1   Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi\ \hat{\mathbf{r}} \\ u_\phi\ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi\ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi,\nabla^2 A - \frac{2\partial_r\psi}{r}) = \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)$$
$$+ \frac{1}{r}J\left(A,\frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r}$$
$$+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi,u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A,B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \tag{3}$$

$$\partial_t A = \frac{1}{r}J(A,\psi) + \frac{1}{\text{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \tag{4}$$

$$\partial_t B_\phi = \frac{1}{r}J(A,u_\phi) + \frac{1}{r}J(B_\phi,\psi)$$
$$+ \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \tag{5}$$

2   Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u\cdot\nabla u]_z = [(\nabla\times B)\times B]_z + \frac{1}{\text{Re}}\left[\nabla^2 u\right]_z \tag{6}$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\frac{1}{r^2}\partial_t\partial_r\Psi - \frac{1}{r}\partial_t\partial_r^2\Psi - \frac{3}{r^4}\partial_z\Psi\partial_r\Psi + \frac{1}{r^3}\partial_r\left(\partial_z\Psi\partial_r\Psi\right) + \frac{2}{r^3}\partial_z\Psi\partial_r^2\Psi - \frac{1}{r^2}\partial_r\left(\partial_z\Psi\partial_r^2\Psi\right)$$

$$- \frac{2}{r^3}\partial_r\Psi\partial_r\partial_z\Psi + \frac{1}{r^2}\partial_r\left(\partial_r\Psi\partial_r\partial_z\Psi\right) =$$

$$\partial_r\left(B_\phi\partial_z B_\phi\right) + \frac{2}{r^3}\partial_z^2 A\partial_z A - \frac{1}{r^2}\partial_r\left(\partial_z^2 A\partial_z A\right) + \frac{3}{r^4}\partial_z A\partial_r A - \frac{1}{r^3}\partial_r\left(\partial_z A\partial_r A\right) - \frac{2}{r^3}\partial_z A\partial_r^2 A$$

$$+ \frac{1}{r^2}\partial_r\left(\partial_z A\partial_r^2 A\right) + \frac{1}{\text{Re}}\left[\frac{3}{r^4}\partial_r\Psi - \frac{3}{r^3}\partial_r^2\Psi + \frac{2}{r^2}\partial_r^3\Psi - \frac{1}{r}\partial_r^4\Psi\right] \quad (7)$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r + [u\cdot\nabla u]_r = [(\nabla\times B)\times B]_r + \frac{1}{\text{Re}}\left[\nabla^2 u\right]_r \quad (8)$$

and take ∂_z of the expanded equation to obtain

$$\frac{1}{r}\partial_t\partial_z^2\Psi - \frac{1}{r^3}\partial_z\left(\partial_z\Psi\partial_z\Psi\right) + \frac{1}{r^2}\partial_z\left(\partial_z\Psi\partial_z\partial_r\Psi\right) - \frac{1}{r^2}\partial_z\left(\partial_r\Psi\partial_z^2\Psi\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$

$$= -\frac{1}{r^2}\partial_z^3 A\partial_r A - \frac{1}{r^2}\partial_z^2 A\partial_r\partial_z A + \frac{2}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2 A\partial_r\partial_z A$$

$$+ \frac{1}{\text{Re}}\left[-\frac{1}{r^2}\partial_z^2\partial_r\Psi + \frac{1}{r}\partial_z^2\partial_r^2\Psi + \frac{1}{r}\partial_z^4\Psi\right] \quad (9)$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi \quad (10)$$

Note that the relevant quantity appears to be $\nabla^2\Psi - \frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\text{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \quad (11)$$

Full Ψ equation according to Susan:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$

$$= J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi$$

$$+ \frac{1}{\text{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \quad (12)$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

The derivation of the non-viscous term on the righthand side of the momentum equation $(\mathbf{J} \times \mathbf{B})$ is as follows.

$$\partial_z\left([(\nabla \times B) \times B]_r\right) - \partial_r\left([(\nabla \times B) \times B]_z\right) \quad (13)$$

$$= \partial_z\left(\left[(\partial_z B_r - \partial_r B_z) B_z - \left(\frac{1}{r}\partial_r(rB_\phi)\right) B_\phi\right]\right) - \partial_r\left([(-\partial_z B_\phi) B_\phi - (\partial_z B_r - \partial_r B_z) B_r]\right)$$

$$(14)$$

$$= -\frac{1}{r^2}\partial_z^3 A\partial_r A + \frac{1}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{2}{r^3}\partial_z^2 A\partial_z A$$

$$+ \frac{1}{r^2}\partial_z^2\partial_r A\partial_z A + \frac{3}{r^4}\partial_r A\partial_z A - \frac{3}{r^3}\partial_r^2 A\partial_z A + \frac{1}{r^2}\partial_r^3 A\partial_z A - \frac{2}{r}B_\phi\partial_z B_\phi \quad (15)$$

This simplifies to

$$J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \quad (16)$$

Full derivation of viscous term:

$$\partial_z\left(\frac{1}{\text{Re}}\left[\nabla^2 u\right]_r\right) - \partial_r\left(\frac{1}{\text{Re}}\left[\nabla^2 u\right]_z\right) \quad (17)$$

$$= \frac{1}{\text{Re}}\left[\partial_z\left(\nabla^2 u_r - \frac{1}{r^2}u_r\right) - \partial_r\left(\nabla^2 u_z\right)\right] \quad (18)$$

$$= \frac{1}{\text{Re}}\left[-\frac{2}{r^2}\partial_z^2\partial_r\Psi + \frac{2}{r}\partial_z^2\partial_r^2\Psi + \frac{1}{r}\partial_z^4\Psi - \frac{3}{r^4}\partial_r\Psi + \frac{3}{r^3}\partial_r^2\Psi - \frac{2}{r^2}\partial_r^3\Psi + \frac{1}{r}\partial_r^4\Psi\right]$$

$$(19)$$

3 Recovery of Narrow Gap Equations

4 Nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$
\begin{aligned}
\widetilde{u} &= \Omega_0 r_0 \delta u \\
\widetilde{x} &= \delta r_0 x \\
\widetilde{\nabla} &= \frac{\nabla}{\delta r_0} \\
\widetilde{t} &= \frac{t}{\Omega_0} \\
\widetilde{B} &= B_0 B \\
\widetilde{P} &= P_0 P \\
\widetilde{\rho} &= \rho_0 \rho
\end{aligned}
\tag{20}
$$

If we define the velocity scale v_0 as the local sound speed scale $\left(c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}} \right)$, then $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$.

Nondimensionalizing the momentum equation and dividing by a factor of $\Omega_0^2 r_0 \delta$ yields

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\rho_0 \Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{21}
$$

If we define

$$
\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \tag{22}
$$

where $P_{mag} = \frac{B_0^2}{8\pi}$, then the factor in front of $(\nabla \times \mathbf{B}) \times \mathbf{B}$ should be $\frac{2}{\beta}$. Agreed?

5 Perturbed Equations

We perturb the wide gap equations according to

$$
\mathbf{B} = B_0 \hat{z} + \mathbf{B_1}, \tag{23}
$$

which is the same perturbation we used in the thin-gap construction. But wait! We perturbed the thin-gap equations according to

$$
\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u_1} \tag{24}
$$

but I don't think this is valid in the wide-gap case. To be sure, I'm going to perturb instead by

$$\mathbf{u} = r\Omega(r)\hat{\phi} + \mathbf{u_1} \tag{25}$$

where $\Omega(r) = \Omega_0 \left(\frac{r}{r_0}\right)^{-q} = \Omega_0\,(r)^{-q}$ in dimensionless coordinates (but keeping Ω_0 to flag a rotational term...) Agreed?

We also add the Coriolis and centrifugal terms $-2\mathbf{\Omega}\times\mathbf{u} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ which expand as follows:

$$-2\Omega(r)\hat{\mathbf{z}}\times\left(\mathbf{u_1} + r\Omega(r)\hat{\phi}\right) = -2\Omega(r)u_r\hat{\phi} + 2\Omega(r)u_\phi\hat{\mathbf{r}} + 2r\Omega(r)^2\hat{\mathbf{r}} \tag{26}$$

$$-\Omega(r)\hat{\mathbf{z}}\times(\Omega(r)\hat{\mathbf{z}}\times r\hat{\mathbf{r}}) = +\Omega(r)^2 r\hat{\mathbf{r}} \tag{27}$$

The $\hat{\mathbf{r}}$ terms will have ∂_z applied to them, which will destroy the $3\Omega(r)^2 r\hat{\mathbf{r}}$ term and so the righthand side of the Ψ equation will ultimately gain only the term $2\Omega(r)\partial_z u_\phi\hat{\mathbf{r}}$. The $\hat{\phi}$ equation gains the term $-\frac{2}{r}\Omega(r)\partial_z\Psi$ on the righthand side.

The Ψ equation becomes

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi - \partial_z\left(u_\phi\Omega(r)\right)$$
$$= \frac{2}{\beta}J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi + 2\Omega(r)\partial_z u_\phi$$
$$+ \frac{1}{\text{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] + \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \tag{28}$$

Note that one of the terms gained from the base state in the $(\mathbf{u}\cdot\nabla)\mathbf{u}$ term can be combined with the term gained from the Coriolis term, but here I write them separately so we can check the equation. Equation 3 becomes

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} + \partial_z\Psi\left(\frac{2}{r}\Omega(r) + \partial_r\Omega(r)\right) =$$
$$\frac{2}{\beta}\frac{J(A, B_\phi)}{r} + \frac{2}{\beta}\frac{B_\phi\partial_z A}{r^2} + \frac{1}{\text{Re}}\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) + \frac{2}{\beta}B_0\partial_z B_\phi - \frac{2}{r}\Omega(r)\partial_z\Psi \tag{29}$$

Equation 4 picks up $+\Omega(r)B_\phi$ from the term $-(u_0\hat{\phi}\cdot\nabla)\mathbf{B_1}$ and $-\Omega(r)B_\phi$ from the term $+(\mathbf{B_1}\cdot\nabla)\hat{\phi}$ and, so those cancel and the equation becomes

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\text{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) + B_0\partial_z\Psi \tag{30}$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The $\hat{\phi}$ component of the induction equation, Equation 5, becomes

$$\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi)$$
$$+\frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \frac{1}{\text{Rm}}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) + B_0\partial_z u_\phi + \partial_z A\left(\frac{2}{r}\Omega(r) + \partial_r\Omega(r)\right)$$
(31)

6   Matrix Formulation

This is all pending a rigorous check of Equations 28 - 31, which Jeff is working on, but I'll start putting this into a matrix construction.

First, the nonlinear vector, on the lefthand side of the equation, is

$$\mathbf{N} = \begin{bmatrix} J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{2}{\beta}J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}u_\phi\partial_z u_\phi + \frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi \\ \frac{1}{r}J\left(\Psi, u_\phi\right) - \frac{1}{r}\frac{2}{\beta}J\left(A, B_\phi\right) + \frac{1}{r^2}u_\phi\partial_z\Psi - \frac{2}{\beta}\frac{1}{r^2}B_\phi\partial_z A \\ -\frac{1}{r}J\left(A, \Psi\right) \\ -\frac{1}{r}J\left(A, u_\phi\right) - \frac{1}{r}J\left(B_\phi, \Psi\right) - \frac{1}{r^2}B_\phi\partial_z\Psi + \frac{1}{r^2}u_\phi\partial_z A \end{bmatrix}$$
(32)

Note that these differ from the thin-gap nonlinear terms not only because of the curvature terms in the Jacobians, but also because of the additional advective terms in all but the **A** equation.

The ∂_t terms are grouped together into

$$\partial_t D = \partial_t \begin{bmatrix} \frac{1}{r}\nabla^2 - \frac{2}{r^2}\partial_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(33)

For the righthand side of the equation, I'll follow the same process that I did with the thin-gap. That is I'll first separate the matrices by physical meaning, and then regroup terms based on z-dependence. The RHS can be described by three matrices: one representing the rotational terms, one representing the background field terms, and one representing the viscous/resistive terms.

$$
RHS = \begin{bmatrix} 3\Omega(r)\partial_z u_\phi \\ -\frac{4}{r}\Omega(r)\partial_z\Psi - \partial_r\Omega(r)\partial_z\Psi \\ 0 \\ \frac{2}{r}\Omega(r)\partial_z A + \partial_r\Omega(r)\partial_z A \end{bmatrix} + \begin{bmatrix} \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \\ \frac{2}{\beta}B_0\partial_z B_\phi \\ B_0\partial_z\Psi \\ B_0\partial_z u_\phi \end{bmatrix} + \begin{bmatrix} \frac{1}{Re}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \\ \frac{1}{Re}\left(\nabla^2 u_\phi - \frac{1}{r}u_\phi\right) \\ \frac{1}{Rm}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \\ \frac{1}{Rm}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 3\Omega(r)\partial_z & 0 & 0 \\ -\frac{4}{r}\Omega(r)\partial_z - \partial_r\Omega(r)\partial_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{r}\Omega(r)\partial_z + \partial_r\Omega(r)\partial_z & 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 - \frac{2}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta}B_0\partial_z \\ B_0\partial_z & 0 & 0 & 0 \\ 0 & B_0\partial_z & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{Re}\left(\nabla^2\left(\frac{1}{r}\nabla^2\right) - \frac{1}{r^3}\partial_r^2 - \frac{1}{r^4}\partial_r\right) \\ \frac{1}{Re}\left(\nabla^2 - \frac{1}{r}\right) \\ \frac{1}{Rm}\left(\nabla^2 - \frac{2}{r}\partial_r\right) \\ \frac{1}{Rm}\left(\nabla^2 - \frac{1}{r^2}\right) \end{bmatrix}
$$

$$
\tag{34}
$$

We separate the viscous/resistive matrix out in terms of ∂_z dependence.

$$
\text{viscous terms} =
$$

$$
\begin{bmatrix} \frac{1}{Re}\left(-\frac{3}{r^4}\partial_r + \frac{3}{r^3}\partial_r^2 - \frac{2}{r^2}\partial_r^3 + \frac{1}{r}\partial_r^4\right) & 0 & 0 & 0 \\ 0 & \frac{1}{Re}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{Rm}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{1}{Rm}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) \end{bmatrix}
$$

$$
+ \partial_z^2 \begin{bmatrix} \frac{1}{Re}\left(-\frac{2}{r^2}\partial_r + \frac{2}{r}\partial_r^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{Re} & 0 & 0 \\ 0 & 0 & \frac{1}{Rm} & 0 \\ 0 & 0 & 0 & \frac{1}{Rm} \end{bmatrix}
$$

$$
+ \partial_z^4 \begin{bmatrix} \frac{1}{Re}\frac{1}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}
$$

Note that the form of each of the terms is different in the first matrix.
We also separate the terms in the $B0$ matrix by ∂_z dependence:

$$
B_0 \text{ terms} = \partial_z \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta}B_0 \\ B_0 & 0 & 0 & 0 \\ 0 & B_0 & 0 & 0 \end{bmatrix} + \partial_z^3 \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\tag{36}
$$

A   Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{37}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{38}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{39}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$\left[\mathbf{u}\cdot\nabla\mathbf{u}\right]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{40}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \tag{41}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \tag{42}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{43}$$

where ∇ on the vector components is given by equation (37).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$