Exploring the saturation of the MRI via weakly nonlinear analysis

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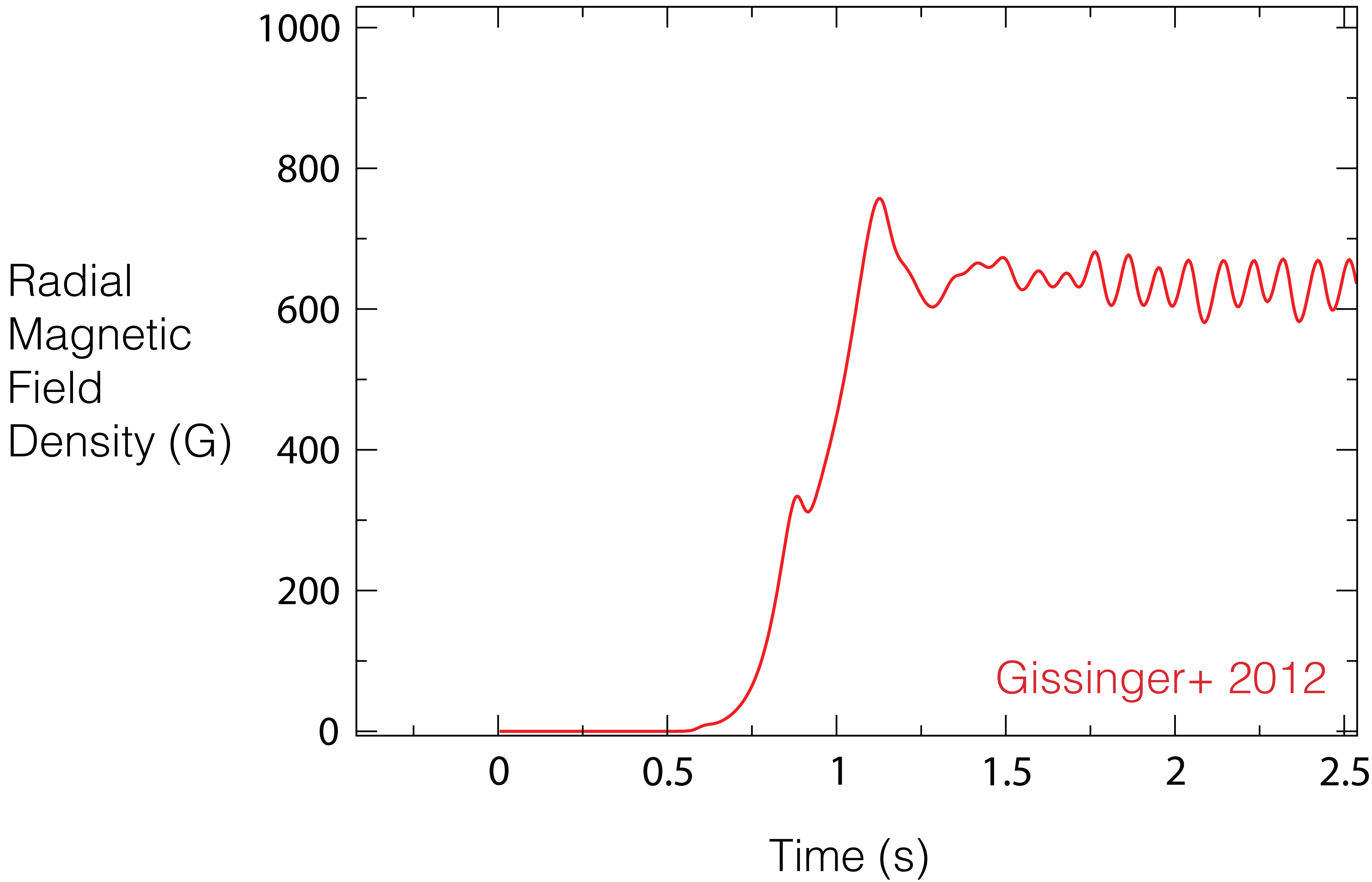
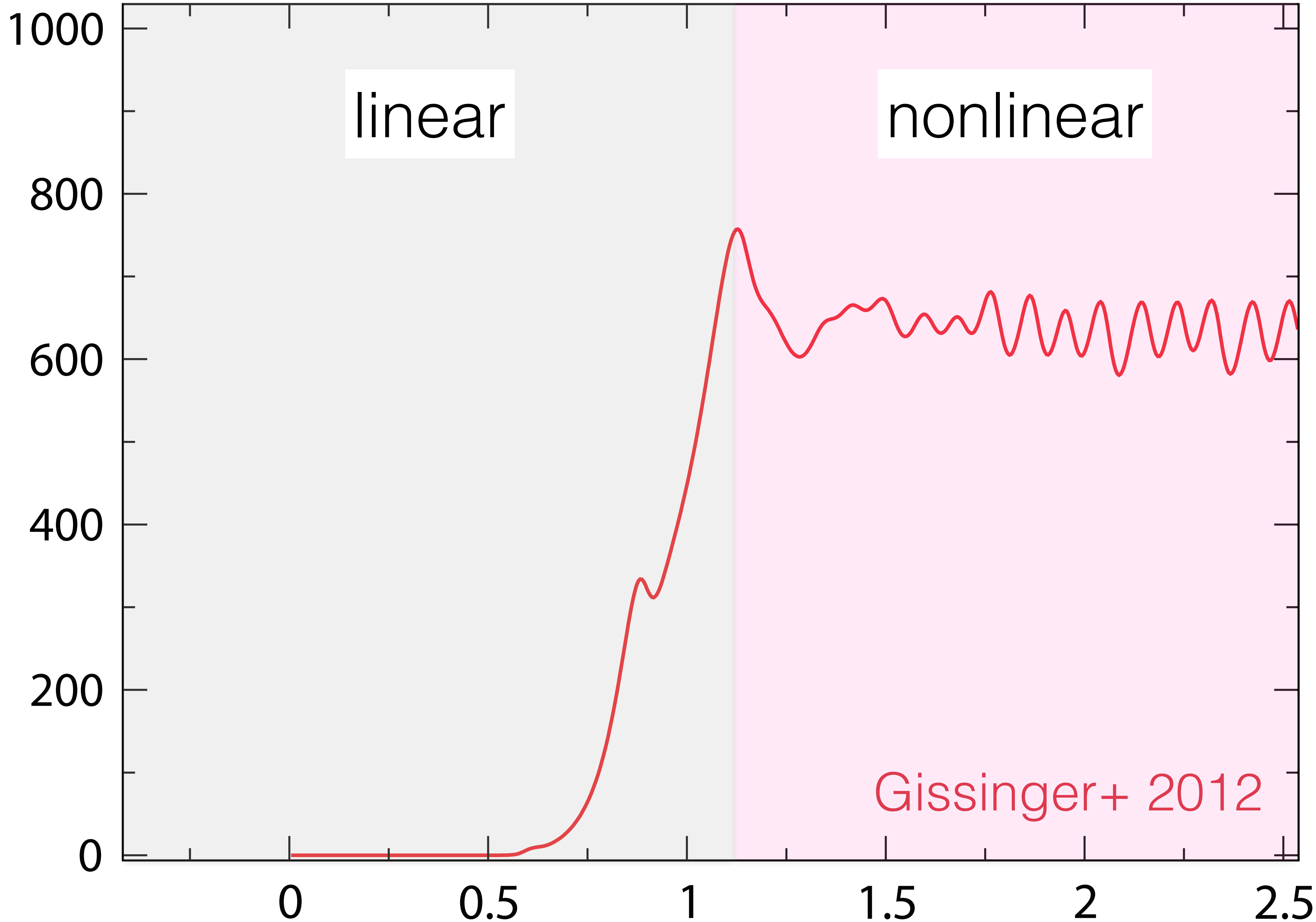
The MRI evolves in stages.



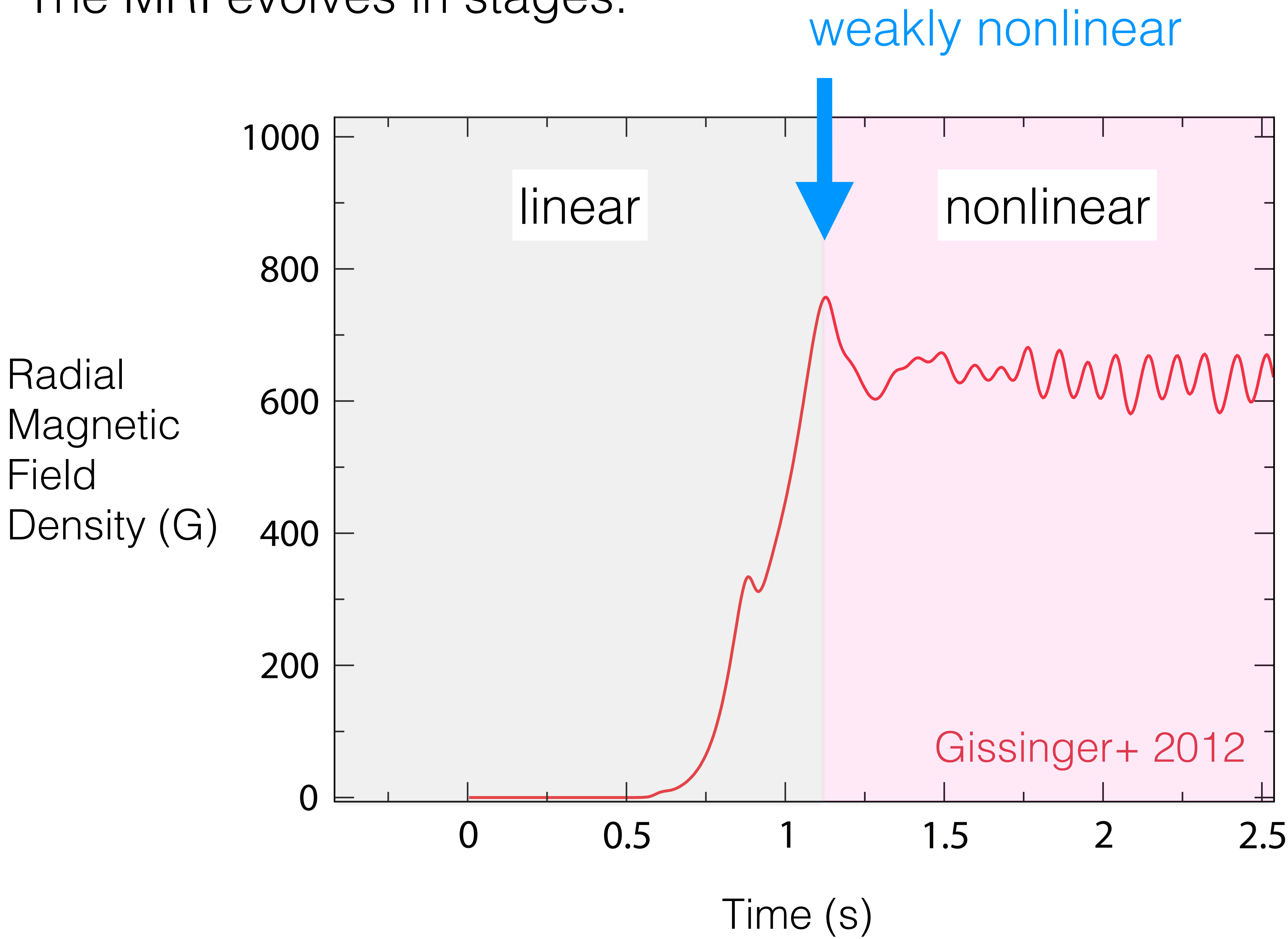
The MRI evolves in stages.

Radial Magnetic Field Density (G)



Time (s)

The MRI evolves in stages.



set-up
boundary conditions
parameter range
open questions, etc

We solve the non-ideal MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \frac{1}{\rho} \left(\mathbf{J} \times \mathbf{B} \right) - 2\Omega \times \mathbf{u} - \Omega \times \left(\Omega \times \mathbf{r} \right) + \nu \nabla^2 \mathbf{u}$$

induction

$$\partial_t \mathbf{B} = \nabla \times \left(\mathbf{u} \times \mathbf{B} \right) + \eta \nabla^2 \mathbf{B}$$

constraints

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

We solve the non-ideal MRI equations.

momentum

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}\left(\mathbf{J}\times\mathbf{B}\right) - 2\Omega\times\mathbf{u} - \Omega\times\left(\Omega\times\mathbf{r}\right) + \nu\nabla^2\mathbf{u}$$

microscopic viscosity

induction

$$\partial_t \mathbf{B} = \nabla\times\left(\mathbf{u}\times\mathbf{B}\right) + \eta\nabla^2\mathbf{B}$$

magnetic resistivity

constraints

$$\nabla\cdot\mathbf{u} = 0$$

$$\nabla\cdot\mathbf{B} = 0$$

We nondimensionalize and perturb the nonlinear MRI equations.

magnetic resistivity

microscopic viscosity

We work in terms of flux and stream functions.

momentum

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \frac{1}{Re} \nabla^4 \Psi$$

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \frac{1}{Re} \nabla^2 u_y$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \frac{1}{Rm} \nabla^2 A$$

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \frac{1}{Rm} \nabla^2 B_y$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re}\nabla^4 \Psi}$$

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - (2-q)\Omega_0 \partial_z \Psi + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re}\nabla^2 u_y}$$

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm}\nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - q\Omega_0 \partial_z A + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm}\nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right) + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right) + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

induction

$$\partial_t A = B_0 \partial_z \Psi + J\left(A, \Psi\right) + \boxed{\frac{1}{Rm} \nabla^2 A}$$

resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + J\left(A, u_y\right) - J\left(\Psi, B_y\right) + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

We work in terms of flux and stream functions.

momentum

nonlinear

viscous

$$\partial_t \nabla^2 \Psi = \frac{2}{\beta} B_0 \partial_z \nabla^2 A + 2\partial_z u_y + \boxed{\frac{2}{\beta} J\left(A, \nabla^2 A\right) - J\left(\Psi, \nabla^2 \Psi\right)} + \boxed{\frac{1}{Re} \nabla^4 \Psi}$$

$$\partial_t u = \frac{2}{\beta} B_0 \partial_z B_y - \boxed{(2-q)\Omega_0 \partial_z \Psi} + \boxed{\frac{2}{\beta} J\left(A, B_y\right) - J\left(\Psi, u_y\right)} + \boxed{\frac{1}{Re} \nabla^2 u_y}$$

shear

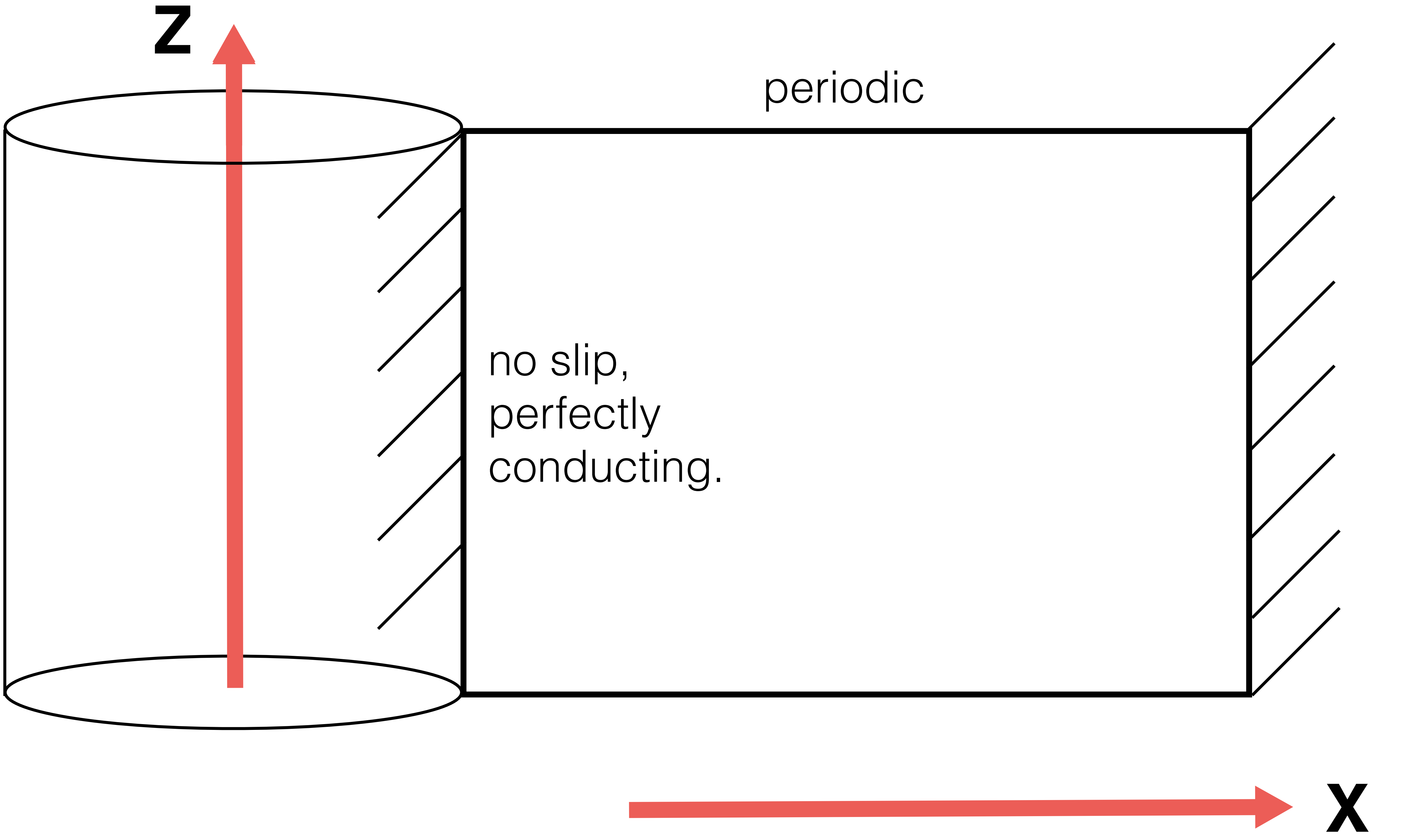
induction

$$\partial_t A = B_0 \partial_z \Psi + \boxed{J\left(A, \Psi\right)} + \boxed{\frac{1}{Rm} \nabla^2 A}$$

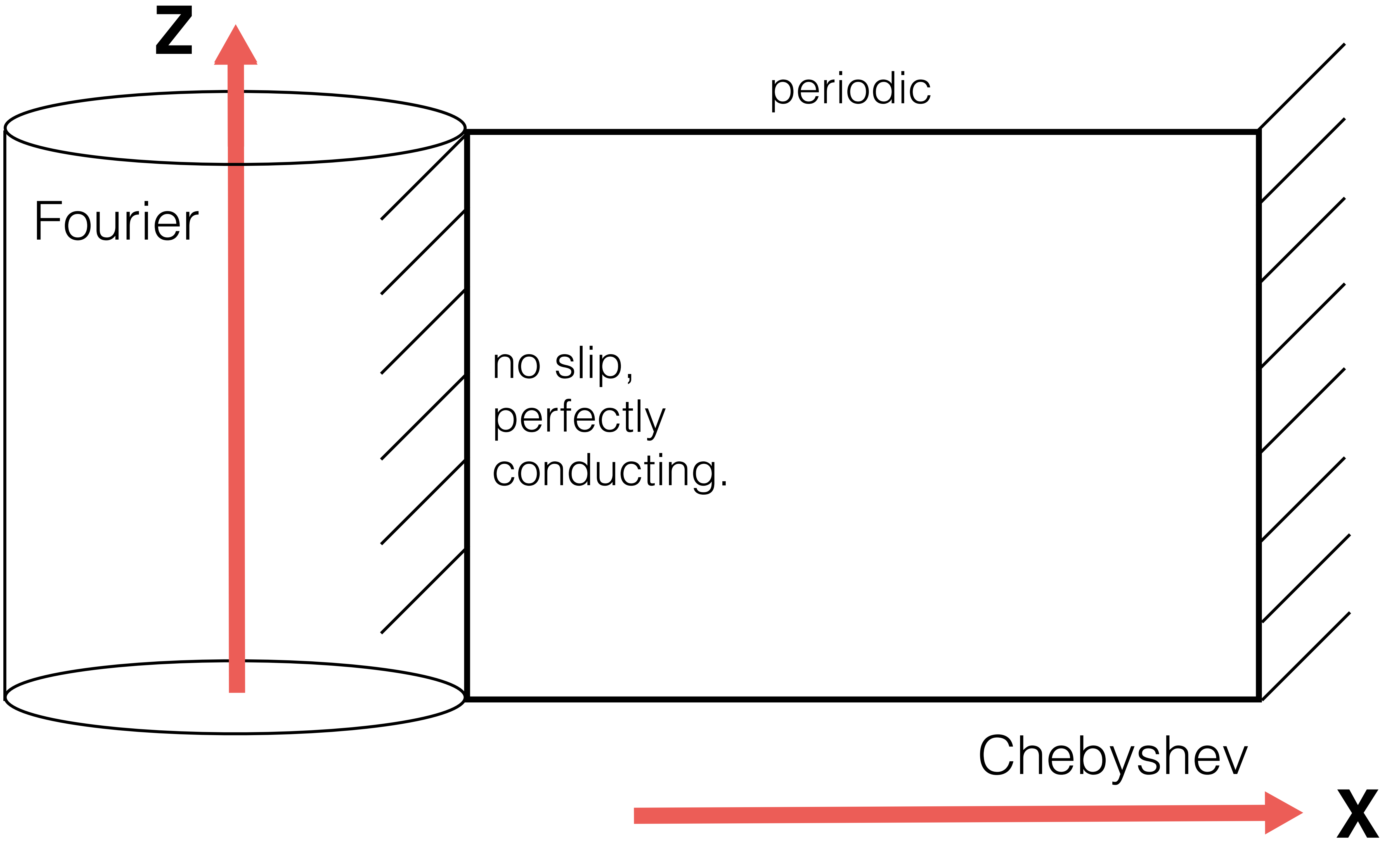
resistive

$$\partial_t B_y = B_0 \partial_z u_y - \boxed{q\Omega_0 \partial_z A} + \boxed{J\left(A, u_y\right) - J\left(\Psi, B_y\right)} + \boxed{\frac{1}{Rm} \nabla^2 B_y}$$

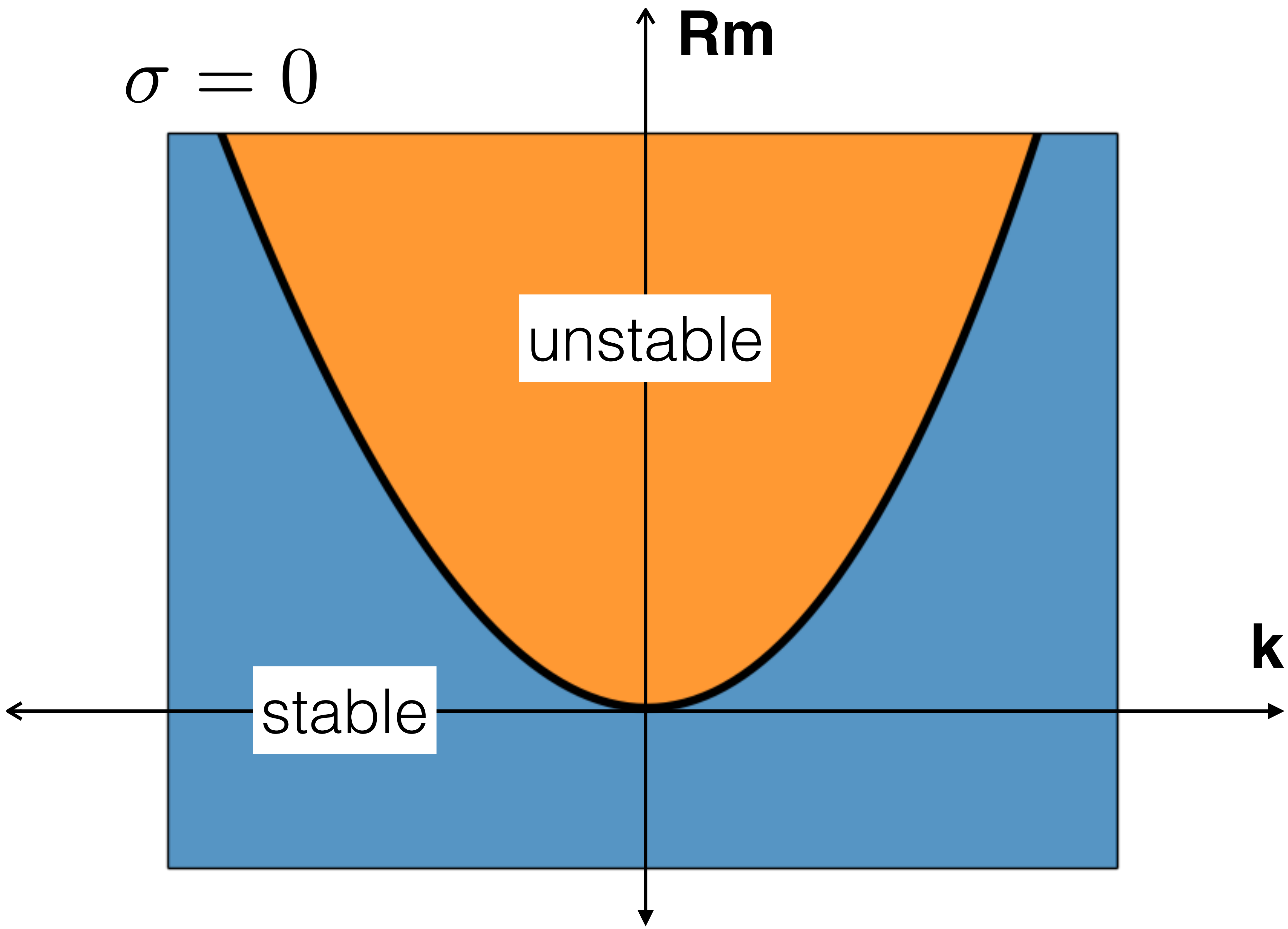
We use experimentally relevant boundary conditions.



Dedalus is a general-purpose spectral code.



Weakly nonlinear analysis explores behavior at the margin of instability.



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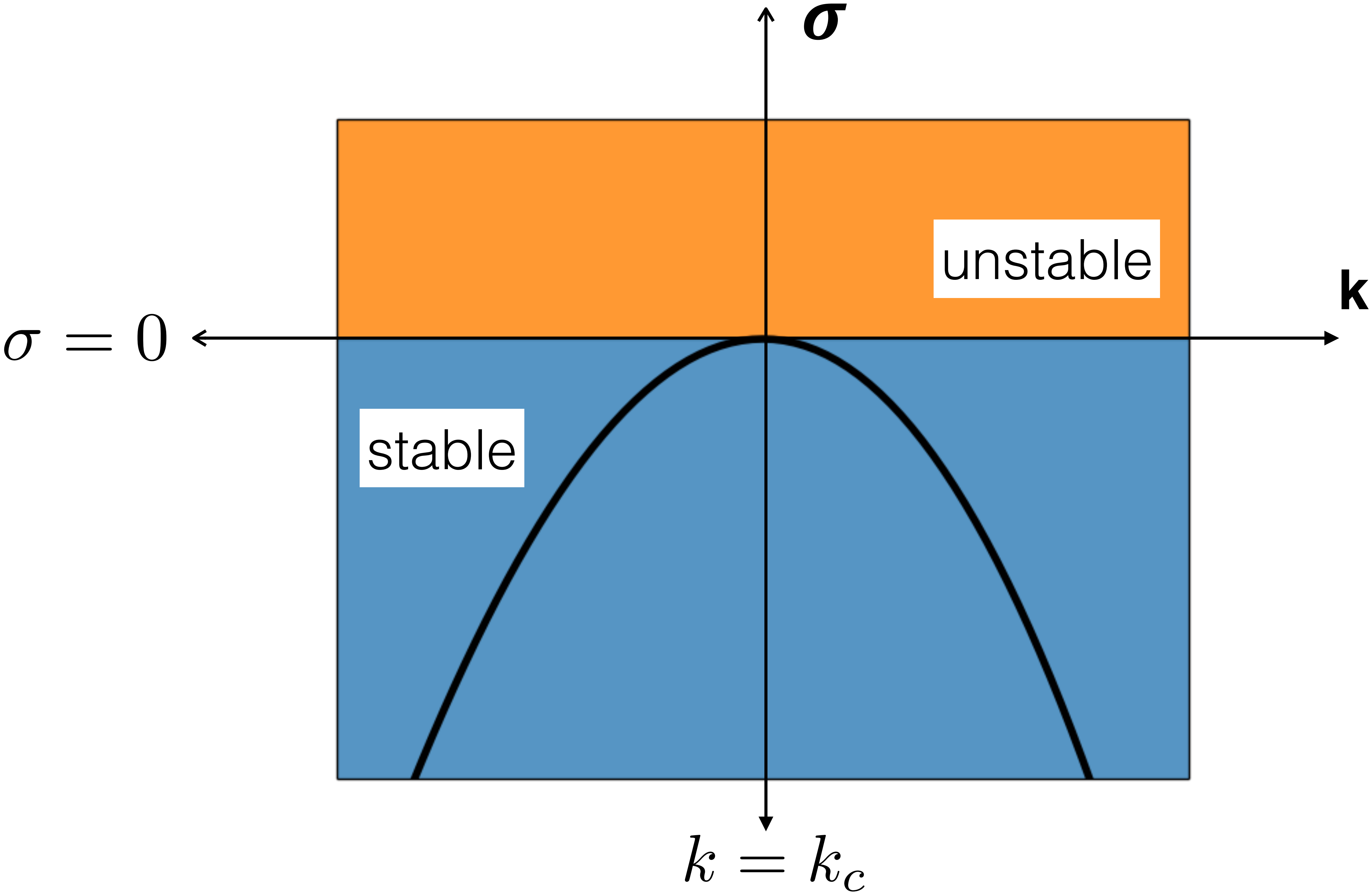

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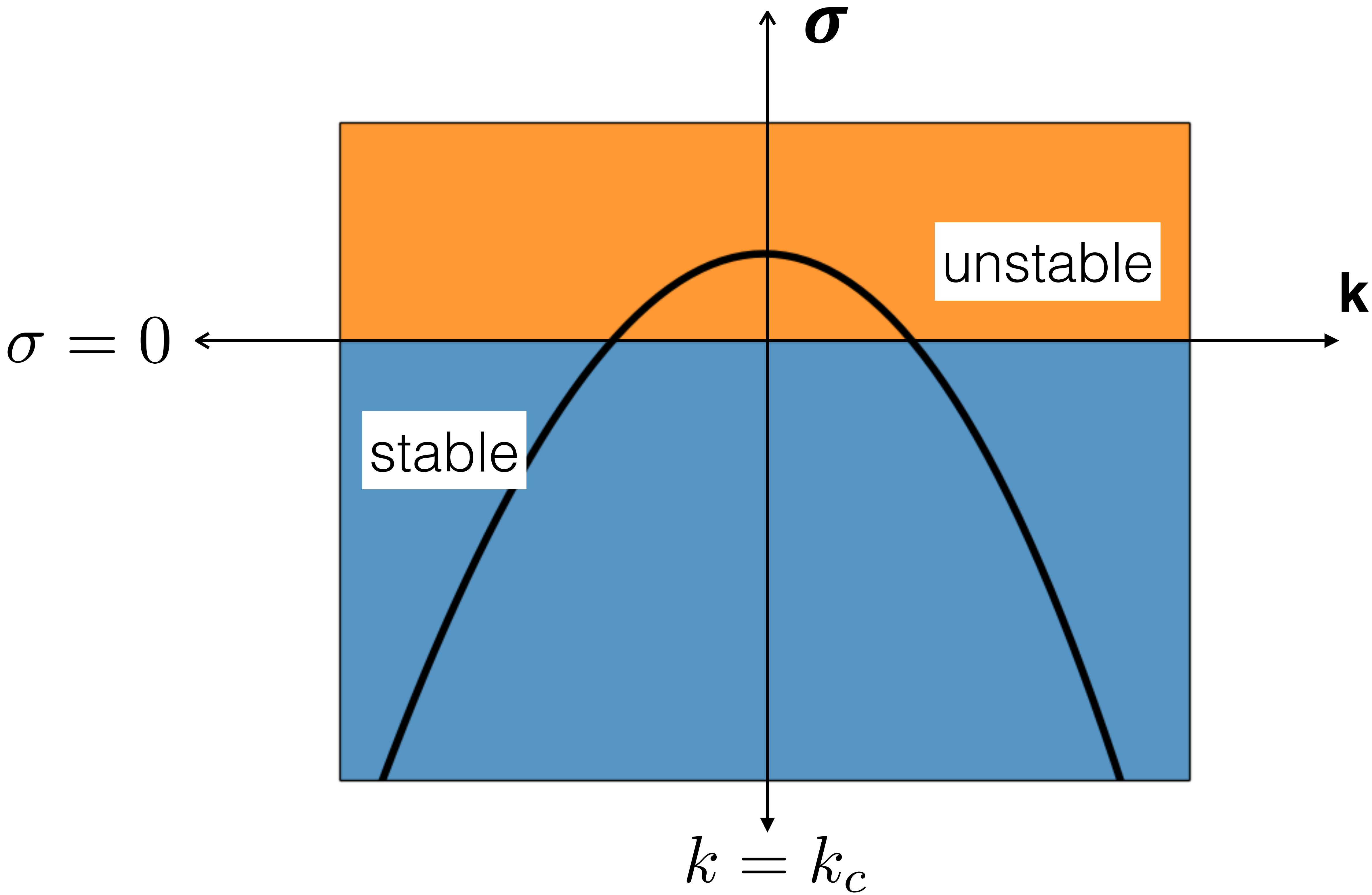
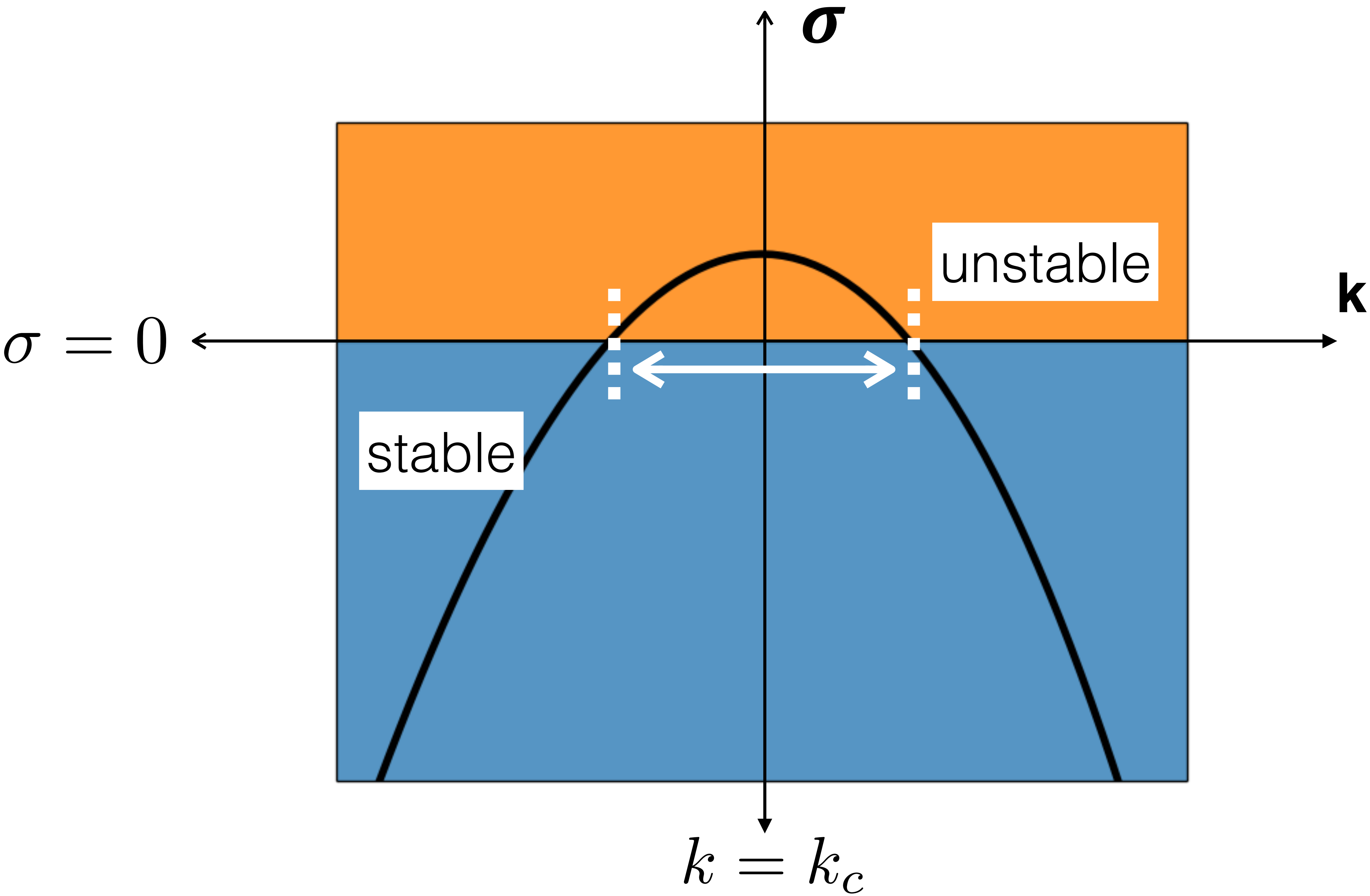


Weakly nonlinear analysis explores behavior at the margin of instability.

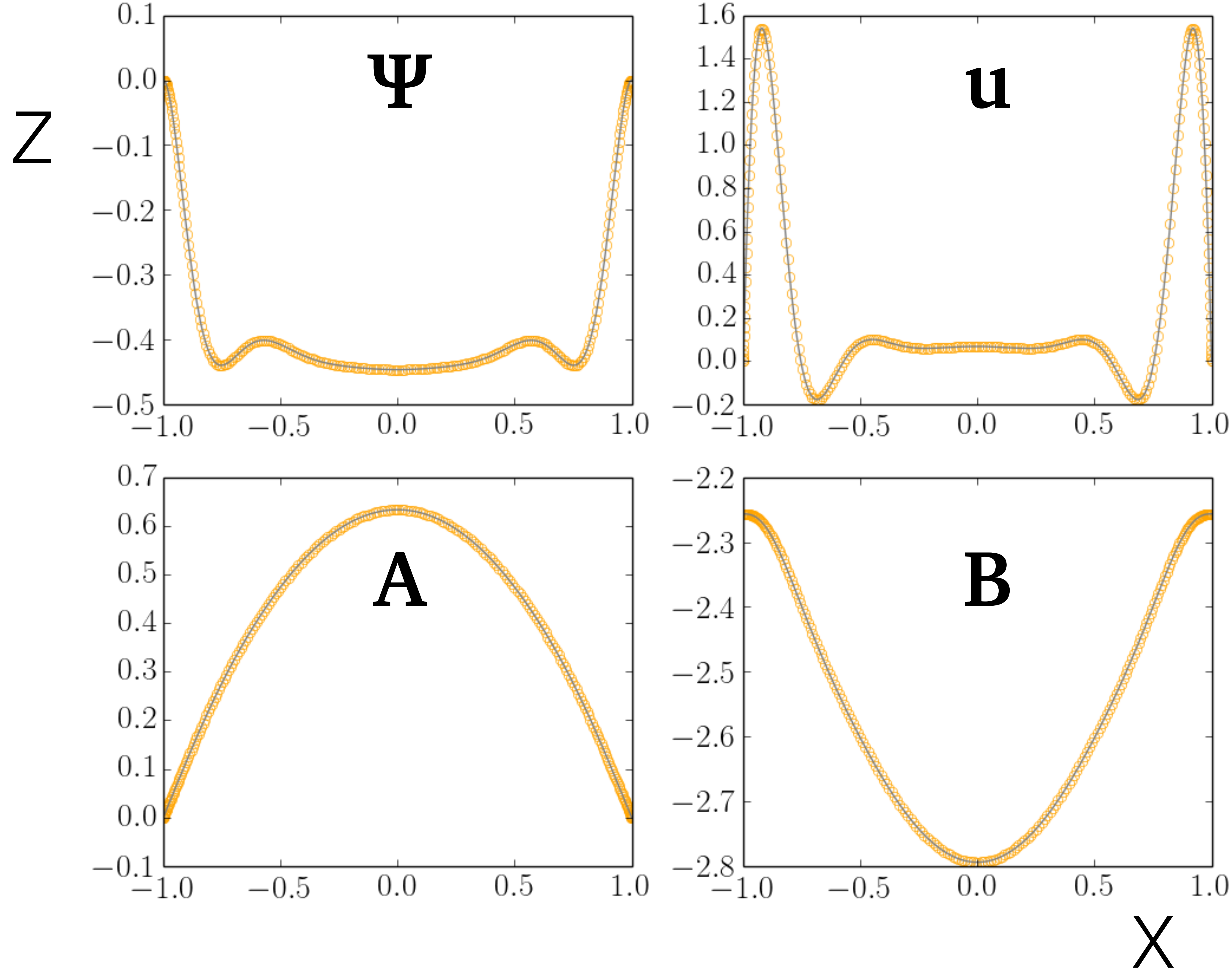


Tune the most unstable mode just over the threshold of instability.

$$\epsilon^2 \equiv 1 - B_0$$

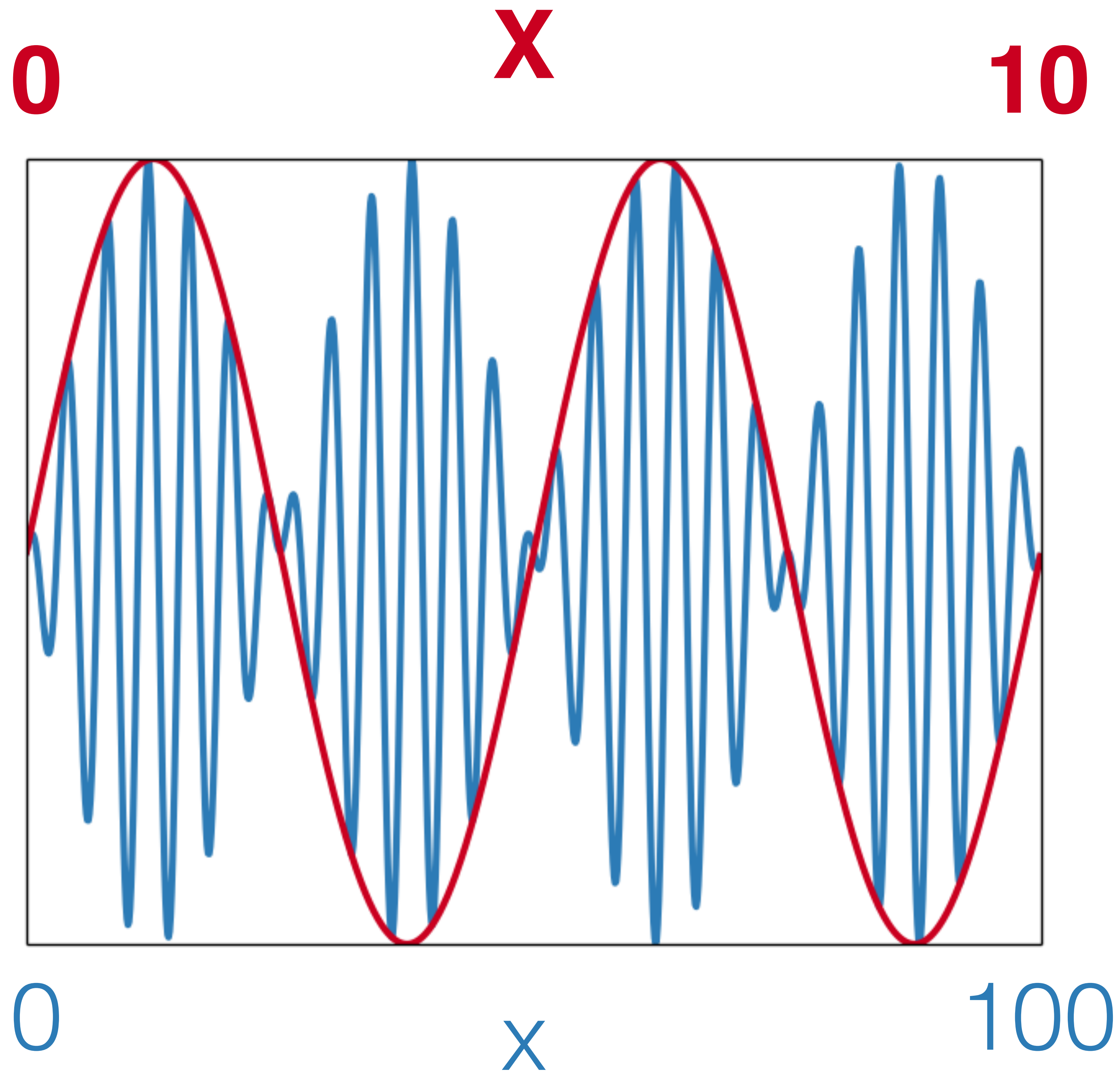
small parameter

Identify the most unstable mode
of the linear MRI.



Multiscale analysis tracks the evolution of fast and slow variables.

$$X \equiv \epsilon x$$

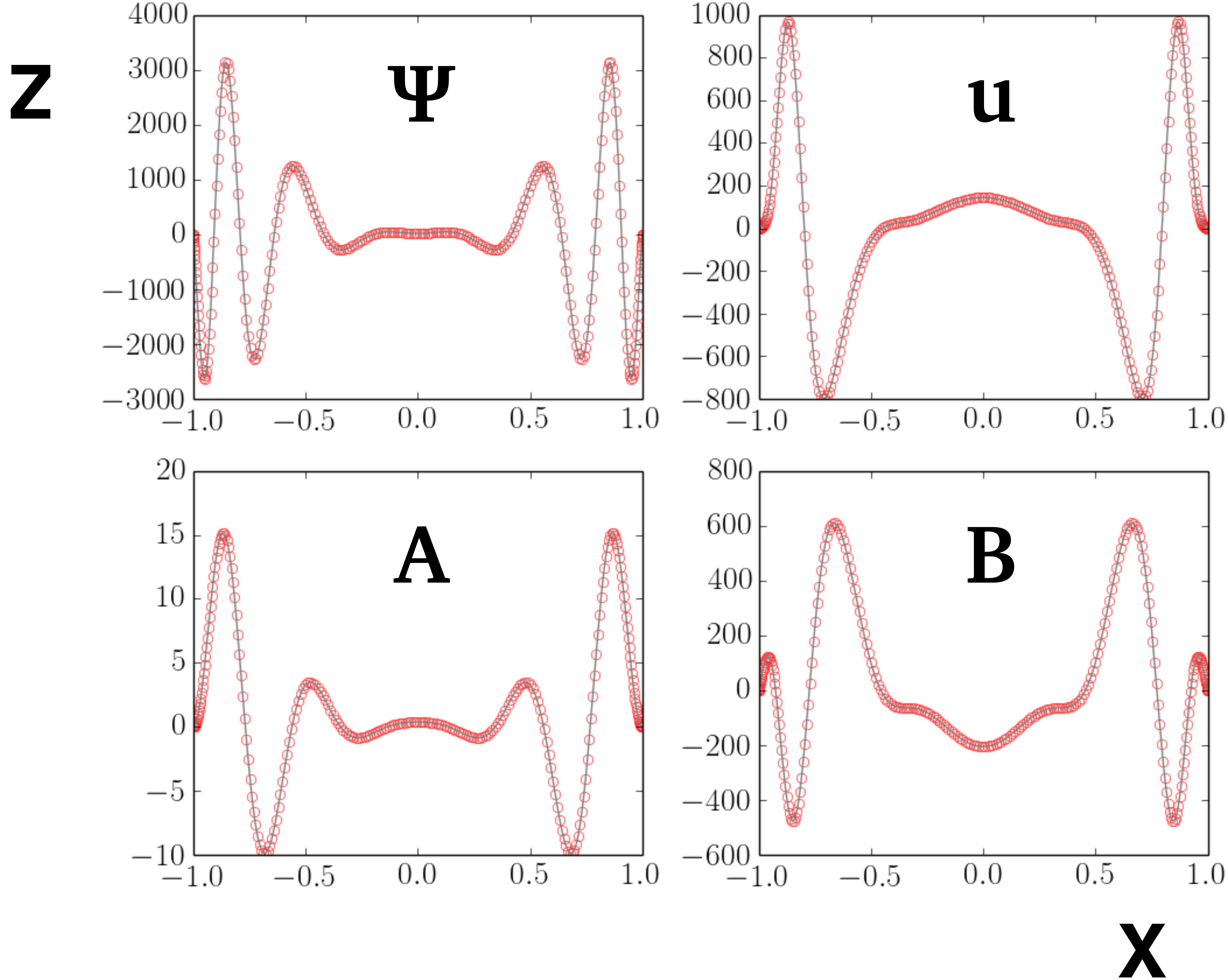


Equations are solved in a matrix formulation.

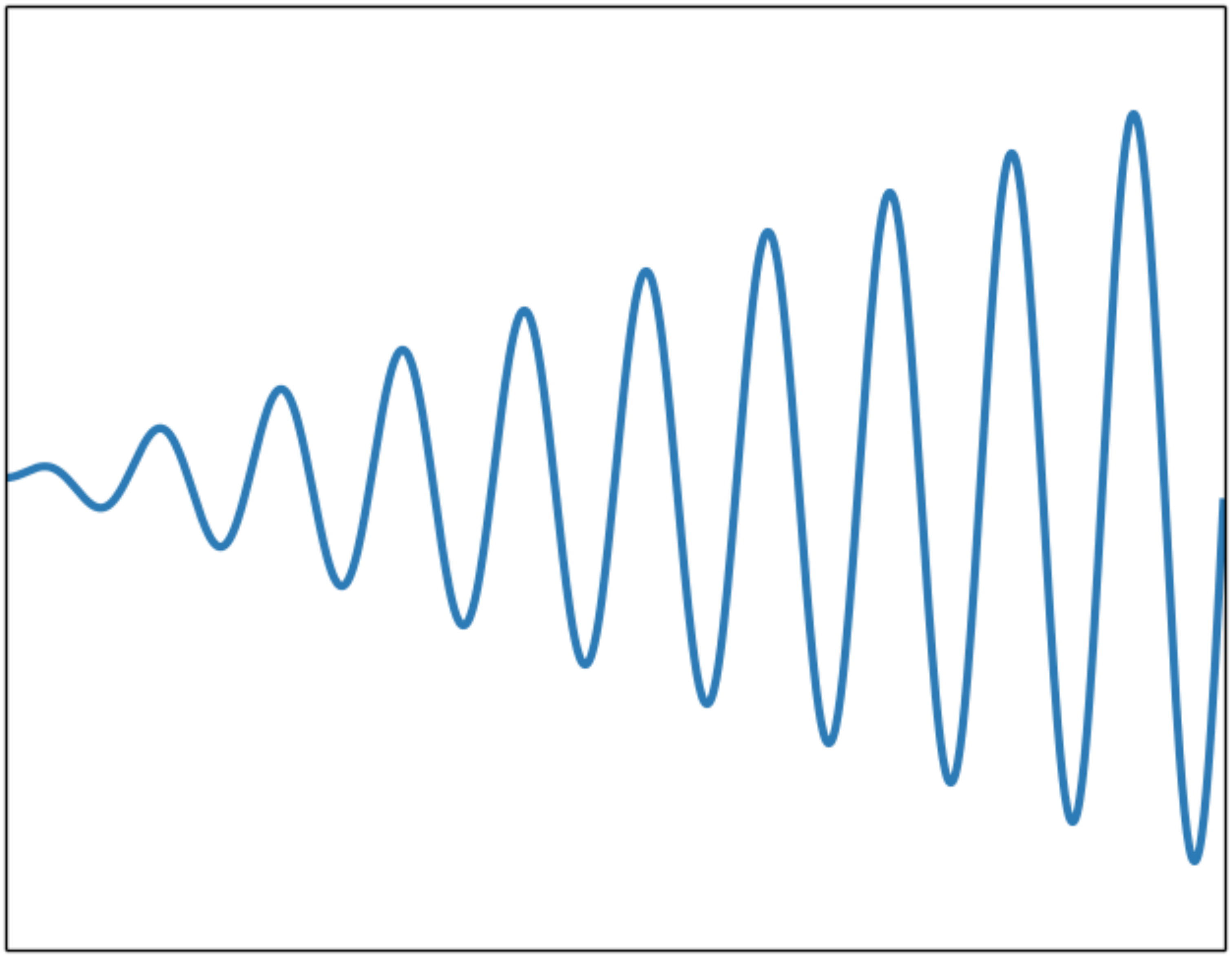
The fluid quantities are expanded in a perturbation series.

$$\mathbf{V} = \epsilon \mathbf{V_1} + \epsilon^2 \mathbf{V_2} + \epsilon^3 \mathbf{V_3} + ...$$

something about boundary layers?



The removal of secular terms yields solvability criteria.



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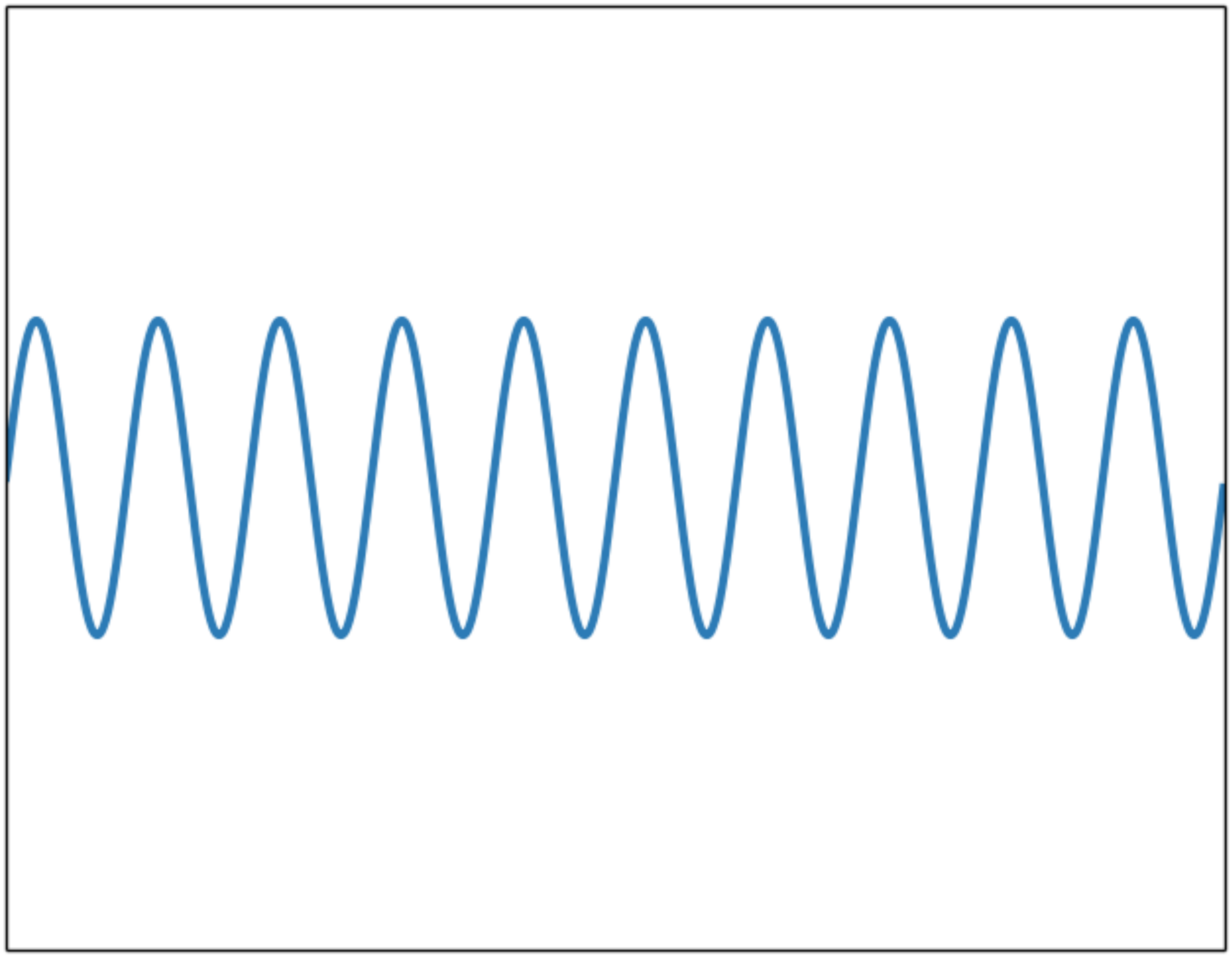
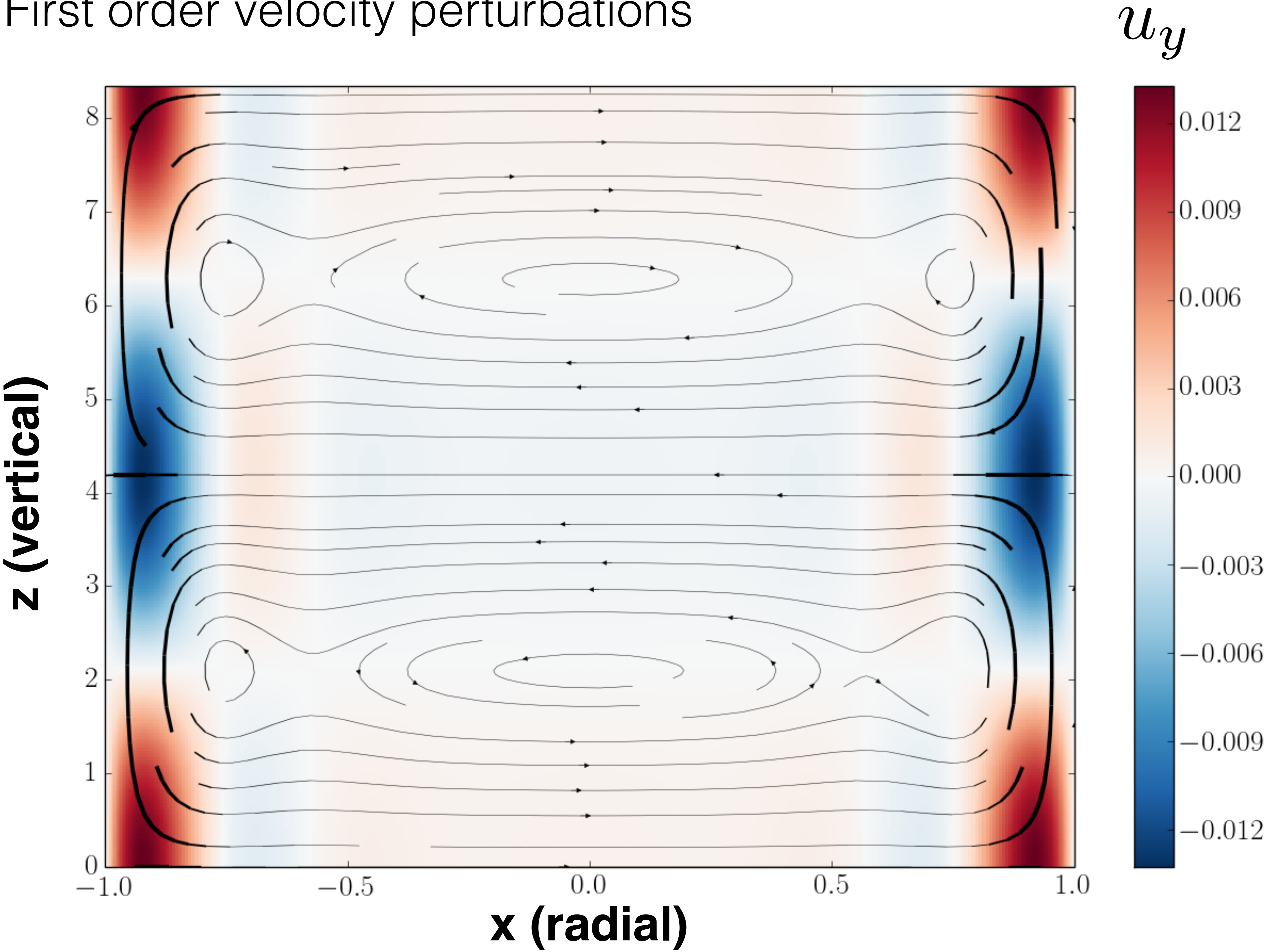


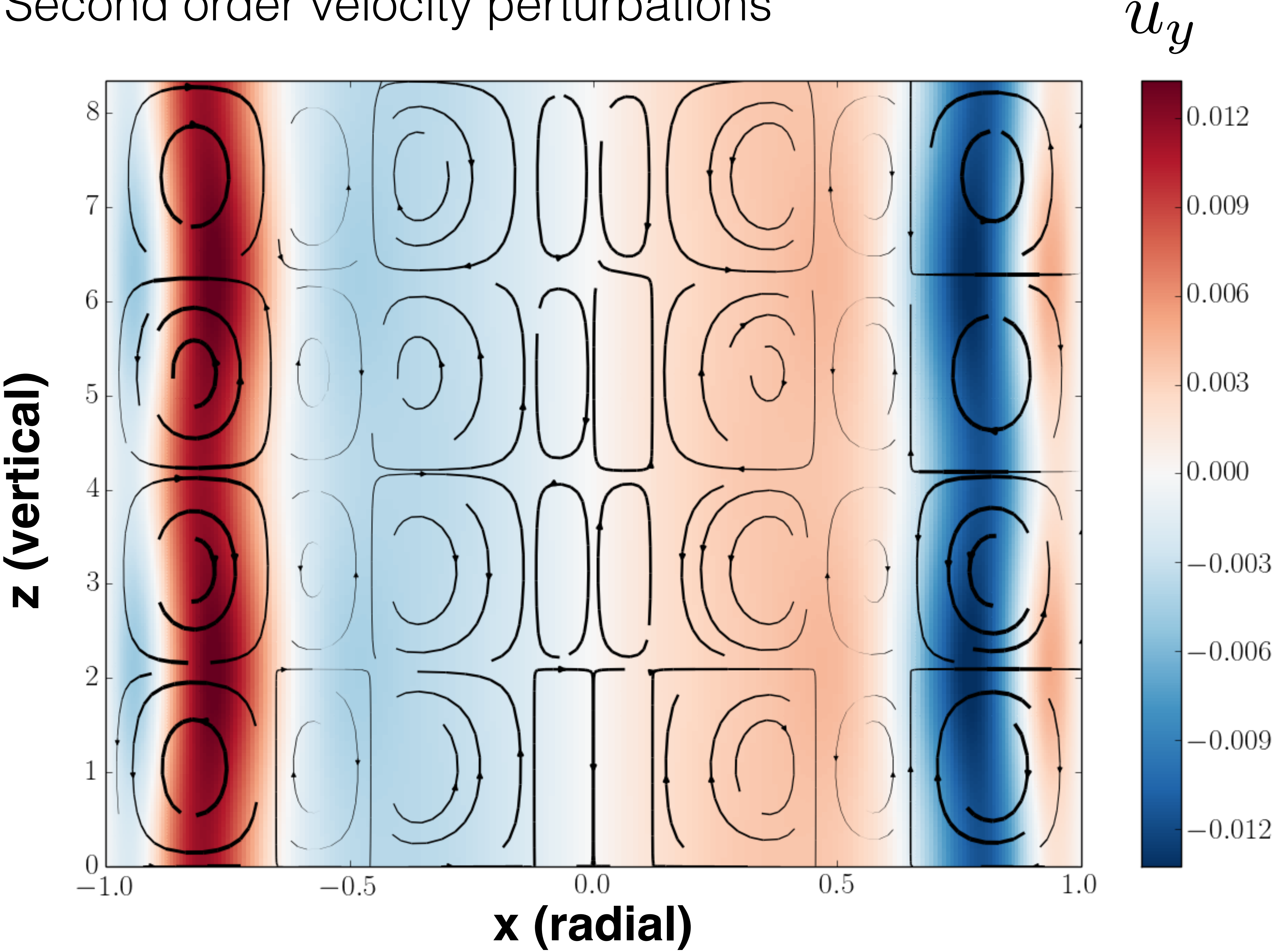
The result is an amplitude equation.

$$a\partial_T \alpha = -b\partial_Z \alpha - c\alpha \left|\alpha^2\right| + h\partial_Z^2 \alpha + giQ^3 \alpha$$

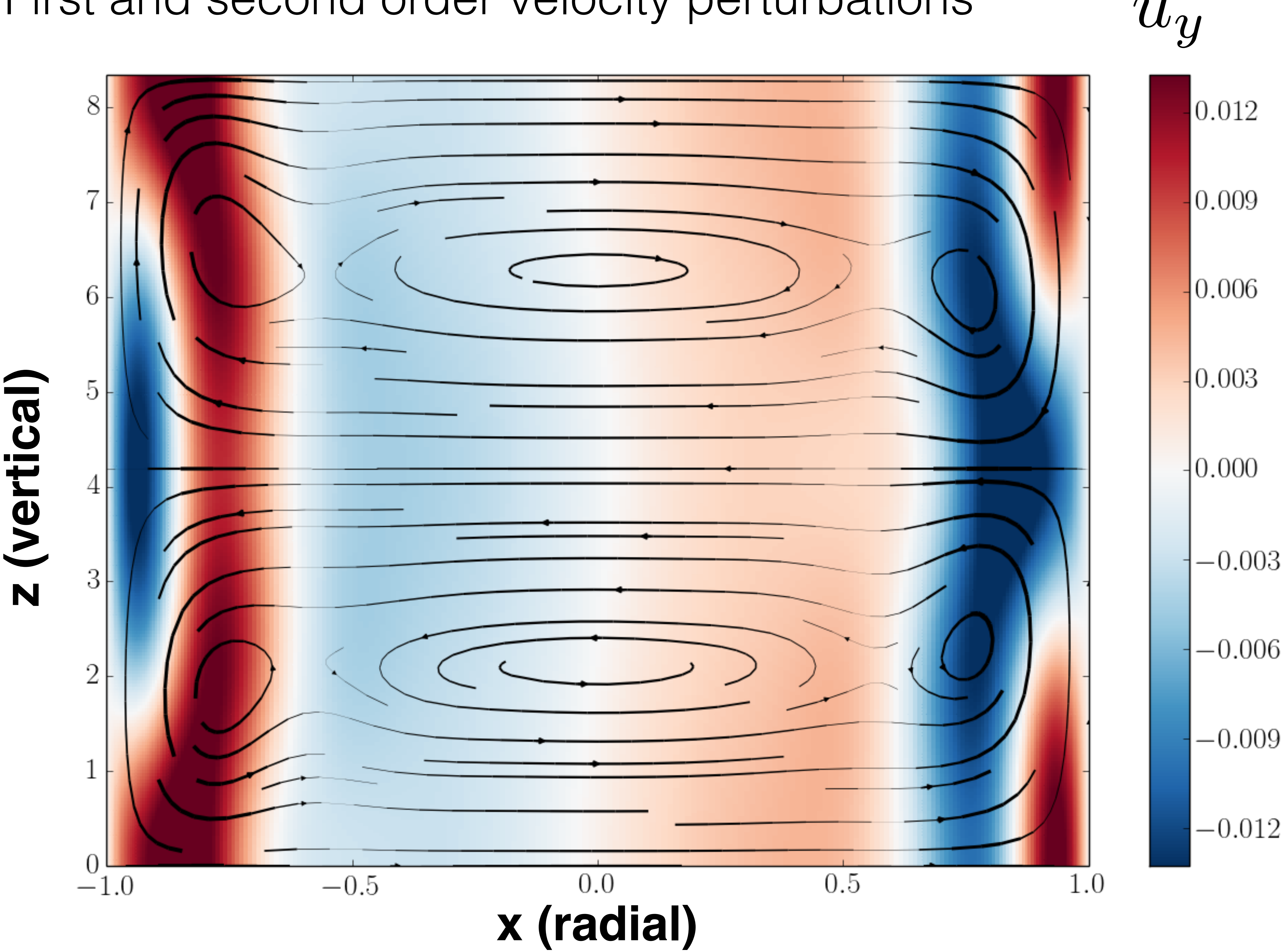
First order velocity perturbations



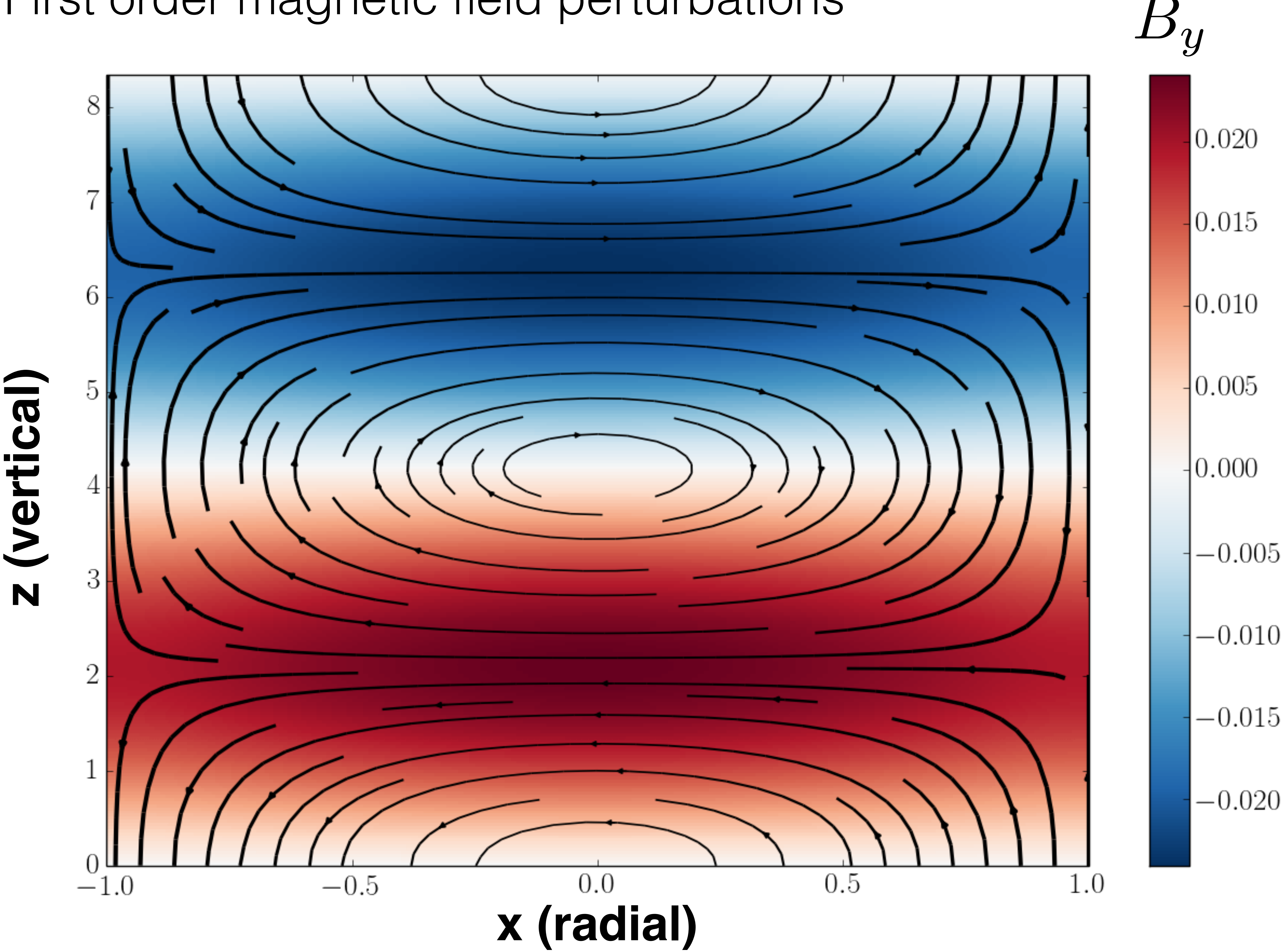
Second order velocity perturbations



First and second order velocity perturbations

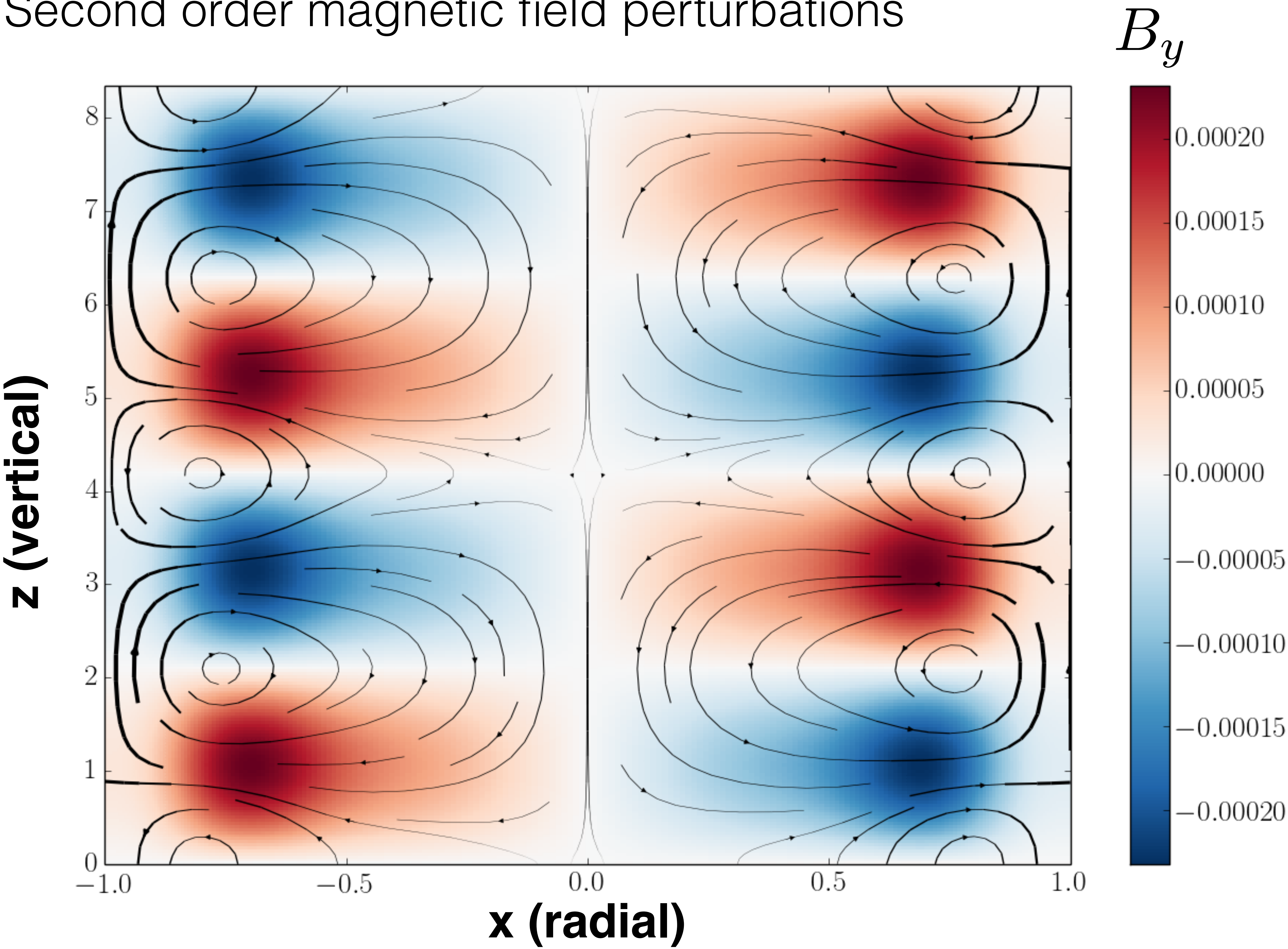


First order magnetic field perturbations

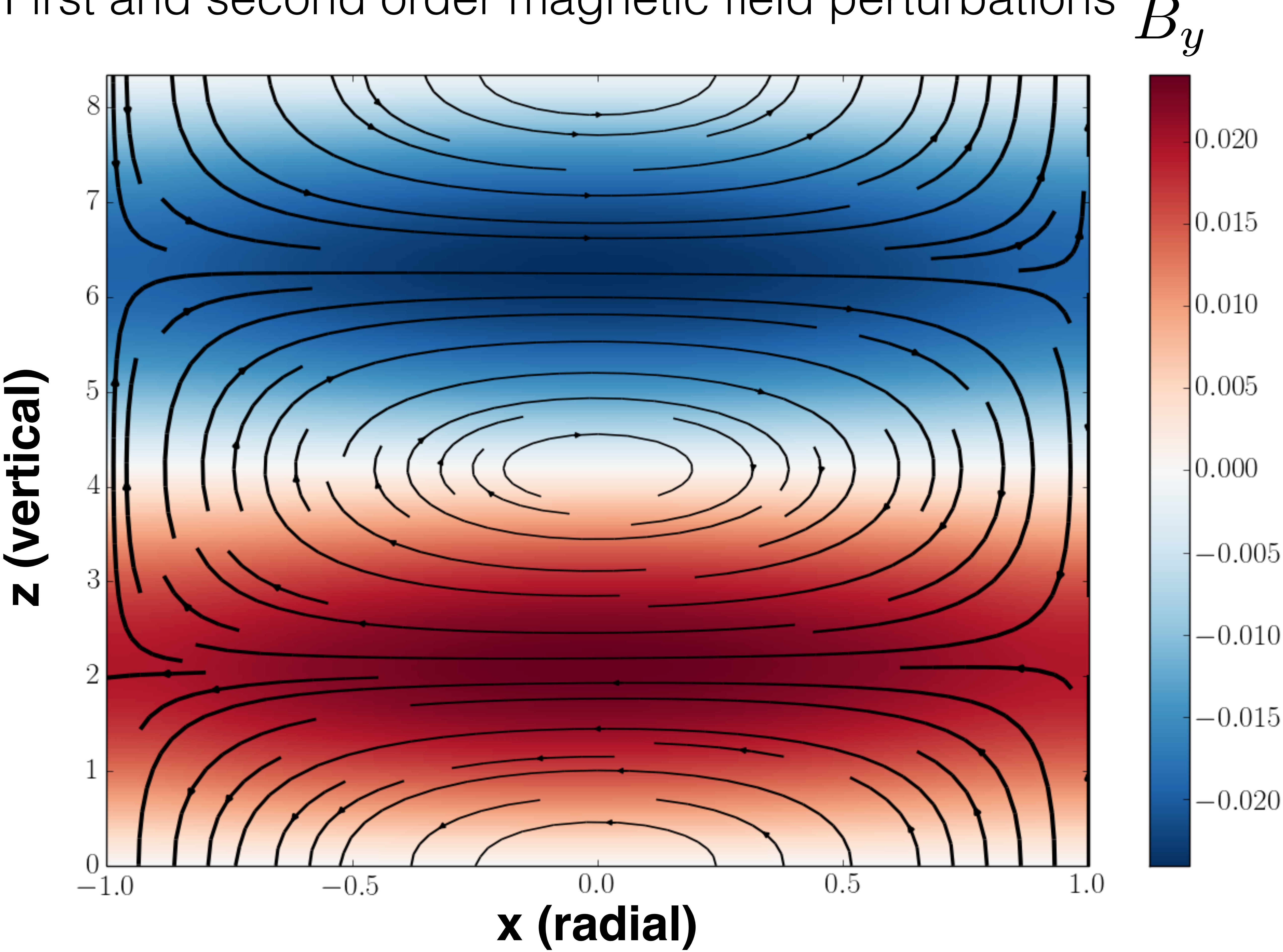


Second order magnetic field perturbations

two OOM smaller!



First and second order magnetic field perturbations



Future work:

non-thin gap approximation
helical MRI
explore parameter space
comparison to experiment