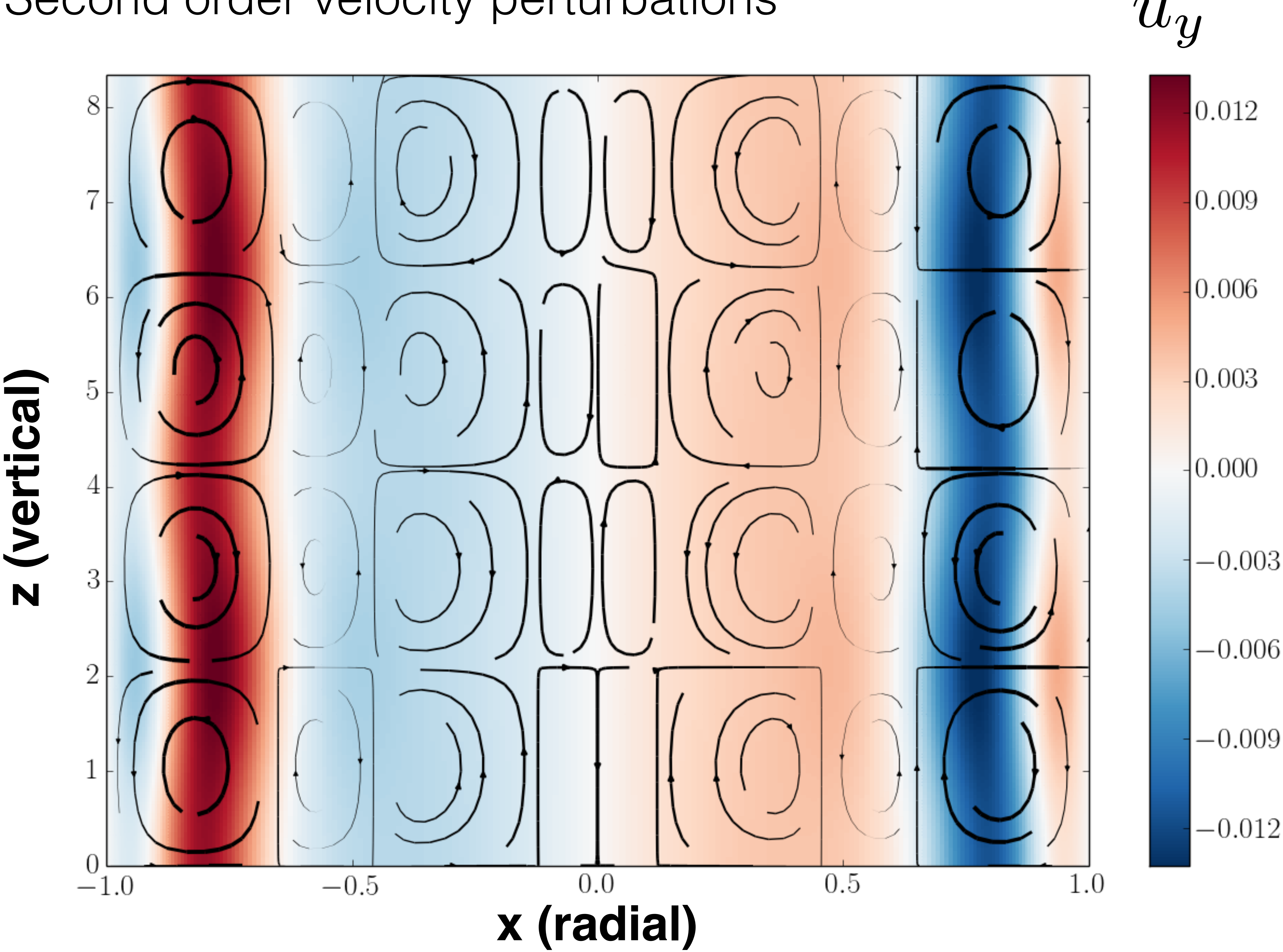
nonlinear term

linear growth

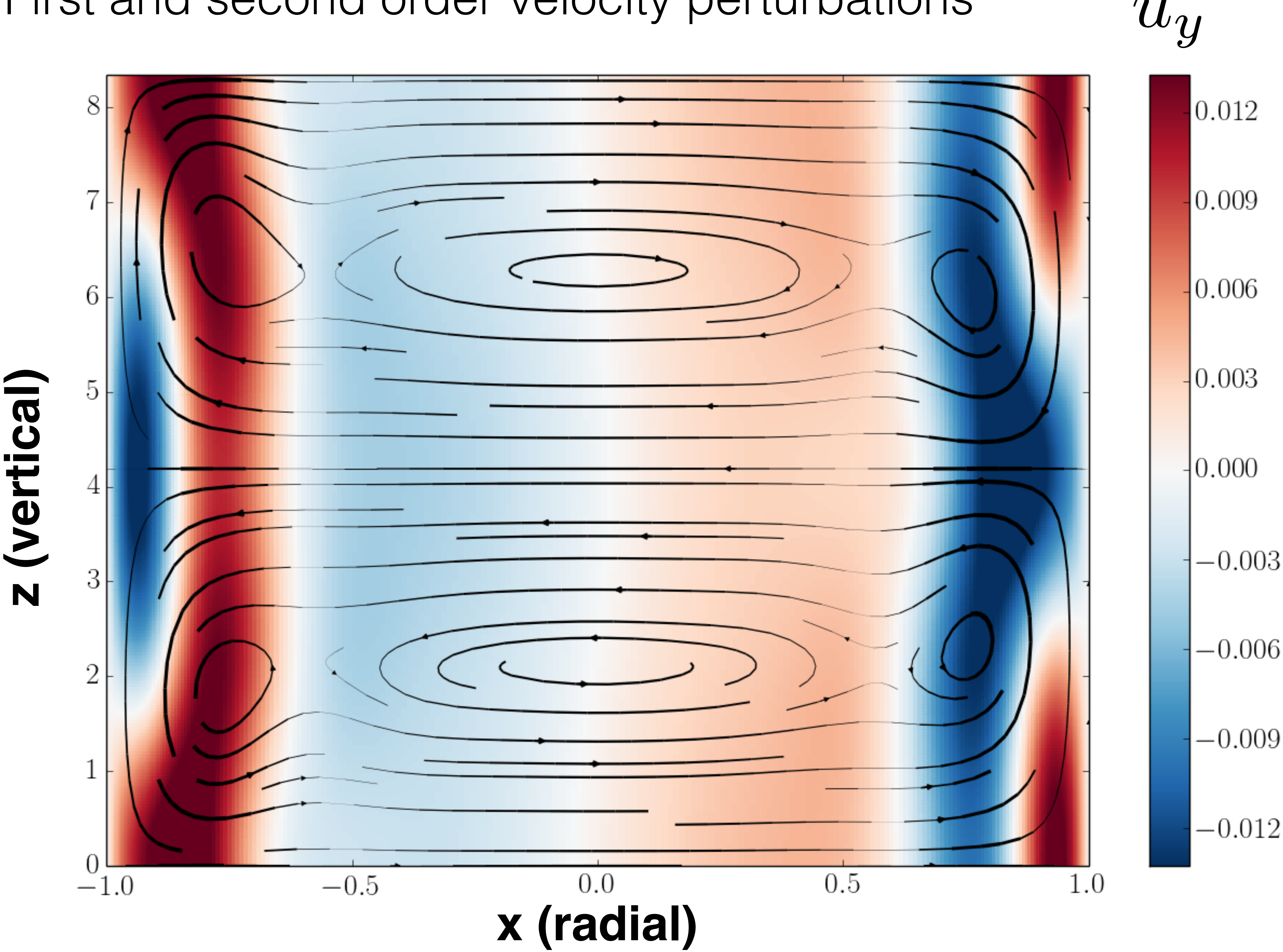
First order velocity perturbations



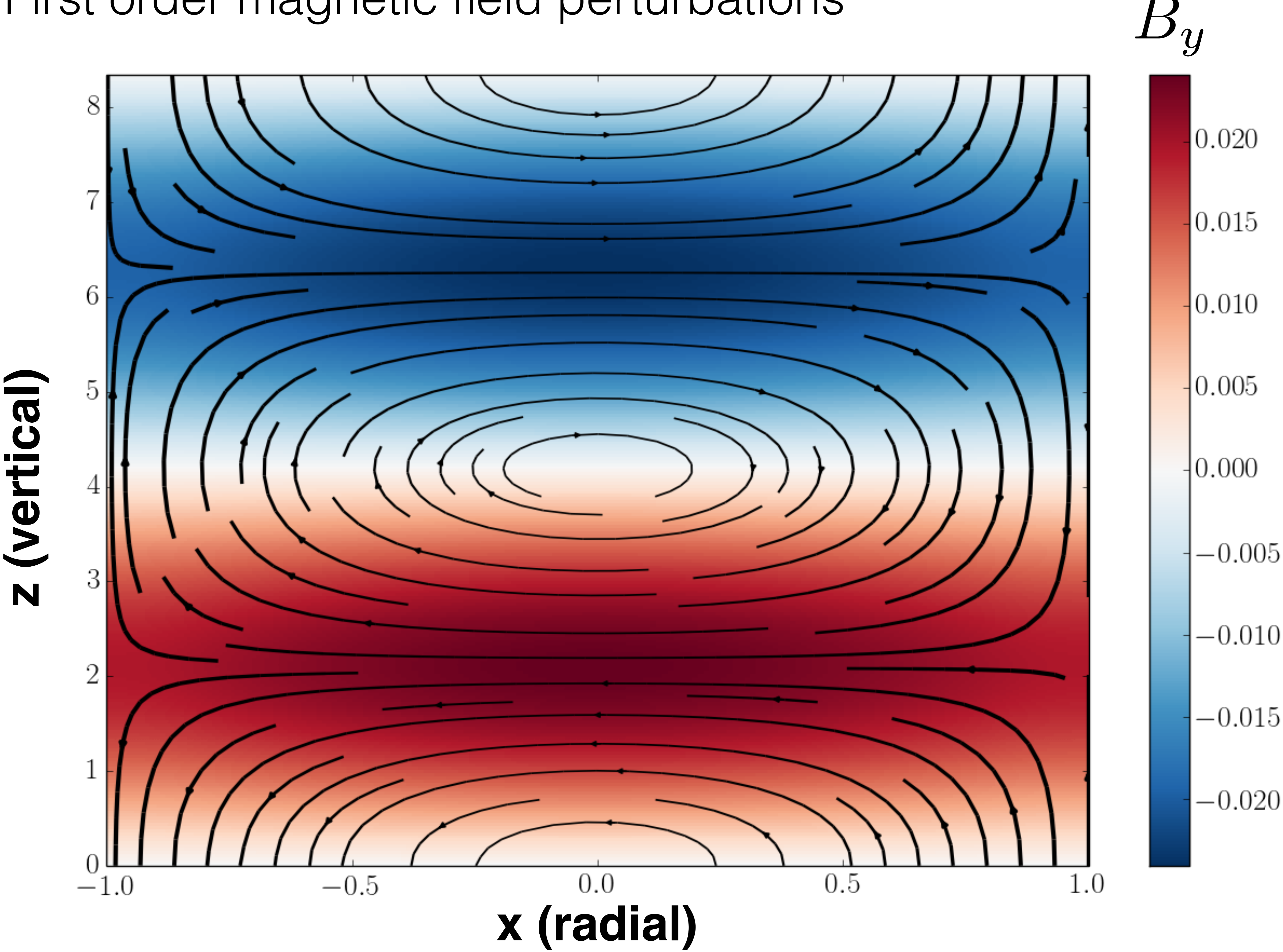
Second order velocity perturbations



First and second order velocity perturbations

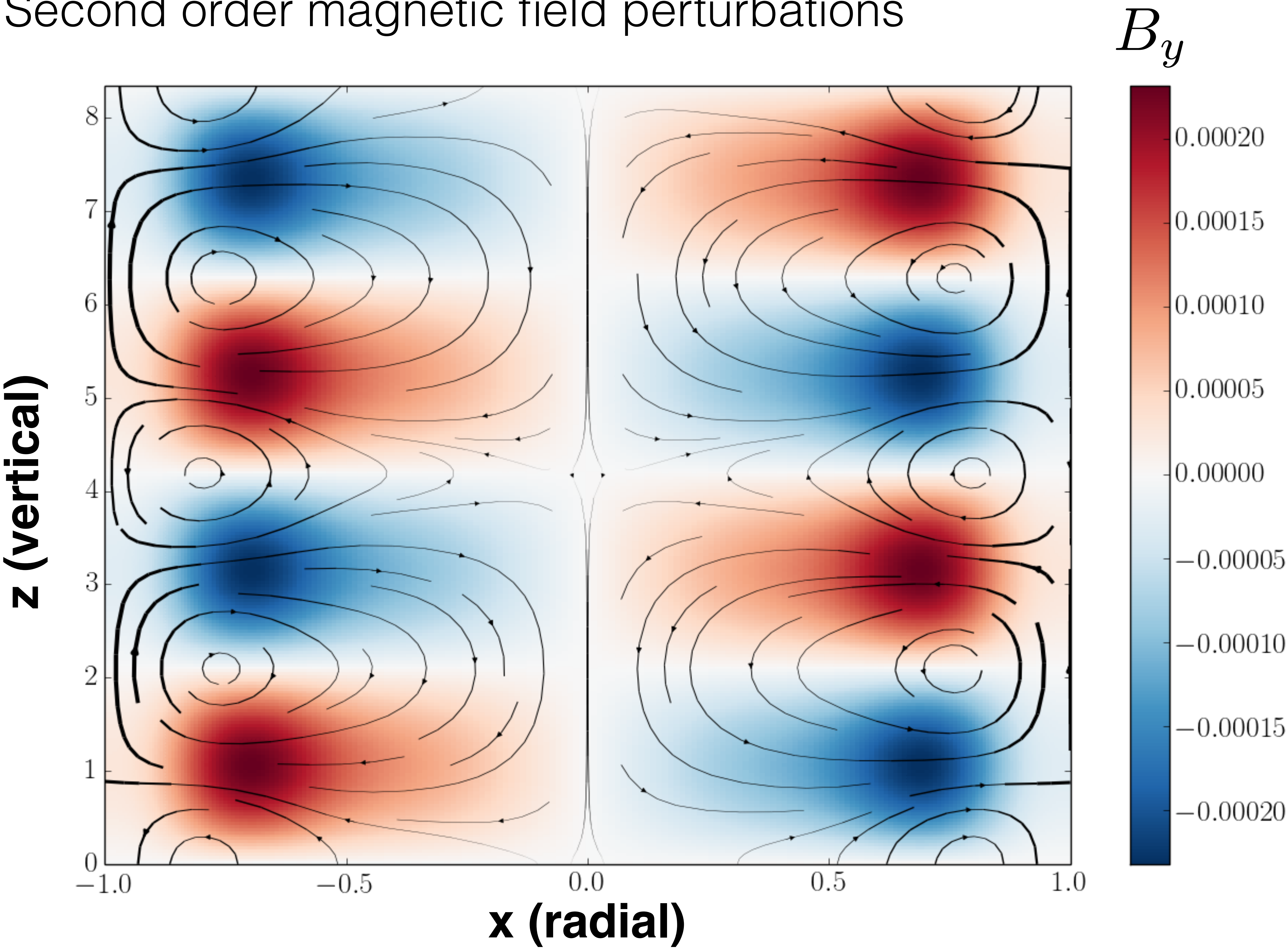


First order magnetic field perturbations



Second order magnetic field perturbations

two OOM smaller!



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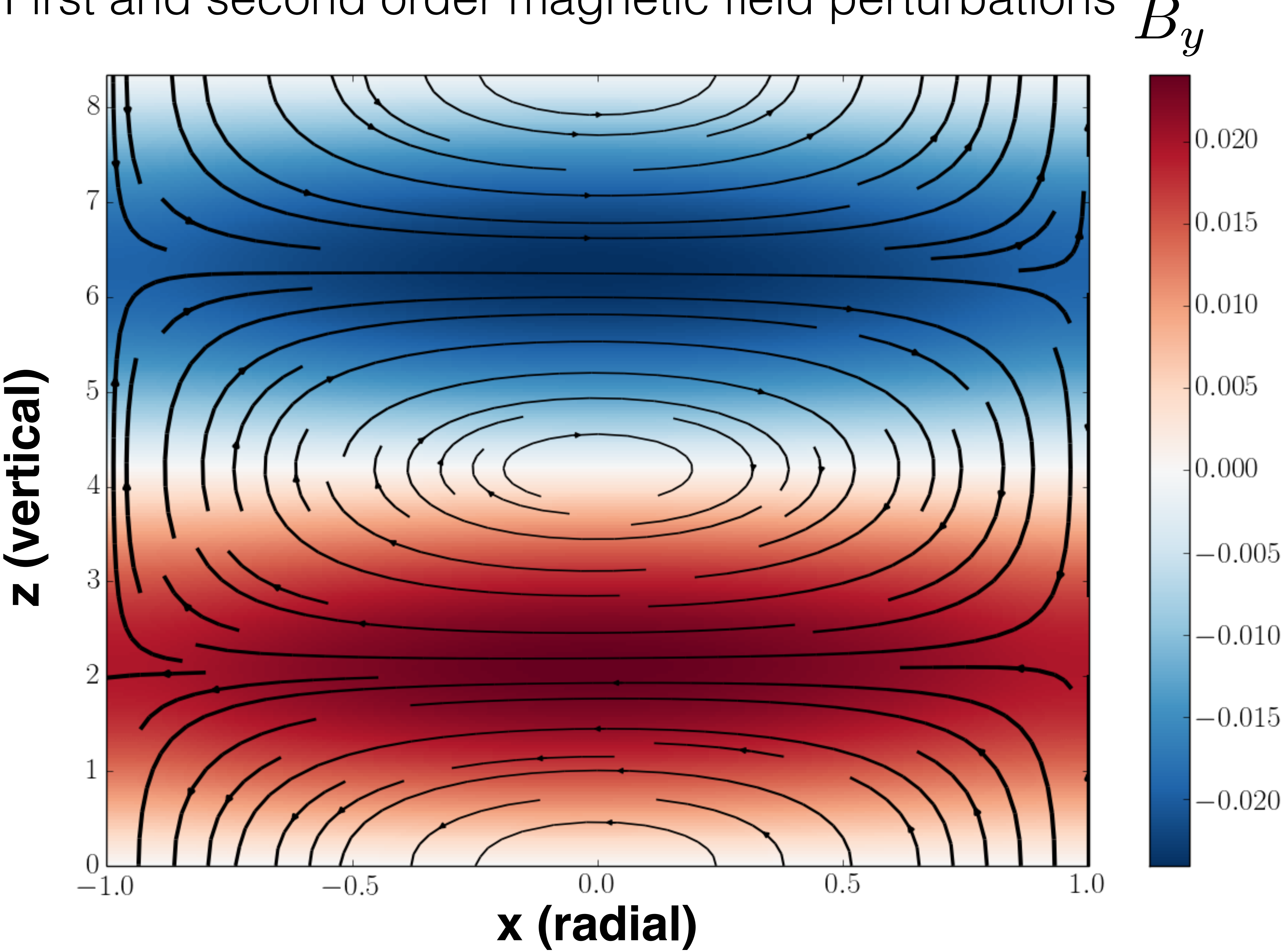

Future work:

non-thin gap approximation
helical MRI
explore parameter space
comparison to experiment