1 Basic Equations

The so-called Stokes stream function, used in axisymmetric situations, is given by

$$\mathbf{u} = \begin{bmatrix} \frac{1}{r}\partial_z \psi \ \hat{\mathbf{r}} \\ u_\phi \ \hat{\phi} \\ -\frac{1}{r}\partial_r \psi \ \hat{\mathbf{z}} \end{bmatrix}; \tag{1}$$

here we define A in the same way.

Using the definitions in

$$\begin{aligned} \partial_t \left[\frac{1}{r} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] + \frac{1}{r^2} J(\psi, \nabla^2 A - \frac{2\partial_r \psi}{r}) = \frac{\partial_z A}{r^3} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \\ + \frac{1}{r} J \left(A, \frac{1}{r} \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \right) - \frac{2B_\phi \partial_z B_\phi}{r} \\ + \nu \left\{ \nabla^2 \left[\frac{1}{r} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right] - \frac{1}{r^2} \left(\nabla^2 \psi - \frac{2\partial_r \psi}{r} \right) \right\} \end{aligned} \tag{2}$$

For the expanded form of the Ψ equation, Susan gets:

$$\begin{aligned} [\hat{\mathbf{r}}] : \partial_t \partial_z \Psi - \frac{1}{r^2} \partial_z^2 \Psi + \frac{1}{r} \partial_z^2 \partial_r \Psi - \frac{1}{r} \partial_r \partial_z^2 \Psi - u_\phi^2 = \\ - \frac{1}{r} \partial_z^2 \partial_r A + \frac{1}{r^2} \partial_z \partial_r^2 A - \frac{1}{r} \partial_z \partial_r^3 A + \frac{1}{\mathrm{Re}} \left[-\frac{1}{r} \partial_z \partial_r \Psi + \partial_z \partial_r^2 \Psi + \partial_z^3 \Psi \right] \end{aligned} \tag{3}$$

$$\begin{aligned} [\hat{\mathbf{z}}] : \partial_t \partial_r \Psi - \frac{1}{r^3} \partial_z \partial_r \Psi + \frac{1}{r^2} \partial_z \partial_r^2 \Psi - \frac{1}{r^2} \partial_r^2 \partial_z \Psi = \\ B_\phi \partial_z B_\phi + \frac{1}{r^2} \partial_z^3 A - \frac{1}{r^3} \partial_r \partial_z A + \frac{1}{r^2} \partial_r^2 \partial_z A \\ + \frac{1}{\mathrm{Re}} \left[\frac{1}{r^3} \partial_r \Psi - \frac{1}{r^2} \partial_r^2 \Psi + \frac{1}{r} \partial_r^2 \Psi + \frac{1}{r} \partial_z^2 \partial_r \Psi \right] \end{aligned} \tag{4}$$

$$\partial_t u_\phi + \frac{J(\psi, u_\phi)}{r} + \frac{u_\phi \partial_z \psi}{r^2} = \frac{J(A, B_\phi)}{r} + \frac{B_\phi \partial_z A}{r^2} + \nu \left(\nabla^2 u_\phi - \frac{u_\phi}{r} \right) \tag{5}$$

$$\partial_t A = \frac{1}{r} J(A, \psi) + \eta \left(\nabla^2 A - \frac{2\partial_r A}{r} \right) \tag{6}$$

Susan gets:

$$\partial_t A = \frac{1}{r} J(A, \Psi) + \frac{1}{Rm} \left[-\frac{1}{r} \partial_r A + \partial_r^2 A + \partial_z^2 A \right] \tag{7}$$

$$\partial_t B_\phi = \frac{1}{r}J(A, u_\phi) + \frac{1}{r}J(B_\phi, \psi) + \frac{1}{r^2}B_\phi \partial_z \psi - \frac{1}{r^2}u_\phi \partial_z A + \eta\left(\nabla^2 B_\phi - \frac{1}{r^2}B_\phi\right) \quad (8)$$

2 Recovery of Narrow Gap Equations

A Cylindrical derivatives

Everything here follows http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node177.html#scyl.

For a scalar field ψ,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}. \quad (9)$$

However, for a *vector* field $\mathbf{u}$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \quad (10)$$

and

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial(ru_\phi)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\hat{\mathbf{z}}. \quad (11)$$

We also need the ϕ component of the convective derivative $\mathbf{u}\cdot\nabla\mathbf{u}$,

$$[\mathbf{u}\cdot\nabla\mathbf{u}]_\phi = \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}, \quad (12)$$

and finally, the vector Laplacian,

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} \quad (13)$$

$$(\nabla^2\mathbf{u})_\phi = \nabla^2 u_\phi + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2} \quad (14)$$

$$(\nabla^2\mathbf{u})_z = \nabla^2 u_z, \quad (15)$$

where ∇ on the vector components is given by equation (9).

Note that, expanding the definition of the vector Laplacian, where the cylindrical scalar Laplacian is substituted in for $\nabla^2 u_r$ and $\nabla^2 u_z$