

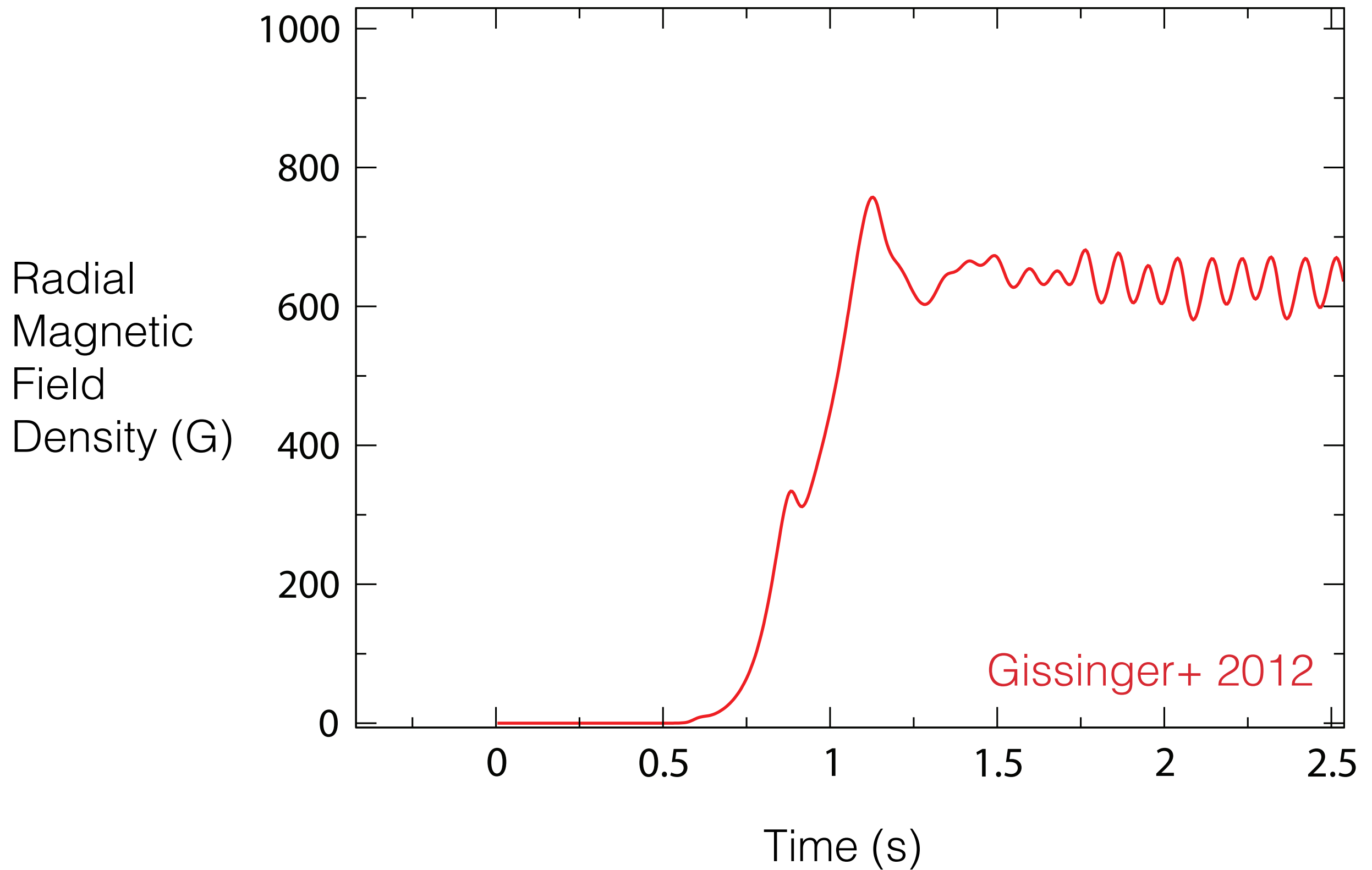
# Exploring the saturation of the MRI via weakly nonlinear analysis

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The MRI evolves in stages.



set-up

boundary conditions

parameter range

open questions, etc

We solve the non-ideal MRI equations.

## momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\Omega \times \mathbf{u} - \Omega \times (\Omega \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

## induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

## constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal MRI equations.

## momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \nu \nabla^2 \mathbf{u}$$

## induction

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

microscopic  
viscosity



magnetic  
resistivity



## constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We nondimensionalize and perturb  
the nonlinear MRI equations.

magnetic  
resistivity

microscopic  
viscosity

We work in terms of flux and stream functions.

## momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - (2 - q) \Omega_0 \partial_z \Psi + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \frac{1}{Re} \nabla^2 u_y$$

## induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

## momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

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## induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A} \quad \text{resistive}$$

$$\partial_t B_y = B_0 \partial_z u_y - q \Omega_0 \partial_z A + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$



We work in terms of flux and stream functions.

## momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

viscous

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2 - q) \Omega_0 \partial_z \Psi} + \frac{2}{\beta} J(A, B_y) - J(\Psi, u_y) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

## induction

$$\partial_t A = B_0 \partial_z \Psi + J(A, \Psi) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q \Omega_0 \partial_z A} + J(A, u_y) - J(\Psi, B_y) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

## momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2 \partial_z u_y + \boxed{\frac{2}{\beta} J(A, \nabla^2 A) - J(\Psi, \nabla^2 \Psi)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

nonlinear

viscous

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2 - q) \Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J(A, B_y) - J(\Psi, u_y)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

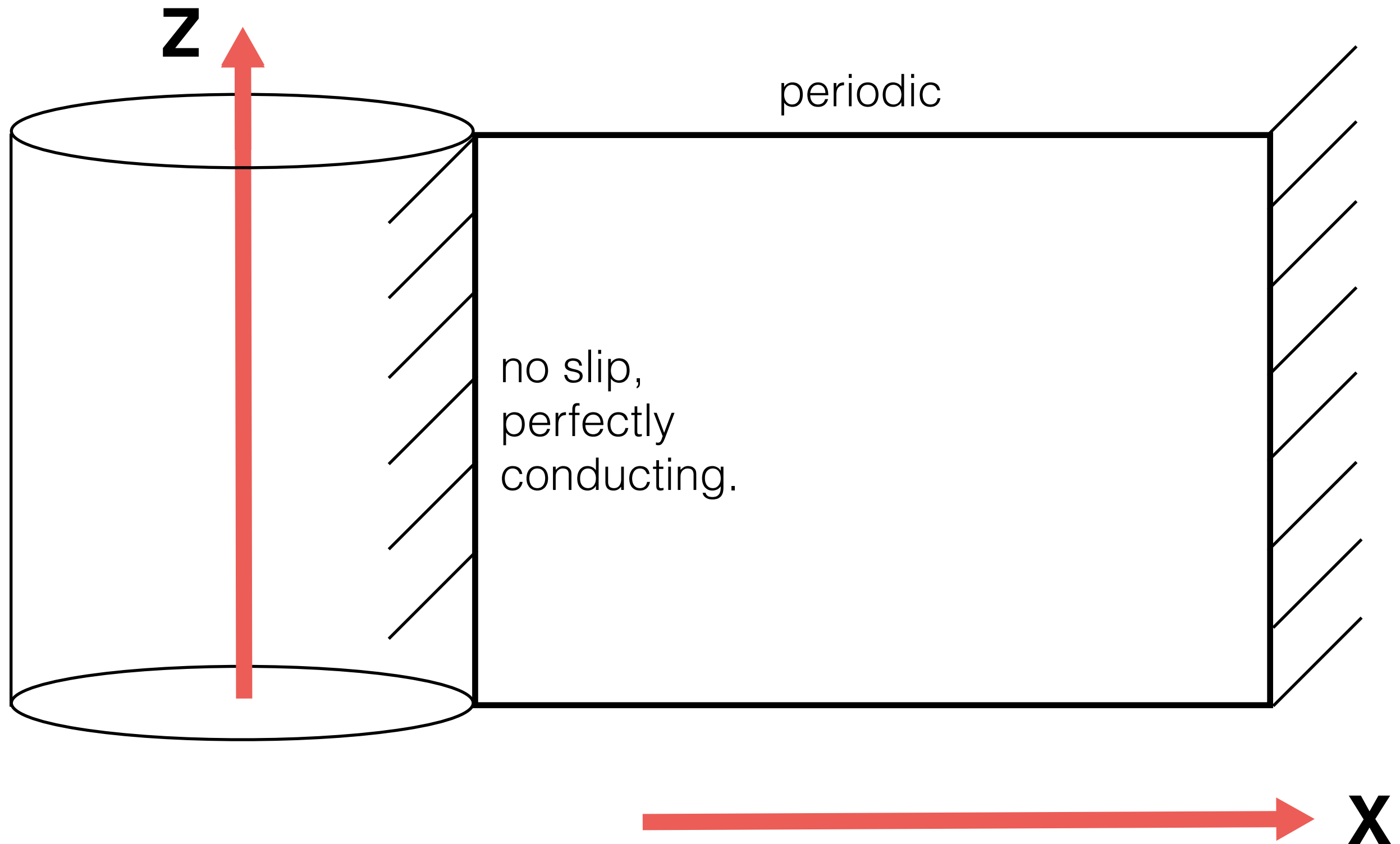
shear

## induction

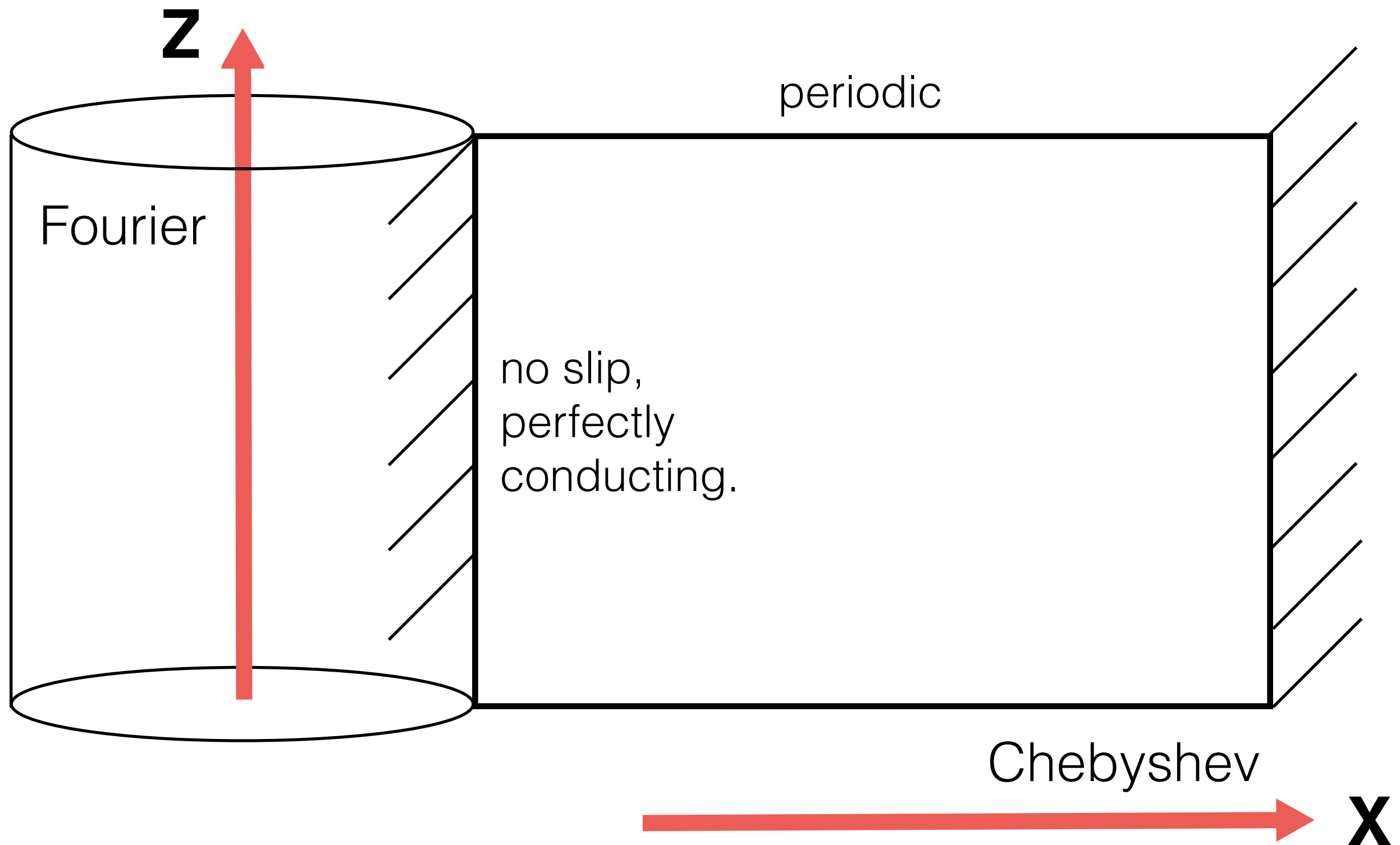
$$\partial_t A = B_0 \partial_z \Psi + \boxed{J(A, \Psi)} + \boxed{\frac{1}{Rm} \nabla^2 A} \quad \text{resistive}$$

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q \Omega_0 \partial_z A} + \boxed{J(A, u_y) - J(\Psi, B_y)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

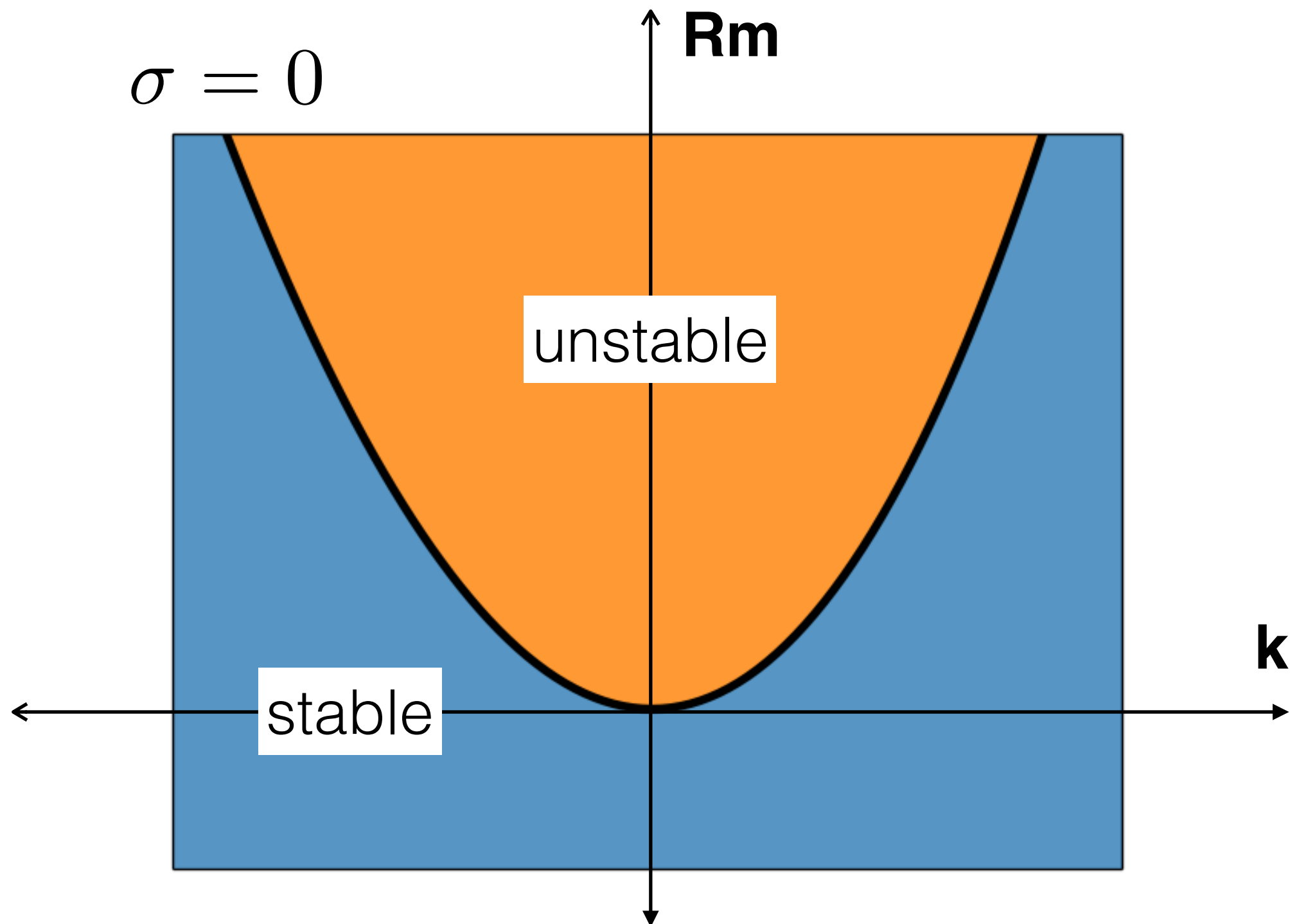
We use experimentally relevant boundary conditions.



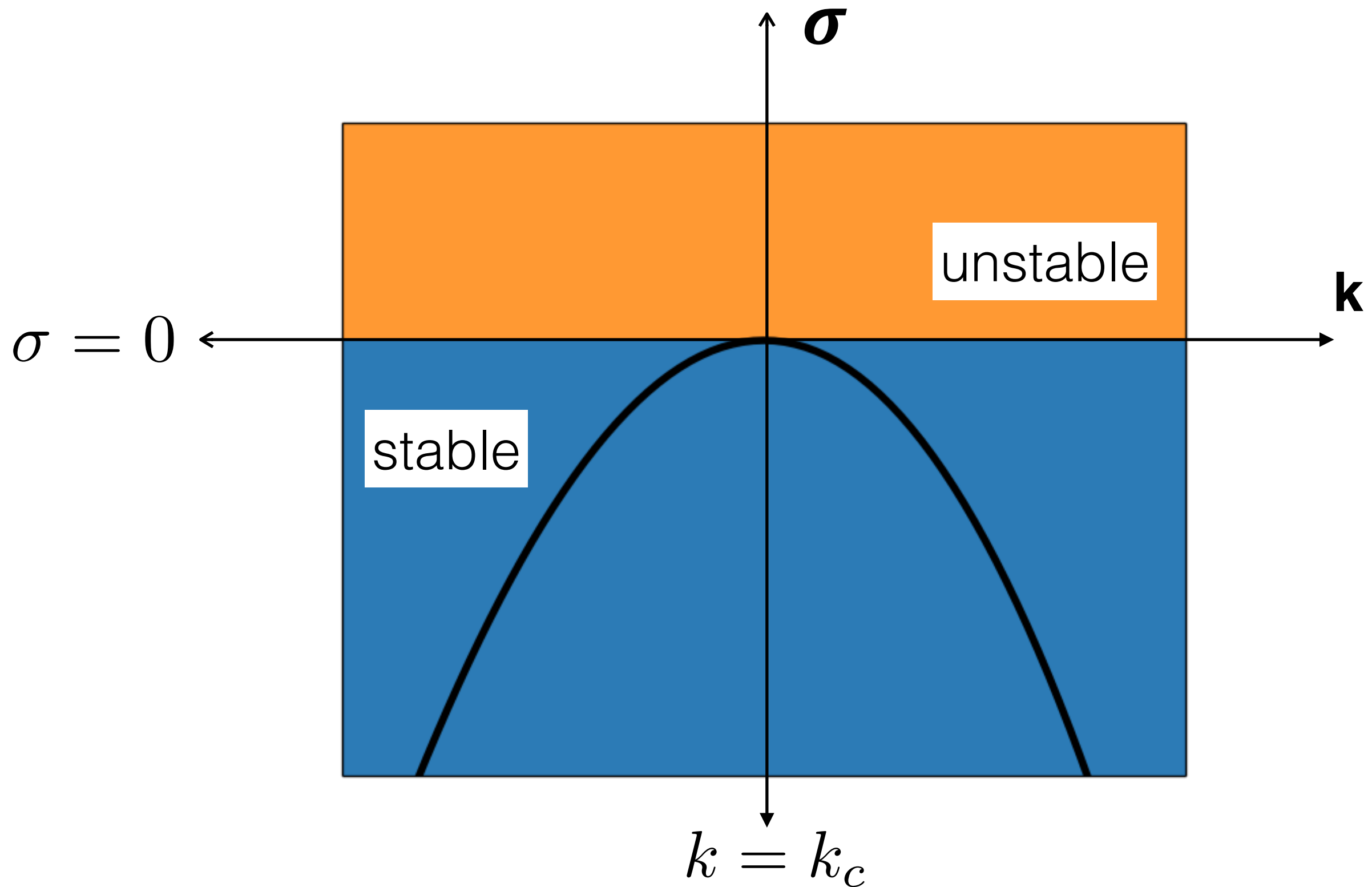
Dedalus is a general-purpose spectral code.



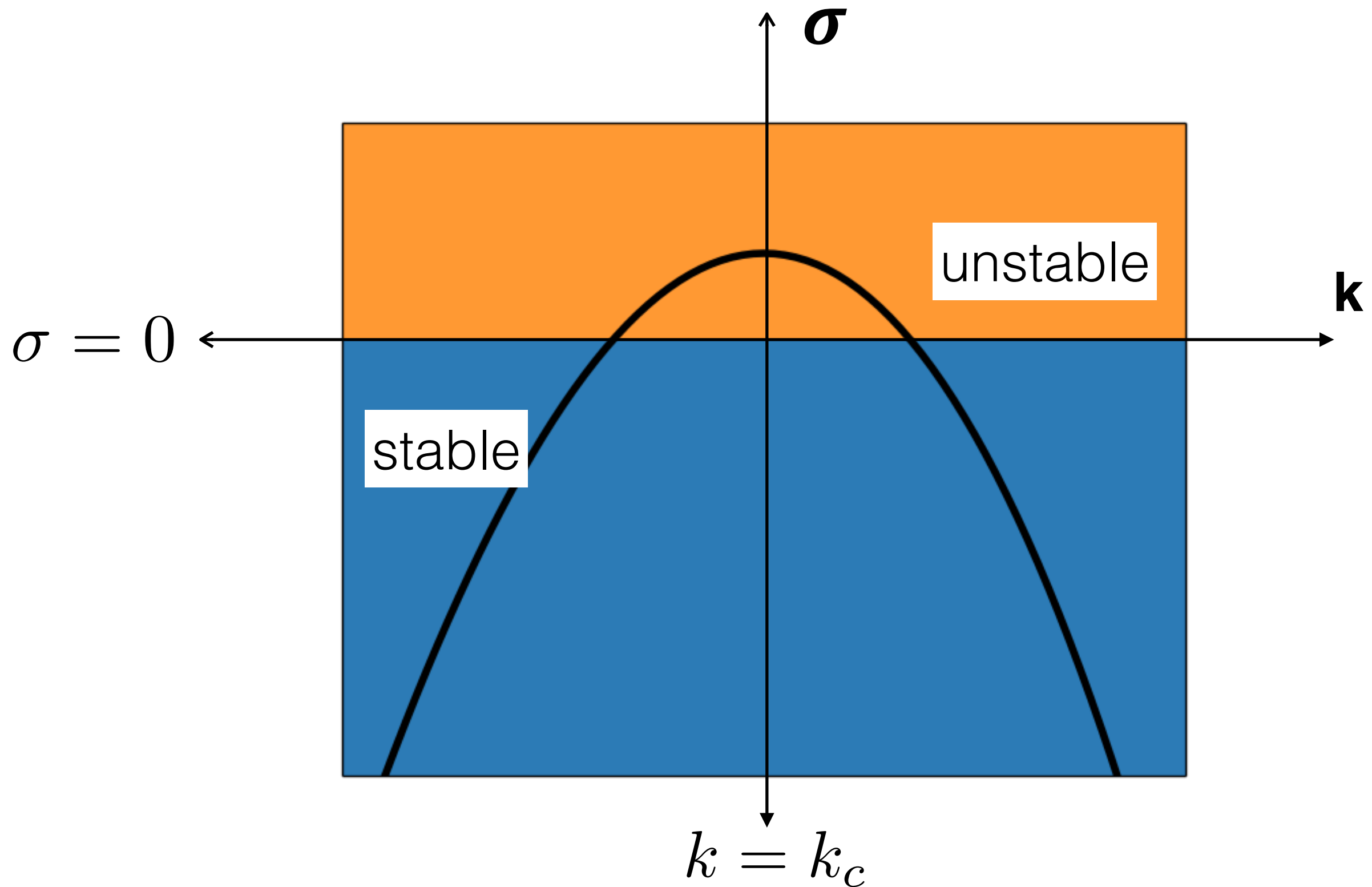
Weakly nonlinear analysis explores behavior at the margin of instability.



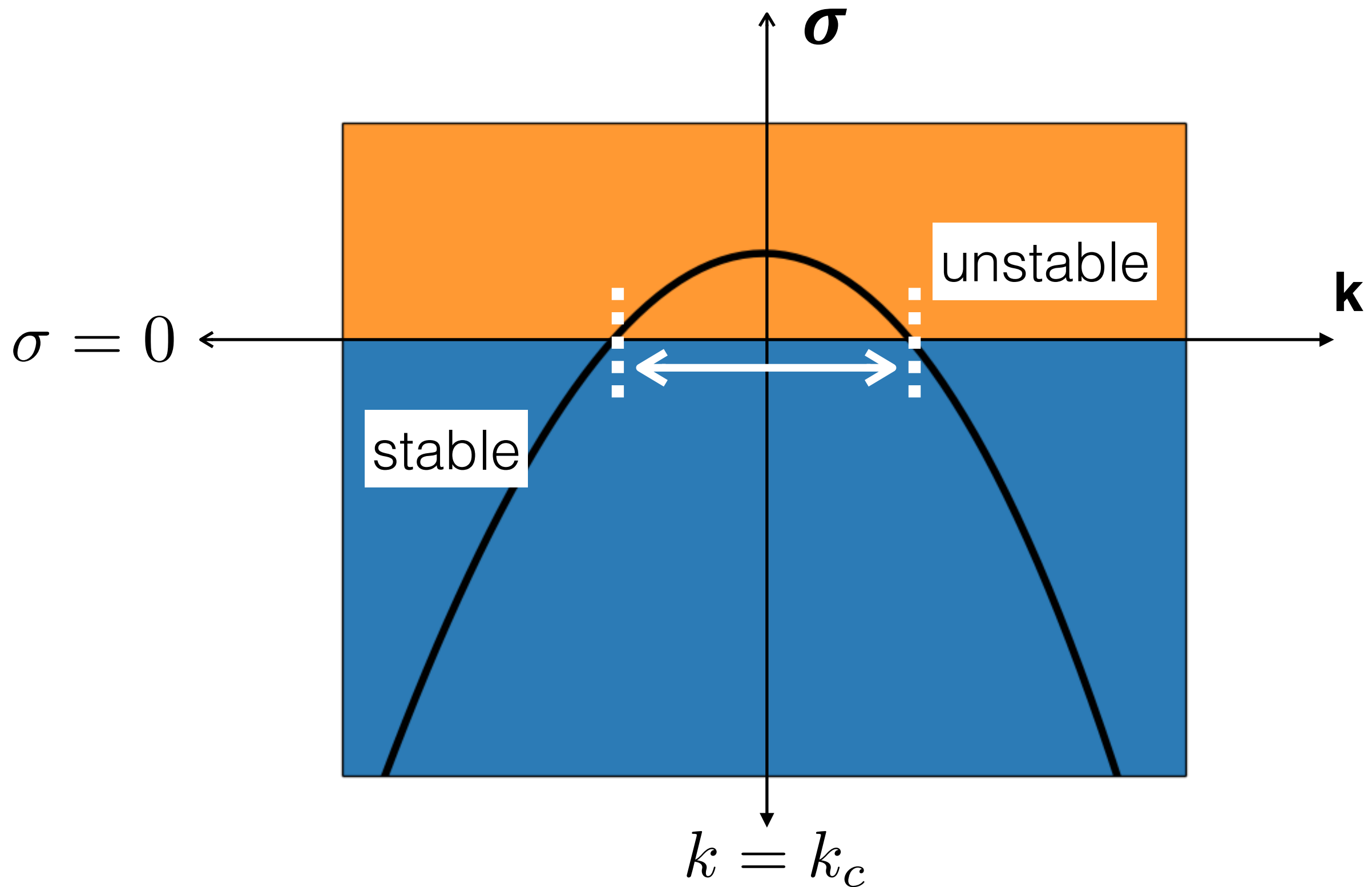
Weakly nonlinear analysis explores behavior at the margin of instability.



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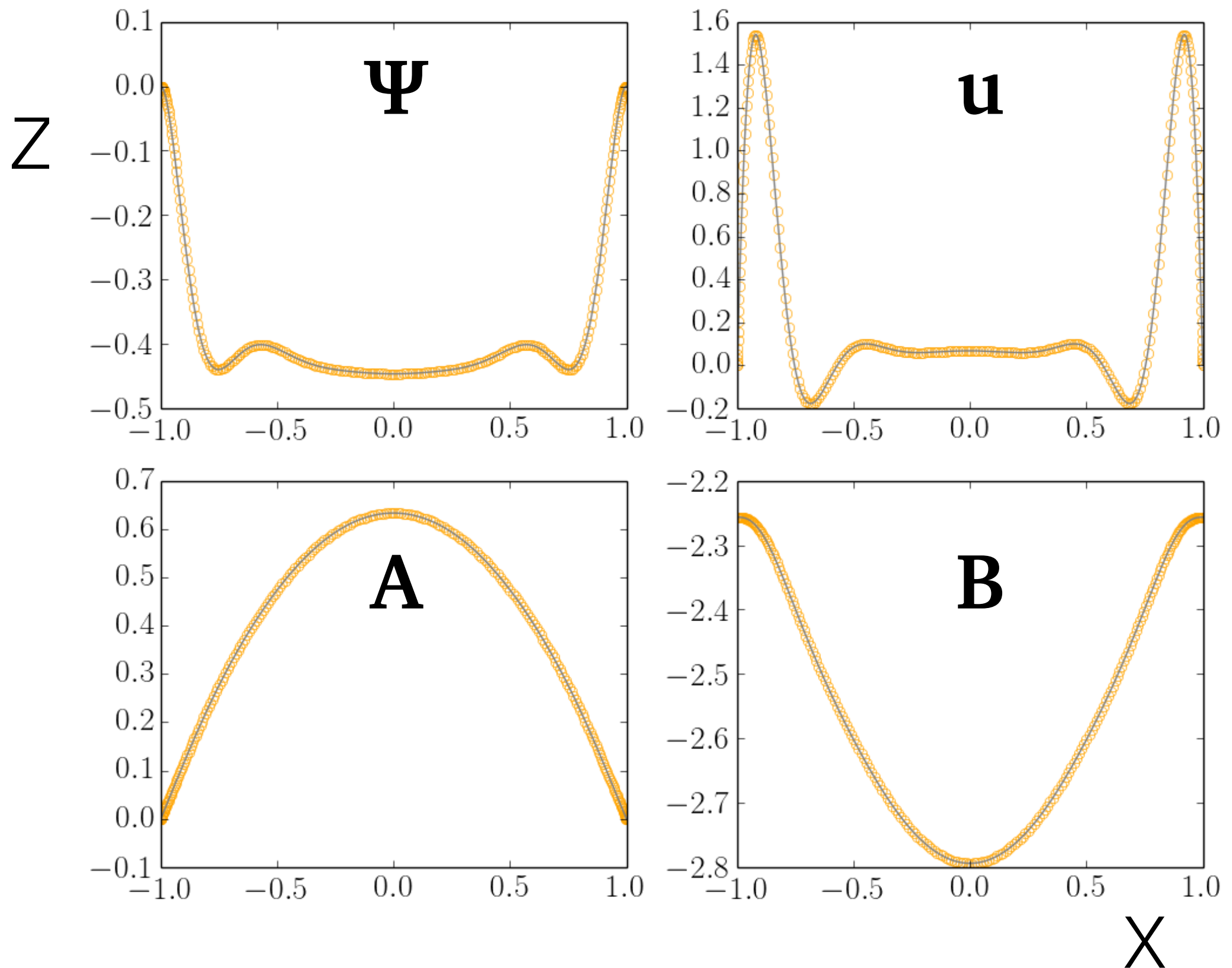
Tune the most unstable mode just over the threshold of instability.

$$\epsilon^2 \equiv 1 - B_0$$



**small  
parameter**

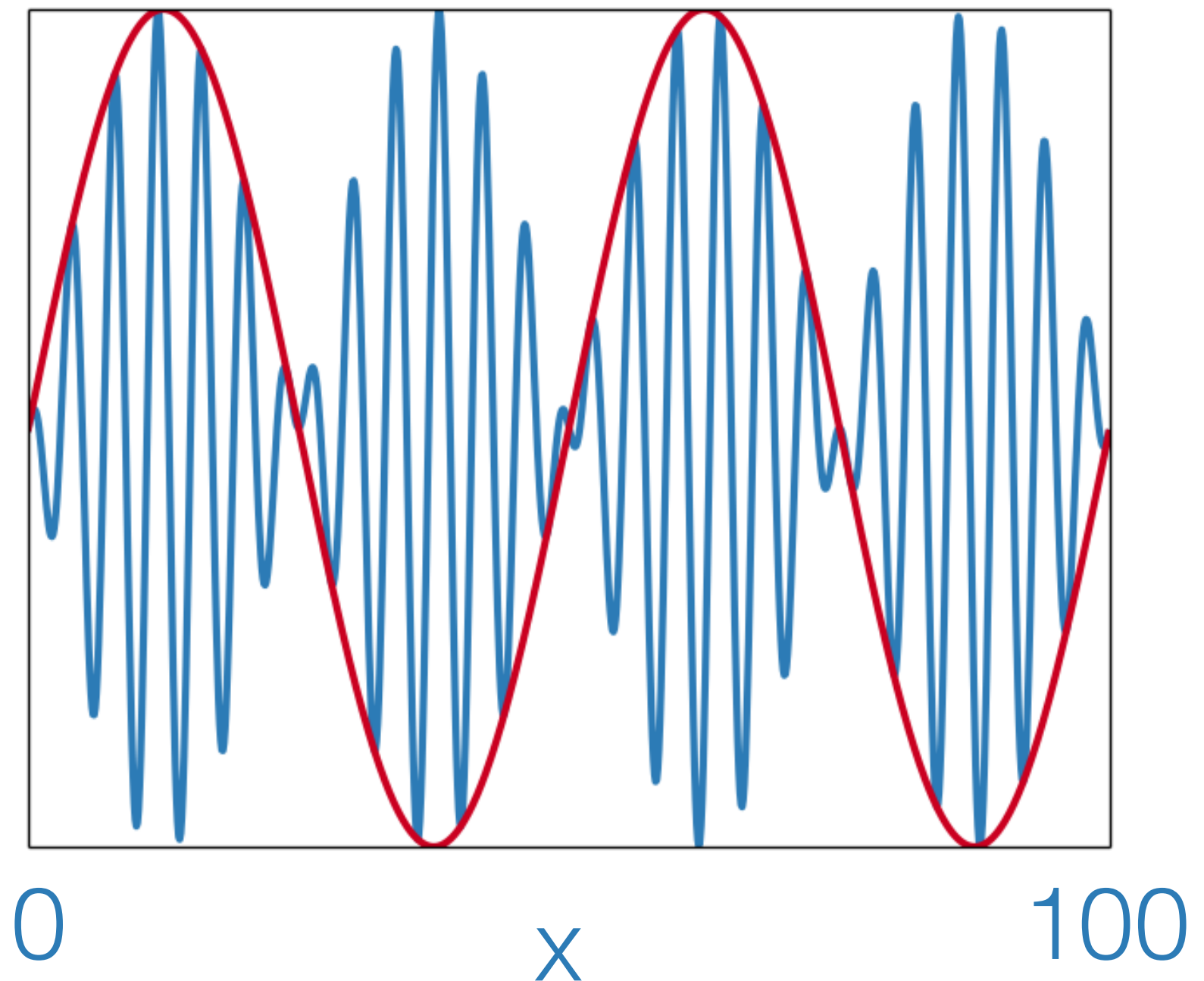
Identify the most unstable mode of the linear MRI.



Multiscale analysis tracks the evolution of fast and slow variables.

**0** **X** **10**

$$X \equiv \epsilon x$$

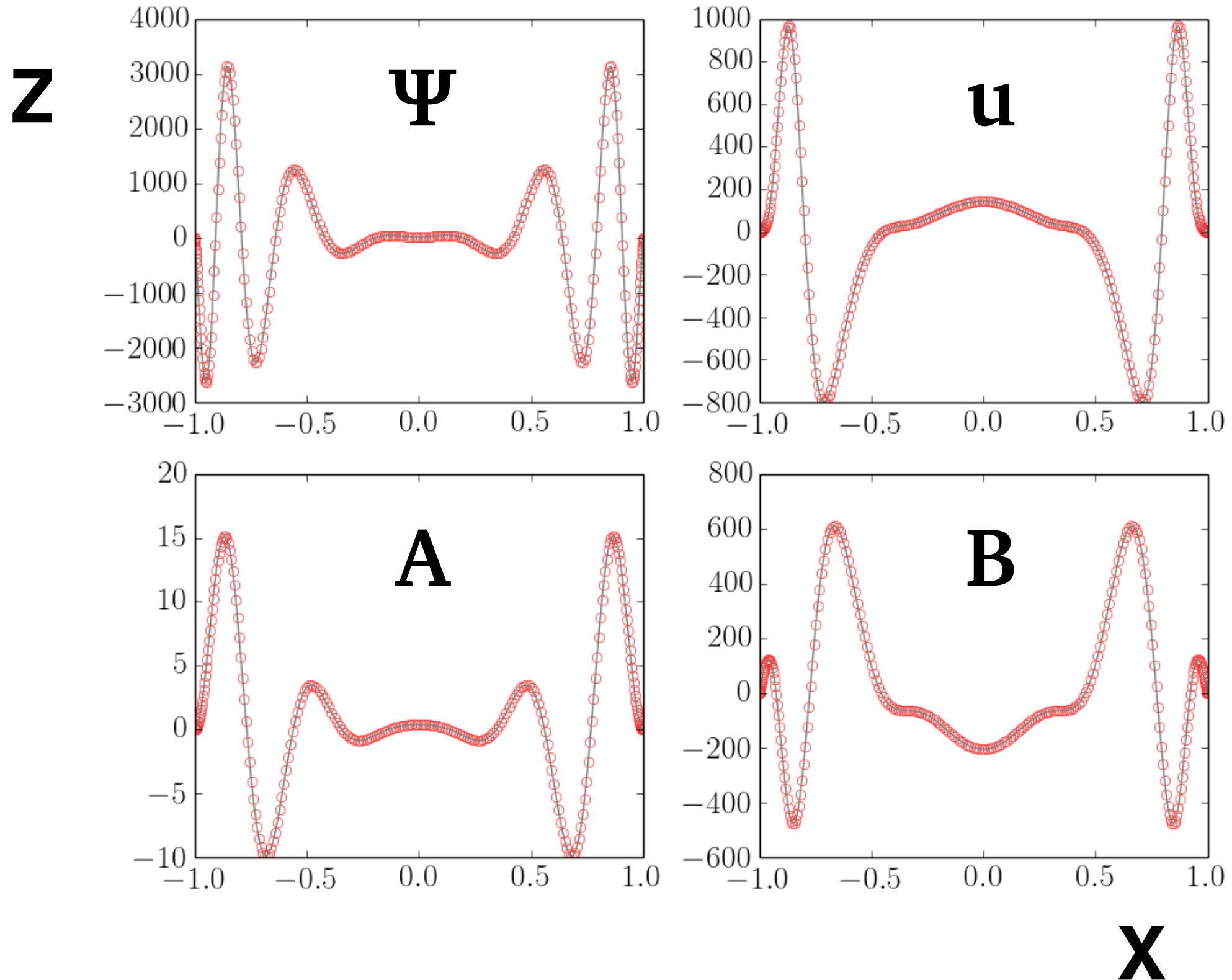


Equations are solved in a  
matrix formulation.

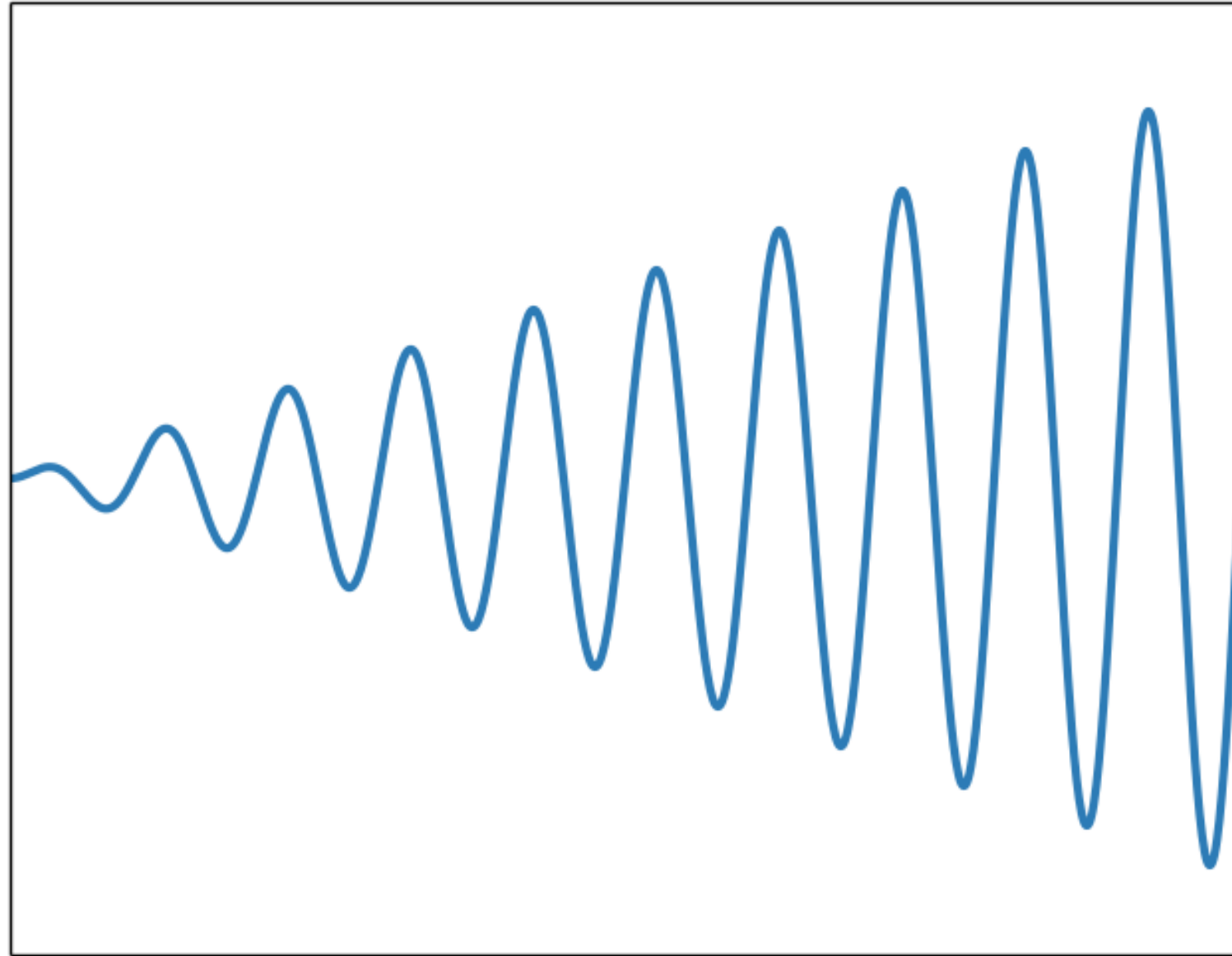
The fluid quantities are expanded  
in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + \epsilon^3 \mathbf{V}_3 + \dots$$

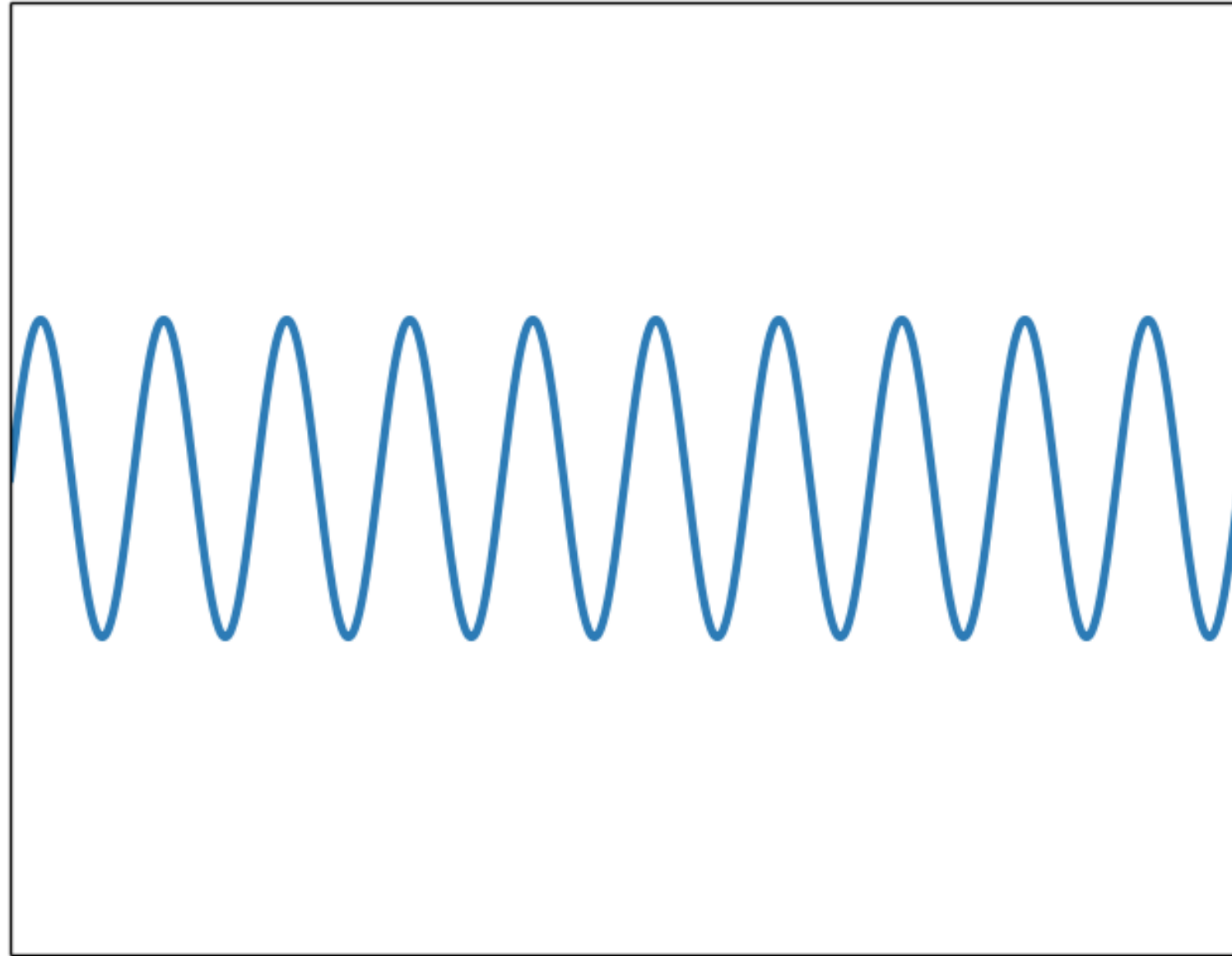
something about boundary layers?



The removal of secular terms yields solvability criteria.



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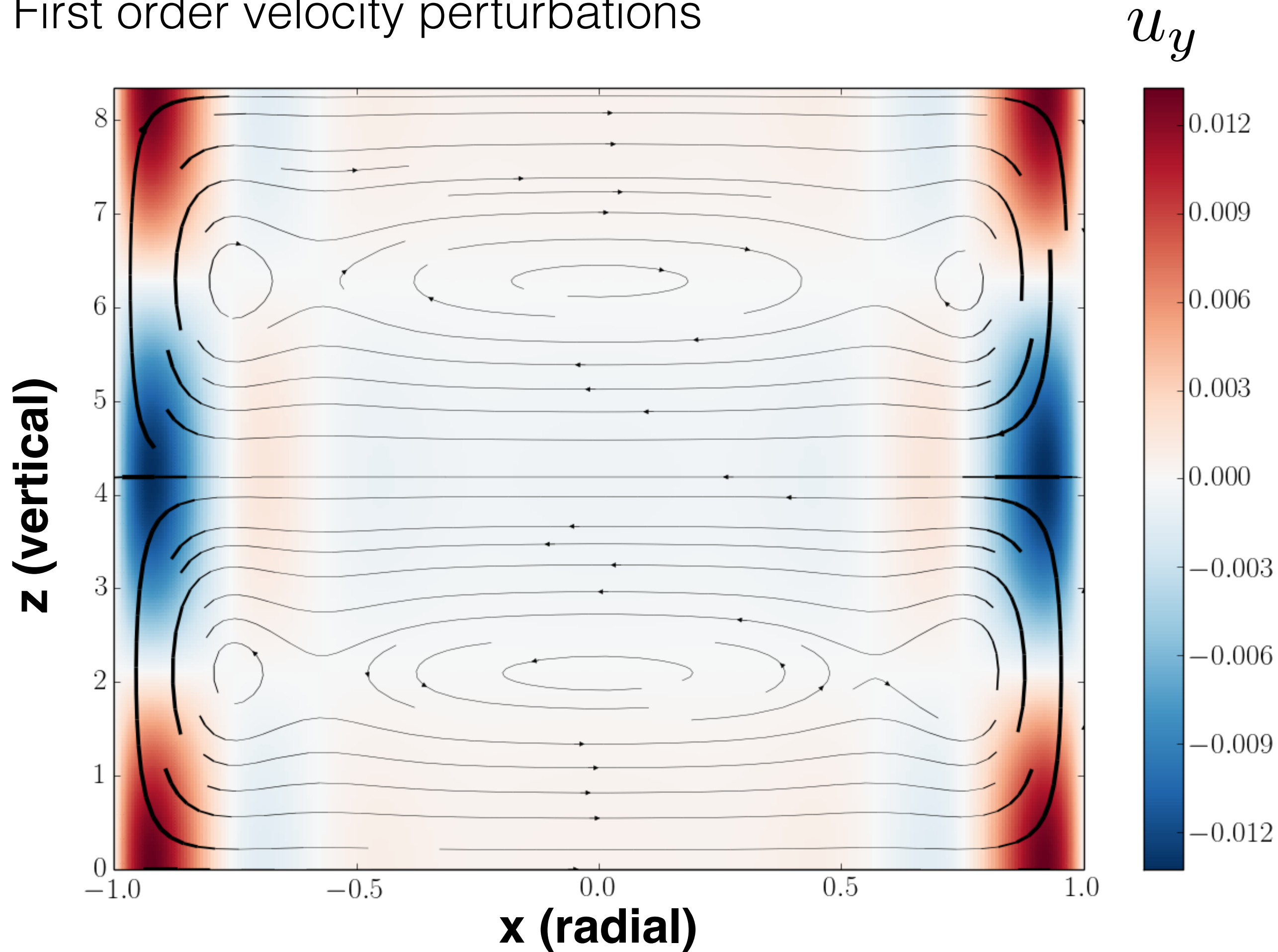




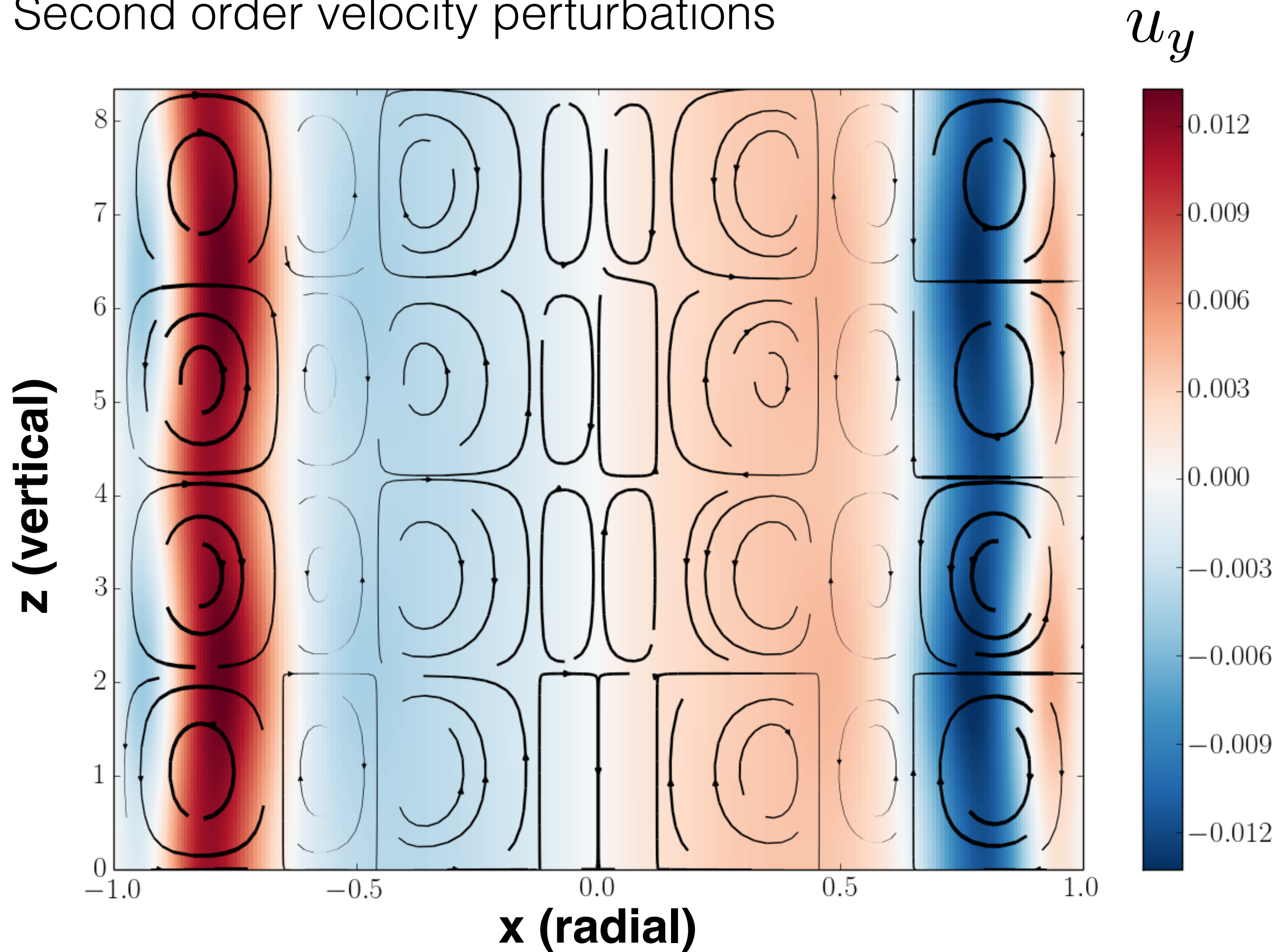
The result is an amplitude equation.

$$a\partial_T\alpha = -b\partial_Z\alpha - c\alpha|\alpha|^2 + h\partial_Z^2\alpha + giQ^3\alpha$$

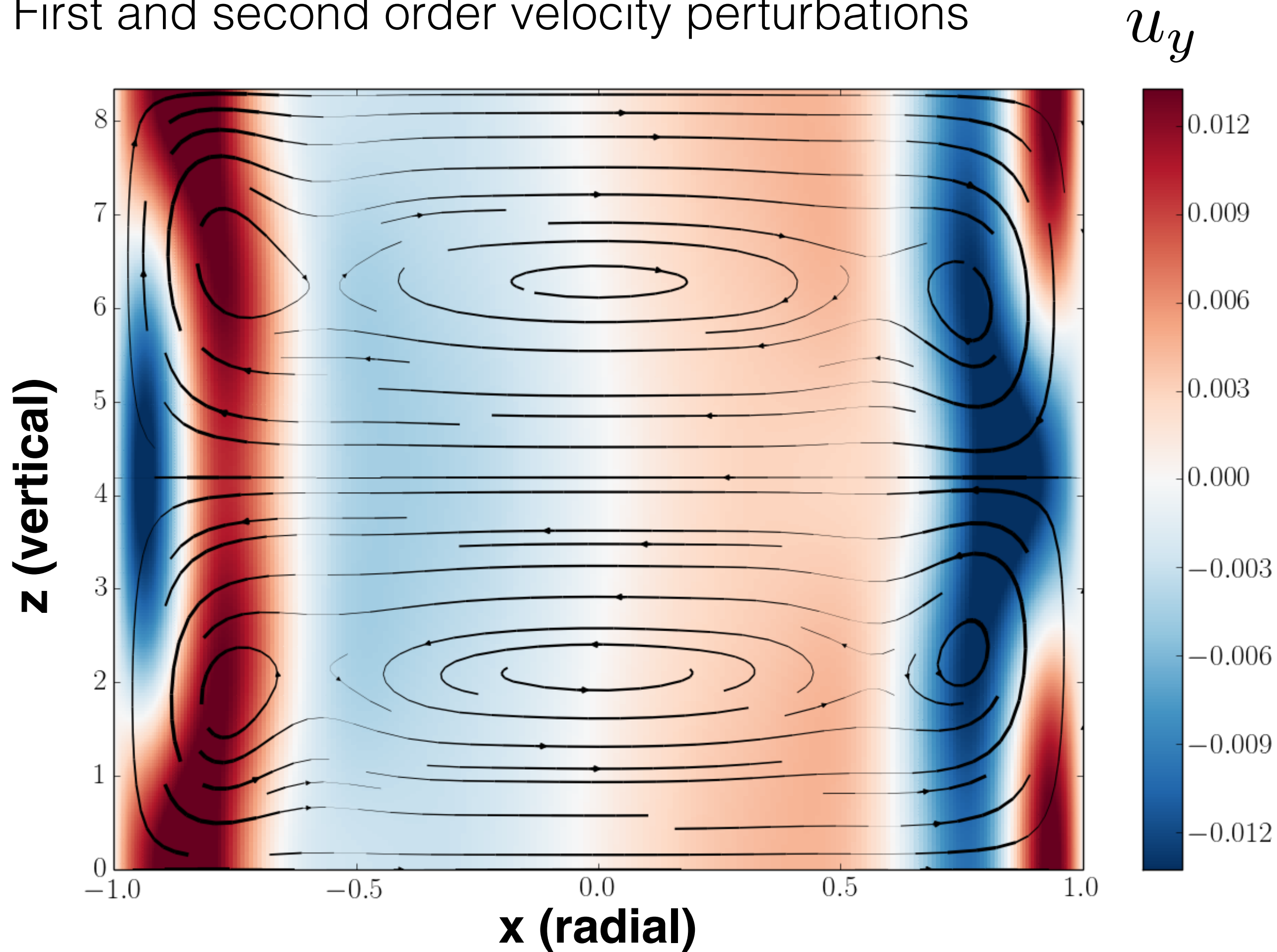
# First order velocity perturbations



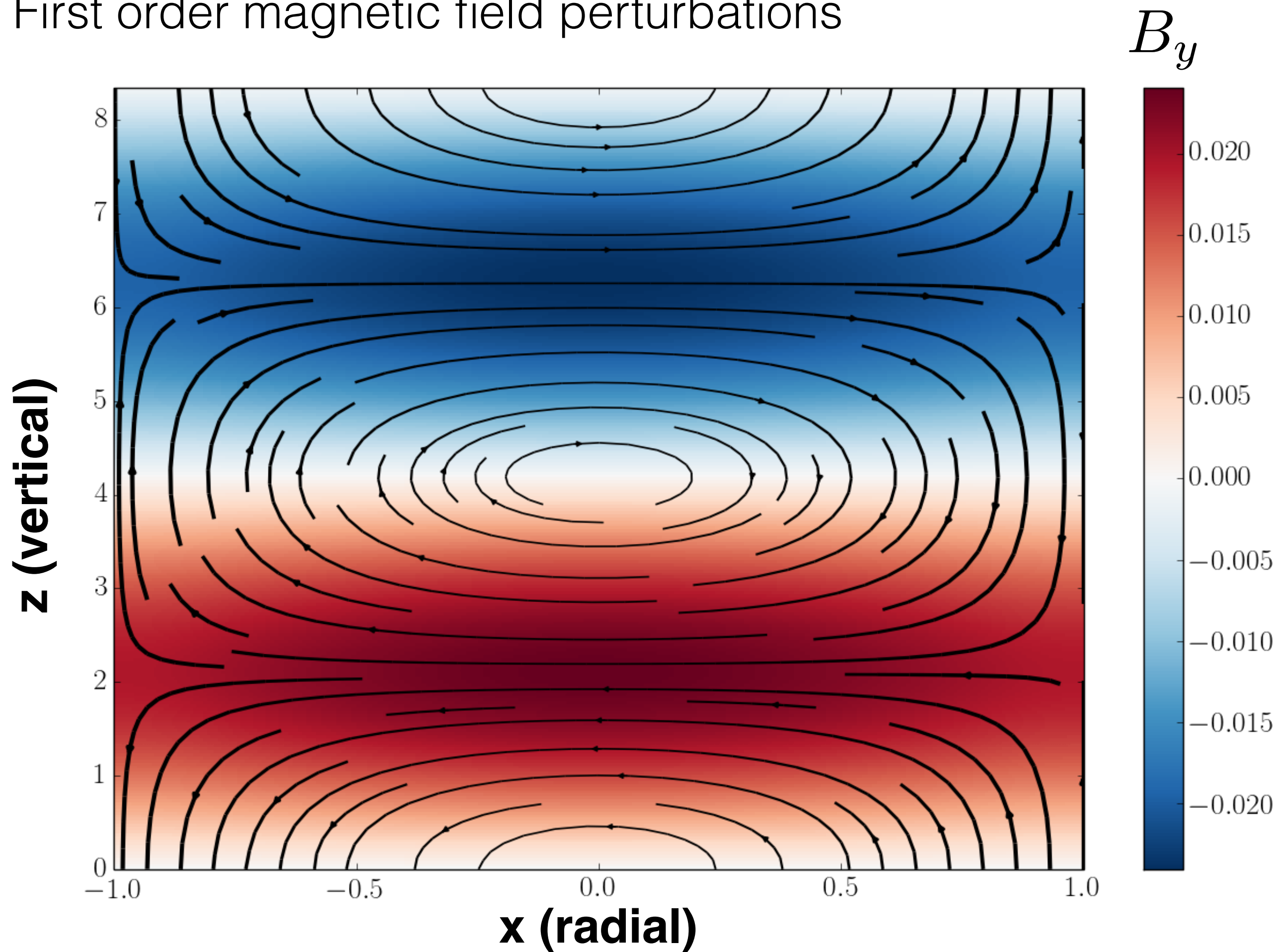
# Second order velocity perturbations



# First and second order velocity perturbations



# First order magnetic field perturbations

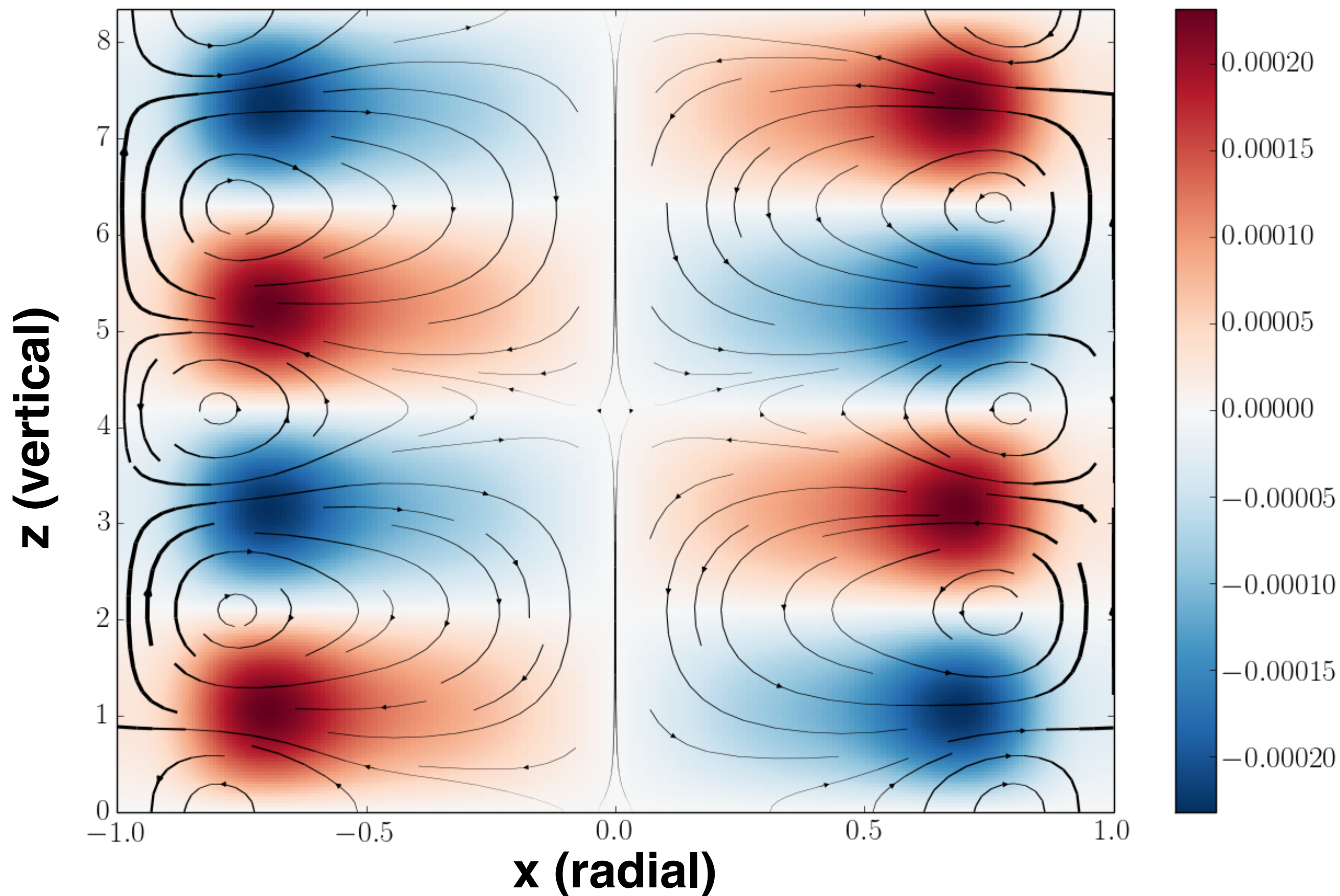




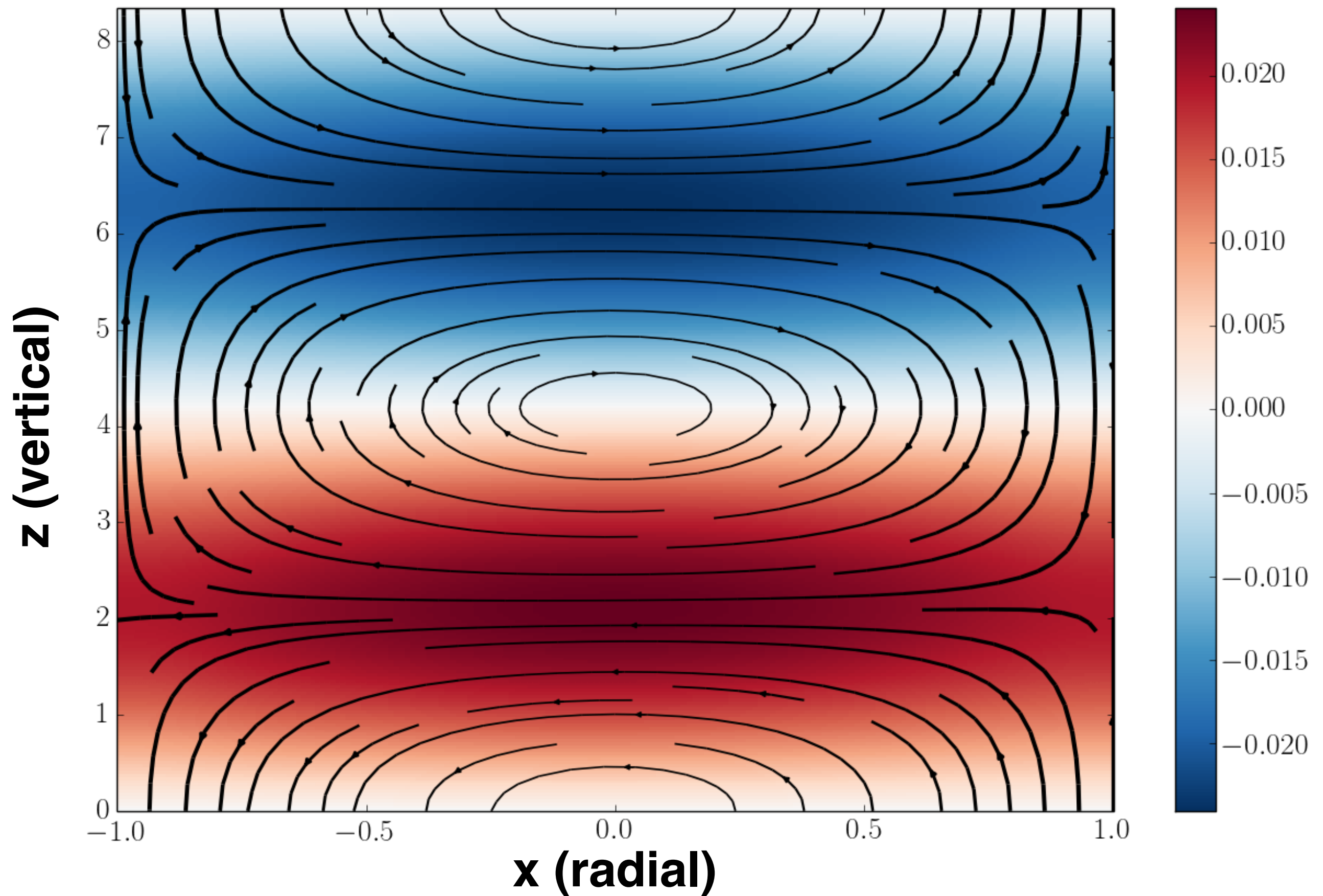
# Second order magnetic field perturbations

two OOM smaller!

$B_y$



First and second order magnetic field perturbations  $B_y$



Future work:

non-thin gap approximation

helical MRI

explore parameter space

comparison to experiment