1   Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z\psi \ \hat{\mathbf{r}} \\ u_\phi \ \hat{\phi} \\ -\frac{1}{r}\partial_r\psi \ \hat{\mathbf{z}} \end{bmatrix};$$

(1)

here we define A in the same way.

Using the definitions in

$$\partial_t\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] + \frac{1}{r^2}J(\psi, \nabla^2 A - \frac{2\partial_r\psi}{r}) = \frac{\partial_z A}{r^3}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)$$
$$+ \frac{1}{r}J\left(A, \frac{1}{r}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right)\right) - \frac{2B_\phi\partial_z B_\phi}{r}$$
$$+ \nu\left\{\nabla^2\left[\frac{1}{r}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right] - \frac{1}{r^2}\left(\nabla^2\psi - \frac{2\partial_r\psi}{r}\right)\right\} \quad (2)$$

For the expanded form of the Ψ equation, Susan gets:

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi\partial_z A}{r^2} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) \quad (3)$$

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\text{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) \quad (4)$$

$$\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi)$$
$$+ \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \quad (5)$$

2   Detailed Derivation of Ψ Equation

The Ψ equation, governing the x- and z-components of the velocity, is particularly tricky to derive so I will write out the steps here.

1. Find $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components of the momentum equation, i.e.:

$$\partial_t u_z + [u \cdot \nabla u]_z = [(\nabla \times B) \times B]_z + \frac{1}{\text{Re}}\left[\nabla^2 u\right]_z \quad (6)$$

We sub in our stream/flux function notation and expand the operators in cylindrical coordinates. Then take ∂_r of the resulting equation to obtain:

$$\frac{1}{r^2}\partial_t\partial_r\Psi - \frac{1}{r}\partial_t\partial_r^2\Psi - \frac{3}{r^4}\partial_z\Psi\partial_r\Psi + \frac{1}{r^3}\partial_r\left(\partial_z\Psi\partial_r\Psi\right) + \frac{2}{r^3}\partial_z\Psi\partial_r^2\Psi - \frac{1}{r^2}\partial_r\left(\partial_z\Psi\partial_r^2\Psi\right)$$
$$- \frac{2}{r^3}\partial_r\Psi\partial_r\partial_z\Psi + \frac{1}{r^2}\partial_r\left(\partial_r\Psi\partial_r\partial_z\Psi\right) =$$
$$\partial_r\left(B_\phi\partial_z B_\phi\right) + \frac{2}{r^3}\partial_z^2 A\partial_z A - \frac{1}{r^2}\partial_r\left(\partial_z^2 A\partial_z A\right) + \frac{3}{r^4}\partial_z A\partial_r A - \frac{1}{r^3}\partial_r\left(\partial_z A\partial_r A\right) - \frac{2}{r^3}\partial_z A\partial_r^2 A$$
$$+ \frac{1}{r^2}\partial_r\left(\partial_z A\partial_r^2 A\right) + \frac{1}{\mathrm{Re}}\left[\frac{3}{r^4}\partial_r\Psi - \frac{3}{r^3}\partial_r^2\Psi + \frac{2}{r^2}\partial_r^3\Psi - \frac{1}{r}\partial_r^4\Psi\right] \quad (7)$$

Repeat this process for the $\hat{\mathbf{r}}$ component of the momentum equation,

$$\partial_t u_r + [u\cdot\nabla u]_r = [(\nabla\times B)\times B]_r + \frac{1}{\mathrm{Re}}\left[\nabla^2 u\right]_r \quad (8)$$

and take ∂_z of the expanded equation to obtain

$$\frac{1}{r}\partial_t\partial_z^2\Psi - \frac{1}{r^3}\partial_z\left(\partial_z\Psi\partial_z\Psi\right) + \frac{1}{r^2}\partial_z\left(\partial_z\Psi\partial_z\partial_r\Psi\right) - \frac{1}{r^2}\partial_z\left(\partial_r\Psi\partial_z^2\Psi\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$
$$= -\frac{1}{r^2}\partial_z^3 A\partial_r A - \frac{1}{r^2}\partial_z^2 A\partial_r\partial_z A + \frac{2}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2 A\partial_r\partial_z A$$
$$+ \frac{1}{\mathrm{Re}}\left[-\frac{1}{r^2}\partial_z^2\partial_r\Psi + \frac{1}{r}\partial_z^2\partial_r^2\Psi + \frac{1}{r}\partial_z^4\Psi\right] \quad (9)$$

It is clear from the ∂_t terms that we must combine these equations by subtracting the $\hat{\mathbf{z}}$ equation from the $\hat{\mathbf{r}}$ equation.

When we do, we can simplify the LHS of the equation to:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi,\frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi \quad (10)$$

Note that the relevant quantity appears to be $\nabla^2\Psi - \frac{2}{r}\partial_r\Psi$, and that the $\frac{1}{r^2}$ in the second term cannot come out of the Jacobian (a point of disagreement with Jeff's equation above). Also I'm confused why Jeff's has no u_ϕ term. The RHS of this equation is significantly more complicated.

RHS viscous term:

$$\frac{1}{\mathrm{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \quad (11)$$

Full Ψ equation according to Susan:

$$\frac{1}{r}\partial_t\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$

$$= J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi$$

$$+ \frac{1}{\text{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] \quad (12)$$

Note that this is actually beautifully symmetric. Except the viscous term which still seems clunky.....

The derivation of the non-viscous term on the righthand side of the momentum equation ($\mathbf{J} \times \mathbf{B}$) is as follows.

$$\partial_z\left([(\nabla \times B) \times B]_r\right) - \partial_r\left([(\nabla \times B) \times B]_z\right) \quad (13)$$

$$= \partial_z\left(\left[(\partial_z B_r - \partial_r B_z)B_z - \left(\frac{1}{r}\partial_r(rB_\phi)\right)B_\phi\right]\right) - \partial_r\left([(-\partial_z B_\phi)B_\phi - (\partial_z B_r - \partial_r B_z)B_r]\right)$$
$$(14)$$

$$= -\frac{1}{r^2}\partial_z^3 A\partial_r A + \frac{1}{r^3}\partial_r\partial_z A\partial_r A - \frac{1}{r^2}\partial_r^2\partial_z A\partial_r A - \frac{2}{r^3}\partial_z^2 A\partial_z A$$
$$+ \frac{1}{r^2}\partial_z^2\partial_r A\partial_z A + \frac{3}{r^4}\partial_r A\partial_z A - \frac{3}{r^3}\partial_r^2 A\partial_z A + \frac{1}{r^2}\partial_r^3 A\partial_z A - \frac{2}{r}B_\phi\partial_z B_\phi \quad (15)$$

This simplifies to

$$J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{r}B_\phi\partial_z B_\phi \quad (16)$$

Full derivation of viscous term:

$$\partial_z\left(\frac{1}{\text{Re}}\left[\nabla^2 u\right]_r\right) - \partial_r\left(\frac{1}{\text{Re}}\left[\nabla^2 u\right]_z\right) \quad (17)$$

$$= \frac{1}{\text{Re}}\left[\partial_z\left(\nabla^2 u_r - \frac{1}{r^2}u_r\right) - \partial_r\left(\nabla^2 u_z\right)\right] \quad (18)$$

$$= \frac{1}{\text{Re}}\left[-\frac{2}{r^2}\partial_z^2\partial_r\Psi + \frac{2}{r}\partial_z^2\partial_r^2\Psi + \frac{1}{r}\partial_z^4\Psi - \frac{3}{r^4}\partial_r\Psi + \frac{3}{r^3}\partial_r^2\Psi - \frac{2}{r^2}\partial_r^3\Psi + \frac{1}{r}\partial_r^4\Psi\right]$$
$$(19)$$

3   Recovery of Narrow Gap Equations

4   Nondimensionalization

The momentum equation must be nondimensionalized. Here are the definitions of all of the dimensional components:

$$
\begin{aligned}
\widetilde{u} &= \Omega_0 r_0 \delta u \\
\widetilde{x} &= \delta r_0 x \\
\widetilde{\nabla} &= \frac{\nabla}{\delta r_0} \\
\widetilde{t} &= \frac{t}{\Omega_0} \\
\widetilde{B} &= B_0 B \\
\widetilde{P} &= P_0 P \\
\widetilde{\rho} &= \rho_0 \rho
\end{aligned}
\tag{20}
$$

If we define the velocity scale v_0 as the local sound speed scale $\left(c_{s0} \equiv \sqrt{\frac{P_0}{\rho_0}} \right)$, then $\Omega_0 r_0 \delta = \sqrt{\frac{P_0}{\rho_0}}$.

Nondimensionalizing the momentum equation and dividing by a factor of $\Omega_0^2 r_0 \delta$ yields

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{4\pi\rho} \frac{B_0^2}{\rho_0 \Omega_0^2 r_0^2 \delta^2} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{21}
$$

If we define

$$
\beta \equiv \frac{P}{P_{mag}} = \frac{\rho_0 \Omega_0^2 r_0^2 8\pi}{B_0^2} \tag{22}
$$

where $P_{mag} = \frac{B_0^2}{8\pi}$, then the factor in front of $(\nabla \times \mathbf{B}) \times \mathbf{B}$ should be $\frac{2}{\beta}$. Agreed?

5   Perturbed Equations

Note: currently missing the Coriolis term in the momentum equation.

We perturb the wide gap equations according to

$$
\mathbf{B} = B_0 \hat{z} + \mathbf{B_1}, \tag{23}
$$

which is the same perturbation we used in the thin-gap construction. But wait! We perturbed the thin-gap equations according to

$$
\mathbf{u} = -q\Omega_0 r \hat{\phi} + \mathbf{u_1} \tag{24}
$$

but I don't think this is valid in the wide-gap case. To be sure, I'm going to perturb instead by

$$\mathbf{u} = r\Omega(r)\hat{\phi} + \mathbf{u_1} \tag{25}$$

where $\Omega(r) = \Omega_0 \left(\frac{r}{r_0}\right)^{-q} = \Omega_0 \left(r\right)^{-q}$ in dimensionless coordinates (but keeping Ω_0 to flag a rotational term...) Agreed?

The Ψ equation becomes

$$\frac{1}{r}\partial_t \left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right) + J\left(\Psi, \frac{1}{r^2}\left(\nabla^2\Psi - \frac{2}{r}\partial_r\Psi\right)\right) - \frac{1}{r}2u_\phi\partial_z u_\phi$$

$$= \frac{2}{\beta}J\left(A, \frac{1}{r^2}\left(\nabla^2 A - \frac{2}{r}\partial_r A\right)\right) - \frac{2}{\beta}\frac{2}{r}B_\phi\partial_z B_\phi$$

$$+ \frac{1}{\text{Re}}\left[\nabla^2\left(\frac{1}{r}\nabla^2\Psi\right) - \frac{1}{r^3}\partial_r^2\Psi - \frac{1}{r^4}\partial_r\Psi\right] + \frac{2}{\beta}\frac{1}{r}B_0\partial_z\left(\nabla^2 A - \frac{2}{r}\partial_r A\right) \tag{26}$$

Equation 3 becomes

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi\partial_z\psi}{r^2} - \frac{2}{r}q\Omega_0\partial_z\Psi = \frac{2}{\beta}\frac{J(A, B_\phi)}{r} + \frac{2}{\beta}\frac{B_\phi\partial_z A}{r^2} + \frac{1}{\text{Re}}\left(\nabla^2 u_\phi - \frac{u_\phi}{r}\right) + \frac{2}{\beta}B_0\partial_z B_\phi \tag{27}$$

The only term gained in the $(\hat{r}, \hat{z})$ component of the induction equation is $(B_0\hat{z}\cdot\nabla)\mathbf{u_1}$, and Equation 4 becomes

$$\partial_t A = \frac{1}{r}J(A, \psi) + \frac{1}{\text{Rm}}\left(\nabla^2 A - \frac{2\partial_r A}{r}\right) + B_0\partial_z\Psi \tag{28}$$

Note that this is perfectly analogous to the thin-gap version of this equation.

The $\hat{\phi}$ component of the induction equation, Equation 5, becomes

$$\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi)$$

$$+ \frac{1}{r^2}B_\phi\partial_z\psi - \frac{1}{r^2}u_\phi\partial_z A + \frac{1}{\text{Rm}}\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) + B_0\partial_z u_\phi - \frac{1}{r}q\Omega_0\partial_z A \tag{29}$$

A   Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \tag{30}$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \tag{31}$$

and

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right)\,\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\,\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \phi}\right)\,\hat{\mathbf{z}}. \tag{32}$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \tag{33}$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi} \tag{34}$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2} \tag{35}$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \tag{36}$$

where ∇ on the vector components is given by equation (30).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$