1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi\ \hat{\mathbf{r}} \\ u_\phi\ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi\ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t \left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) &= \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ &+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ &+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \end{aligned} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{3}$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\mathrm{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{4}$$

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r}J(A, u_\phi) &+ \frac{1}{r}J(B_\phi, \psi) \\ &+ \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \end{aligned} \tag{5}$$

2 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_z \tag{6}$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\frac{1}{r^2}\partial_t\partial_r\Psi-\frac{1}{r}\partial_t\partial_r^2\Psi-\frac{3}{r^4}\partial_z\Psi\partial_r\Psi+\frac{1}{r^3}\partial_r\left(\partial_z\Psi\partial_r\Psi\right)+\frac{2}{r^3}\partial_z\Psi\partial_r^2\Psi-\frac{1}{r^2}\partial_r\left(\partial_z\Psi\partial_r^2\Psi\right)$$
$$-\frac{2}{r^3}\partial_r\Psi\partial_r\partial_z\Psi+\frac{1}{r^2}\partial_r\left(\partial_r\Psi\partial_r\partial_z\Psi\right)=$$
$$\partial_r\left(B_\phi\partial_z B_\phi\right)+\frac{2}{r^3}\partial_z^2 A\partial_z A-\frac{1}{r^2}\partial_r\left(\partial_z^2 A\partial_z A\right)+\frac{3}{r^4}\partial_z A\partial_r A-\frac{1}{r^3}\partial_r\left(\partial_z A\partial_r A\right)-\frac{2}{r^3}\partial_z A\partial_r^2 A$$
$$+\frac{1}{r^2}\partial_r\left(\partial_z A\partial_r^2 A\right)+\frac{1}{\mathrm{Re}}\left[\frac{3}{r^4}\partial_r\Psi-\frac{3}{r^3}\partial_r^2\Psi+\frac{2}{r^2}\partial_r^3\Psi-\frac{1}{r}\partial_r^4\Psi\right] \quad (7)$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r+\left[u\cdot\nabla u\right]_r=\left[(\nabla\times B)\times B\right]_r+\frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_r \quad (8)$$

and take ∂_z of the expanded equation to obtain

$$\frac{1}{r}\partial_t\partial_z^2\Psi-\frac{1}{r^3}\partial_z\left(\partial_z\Psi\partial_z\Psi\right)+\frac{1}{r^2}\partial_z\left(\partial_z\Psi\partial_z\partial_r\Psi\right)-\frac{1}{r^2}\partial_z\left(\partial_r\Psi\partial_z^2\Psi\right)-\frac{1}{r}2u_\phi\partial_z u_\phi$$
$$=-\frac{1}{r^2}\partial_z^3 A\partial_r A-\frac{1}{r^2}\partial_z^2 A\partial_r\partial_z A+\frac{2}{r^3}\partial_r\partial_z A\partial_r A-\frac{1}{r^2}\partial_r^2\partial_z A\partial_r A-\frac{1}{r^2}\partial_r^2 A\partial_r\partial_z A$$
$$+\frac{1}{\mathrm{Re}}\left[-\frac{1}{r^2}\partial_z^2\partial_r\Psi+\frac{1}{r}\partial_z^2\partial_r^2\Psi+\frac{1}{r}\partial_z^4\Psi\right] \quad (9)$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\phi\partial_z u_\phi \quad (10)$$

Note that the relevant quantity appears to be $\nabla^2\Psi-\frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right] \quad (11)$$

Full Ψ equation according to Susan:

$$
\begin{aligned}
\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\phi\partial_z u_\phi \\
=J\left(A,\frac{1}{r^2}\left(\nabla^2A-\frac{2}{r}\partial_rA\right)\right)-\frac{2}{r}B_\phi\partial_zB_\phi \\
+\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right]
\end{aligned}
\tag{12}
$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

The derivation of the non-viscous term on the righthand side of the momentum equation ($\mathbf{J}\times\mathbf{B}$) is as follows.

$$\partial_z\left([(\nabla\times B)\times B]_r\right)-\partial_r\left([(\nabla\times B)\times B]_z\right) \tag{13}$$

$$=\partial_z\left(\left[\left(\partial_zB_r-\partial_rB_z\right)B_z-\left(\frac{1}{r}\partial_r\left(rB_\phi\right)\right)B_\phi\right]\right)-\partial_r\left(\left[\left(-\partial_zB_\phi\right)B_\phi-\left(\partial_zB_r-\partial_rB_z\right)B_r\right]\right) \tag{14}$$

$$
\begin{aligned}
=&-\frac{1}{r^2}\partial_z^3A\partial_rA+\frac{1}{r^3}\partial_r\partial_zA\partial_rA-\frac{1}{r^2}\partial_r^2\partial_zA\partial_rA-\frac{2}{r^3}\partial_z^2A\partial_zA \\
&+\frac{1}{r^2}\partial_z^2\partial_rA\partial_zA+\frac{3}{r^4}\partial_rA\partial_zA-\frac{3}{r^3}\partial_r^2A\partial_zA+\frac{1}{r^2}\partial_r^3A\partial_zA-\frac{2}{r}B_\phi\partial_zB_\phi
\end{aligned}
\tag{15}
$$

This simplifies to

$$J\left(A,\frac{1}{r^2}\left(\nabla^2A-\frac{2}{r}\partial_rA\right)\right)-\frac{2}{r}B_\phi\partial_zB_\phi \tag{16}$$

Full derivation of viscous term:

$$\partial_z\left(\frac{1}{\mathrm{Re}}\left[\nabla^2u\right]_r\right)-\partial_r\left(\frac{1}{\mathrm{Re}}\left[\nabla^2u\right]_z\right) \tag{17}$$

$$=\frac{1}{\mathrm{Re}}\left[\partial_z\left(\nabla^2u_r-\frac{1}{r^2}u_r\right)-\partial_r\left(\nabla^2u_z\right)\right] \tag{18}$$

$$=\frac{1}{\mathrm{Re}}\left[-\frac{2}{r^2}\partial_z^2\partial_r\Psi+\frac{2}{r}\partial_z^2\partial_r^2\Psi+\frac{1}{r}\partial_z^4\Psi-\frac{3}{r^4}\partial_r\Psi+\frac{3}{r^3}\partial_r^2\Psi-\frac{2}{r^2}\partial_r^3\Psi+\frac{1}{r}\partial_r^4\Psi\right] \tag{19}$$

3 Recovery of Narrow Gap Equations

4 Perturbed Equations

Note: two things are currently missing. The constant β from the nondimensionalization, and the Coriolis term in the momentum equation.

We perturb the wide gap equations according to

$$\mathbf{B} = B_0 \hat{z} + \mathbf{B_1} \tag{20}$$

$$\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u_1} \tag{21}$$

The Ψ equation becomes

$$\begin{aligned} \frac{1}{r}\partial_t \left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi \\ = J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \\ + \frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] + \frac{1}{r}B_0\partial_z\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \end{aligned} \tag{22}$$

Equation 3 becomes

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} - \frac{2}{r}q\Omega_0\partial_z\Psi = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \frac{1}{\mathrm{Re}}\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) + B_0\partial_z B_\phi \tag{23}$$

The only term gained in the $(\hat{r}, \hat{z})$ component of the induction equation is $(B_0\hat{z}\cdot\nabla)\mathbf{u_1}$, and Equation 4 becomes

$$\partial_t A = \frac{1}{r}J(A,\psi) + \frac{1}{\mathrm{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) + B_0\partial_z\Psi \tag{24}$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The $\hat{\phi}$ component of the induction equation, Equation 5, becomes

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi) \\ + \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \frac{1}{\mathrm{Rm}}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) + B_0\partial_z u_\phi - \frac{1}{r}q\Omega_0\partial_z A \end{aligned} \tag{25}$$

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{26}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{27}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{28}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{29}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \tag{30}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \tag{31}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{32}$$

where ∇ on the vector components is given by equation (26).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$