

## 1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r} \partial_z \psi \hat{\mathbf{r}} \\ u_\phi \hat{\phi} \\ -\frac{1}{r} \partial_r \psi \hat{\mathbf{z}} \end{bmatrix}; \quad (1)$$

here we define  $A$  in the same way.

Using the definitions in

$$\begin{aligned} \partial_t \left[ \frac{1}{r} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] + \frac{1}{r^2} J(\psi, \nabla^2 A - \frac{2\partial_r \psi}{r}) &= \frac{\partial_z A}{r^3} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \\ &+ \frac{1}{r} J \left( A, \frac{1}{r} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \right) - \frac{2B_\phi \partial_z B_\phi}{r} \\ &+ \nu \left\{ \nabla^2 \left[ \frac{1}{r} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] - \frac{1}{r^2} \left( \nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right\} \end{aligned} \quad (2)$$

For the expanded form of the  $\Psi$  equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \nu \left( \nabla^2 u_\phi - \frac{u_\phi}{r} \right) \quad (3)$$

$$\partial_t A = \frac{1}{r} J(A, \psi) + \frac{1}{\text{Rm}} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) \quad (4)$$

$$\begin{aligned} \partial_t B_\phi &= \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ &+ \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \eta \left( \nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) \end{aligned} \quad (5)$$

## 2 Detailed Derivation of $\Psi$ Equation

The  $\Psi$  equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find  $\hat{\mathbf{r}}$  and  $\hat{\mathbf{z}}$  components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\text{Re}} [\nabla^2 u]_z \quad (6)$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take  $\partial_r$  of the resulting equation to obtain:

$$\begin{aligned}
& \frac{1}{r^2} \partial_t \partial_r \Psi - \frac{1}{r} \partial_t \partial_r^2 \Psi - \frac{3}{r^4} \partial_z \Psi \partial_r \Psi + \frac{1}{r^3} \partial_r (\partial_z \Psi \partial_r \Psi) + \frac{2}{r^3} \partial_z \Psi \partial_r^2 \Psi - \frac{1}{r^2} \partial_r (\partial_z \Psi \partial_r^2 \Psi) \\
& \quad - \frac{2}{r^3} \partial_r \Psi \partial_r \partial_z \Psi + \frac{1}{r^2} \partial_r (\partial_r \Psi \partial_r \partial_z \Psi) = \\
& \partial_r (B_\phi \partial_z B_\phi) + \frac{2}{r^3} \partial_z^2 A \partial_z A - \frac{1}{r^2} \partial_r (\partial_z^2 A \partial_z A) + \frac{3}{r^4} \partial_z A \partial_r A - \frac{1}{r^3} \partial_r (\partial_z A \partial_r A) - \frac{2}{r^3} \partial_z A \partial_r^2 A \\
& \quad + \frac{1}{r^2} \partial_r (\partial_z A \partial_r^2 A) + \frac{1}{\text{Re}} \left[ \frac{3}{r^4} \partial_r \Psi - \frac{3}{r^3} \partial_r^2 \Psi + \frac{2}{r^2} \partial_r^3 \Psi - \frac{1}{r} \partial_r^4 \Psi \right] \quad (7)
\end{aligned}$$

Repeat this process for the  $\hat{\mathbf{r}}$  component of the momentum equation,

$$\partial_t u_r + [u \cdot \nabla u]_r = [(\nabla \times B) \times B]_r + \frac{1}{\text{Re}} [\nabla^2 u]_r \quad (8)$$

and take  $\partial_z$  of the expanded equation to obtain

$$\begin{aligned}
& \frac{1}{r} \partial_t \partial_z^2 \Psi - \frac{1}{r^3} \partial_z (\partial_z \Psi \partial_z \Psi) + \frac{1}{r^2} \partial_z (\partial_z \Psi \partial_z \partial_r \Psi) - \frac{1}{r^2} \partial_z (\partial_r \Psi \partial_z^2 \Psi) - \frac{1}{r} 2u_\phi \partial_z u_\phi \\
& = -\frac{1}{r^2} \partial_z^3 A \partial_r A - \frac{1}{r^2} \partial_z^2 A \partial_r \partial_z A + \frac{2}{r^3} \partial_r \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 A \partial_r \partial_z A \\
& \quad + \frac{1}{\text{Re}} \left[ -\frac{1}{r^2} \partial_z^2 \partial_r \Psi + \frac{1}{r} \partial_z^2 \partial_r^2 \Psi + \frac{1}{r} \partial_z^4 \Psi \right] \quad (9)
\end{aligned}$$

It is clear from the  $\partial_t$  terms that we must combine these equations by subtracting the  $\hat{\mathbf{z}}$  equation from the  $\hat{\mathbf{r}}$  equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r} \partial_t \left( \nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) + J \left( \Psi, \frac{1}{r^2} \left( \nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) \right) - \frac{1}{r} 2u_\phi \partial_z u_\phi \quad (10)$$

Note that the relevant quantity appears to be  $\nabla^2 \Psi - \frac{2}{r} \partial_r \Psi$ , and that the  $\frac{1}{r^2}$  in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no  $u_\phi$  term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\text{Re}} \left[ \nabla^2 \left( \frac{1}{r} \nabla^2 \Psi \right) - \frac{1}{r^3} \partial_r^2 \Psi - \frac{1}{r^4} \partial_r \Psi \right] \quad (11)$$

Full  $\Psi$  equation according to Susan:

$$\begin{aligned}
& \frac{1}{r} \partial_t \left( \nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) + J \left( \Psi, \frac{1}{r^2} \left( \nabla^2 \Psi - \frac{2}{r} \partial_r \Psi \right) \right) - \frac{1}{r} 2u_\phi \partial_z u_\phi \\
& = J \left( A, \frac{1}{r^2} \left( \nabla^2 A - \frac{2}{r} \partial_r A \right) \right) - \frac{2}{r} B_\phi \partial_z B_\phi \\
& \quad + \frac{1}{\text{Re}} \left[ \nabla^2 \left( \frac{1}{r} \nabla^2 \Psi \right) - \frac{1}{r^3} \partial_r^2 \Psi - \frac{1}{r^4} \partial_r \Psi \right] \quad (12)
\end{aligned}$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky....

The derivation of the non-viscous term on the righthand side of the momentum equation ( $\mathbf{J} \times \mathbf{B}$ ) is as follows.

$$\partial_z ([(\nabla \times B) \times B]_r) - \partial_r ([(\nabla \times B) \times B]_z) \quad (13)$$

$$\begin{aligned}
& = \partial_z \left( \left[ (\partial_z B_r - \partial_r B_z) B_z - \left( \frac{1}{r} \partial_r (r B_\phi) \right) B_\phi \right] \right) - \partial_r \left( [(-\partial_z B_\phi) B_\phi - (\partial_z B_r - \partial_r B_z) B_r] \right) \\
& \quad (14)
\end{aligned}$$

$$\begin{aligned}
& = -\frac{1}{r^2} \partial_z^3 A \partial_r A + \frac{1}{r^3} \partial_r \partial_z A \partial_r A - \frac{1}{r^2} \partial_r^2 \partial_z A \partial_r A - \frac{2}{r^3} \partial_z^2 A \partial_z A \\
& \quad + \frac{1}{r^2} \partial_z^2 \partial_r A \partial_z A + \frac{3}{r^4} \partial_r A \partial_z A - \frac{3}{r^3} \partial_r^2 A \partial_z A + \frac{1}{r^2} \partial_r^3 A \partial_z A - \frac{2}{r} B_\phi \partial_z B_\phi \quad (15)
\end{aligned}$$

This simplifies to

$$J \left( A, \frac{1}{r^2} \left( \nabla^2 A - \frac{2}{r} \partial_r A \right) \right) - \frac{2}{r} B_\phi \partial_z B_\phi \quad (16)$$

Full derivation of viscous term:

$$\partial_z \left( \frac{1}{\text{Re}} [\nabla^2 u]_r \right) - \partial_r \left( \frac{1}{\text{Re}} [\nabla^2 u]_z \right) \quad (17)$$

$$= \frac{1}{\text{Re}} \left[ \partial_z \left( \nabla^2 u_r - \frac{1}{r^2} u_r \right) - \partial_r \left( \nabla^2 u_z \right) \right] \quad (18)$$

$$\begin{aligned}
& = \frac{1}{\text{Re}} \left[ -\frac{2}{r^2} \partial_z^2 \partial_r \Psi + \frac{2}{r} \partial_z^2 \partial_r^2 \Psi + \frac{1}{r} \partial_z^4 \Psi - \frac{3}{r^4} \partial_r \Psi + \frac{3}{r^3} \partial_r^2 \Psi - \frac{2}{r^2} \partial_r^3 \Psi + \frac{1}{r} \partial_r^4 \Psi \right] \\
& \quad (19)
\end{aligned}$$

### 3 Recovery of Narrow Gap Equations

#### 4 Nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$\begin{aligned}
 \tilde{u} &= \Omega_0 r_0 \delta u \\
 \tilde{x} &= \delta r_0 x \\
 \tilde{\nabla} &= \frac{\nabla}{\delta r_0} \\
 \tilde{t} &= \frac{t}{\Omega_0} \\
 \tilde{B} &= B_0 B \\
 \tilde{P} &= P_0 P \\
 \tilde{\rho} &= \rho_0 \rho
 \end{aligned} \tag{20}$$

If we define the velocity scale  $v_0$  as the local sound speed scale ( $c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}}$ ), then  $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$ .

Nondimensionalizing the momentum equation and dividing by a factor of  $\Omega_0^2 r_0 \delta$  yields

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{21}$$

If we define

$$\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \tag{22}$$

where  $P_{mag} = \frac{B_0^2}{8\pi}$ , then the factor in front of  $(\nabla \times \mathbf{B}) \times \mathbf{B}$  should be  $\frac{2}{\beta}$ . Agreed?

### 5 Perturbed Equations

We perturb the wide gap equations according to

$$\mathbf{B} = B_0 \hat{z} + \mathbf{B}_1, \tag{23}$$

which is the same perturbation we used in the thin-gap construction. But wait! We perturbed the thin-gap equations according to

$$\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u}_1 \tag{24}$$

but I don't think this is valid in the wide-gap case. To be sure, I'm going to perturb instead by

$$\mathbf{u} = r\Omega(r)\hat{\phi} + \mathbf{u}_1 \quad (25)$$

where  $\Omega(r) = \Omega_0 \left(\frac{r}{r_0}\right)^{-q} = \Omega_0(r)^{-q}$  in dimensionless coordinates (but keeping  $\Omega_0$  to flag a rotational term...) Agreed?

We also add the Coriolis and centrifugal terms  $-2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$  which expand as follows:

$$-2\Omega(r)\hat{\mathbf{z}} \times (\mathbf{u}_1 + r\Omega(r)\hat{\phi}) = -2\Omega(r)u_r\hat{\phi} + 2\Omega(r)u_\phi\hat{\mathbf{r}} + 2r\Omega(r)^2\hat{\mathbf{r}} \quad (26)$$

$$-\Omega(r)\hat{\mathbf{z}} \times (\Omega(r)\hat{\mathbf{z}} \times r\hat{\mathbf{r}}) = +\Omega(r)^2r\hat{\mathbf{r}} \quad (27)$$

The  $\hat{\mathbf{r}}$  terms will have  $\partial_z$  applied to them, which will destroy the  $3\Omega(r)^2r\hat{\mathbf{r}}$  term and so the righthand side of the  $\Psi$  equation will ultimately gain only the term  $2\Omega(r)\partial_z u_\phi\hat{\mathbf{r}}$ .

The  $\Psi$  equation becomes

$$\begin{aligned} \frac{1}{r}\partial_t \left( \nabla^2 \Psi - \frac{2}{r}\partial_r \Psi \right) + J \left( \Psi, \frac{1}{r^2} \left( \nabla^2 \Psi - \frac{2}{r}\partial_r \Psi \right) \right) - \frac{1}{r}2u_\phi\partial_z u_\phi - \partial_z (u_\phi\Omega(r)) \\ = \frac{2}{\beta}J \left( A, \frac{1}{r^2} \left( \nabla^2 A - \frac{2}{r}\partial_r A \right) \right) - \frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi + 2\Omega(r)\partial_z u_\phi \\ + \frac{1}{\text{Re}} \left[ \nabla^2 \left( \frac{1}{r}\nabla^2 \Psi \right) - \frac{1}{r^3}\partial_r^2 \Psi - \frac{1}{r^4}\partial_r \Psi \right] + \frac{2}{\beta}\frac{1}{r}B_0\partial_z \left( \nabla^2 A - \frac{2}{r}\partial_r A \right) \end{aligned} \quad (28)$$

Note that one of the terms gained from the base state in the  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  term can be combined with the term gained from the Coriolis term, but here I write them separately so we can check the equation. Equation 3 becomes

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z \psi}{r^2} + \partial_z \Psi \left( \frac{2}{r}\Omega(r) + \partial_r \Omega(r) \right) = \frac{2}{\beta}\frac{J(A, B_\phi)}{r} + \frac{2}{\beta}\frac{B_\phi\partial_z A}{r^2} + \frac{1}{\text{Re}} \left( \nabla^2 u_\phi - \frac{u_\phi}{r} \right) + \frac{2}{\beta}B_0\partial_z B_\phi \quad (29)$$

Equation 4 picks up  $+\Omega(r)B_\phi$  from the term  $-(u_0\hat{\phi} \cdot \nabla)\mathbf{B}_1$  and  $-\Omega(r)B_\phi$  from the term  $+(\mathbf{B}_1 \cdot \nabla)\hat{\phi}$  and, so those cancel and the equation becomes

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\text{Rm}} \left( \nabla^2 A - \frac{2\partial_r A}{r} \right) + B_0\partial_z \Psi \quad (30)$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The  $\hat{\phi}$  component of the induction equation, Equation 5, becomes

$$\begin{aligned} \partial_t B_\phi &= \frac{1}{r} J(A, u_\phi) + \frac{1}{r} J(B_\phi, \psi) \\ &+ \frac{1}{r^2} B_\phi \partial_z \psi - \frac{1}{r^2} u_\phi \partial_z A + \frac{1}{\text{Rm}} \left( \nabla^2 B_\phi - \frac{1}{r^2} B_\phi \right) + B_0 \partial_z u_\phi + \partial_z A \left( \frac{2}{r} \Omega(r) + \partial_r \Omega(r) \right) \end{aligned} \quad (31)$$

## A Cylindrical derivatives

Everything here follows <http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl>.

For a scalar field  $\psi$ ,

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \hat{\phi} + \frac{\partial \psi}{\partial z} \hat{\mathbf{z}}. \quad (32)$$

However, for a *vector* field  $\mathbf{u}$ ,

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \quad (33)$$

and

$$\nabla \times \mathbf{u} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\phi} + \left( \frac{1}{r} \frac{\partial(r u_\phi)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right) \hat{\mathbf{z}}. \quad (34)$$

We also need the  $\phi$  component of the convective derivative  $\mathbf{u} \cdot \nabla \mathbf{u}$ ,

$$[\mathbf{u} \cdot \nabla \mathbf{u}]_\phi = \mathbf{u} \cdot \nabla u_\phi + \frac{u_r u_\phi}{r}, \quad (35)$$

and finally, the vector Laplacian,

$$(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \quad (36)$$

$$(\nabla^2 \mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \quad (37)$$

$$(\nabla^2 \mathbf{u})_z = \nabla^2 u_z, \quad (38)$$

where  $\nabla$  on the vector components is given by equation (32).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for  $\nabla^2 u_r$  and  $\nabla^2 u_z$