$$\nabla \times \nabla \times (\alpha g \hat{z} T) = \alpha g (\nabla \times \nabla \times (\hat{z} T))$$

$$= \alpha g \, \varepsilon_{kij} \varepsilon_{kmn} \partial_j \partial_m (\hat{z} T)_n$$

$$= \alpha g (\partial_j \partial_i (\hat{z} T)_j - \delta_j \delta_j (\hat{z} T)_i)$$

$$\alpha g (\partial_j \partial_i (\hat{z} T)_j - \delta_j \delta_j (\hat{z} T)_i) \cdot \hat{z} = T(\partial_j, \partial_i)^2$$

Explicit Derivation of vertical velocity amplitudes in Rayleigh-Bénard Convection; Following Newell & Whitehead 1969

1

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) \nabla^2 \omega - \alpha g (\beta_0 + \epsilon^2 \beta_2) \nabla_1^2 \omega$$

$$= -\alpha g \nabla_1^2 (u \cdot \nabla T) + \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)[\hat{z} \cdot (\nabla \times \nabla \times (\vec{\Omega} \times \vec{u}))]$$

$X = \varepsilon x, \; Y = \varepsilon y, \; T = \varepsilon^2 t$ (I'll use R/r for X/x : $R = \varepsilon r$)

$(\omega \to \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2)$

$$\nabla_1^2 = \nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R}$$

$$\begin{cases} \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} \\ \nabla_{1r} \to \nabla_{1r} + \varepsilon \nabla_{1R} \end{cases} \quad \nabla_{1r} = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial y}\right) \quad \nabla_{1R} = \left(\frac{\partial}{\partial R}, \frac{\partial}{\partial Y}\right)$$

$$\therefore \nabla^2 \to \nabla_1^2 + \frac{\partial^2}{\partial z^2} \qquad \therefore \nabla^2 \to \nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2}$$

$$\left.\begin{aligned} &\left.\begin{aligned} &\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} - \nu \left(\nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2}\right)\right) \\ &\cdot \left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} - \kappa \left(\nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2}\right)\right) \\ &\cdot \left(\nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2}\right)(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2) \end{aligned}\right\} \text{1st term} \\ &\left. -\alpha g (\beta_0 + \varepsilon^2 \beta_2)(\nabla_{1r}^2 + \varepsilon^2 \nabla_{1R}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R})(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2) \right\} \text{2nd term} \end{aligned}\right\} \text{LHS}$$

---

* group powers of ε. 0^{th} power is easy:

$$\mathcal{L}_0 = \left(\frac{\partial}{\partial t} - \nu \left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t} - \kappa \left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$

$$- \alpha g \beta_0 \nabla_{1r}^2$$

* 1^{st} order in ε incl. everything multiplied by $2\varepsilon \nabla_{1r} \cdot \nabla_{1R}$ ...

$(\longrightarrow)$

$$\mathcal{L}_1 = \left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)$$
$$+\left(\frac{\partial}{\partial t} - \nu\, 2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial^2}{\partial z^2}\right)$$
$$+\left(\frac{\partial}{\partial t} - \kappa\, 2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial^2}{\partial z^2}\right)$$
$$-\alpha g\beta_0\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)$$
$$= 2\nabla_{1r}\cdot\nabla_{1R}\Big\{\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)$$
$$-\nu\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$-\kappa\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$-\alpha g\beta_0\Big\}$$

second order in ε: everything mult. by $\varepsilon^2\frac{\partial}{\partial T}$, $\varepsilon^2\nabla_{1R}^2$, or $(2\varepsilon\nabla_{1r}\cdot\nabla_{1R})^2$ (also β_2 term)

$\mathcal{L}_2$ 1st term: $$\varepsilon^2\frac{\partial}{\partial T}\Big\{\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$+\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\Big\}$$
$$= \frac{\partial}{\partial T}\Big\{\left[2\frac{\partial}{\partial t} - (\kappa+\nu)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right]\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\Big\}$$

$\mathcal{L}_2$ 2nd term: $$-\nu\varepsilon^2\nabla_{1R}^2\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$-\kappa\varepsilon^2\nabla_{1R}^2\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$+\varepsilon^2\nabla_{1R}^2\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right) - \alpha g\beta_0\varepsilon^2\nabla_{1R}^2$$
$$= \nabla_{1R}^2\Big\{-\nu\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$-\kappa\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$+\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)$$
$$-\alpha g\beta_0\Big\}$$

$\mathcal{L}_2$ 3rd term: $$-\nu\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\left(-\kappa\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\right)\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)$$
$$-\nu\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\left(\frac{\partial}{\partial t} - \kappa\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)$$
$$-\kappa\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)\left(\frac{\partial}{\partial t} - \nu\left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2}\right)\right)\left(2\varepsilon\nabla_{1r}\cdot\nabla_{1R}\right)$$

$(\rightarrow)$

$\mathcal{L}_2$ 3rd term cont'd:

$$4\varepsilon^2(\nabla_{1r}\cdot\nabla_{1R})^2\Big\{\nu\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right) - \nu\left(\frac{\partial}{\partial t}-\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right) - \kappa\left(\frac{\partial}{\partial t}-\nu\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)$$

$$\Rightarrow 4\left[\nu\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)-\kappa\left(\frac{\partial}{\partial t}-\nu\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)-\nu\left(\frac{\partial}{\partial t}-\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)\right](\nabla_{1r}\cdot\nabla_{1R})^2$$

also the term $-\kappa g\beta_2\nabla_{1r}^2$ contributes to $\mathcal{L}_2$.

Thus:

$$\mathcal{L}_2 = \frac{\partial}{\partial T}\left\{\left[2\frac{\partial}{\partial t}-(\kappa+\nu)\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right]\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right\}$$

$$+\nabla_{1R}^2\Big\{-\nu\left(\frac{\partial}{\partial t}-\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right) -\kappa\left(\frac{\partial}{\partial t}-\nu\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right) + \left(\frac{\partial}{\partial t}-\nu\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)\left(\frac{\partial}{\partial t}-\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)-\alpha g\beta_0\Big\}$$

$$+4\left\{\nu\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)-\kappa\left(\frac{\partial}{\partial t}-\nu\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right) -\nu\left(\frac{\partial}{\partial t}-\kappa\left(\nabla_{1r}^2+\frac{\partial^2}{\partial z^2}\right)\right)\right\}\times(\nabla_{1r}\cdot\nabla_{1R})^2$$

and the LHS of the equation becomes:

$$(\mathcal{L}_0+\varepsilon\mathcal{L}_1+\varepsilon^2\mathcal{L}_2)(\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2)$$

The RHS begins as:

$$-\alpha g\nabla_1^2(u\cdot\nabla T)+\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\left[\hat{z}\cdot(\nabla\times\nabla\times(\vec{\Omega}\times\vec{u}))\right]$$

Making the substitutions, this becomes: (w/ $u\to u_0+\varepsilon u_1$ & $\nabla T\to\varepsilon\nabla T_0+\varepsilon^2\nabla T_1$ *)

$$-\alpha g(\nabla_{1r}^2+\varepsilon^2\nabla_{1R}^2+2\varepsilon\nabla_{1r}\cdot\nabla_{1R})\cdot\left[(u_0+\varepsilon u_1)\cdot(\varepsilon\nabla T_0+\varepsilon^2\nabla T_1)\right]$$

$$=-\alpha\varepsilon g(\nabla_{1r}^2+\cancelto{0}{\varepsilon^2\nabla_{1R}^2}+2\varepsilon\nabla_{1r}\cdot\nabla_{1R})\cdot\left[(u_0+\varepsilon u_1)\cdot(\nabla T_0+\varepsilon\nabla T_1)\right]$$ drop ε orders $>\mathcal{O}(2)$

*All vars are expanded $\varepsilon f_0+\varepsilon^2 f_1+\ldots$ but extra ε in u cancels w/ extra ε on RHS in ω!!!

$$=-\alpha g\varepsilon(\nabla_{1r}^2+2\varepsilon\nabla_{1r}\cdot\nabla_{1R})\{u_0\cdot\nabla T_0+\varepsilon(u_1\cdot\nabla T_0+u_0\cdot\nabla T_1)\}$$ 1st term on RHS

$$+\left[\frac{\partial}{\partial t}+\varepsilon^2\frac{\partial}{\partial T}-\kappa\left(\nabla_{1r}^2+\varepsilon^2\nabla_{1R}^2+2\varepsilon\nabla_{1r}\cdot\nabla_{1R}+\frac{\partial^2}{\partial z^2}\right)\right]$$

$$\cdot\left[\tilde{\nabla}\times\tilde{\nabla}\times\{(\varepsilon\Omega_0+\varepsilon^2\Omega_1)\times(\varepsilon u_0+\varepsilon^2u_1)\}\cdot\hat{z}\right]$$ need to pull out an ε to cancel w/ LHS

$$\hookrightarrow\left[\tilde{\nabla}\times\tilde{\nabla}\times\{(\varepsilon\Omega_0\times\varepsilon u_0)+(\varepsilon\Omega_0\times\varepsilon^2u_1)+(\varepsilon^2\Omega_1\times\varepsilon u_0)+(\varepsilon^2\Omega_1\times\varepsilon^2u_1)\}\right]$$

$$\varepsilon^{\cancel{2}}\left[\tilde{\nabla}\times\tilde{\nabla}\times\{(\Omega_0\times u_0)+(\Omega_0\times\varepsilon u_1)+(\Omega_1\times\varepsilon u_0)+(\varepsilon^2\Omega_1\times u_1)\}\right]$$

cancel one ε with LHS.

$(\longrightarrow)$

Now RHS reads:

$$-\alpha g \varepsilon (\nabla_{1r}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R}) \{ u_0 \cdot \nabla T_0 + \varepsilon (u_1 \cdot \nabla T_0 + u_0 \cdot \nabla T_1) \}$$

$$+ \varepsilon \left\{ \left[\frac{\partial}{\partial t} + \cancel{\varepsilon^2 \frac{\partial}{\partial T}}^{0} - \kappa \left(\nabla_{1r}^2 + \cancel{\varepsilon^2 \nabla_{1R}^2}^{0} + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2} \right) \right] \right\}$$

($\mathcal{O} > 3$ terms drop)

$$\cdot \left[\tilde{\nabla} \times \tilde{\nabla} \times \{ (\vec{\Omega}_0 \times \vec{u}_0) + \varepsilon (\vec{\Omega}_0 \times \vec{u}_1) + \varepsilon (\vec{\Omega}_1 \times \vec{u}_0) + \cancel{\varepsilon^2 (\Omega_1 \times u_1)}^{0} \right] \hat{z}$$

So our final equation reads:

$$(\mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2)(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2)$$

$$= -\alpha g \varepsilon (\nabla_{1r}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R}) \{ \vec{u}_0 \cdot \nabla T_0 + \varepsilon (\vec{u}_1 \cdot \nabla T_0 + \vec{u}_0 \cdot \nabla T_1) \}$$

$$+ \varepsilon \left\{ \left[\frac{\partial}{\partial t} - \kappa \left(\nabla_{1r}^2 + 2\varepsilon \nabla_{1r} \cdot \nabla_{1R} + \frac{\partial^2}{\partial z^2} \right) \right] \right\} \left[\tilde{\nabla} \times \tilde{\nabla} \times \{ (\vec{\Omega}_0 \times \vec{u}_0) \right.$$

$$\left. + \varepsilon (\vec{\Omega}_0 \times \vec{u}_1) + \varepsilon (\vec{\Omega}_1 \times \vec{u}_0) \} \cdot \hat{z} \right]$$

Now set terms in $\mathcal{O}(\varepsilon^2)$ equal:

$$\varepsilon^2 \mathcal{L}_0 \omega_2 + \varepsilon^2 \mathcal{L}_1 \omega_1 + \varepsilon^2 \mathcal{L}_2 \omega_0$$

$$= -\alpha g \varepsilon \left[(2\varepsilon \nabla_{1r} \cdot \nabla_{1R})(\vec{u}_0 \cdot \nabla T_0) + \varepsilon \nabla_{1r}^2 (u_1 \cdot \nabla T_0 + u_0 \cdot \nabla T_1) \right]$$

$$+ \varepsilon \left[\left(\frac{\partial}{\partial t} - \kappa \left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2} \right) \right) \cdot \left[\tilde{\nabla} \times \tilde{\nabla} \times \{ \varepsilon (\vec{\Omega}_0 \times \vec{u}_1) + \varepsilon (\vec{\Omega}_1 \times \vec{u}_0) \} \cdot \hat{z} \right] \right.$$

$$- \varepsilon^2 \kappa 2 \nabla_{1r} \cdot \nabla_{1R} \left[\tilde{\nabla} \times \tilde{\nabla} \times (\vec{\Omega}_0 \times \vec{u}_0) \right] \cdot \hat{z}$$

$$\mathcal{L}_0 \omega_2 + \mathcal{L}_1 \omega_1 + \mathcal{L}_2 \omega_0$$

$$= -\alpha g \left[(2\nabla_{1r} \cdot \nabla_{1R})(u_0 \cdot \nabla T_0) + \nabla_{1r}^2 (u_1 \cdot \nabla T_0 + u_0 \cdot \nabla T_1) \right]$$

$$+ \left(\frac{\partial}{\partial t} - \kappa \left(\nabla_{1r}^2 + \frac{\partial^2}{\partial z^2} \right) \right) \cdot \left[\tilde{\nabla} \times \tilde{\nabla} \times \{ (\vec{\Omega}_0 \times \vec{u}_1) + (\vec{\Omega}_1 \times \vec{u}_0) \} \cdot \hat{z} \right]$$

$$- \kappa 2 \nabla_{1r} \cdot \nabla_{1R} \left[\tilde{\nabla} \times \tilde{\nabla} \times (\vec{\Omega}_0 \times \vec{u}_0) \right] \cdot \hat{z}$$

From soln. to $\mathcal{O}(\varepsilon)$ balance: $\omega_1 = u_1 = v_1 = 0$

$$T_1 = -\frac{\beta_0 d^3}{2\pi \kappa^2 (\pi^2 + k^2 d^2)} W W^* \sin\left(\frac{2\pi z}{d}\right)$$

And from modified 'neutral' solutions:

$$T_0 = \frac{\beta_0 d^2}{\kappa (\pi^2 + k^2 d^2)} (W e^{ik \cdot x} + W^* e^{-ik \cdot x}) \sin\left(\frac{\pi z}{d}\right)$$

$$u_0 = \frac{i k_x \pi}{k^2 d} (W e^{ik \cdot x} - W^* e^{-ik \cdot x}) \cos\left(\frac{\pi z}{d}\right)$$