

1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r} \partial_z \psi \hat{\mathbf{r}} \\ u_\phi \hat{\phi} \\ -\frac{1}{r} \partial_r \psi \hat{\mathbf{z}} \end{bmatrix}; \quad (1)$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t \left[\frac{1}{r} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] + \frac{1}{r^2} J(\psi, \nabla^2 A - \frac{2\partial_r \psi}{r}) &= \frac{\partial_z A}{r^3} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \\ &+ \frac{1}{r} J \left(A, \frac{1}{r} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \right) - \frac{2B_\phi \partial_z B_\phi}{r} \\ &+ \nu \left\{ \nabla^2 \left[\frac{1}{r} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] - \frac{1}{r^2} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right\} \end{aligned} \quad (2)$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \nu \left(\nabla^2 u_\phi - \frac{u_\phi}{r} \right) \quad (3)$$

$$\partial_t A = \frac{1}{r} J(A, \psi) + \frac{1}{\text{Rm}} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \quad (4)$$

$$\begin{aligned} \partial_t B_\phi &= \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ &+ \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left(\nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \end{aligned} \quad (5)$$

2 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\text{Re}} [\nabla^2 u]_z \quad (6)$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\begin{aligned}
& \frac{1}{r^2} \partial_t \partial_r \Psi - \frac{1}{r} \partial_t \partial_r^2 \Psi - \frac{3}{r^4} \partial_z \Psi \partial_r \Psi + \frac{1}{r^3} \partial_r (\partial_z \Psi \partial_r \Psi) + \frac{2}{r^3} \partial_z \Psi \partial_r^2 \Psi - \frac{1}{r^2} \partial_r (\partial_z \Psi \partial_r^2 \Psi) \\
& \quad - \frac{2}{r^3} \partial_r \Psi \partial_r \partial_z \Psi + \frac{1}{r^2} \partial_r (\partial_r \Psi \partial_r \partial_z \Psi) = \\
& \partial_r (B_\phi \partial_z B_\phi) + \frac{2}{r^3} \partial_z^2 A \partial_z A - \frac{1}{r^2} \partial_r (\partial_z^2 A \partial_z A) + \frac{3}{r^4} \partial_z A \partial_r A - \frac{1}{r^3} \partial_r (\partial_z A \partial_r A) - \frac{2}{r^3} \partial_z A \partial_r^2 A \\
& \quad + \frac{1}{r^2} \partial_r (\partial_z A \partial_r^2 A) + \frac{1}{\text{Re}} \left[\frac{3}{r^4} \partial_r \Psi - \frac{3}{r^3} \partial_r^2 \Psi + \frac{2}{r^2} \partial_r^3 \Psi - \frac{1}{r} \partial_r^4 \Psi \right] \quad (7)
\end{aligned}$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r + [u \cdot \nabla u]_r = [(\nabla \times B) \times B]_r + \frac{1}{\text{Re}} [\nabla^2 u]_r \quad (8)$$

and take ∂_z of the expanded equation to obtain

$$\begin{aligned}
& \frac{1}{r} \partial_t \partial_z^2 \Psi - \frac{1}{r^3} \partial_z (\partial_z \Psi \partial_z \Psi) + \frac{1}{r^2} \partial_z (\partial_z \Psi \partial_z \partial_r \Psi) - \frac{1}{r^2} \partial_z (\partial_r \Psi \partial_z^2 \Psi) - \frac{1}{r} 2u_\phi \partial_z u_\phi \\
& = -\frac{1}{r^2} \partial_z^3 A \partial_r A - \frac{1}{r^2} \partial_z^2 A \partial_r \partial_z A + \frac{2}{r^3} \partial_r \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 A \partial_r \partial_z A \\
& \quad + \frac{1}{\text{Re}} \left[-\frac{1}{r^2} \partial_z^2 \partial_r \Psi + \frac{1}{r} \partial_z^2 \partial_r^2 \Psi + \frac{1}{r} \partial_z^4 \Psi \right] \quad (9)
\end{aligned}$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r} \partial_t \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) + J \left(\Psi, \frac{1}{r^2} \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) \right) - \frac{1}{r} 2u_\phi \partial_z u_\phi \quad (10)$$

Note that the relevant quantity appears to be $\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\text{Re}} \left[\nabla^2 \left(\frac{1}{r} \nabla^2 \Psi \right) - \frac{1}{r^3} \partial_r^2 \Psi - \frac{1}{r^4} \partial_r \Psi \right] \quad (11)$$

Full Ψ equation according to Susan:

$$\begin{aligned}
& \frac{1}{r} \partial_t \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) + J \left(\Psi, \frac{1}{r^2} \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) \right) - \frac{1}{r} 2u_\phi \partial_z u_\phi \\
& = J \left(A, \frac{1}{r^2} \left(\nabla^2 A - \frac{2}{r} \partial_r A \right) \right) - \frac{2}{r} B_\phi \partial_z B_\phi \\
& \quad + \frac{1}{\text{Re}} \left[\nabla^2 \left(\frac{1}{r} \nabla^2 \Psi \right) - \frac{1}{r^3} \partial_r^2 \Psi - \frac{1}{r^4} \partial_r \Psi \right] \quad (12)
\end{aligned}$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky....

The derivation of the non-viscous term on the righthand side of the momentum equation ($\mathbf{J} \times \mathbf{B}$) is as follows.

$$\partial_z ([(\nabla \times B) \times B]_r) - \partial_r ([(\nabla \times B) \times B]_z) \quad (13)$$

$$\begin{aligned}
& = \partial_z \left(\left[(\partial_z B_r - \partial_r B_z) B_z - \left(\frac{1}{r} \partial_r (r B_\phi) \right) B_\phi \right] \right) - \partial_r \left([(-\partial_z B_\phi) B_\phi - (\partial_z B_r - \partial_r B_z) B_r] \right) \\
& \quad (14)
\end{aligned}$$

$$\begin{aligned}
& = -\frac{1}{r^2} \partial_z^3 A \partial_r A + \frac{1}{r^3} \partial_r \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 \partial_z A \partial_r A - \frac{2}{r^3} \partial_z^2 A \partial_z A \\
& \quad + \frac{1}{r^2} \partial_z^2 \partial_r A \partial_z A + \frac{3}{r^4} \partial_r A \partial_z A - \frac{3}{r^3} \partial_r^2 A \partial_z A + \frac{1}{r^2} \partial_r^3 A \partial_z A - \frac{2}{r} B_\phi \partial_z B_\phi \quad (15)
\end{aligned}$$

This simplifies to

$$J \left(A, \frac{1}{r^2} \left(\nabla^2 A - \frac{2}{r} \partial_r A \right) \right) - \frac{2}{r} B_\phi \partial_z B_\phi \quad (16)$$

Full derivation of viscous term:

$$\partial_z \left(\frac{1}{\text{Re}} [\nabla^2 u]_r \right) - \partial_r \left(\frac{1}{\text{Re}} [\nabla^2 u]_z \right) \quad (17)$$

$$= \frac{1}{\text{Re}} \left[\partial_z \left(\nabla^2 u_r - \frac{1}{r^2} u_r \right) - \partial_r \left(\nabla^2 u_z \right) \right] \quad (18)$$

$$\begin{aligned}
& = \frac{1}{\text{Re}} \left[-\frac{2}{r^2} \partial_z^2 \partial_r \Psi + \frac{2}{r} \partial_z^2 \partial_r^2 \Psi + \frac{1}{r} \partial_z^4 \Psi - \frac{3}{r^4} \partial_r \Psi + \frac{3}{r^3} \partial_r^2 \Psi - \frac{2}{r^2} \partial_r^3 \Psi + \frac{1}{r} \partial_r^4 \Psi \right] \\
& \quad (19)
\end{aligned}$$

3 Recovery of Narrow Gap Equations

4 Nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$\begin{aligned}
 \tilde{u} &= \Omega_0 r_0 \delta u \\
 \tilde{x} &= \delta r_0 x \\
 \tilde{\nabla} &= \frac{\nabla}{\delta r_0} \\
 \tilde{t} &= \frac{t}{\Omega_0} \\
 \tilde{B} &= B_0 B \\
 \tilde{P} &= P_0 P \\
 \tilde{\rho} &= \rho_0 \rho
 \end{aligned} \tag{20}$$

If we define the velocity scale v_0 as the local sound speed scale ($c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}}$), then $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$.

Nondimensionalizing the momentum equation and dividing by a factor of $\Omega_0^2 r_0 \delta$ yields

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{21}$$

If we define

$$\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \tag{22}$$

where $P_{mag} = \frac{B_0^2}{8\pi}$, then the factor in front of $(\nabla \times \mathbf{B}) \times \mathbf{B}$ should be $\frac{2}{\beta}$. Agreed?

5 Perturbed Equations

Note: two things are currently missing. The constant β from the nondimensionalization, and the Coriolis term in the momentum equation.

We perturb the wide gap equations according to

$$\mathbf{B} = B_0 \hat{z} + \mathbf{B}_1 \tag{23}$$

$$\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u}_1 \tag{24}$$

The Ψ equation becomes

$$\begin{aligned}
& \frac{1}{r} \partial_t \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) + J \left(\Psi, \frac{1}{r^2} \left(\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) \right) - \frac{1}{r} 2u_\phi \partial_z u_\phi \\
& = J \left(A, \frac{1}{r^2} \left(\nabla^2 A - \frac{2}{r} \partial_r A \right) \right) - \frac{2}{r} B_\phi \partial_z B_\phi \\
& + \frac{1}{\text{Re}} \left[\nabla^2 \left(\frac{1}{r} \nabla^2 \Psi \right) - \frac{1}{r^3} \partial_r^2 \Psi - \frac{1}{r^4} \partial_r \Psi \right] + \frac{1}{r} B_0 \partial_z \left(\nabla^2 A - \frac{2}{r} \partial_r A \right) \quad (25)
\end{aligned}$$

Equation 3 becomes

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} - \frac{2}{r} q \Omega_0 \partial_z \Psi = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \frac{1}{\text{Re}} \left(\nabla^2 u_\phi - \frac{u_\phi}{r} \right) + B_0 \partial_z B_\phi \quad (26)$$

The only term gained in the (\hat{r}, \hat{z}) component of the induction equation is $(B_0 \hat{z} \cdot \nabla) \mathbf{u}_1$, and Equation 4 becomes

$$\partial_t A = \frac{1}{r} J(A, \psi) + \frac{1}{\text{Rm}} \left(\nabla^2 A - \frac{2 \partial_r A}{r} \right) + B_0 \partial_z \Psi \quad (27)$$

Note that this is perfectly analogous to the thin-gap version of this equation. The $\hat{\phi}$ component of the induction equation, Equation 5, becomes

$$\begin{aligned}
& \partial_t B_\phi = \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\
& + \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \frac{1}{\text{Rm}} \left(\nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) + B_0 \partial_z u_\phi - \frac{1}{r} q \Omega_0 \partial_z A \quad (28)
\end{aligned}$$

A Cylindrical derivatives

Everything here follows <http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl>.

For a scalar field ψ ,

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\phi} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}. \quad (29)$$

However, for a *vector* field \mathbf{u} ,

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \quad (30)$$

and

$$\nabla \times \mathbf{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\phi} + \left(\frac{1}{r} \frac{\partial(r u_\phi)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right) \hat{\mathbf{z}}. \quad (31)$$

We also need the ϕ component of the convective derivative $\mathbf{u} \cdot \nabla \mathbf{u}$,

$$[\mathbf{u} \cdot \nabla \mathbf{u}]_\phi = \mathbf{u} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}, \quad (32)$$

and finally, the vector Laplacian,

$$(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \quad (33)$$

$$(\nabla^2 \mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \quad (34)$$

$$(\nabla^2 \mathbf{u})_z = \nabla^2 u_z, \quad (35)$$

where ∇ on the vector components is given by equation (29).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$