1 Base Flow

In a viscous, Taylor-Couette device, there is only one base state that satisfies no-slip boundary conditions and the Navier-Stokes equations. This is called Couette flow, and can be written as

$$\Omega(r) = a + \frac{b}{r^2}, \tag{1}$$

where $a = (\Omega_2 r_2^2 - \Omega_1 r_1^2)/(r_2^2 - r_1^2)$ and $b = r_1^2 r_2^2(\Omega_1 - \Omega_2)/(r_2^2 - r_1^2)$. Here, Ω_1 is the rotation rate at the inner cylinder radius r_1, and likewise for the outer cylinder r_2.

Keplerian flow is $\Omega \propto r^{-3/2}$; these appear to be incompatible. However, in Couette flow, we can approximate Keplerian flow by creating what is called "quasi-Keplerian" flow.

Defining $q \equiv -d\ln\Omega/d\ln r$, we can see that for Couette flow,

$$q \equiv -\frac{d\ln\Omega}{d\ln r} = \frac{2b}{ar^2 + b}. \tag{2}$$

Thus, assuming r_1 and r_2 are fixed by the experiment, through judicious choices of Ω_1 and Ω_2, we can make $q(r_0) = 3/2$, where r_0 is some reference radius. Goodman & Ji do exactly this with $\bar{\zeta}$.

So, we have to choose $\Omega(r) = a + b/r^2$; everywhere it appears in our equations.

2 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi\ \hat{\mathbf{r}} \\ u_\phi\ \hat{\boldsymbol{\phi}} \\ -\frac{1}{r}\partial_r\psi\ \hat{\mathbf{z}} \end{bmatrix}; \tag{3}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) &= \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \\ &+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r} \\ &+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \end{aligned} \tag{4}$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \nu \left(\nabla^2 u_\phi - \frac{u_\phi}{r} \right) \tag{5}$$

$$\partial_t A = \frac{1}{r} J(A, \psi) + \frac{1}{\mathrm{Rm}} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \tag{6}$$

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ + \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left(\nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \end{aligned} \tag{7}$$

3 Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\mathrm{Re}} \left[\nabla^2 u \right]_z \tag{8}$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\begin{aligned} \frac{1}{r^2}\partial_t \partial_r \Psi - \frac{1}{r}\partial_t \partial_r^2 \Psi - \frac{3}{r^4}\partial_z \Psi \partial_r \Psi + \frac{1}{r^3}\partial_r \left(\partial_z \Psi \partial_r \Psi \right) + \frac{2}{r^3}\partial_z \Psi \partial_r^2 \Psi - \frac{1}{r^2}\partial_r \left(\partial_z \Psi \partial_r^2 \Psi \right) \\ - \frac{2}{r^3}\partial_r \Psi \partial_r \partial_z \Psi + \frac{1}{r^2}\partial_r \left(\partial_r \Psi \partial_r \partial_z \Psi \right) = \\ \partial_r \left(B_\phi \partial_z B_\phi \right) + \frac{2}{r^3}\partial_z^2 A \partial_z A - \frac{1}{r^2}\partial_r \left(\partial_z^2 A \partial_z A \right) + \frac{3}{r^4}\partial_z A \partial_r A - \frac{1}{r^3}\partial_r \left(\partial_z A \partial_r A \right) - \frac{2}{r^3}\partial_z A \partial_r^2 A \\ + \frac{1}{r^2}\partial_r \left(\partial_z A \partial_r^2 A \right) + \frac{1}{\mathrm{Re}} \left[\frac{3}{r^4}\partial_r \Psi - \frac{3}{r^3}\partial_r^2 \Psi + \frac{2}{r^2}\partial_r^3 \Psi - \frac{1}{r}\partial_r^4 \Psi \right] \end{aligned} \tag{9}$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r + [u \cdot \nabla u]_r = [(\nabla \times B) \times B]_r + \frac{1}{\mathrm{Re}} \left[\nabla^2 u \right]_r \tag{10}$$

and take ∂_z of the expanded equation to obtain

$$\begin{aligned} \frac{1}{r}\partial_t \partial_z^2 \Psi - \frac{1}{r^3}\partial_z \left(\partial_z \Psi \partial_z \Psi \right) + \frac{1}{r^2}\partial_z \left(\partial_z \Psi \partial_z \partial_r \Psi \right) - \frac{1}{r^2}\partial_z \left(\partial_r \Psi \partial_z^2 \Psi \right) - \frac{1}{r} 2 u_\phi \partial_z u_\phi \\ = -\frac{1}{r^2}\partial_z^3 A \partial_r A - \frac{1}{r^2}\partial_z^2 A \partial_r \partial_z A + \frac{2}{r^3}\partial_r \partial_z A \partial_r A - \frac{1}{r^2}\partial_r^2 \partial_z A \partial_r A - \frac{1}{r^2}\partial_r^2 A \partial_r \partial_z A \\ + \frac{1}{\mathrm{Re}} \left[-\frac{1}{r^2}\partial_z^2 \partial_r \Psi + \frac{1}{r}\partial_z^2 \partial_r^2 \Psi + \frac{1}{r}\partial_z^4 \Psi \right] \end{aligned} \tag{11}$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi \tag{12}$$

Note that the relevant quantity appears to be $\nabla^2\Psi - \frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \tag{13}$$

Full Ψ equation according to Susan:

$$\begin{aligned}\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) &+ J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi \\ &= J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \\ &\quad + \frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right]\end{aligned} \tag{14}$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

The derivation of the non-viscous term on the righthand side of the momentum equation ($\mathbf{J}\times\mathbf{B}$) is as follows.

$$\partial_z\left([(\nabla\times B)\times B]_r\right) - \partial_r\left([(\nabla\times B)\times B]_z\right) \tag{15}$$

$$= \partial_z\left(\left[\left(\partial_z B_r - \partial_r B_z\right)B_z - \left(\frac{1}{r}\partial_r\left(rB_\phi\right)\right)B_\phi\right]\right) - \partial_r\left(\left[\left(-\partial_z B_\phi\right)B_\phi - \left(\partial_z B_r - \partial_r B_z\right)B_r\right]\right) \tag{16}$$

$$\begin{aligned}= &-\frac{1}{r^2}\partial_z^3 A\partial_r A + \frac{1}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{2}{r^3}\partial_z^2 A\partial_z A \\ &+ \frac{1}{r^2}\partial_z^2\partial_r A\partial_z A + \frac{3}{r^4}\partial_r A\partial_z A - \frac{3}{r^3}\partial_r^2 A\partial_z A + \frac{1}{r^2}\partial_r^3 A\partial_z A - \frac{2}{r}B_\phi\partial_z B_\phi\end{aligned} \tag{17}$$

This simplifies to

$$J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \tag{18}$$

Full derivation of viscous term:

$$\partial_z \left(\frac{1}{\mathrm{Re}} \left[\nabla^2 u \right]_r \right) - \partial_r \left(\frac{1}{\mathrm{Re}} \left[\nabla^2 u \right]_z \right) \tag{19}$$

$$= \frac{1}{\mathrm{Re}} \left[\partial_z \left(\nabla^2 u_r - \frac{1}{r^2} u_r \right) - \partial_r \left(\nabla^2 u_z \right) \right] \tag{20}$$

$$= \frac{1}{\mathrm{Re}} \left[-\frac{2}{r^2} \partial_z^2 \partial_r \Psi + \frac{2}{r} \partial_z^2 \partial_r^2 \Psi + \frac{1}{r} \partial_z^4 \Psi - \frac{3}{r^4} \partial_r \Psi + \frac{3}{r^3} \partial_r^2 \Psi - \frac{2}{r^2} \partial_r^3 \Psi + \frac{1}{r} \partial_r^4 \Psi \right] \tag{21}$$

4 Recovery of Narrow Gap Equations

In order to recover the narrow gap, we take the transformation from dimensional, cylindrical coordinates $(\widetilde{r}, \widetilde{z})$ to dimensioneless quantities (x, z). In order to do so, we use $\widetilde{r} = r_0(1 + \delta x)$, where $\delta = (r_{out} - r_{in})/r_0 \to 0$. Then, we recall that we must change all derivatives $\partial_{\widetilde{r}} \psi = \partial_x \psi \partial_{\widetilde{r}} x$ and likewise for $\partial_{\widetilde{z}}$. In order that the x and z parts of the equations come in at the same order, we must choose $\widetilde{z} = r_0 \delta z$. Then, $\partial_{\widetilde{r}} x = \delta^{-1}$ and $\partial_{\widetilde{z}} z = \delta^{-1}$. When you finally expand out all terms, you should see that everything that is not ∂_x^2 and ∂_z^2 will be at higher order in δ, and thus will go to zero. For example, just taking the scalar Laplacian of B_ϕ,

$$\nabla^2 B_\phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} \tag{22}$$

gives

$$\frac{1}{(1 + \delta x)} \delta^{-2} \partial_x \left[(1 + \delta x) \partial_x B_\psi \right] + \delta^{-2} \partial_z^2 B_\phi, \tag{23}$$

and using the binomial expansion,

$$(1 - \delta x) \delta^{-2} \left(\partial_x^2 B_\psi + \delta \partial_x B_\psi + \delta x \partial_x^2 B_\psi \right) + \delta^{-2} \partial_z^2 B_\phi. \tag{24}$$

When $\delta \to 0$, this leaves

$$\partial_x^2 B_\phi + \partial_z^2 B_\phi, \tag{25}$$

as expected. Of course, we'd also do $B_\phi \to B_y$ at the same time.

Another operator we need is the curl $\nabla \times \mathbf{A}$,

$$\nabla \times \mathbf{A} = -\partial_{\widetilde{z}} A_\phi \hat{\mathbf{r}} + \left(\partial_{\widetilde{z}} A_r - \partial_{\widetilde{r}} A_z \right) \hat{\phi} + \frac{1}{\widetilde{r}} \partial_{\widetilde{r}} \left(\widetilde{r} A_\phi \right) \hat{\mathbf{z}} \tag{26}$$

$$= -\frac{1}{\delta} \partial_z A_\phi \hat{\mathbf{r}} + \left(\frac{1}{\delta} \partial_z A_r - \frac{1}{\delta} \partial_x A_z \right) \hat{\phi} + (1 - \delta x) \frac{1}{\delta} \left(\partial_x A_\phi + \delta A_\phi + \delta x \partial_x A_\phi \right) \hat{\mathbf{z}} \tag{27}$$

Taking $\lim\limits_{\delta\to 0}$ of the above and changing variables $\phi \to y$, etc., we get

$$\nabla \times \mathbf{A} = -\partial_z A_y \hat{\mathbf{x}} + (\partial_z A_x - \partial_x A_z)\, \hat{\mathbf{y}} + \partial_x A_y \hat{\mathbf{z}} \tag{28}$$

The u_ϕ equation (Equation 5) reduces as follows:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \nu \left(\nabla^2 u_\phi - \frac{u_\phi}{r} \right) \tag{29}$$

$$\begin{aligned} &\delta^{-2}\partial_t u_\phi + (1-\delta r)\delta^{-2} J(\Psi, u_\phi) + (1-2\delta r)\delta^{-1} u_\phi \partial_z \Psi = \\ &(1-\delta r)\delta^{-2} J(A, B_\phi) + (1-2\delta r)\delta^{-1} B_\phi \partial_z A + \\ &\nu \left[(1-\delta x)\delta^{-2} \left(\partial_x^2 u_\phi + \delta \partial_x u_\phi + \delta x \partial_x^2 u_\phi \right) + \delta^{-2}\partial_z^2 u_\phi \right] - \nu(1-\delta r) u_\phi \end{aligned} \tag{30}$$

$$\partial_t u_\phi + J(\Psi, u_\phi) = J(A, B_\phi) + \nu(\partial_x^2 u_\phi + \partial_z^2 u_\phi) \tag{31}$$

The A equation reduces as follows. We also choose $\widetilde{t} = \delta^2 \Omega_0 t$ so that time comes in at the same order as the other dimensions...

$$\partial_t A = \frac{1}{r} J(A, \Psi) + \frac{1}{\mathrm{Rm}} \left(\nabla^2 A - \frac{2}{r} \partial_r A \right) \tag{32}$$

$$\delta^{-2}\partial_t A = (1-\delta x)\delta^{-2} J(A, \Psi) + \frac{1}{\mathrm{Rm}} \left[(1-\delta x)\delta^{-2} \left(\partial_x^2 A + \delta \partial_x A + \delta x \partial_x^2 A \right) + \delta^{-2} \partial_z^2 A - \frac{2}{\delta}(1-\delta x)\partial_x A \right] \tag{33}$$

This reduces to

$$\partial_t A = J(A, \Psi) + \frac{1}{\mathrm{Rm}} \nabla^2 A, \tag{34}$$

which is the (unperturbed) thin-gap equation, when we drop all terms that are higher order than δ^{-2}. Note that this means we have to drop terms of $\mathcal{O}(\delta^{-1})$, which means that this formalism cannot be applied to the perturbed equations (see Equation 57), because the term $B_0 \partial_z A$ would drop out, but shouldn't. (??)

The B_ϕ equation

$$\begin{aligned} \partial_t B_\phi = \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ + \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left(\nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \end{aligned} \tag{35}$$

$$\begin{aligned} \delta^{-2}\partial_t B_\phi = {} & (1-\delta r)\delta^{-2} J(A, u_\phi) + (1-\delta r)\delta^{-2} J(B_\phi, \Psi) \\ & + (1-2\delta r)\delta^{-1} B_\phi \partial_z \Psi - (1-2\delta r)\delta^{-1} u_\phi \partial_z A + \\ & \eta \left[(1-\delta x)\delta^{-2} \left(\partial_x^2 B_\phi + \delta \partial_x B_\phi + \delta x \partial_x^2 B_\phi \right) + \delta^{-2} \partial_z^2 B_\phi \right] - \eta(1-2\delta r) B_\phi \end{aligned} \tag{36}$$

$$\partial_t B_\phi = J(A, u_\phi) + J(B_\phi, \Psi) + \eta(\partial_x^2 B_\phi + \partial_z^2 B_\phi) \tag{37}$$

The Ψ equation (Equation 14) reduces as follows

$$\begin{aligned}
&(1-\delta x)\delta^{-2}\left[(1-\delta x)\delta^{-2}\left(\partial_x^2\Psi + \delta\partial_x\Psi + \delta x\partial_x^2\Psi\right) + \delta^{-2}\partial_z^2\Psi - 2(1-\delta x)\delta^{-1}\partial_x\Psi\right]\\
&+\delta^{-2}J(\Psi,(1-2\delta x)\left((1-\delta x)\delta^{-2}\left(\partial_x^2\Psi + \delta\partial_x\Psi + \delta x\partial_x^2\Psi\right) + \delta^{-2}\partial_z^2\Psi - 2(1-\delta x)\delta^{-1}\partial_x\Psi\right))\\
&-(1-\delta x)2\delta^{-1}u_\phi\partial_z u_\phi\\
&=\delta^{-2}J(A,(1-2\delta)\left((1-\delta x)\delta^{-2}\left(\partial_x^2\Psi + \delta\partial_x\Psi + \delta x\partial_x^2\Psi\right) + \delta^{-2}\partial_z^2\Psi - 2(1-\delta x)\delta^{-1}A\right))\\
&-2(1-\delta x)\delta^{-1}B_\phi\partial_z B_\phi + \frac{1}{\mathrm{Re}}\left[\delta^{-2}(\partial_x^2 + \delta\partial_x + \delta x\partial_x^2 + \partial_z^2)((1-\delta x)(\partial_x^2\Psi + \delta\partial_x\Psi + \delta x\partial_x^2\Psi + \partial_z^2\Psi))\right]\\
&+\frac{1}{\mathrm{Re}}\left[(1-3\delta x)\delta^{-2}\partial_x^2\Psi - (1-4\delta x)\delta^{-1}\partial_x\Psi\right]
\end{aligned} \tag{38}$$

The viscous term is

$$\frac{1}{\mathrm{Re}}(1- \tag{39}$$

5 Nondimensionalization

5.1 Controlled gap width nondimensionalization

The idea is to parameterize the gap width rather than simply nondimensionalizing lengths by a point in the center of the channel r_0, as we did below.

There are many ways to do this but one option is to define

$$\widetilde{r} = r_0 + \widetilde{x} \tag{40}$$

Where tildes indicate dimensional quantities. Then for $\widetilde{x}$ to stay Cartesian, we need

$$\widetilde{r} - r_0 \ll r_2 - r_1 = d \tag{41}$$

where $r_2 = r_1 = d$ is the gap width. Also define $\widetilde{x} \ll d$ and $d/r_0 << 1$ for the small gap approximation. Thus when we nondimensionalize,

$$r = \frac{\widetilde{r}}{r_0} = 1 + \frac{\widetilde{x}}{r_0} \tag{42}$$

Note that we could control the gap width in terms of a small parameter ϵx instead of a fraction $\frac{x}{r_0}$, but I believe both should work the same way.

In the example Jeff worked out before, we looked at the base velocity profile in the thin gap limit. The rotation profile is given by $\Omega(r) = \Omega_0 r^{-q}$ in dimensionless quantities, and the background flow $u = r\Omega(r)$, so in our new notation:

$$u = \left(1 + \frac{\widetilde{x}}{r_0}\right) \Omega_0 \left(1 + \frac{\widetilde{x}}{r_0}\right)^{-q} = \Omega_0 \left(1 + \frac{\widetilde{x}}{r_0}\right)^{1-q} \tag{43}$$

Taylor expanding the exp(1-q) quantity, we arrive at

$$u = \Omega_0 \left(1 + \frac{\widetilde{x}}{r_0}\right) - \Omega_0 q \frac{\widetilde{x}}{r_0} \tag{44}$$

Then if we define this with respect to a rest frame $\Omega_0 \left(1 + \frac{\widetilde{x}}{r_0}\right)$ and define a dimensionless $x = \widetilde{x}/r_0$, we get the Umurhan+ base flow $u = -\Omega_0 q x$.

This makes sense to me but I'm having trouble extending it to the actual equations. This implies, for example, that the term $\frac{1}{\eta}\nabla^2 B$, which in the $\hat{\phi}$ equation is

$$\left[\nabla^2 B\right]_\phi = \nabla^2 B_\phi - \frac{B_\phi}{r^2}, \tag{45}$$

should become

$$\nabla^2 B_\phi - B_\phi \left(1 - \frac{2\widetilde{x}}{r_0}\right) \tag{46}$$

after substitution and Taylor expansion of the r^{-2}. But clearly there's no $-B_\phi \left(1 - \frac{2\widetilde{x}}{r_0}\right)$ in the thin gap equation. What am I missing?? Is this not the right place to be substituting in the new definition of r??

5.2 Typical nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$\begin{aligned} \widetilde{u} &= \Omega_0 r_0 \delta u \\ \widetilde{x} &= \delta r_0 x \\ \widetilde{\nabla} &= \frac{\nabla}{\delta r_0} \\ \widetilde{t} &= \frac{t}{\Omega_0} \\ \widetilde{B} &= B_0 B \\ \widetilde{P} &= P_0 P \\ \widetilde{\rho} &= \rho_0 \rho \end{aligned} \tag{47}$$

If we define the velocity scale v_0 as the local sound speed scale $\left(c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}}\right)$, then $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$.

Nondimensionalizing the momentum equation and dividing by a factor of $\Omega_0^2 r_0 \delta$ yields

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\rho_0 \Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - 2\mathbf{\Omega} \times \mathbf{u} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \quad (48)$$

If we define

$$\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \quad (49)$$

where $P_{mag} = \frac{B_0^2}{8\pi}$, then the factor in front of $(\nabla \times \mathbf{B}) \times \mathbf{B}$ should be $\frac{2}{\beta}$. Agreed?

6 Perturbed Equations

We perturb the wide gap equations according to

$$\mathbf{B} = B_0 \hat{z} + \mathbf{B_1}, \quad (50)$$

which is the same perturbation we used in the thin-gap construction. But wait! We perturbed the thin-gap equations according to

$$\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u_1} \quad (51)$$

but I don't think this is valid in the wide-gap case. To be sure, I'm going to perturb instead by

$$\mathbf{u} = r\Omega(r)\hat{\phi} + \mathbf{u_1} \quad (52)$$

where $\Omega(r) = \Omega_0 \left(\frac{r}{r_0}\right)^{-q} = \Omega_0 \left(r\right)^{-q}$ in dimensionless coordinates (but keeping Ω_0 to flag a rotational term...) Agreed?

We also add the Coriolis and centrifugal terms $-2\mathbf{\Omega} \times \mathbf{u} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ which expand as follows:

$$-2\Omega(r)\hat{\mathbf{z}} \times \left(\mathbf{u_1} + r\Omega(r)\hat{\phi}\right) = -2\Omega(r) u_r \hat{\phi} + 2\Omega(r) u_\phi \hat{\mathbf{r}} + 2r\Omega(r)^2 \hat{\mathbf{r}} \quad (53)$$

$$-\Omega(r)\hat{\mathbf{z}} \times (\Omega(r)\hat{\mathbf{z}} \times r\hat{\mathbf{r}}) = +\Omega(r)^2 r \hat{\mathbf{r}} \quad (54)$$

The $\hat{\mathbf{r}}$ terms will have ∂_z applied to them, which will destroy the $3\Omega(r)^2 r \hat{\mathbf{r}}$ term and so the righthand side of the Ψ equation will ultimately gain only the term $2\Omega(r)\partial_z u_\phi \hat{\mathbf{r}}$. The $\hat{\phi}$ equation gains the term $-\frac{2}{r}\Omega(r)\partial_z \Psi$ on the righthand side.

The Ψ equation becomes

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)+J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{1}{r}2u_\phi\partial_z u_\phi-\partial_z\left(u_\phi\Omega(r)\right)$$
$$=\frac{2}{\beta}J\left(A,\frac{1}{r^2}\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\right)-\frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi+2\Omega(r)\partial_z u_\phi$$
$$+\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right)-\frac{1}{r^3}\partial_r^2\Psi-\frac{1}{r^4}\partial_r\Psi\right]+\frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\quad(55)$$

Note that one of the terms gained from the base state in the $(\mathbf{u}\cdot\nabla)\mathbf{u}$ term can be combined with the term gained from the Coriolis term, but here I write them separately so we can check the equation. Equation 5 becomes

$$\partial_t u_\phi+\frac{J(\psi,u_\phi)}{r}+\frac{u_\phi\partial_z\psi}{r^2}+\partial_z\Psi\left(\frac{2}{r}\Omega(r)+\partial_r\Omega(r)\right)=$$
$$\frac{2}{\beta}\frac{J(A,B_\phi)}{r}+\frac{2}{\beta}\frac{B_\phi\partial_z A}{r^2}+\frac{1}{\mathrm{Re}}\left(\nabla^2 u_\phi-\frac{u_\phi}{r}\right)+\frac{2}{\beta}B_0\partial_z B_\phi-\frac{2}{r}\Omega(r)\partial_z\Psi\quad(56)$$

Equation 6 picks up $+\Omega(r)B_\phi$ from the term $-(u_0\hat{\phi}\cdot\nabla)\mathbf{B_1}$ and $-\Omega(r)B_\phi$ from the term $+(\mathbf{B_1}\cdot\nabla)\hat{\phi}$ and, so those cancel and the equation becomes

$$\partial_t A=\frac{1}{r}J(A,\psi)+\frac{1}{\mathrm{Rm}}\left(\nabla^2 A-\frac{2\partial_r A}{r}\right)+B_0\partial_z\Psi\quad(57)$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The $\hat{\phi}$ component of the induction equation, Equation 7, becomes

$$\partial_t B_\phi=\frac{1}{r}J(A,u_\phi)+\frac{1}{r}J(B_\phi,\psi)$$
$$+\frac{1}{r^2}B_\phi\partial_z\psi-\frac{1}{r^2}u_\phi\partial_z A+\frac{1}{\mathrm{Rm}}\left(\nabla^2 B_\phi-\frac{1}{r^2}B_\phi\right)+B_0\partial_z u_\phi+\partial_z A\left(\frac{2}{r}\Omega(r)+\partial_r\Omega(r)\right)\quad(58)$$

7 Matrix Formulation

This is all pending a rigorous check of Equations 55 - 58, which Jeff is working on, but I'll start putting this into a matrix construction.

First, the nonlinear vector, on the lefthand side of the equation, is

$$\mathbf{N}=\begin{bmatrix} J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi-\frac{2}{r}\partial_r\Psi\right)\right)-\frac{2}{\beta}J\left(A,\frac{1}{r^2}\left(\nabla^2 A-\frac{2}{r}\partial_r A\right)\right)-\frac{2}{r}u_\phi\partial_z u_\phi+\frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi \\ \frac{1}{r}J\left(\Psi,u_\phi\right)-\frac{1}{r}\frac{2}{\beta}J\left(A,B_\phi\right)+\frac{1}{r^2}u_\phi\partial_z\Psi-\frac{2}{\beta}\frac{1}{r^2}B_\phi\partial_z A \\ -\frac{1}{r}J\left(A,\Psi\right) \\ -\frac{1}{r}J\left(A,u_\phi\right)-\frac{1}{r}J\left(B_\phi,\Psi\right)-\frac{1}{r^2}B_\phi\partial_z\Psi+\frac{1}{r^2}u_\phi\partial_z A \end{bmatrix}\quad(59)$$

Note that these differ from the thin-gap nonlinear terms not only because of the curvature terms in the Jacobians, but also because of the additional advective terms in all but the **A** equation.

The ∂_t terms are grouped together into

$$\partial_t D = \partial_t \begin{bmatrix} \frac{1}{r}\nabla^2 - \frac{2}{r^2}\partial_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{60}$$

For the righthand side of the equation, I'll follow the same process that I did with the thin-gap. That is I'll first separate the matrices by physical meaning, and then regroup terms based on z-dependence. The RHS can be described by three matrices: one representing the rotational terms, one representing the background field terms, and one representing the viscous/resistive terms.

$$\begin{aligned} RHS = & \begin{bmatrix} 3\Omega(r)\partial_z u_\phi \\ -\frac{4}{r}\Omega(r)\partial_z\Psi - \partial_r\Omega(r)\partial_z\Psi \\ 0 \\ \frac{2}{r}\Omega(r)\partial_z A + \partial_r\Omega(r)\partial_z A \end{bmatrix} + \begin{bmatrix} \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \\ \frac{2}{\beta}B_0\partial_z B_\phi \\ B_0\partial_z\Psi \\ B_0\partial_z u_\phi \end{bmatrix} + \begin{bmatrix} \frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \\ \frac{1}{\mathrm{Re}}\left(\nabla^2 u_\phi - \frac{1}{r}u_\phi\right) \\ \frac{1}{\mathrm{Rm}}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \\ \frac{1}{\mathrm{Rm}}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \end{bmatrix} \\ = & \begin{bmatrix} 0 & 3\Omega(r)\partial_z & 0 & 0 \\ -\frac{4}{r}\Omega(r)\partial_z - \partial_r\Omega(r)\partial_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{r}\Omega(r)\partial_z + \partial_r\Omega(r)\partial_z & 0 \end{bmatrix} + \\ & \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 - \frac{2}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta}B_0\partial_z \\ B_0\partial_z & 0 & 0 & 0 \\ 0 & B_0\partial_z & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\mathrm{Re}}\left(\nabla^2\left(\frac{1}{r}\nabla^2\right) - \frac{1}{r^3}\partial_r^2 - \frac{1}{r^4}\partial_r\right) \\ \frac{1}{\mathrm{Re}}\left(\nabla^2 - \frac{1}{r}\right) \\ \frac{1}{\mathrm{Rm}}\left(\nabla^2 - \frac{2}{r}\partial_r\right) \\ \frac{1}{\mathrm{Rm}}\left(\nabla^2 - \frac{1}{r^2}\right) \end{bmatrix} \end{aligned} \tag{61}$$

We separate the viscous/resistive matrix out in terms of ∂_z dependence.

$$\text{viscous terms} =$$
$$\begin{bmatrix} \frac{1}{\mathrm{Re}}\left(-\frac{3}{r^4}\partial_r + \frac{3}{r^3}\partial_r^2 - \frac{2}{r^2}\partial_r^3 + \frac{1}{r}\partial_r^4\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\mathrm{Re}}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{Rm}}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{Rm}}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) \end{bmatrix}$$
$$+ \partial_z^2 \begin{bmatrix} \frac{1}{\mathrm{Re}}\left(-\frac{2}{r^2}\partial_r + \frac{2}{r}\partial_r^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\mathrm{Re}} & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{Rm}} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{Rm}} \end{bmatrix}$$
$$+ \partial_z^4 \begin{bmatrix} \frac{1}{\mathrm{Re}}\frac{1}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{62}$$

Note that the form of each of the terms is different in the first matrix. We also separate the terms in the $B0$ matrix by ∂_z dependence:

$$B_0 \text{ terms} = \partial_z \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta}B_0 \\ B_0 & 0 & 0 & 0 \\ 0 & B_0 & 0 & 0 \end{bmatrix} + \partial_z^3 \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}B_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{63}$$

We replace B_0 with our lower magnetic field strength $(1-\epsilon^2)$:

$$B_0 \text{ terms} = (1-\epsilon^2)\partial_z \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + (1-\epsilon^2)\partial_z^3 \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{64}$$

We now define, in analogy to the thin-gap matrices, the following:

$$\mathcal{L}_0 = \begin{bmatrix} \frac{1}{\mathrm{Re}}\left(-\frac{3}{r^4}\partial_r + \frac{3}{r^3}\partial_r^2 - \frac{2}{r^2}\partial_r^3 + \frac{1}{r}\partial_r^4\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\mathrm{Re}}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{Rm}}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{Rm}}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) \end{bmatrix} \tag{65}$$

$$\mathcal{L}_1 = \begin{bmatrix} 0 & 3\Omega(r) & \frac{2}{\beta}\frac{1}{r}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ -\frac{4}{r}\Omega(r) - \partial_r\Omega(r) & 0 & 0 & \frac{2}{\beta} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{r}\Omega(r) + \partial_r\Omega(r) & 0 \end{bmatrix} \tag{66}$$

$$\mathcal{L}_2 = \begin{bmatrix} \frac{1}{\mathrm{Re}}\left(-\frac{2}{r^2}\partial_r + \frac{2}{r}\partial_r^2\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\mathrm{Re}} & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{Rm}} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{Rm}} \end{bmatrix} \tag{67}$$

$$\mathcal{L}_3 = \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{68}$$

$$\mathcal{L}_4 = \begin{bmatrix} \frac{1}{\mathrm{Re}}\frac{1}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{69}$$

$$\mathcal{G} = \begin{bmatrix} 0 & 0 & \frac{2}{\beta}\frac{1}{r}\left(\partial_r^2 - \frac{1}{r}\partial_r\right) & 0 \\ 0 & 0 & 0 & \frac{2}{\beta} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{70}$$

Defining

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1\partial_z + \mathcal{L}_2\partial_z^2 + \mathcal{L}_3\partial_z^3 + \mathcal{L}_4\partial_z^4 \tag{71}$$

and

$$\widetilde{\mathcal{G}} = -\mathcal{G}\partial_z - \mathcal{L}_3\partial_z^3, \tag{72}$$

we have

$$\mathcal{D}\partial_t\mathbf{V} + \mathbf{N} = \mathcal{L}\mathbf{V} + \epsilon^2\widetilde{\mathcal{G}}\mathbf{V} \tag{73}$$

which we then expand:

$$\begin{aligned} \partial_t &\to \epsilon^2\partial_T \\ \partial_z &\to \partial_z + \epsilon\partial_Z \end{aligned} \tag{74}$$

and perturb:

$$\mathbf{V} = \epsilon\mathbf{V}_1 + \epsilon^2\mathbf{V}_2 + \epsilon^3\mathbf{V}_3 \tag{75}$$

Note that as we have defined these matrices, the matrices are of course different than in the thin-gap case, and the nonlinear terms are different, but the equations for behavior at each order in ϵ are identical, that is

$$\mathcal{L}\mathbf{V}_1 = 0 \tag{76}$$

$$\mathcal{L}\mathbf{V}_2 = \mathbf{N}_2 - \widetilde{\mathcal{L}}_1\partial_Z\mathbf{V}_1 \tag{77}$$

$$\mathcal{D}\partial_T \mathbf{V}_1 + \mathbf{N}_3 = \mathcal{L}\mathbf{V}_3 + \widetilde{\mathcal{L}}_1 \partial_Z \mathbf{V}_2 + \widetilde{\mathcal{L}}_2 \partial_Z^2 \mathbf{V}_1 + \widetilde{\mathcal{G}}\mathbf{V}_1 \tag{78}$$

are still the correct equations to orders ϵ, ϵ^2, and ϵ^3, respectively.

8 Expansion of Nonlinear Terms

Expanding the terms in Equation 59, we find.

$$\mathbf{N} = \epsilon^2 \mathbf{N}_2 + \epsilon^3 \mathbf{N}_3 \tag{79}$$

$$\begin{aligned} N_2^\Psi = {} & J\left(\Psi_1, \frac{1}{r^2}\nabla^2\Psi_1\right) + J\left(\Psi_1, -\frac{2}{r^3}\partial_r\Psi_1\right) \\ & - \frac{2}{\beta}J\left(A_1, \frac{1}{r^2}\nabla^2 A_1\right) - \frac{2}{\beta}J\left(A_1, -\frac{2}{r^3}\partial_r A_1\right) - \frac{2}{r}u_1\partial_z u_1 + \frac{2}{\beta}\frac{2}{r}B_1\partial_z B_1 \end{aligned} \tag{80}$$

$$\begin{aligned} N_3^\Psi = {} & J\left(\Psi_1, \frac{1}{r^2}\nabla^2\Psi_2\right) + J\left(\Psi_2, \frac{1}{r^2}\nabla^2\Psi_1\right) + 2J\left(\Psi_1, \frac{1}{r^2}\partial_Z\partial_z\Psi_1\right) \\ & + J\left(\Psi_1, -\frac{2}{r^3}\partial_r\Psi_2\right) + J\left(\Psi_2, -\frac{2}{r^3}\partial_r\Psi_1\right) + \widetilde{J}\left(\Psi_1, \frac{1}{r^2}\nabla^2\Psi_1\right) + \widetilde{J}\left(\Psi_1, -\frac{2}{r^3}\partial_r\Psi_1\right) \\ & - \frac{2}{\beta}J\left(A_1, \frac{1}{r^2}\nabla^2 A_2\right) - \frac{2}{\beta}J\left(A_2, \frac{1}{r^2}\nabla^2 A_1\right) - \frac{4}{\beta}J\left(A_1, \frac{1}{r^2}\partial_Z\partial_z A_1\right) - \frac{2}{\beta}J\left(A_1, -\frac{2}{r^3}\partial_r A_2\right) \\ & - \frac{2}{\beta}J\left(A_2, -\frac{2}{r^3}\partial_r A_1\right) - \frac{2}{\beta}\widetilde{J}\left(A_1, \frac{1}{r^2}\nabla^2 A_1\right) - \frac{2}{\beta}\widetilde{J}\left(A_1, -\frac{2}{r^3}\partial_r A_1\right) \\ & - \frac{2}{r}u_1\partial_z u_2 - \frac{2}{r}u_2\partial_z u_1 - \frac{2}{r}u_1\partial_Z u_1 + \frac{2}{\beta}\frac{2}{r}B_1\partial_z B_2 + \frac{2}{\beta}\frac{2}{r}B_2\partial_z B_1 + \frac{2}{\beta}\frac{2}{r}B_1\partial_Z B_1 \end{aligned} \tag{81}$$

$$N_2^u = \frac{1}{r}J(\Psi_1, u_1) - \frac{1}{r}\frac{2}{\beta}J(A_1, B_1) + \frac{1}{r^2}u_1\partial_z\Psi_1 - \frac{2}{\beta}\frac{1}{r^2}B_1\partial_z A_1 \tag{82}$$

$$\begin{aligned} N_3^u = {} & \frac{1}{r}J(\Psi_1, u_2) + \frac{1}{r}J(\Psi_2, u_1) + \frac{1}{r}\widetilde{J}(\Psi_1, u_1) \\ & - \frac{1}{r}\frac{2}{\beta}J(A_1, B_2) - \frac{1}{r}\frac{2}{\beta}J(A_2, B_1) - \frac{1}{r}\frac{2}{\beta}\widetilde{J}(A_1, B_1) \\ & + \frac{1}{r^2}u_1\partial_z\Psi_2 + \frac{1}{r^2}u_2\partial_z\Psi_1 + \frac{1}{r^2}u_1\partial_Z\Psi_1 \\ & - \frac{2}{\beta}\frac{1}{r^2}B_1\partial_z A_2 - \frac{2}{\beta}\frac{1}{r^2}B_2\partial_z A_1 - \frac{2}{\beta}\frac{1}{r^2}B_1\partial_Z A_1 \end{aligned} \tag{83}$$

$$N_2^A = -\frac{1}{r} J\left(A_1, \Psi_1\right) \tag{84}$$

$$N_3^A = -\frac{1}{r} J\left(A_1, \Psi_2\right) - \frac{1}{r} J\left(A_2, \Psi_1\right) - \frac{1}{r} \widetilde{J}\left(A_1, \Psi_1\right) \tag{85}$$

$$N_2^B = -\frac{1}{r} J\left(A_1, u_1\right) - \frac{1}{r} J\left(B_1, \Psi_1\right) - \frac{1}{r^2} B_1 \partial_z \Psi_1 + \frac{1}{r^2} u_1 \partial_z A_1 \tag{86}$$

$$\begin{aligned} N_3^B = &-\frac{1}{r} J\left(A_1, u_2\right) - \frac{1}{r} J\left(A_2, u_1\right) - \frac{1}{r} \widetilde{J}\left(A_1, u_1\right) \\ &- \frac{1}{r} J\left(B_1, \Psi_2\right) - \frac{1}{r} J\left(B_2, \Psi_1\right) - \frac{1}{r} \widetilde{J}\left(B_1, u_1\right) \\ &- \frac{1}{r^2} B_1 \partial_z \Psi_2 - \frac{1}{r^2} B_2 \partial_z \Psi_1 - \frac{1}{r^2} B_1 \partial_Z \Psi_1 \\ &+ \frac{1}{r^2} u_1 \partial_z A_2 + \frac{1}{r^2} u_2 \partial_z A_1 + \frac{1}{r^2} u_1 \partial_Z A_1 \end{aligned} \tag{87}$$

9 Boundary Conditions

We use boundary conditions that are no-slip and perfectly conducting. No-slip means that all components of velocity are zero at the boundary, and perfectly conducting means that the radial component of the magnetic field and the axial component of the current are both zero at the boundary.

No-slip means $u_r = u_\phi = u_z = 0$ and therefore $u_z = -\frac{1}{r}\partial_r \Psi = 0$ and $u_r = \frac{1}{r}\partial_z \Psi = 0$.

Perfectly conducting means $J_z = (\nabla \times \mathbf{B})_z = B_\phi + r\partial_r B_\phi = 0$

and also that $B_r = \frac{1}{r}\partial_z A = 0$.

Thus our final boundary conditions are $\Psi = u = A = \partial_r \Psi = \partial_r(rB_\phi) = 0$, for a total of ten boundary condition equations.

10 Notes for paper

– Comment on channel mode formation w.r.t. parasitic modes?

Discussion:

The complex nature of the MRI means its properties are best parsed in idealized regimes or in simplified geometries.

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node257.html.

For a scalar field ψ,

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\phi} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}, \tag{88}$$

and

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{89}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \tag{90}$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \tag{91}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{92}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \tag{93}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \tag{94}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{95}$$

where ∇ on the vector components is given by equation (88).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$