PART II - QR FACTORIZATION AND LEAST SQUARES

Lecture 6 - *Projectors*

OBJECTIVE:
The theme of Part II is orthogonality.
We introduce the fundamental tool of projection matrices (or *projectors*), both orthogonal and nonorthogonal.

◇ PROJECTORS

Definition 1. *A* projector *is a square matrix* $\mathbf{P}$ *that satisfies*

$$\mathbf{P}^2 = \mathbf{P}$$

(Such a matrix is called *idempotent*.)

This definition does not distinguish between orthogonal and nonorthogonal projectors.

We will call nonorthogonal projectors *oblique projectors*.

Geometrically, you can think of shining a light onto range($\mathbf{P}$).

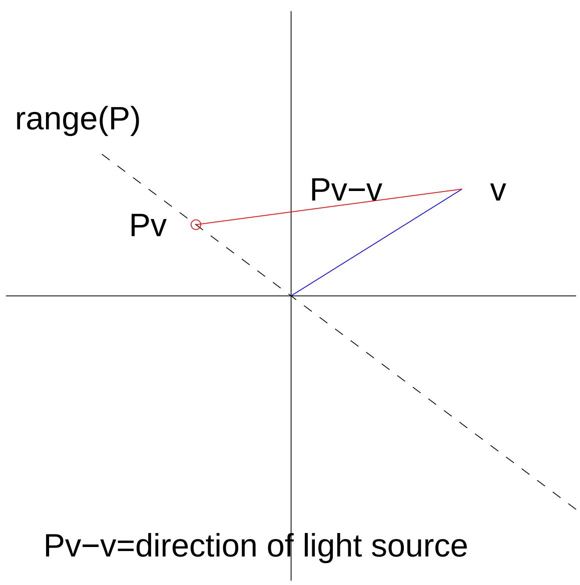
Then $\mathbf{Pv}$ would be the shadow projected onto range$(\mathbf{P})$ by the vector $\mathbf{v}$.

Note 1. *If* $\mathbf{v} \in \mathrm{range}(\mathbf{P})$, *then by definition* $\mathbf{v} = \mathbf{Px}$ *for some* $\mathbf{x}$.
And so,

$$\mathbf{Pv} = \mathbf{P}^2\mathbf{x} = \mathbf{Px} = \mathbf{v}$$

i.e., $\mathbf{v}$ *lies exactly in its own shadow.*

Usually, $\mathbf{v} \neq \mathbf{Pv}$, so we might ask if we determine the direction of the light source given $\mathbf{v}$ and $\mathbf{Pv}$.



Now notice that

$$\begin{aligned}
\mathbf{P}(\mathbf{P}\mathbf{v}-\mathbf{v}) &= \mathbf{P}^2\mathbf{v}-\mathbf{P}\mathbf{v} \\
&= \mathbf{P}\mathbf{v}-\mathbf{P}\mathbf{v} \\
&= \mathbf{0}
\end{aligned}$$

$$\therefore \qquad \mathbf{P}\mathbf{v}-\mathbf{v} \in \mathrm{null}(\mathbf{P})$$

i.e., the light source is always described by a vector in $\mathrm{null}(\mathbf{P})$.

$\diamond$ COMPLEMENTARY PROJECTORS

If $\mathbf{P}$ is a projector, then so is $\mathbf{I}-\mathbf{P}$:

$$\begin{aligned}(\mathbf{I}-\mathbf{P})^2 &= (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) \\ &= \mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P}^2 \\ &= \mathbf{I}-\mathbf{P}\end{aligned}$$

$\mathbf{I}-\mathbf{P}$ is called the *complementary projector* to $\mathbf{P}$.

Onto what space does $\mathbf{I}-\mathbf{P}$ project? null($\mathbf{P}$)!

Proof:

If $\mathbf{P}\mathbf{v}=\mathbf{0}$, then $(\mathbf{I}-\mathbf{P})\mathbf{v}=\mathbf{v}$

$$\therefore \quad \mathrm{null}(\mathbf{P}) \subseteq \mathrm{range}(\mathbf{I}-\mathbf{P})$$

Now, for any $\mathbf{v}$, $(\mathbf{I}-\mathbf{P})\mathbf{v}=\mathbf{v}-\mathbf{P}\mathbf{v}\in \mathrm{null}(\mathbf{P})$

$$\therefore \quad \mathrm{range}(\mathbf{I}-\mathbf{P}) \subseteq \mathrm{null}(\mathbf{P})$$

$$\therefore \quad \text{for any projector } \mathbf{P}, \text{range}(\mathbf{I}-\mathbf{P}) = \text{null}(\mathbf{P})$$

Note 2. *By writing* $\mathbf{P} = \mathbf{I} - (\mathbf{I} - \mathbf{P})$, *we can derive*

$$\text{range}(\mathbf{P}) = \text{null}(\mathbf{I} - \mathbf{P})$$

Also, $\text{null}(\mathbf{I} - \mathbf{P}) \cap \text{null}(\mathbf{P}) = \{0\}$.

Proof:

Let $\mathbf{v}$ be in both $\text{null}(\mathbf{I} - \mathbf{P})$ and $\text{null}(\mathbf{P})$.
Then, $\mathbf{v} = \mathbf{v} - \mathbf{P}\mathbf{v} = (\mathbf{I} - \mathbf{P})\mathbf{v} = 0$.

$$\therefore \quad \text{range}(\mathbf{P}) \cap \text{null}(\mathbf{P}) = \{0\}$$

This says *a projector separates* $\mathbb{R}^m$ *into two spaces*.

Now, let $\mathbf{S}_1, \mathbf{S}_2$ be two subspaces of $\mathbb{R}^m$ such that $\mathbf{S}_1 \cap \mathbf{S}_2 = \{0\}$ and $\mathbf{S}_1 + \mathbf{S}_2 = \mathbb{R}^m$
(these are called *complementary subspaces*).

Note 3. $\mathbf{S}_1 + \mathbf{S}_2 = \mathbb{R}^m$ *really means* $\text{span}(\mathbf{S}_1 + \mathbf{S}_2)$.

Then there is a projector $\mathbf{P}$ such that $\text{range}(\mathbf{P}) = \mathbf{S}_1$ and $\text{null}(\mathbf{P}) = \mathbf{S}_2$.
We say $\mathbf{P}$ projects *onto* $\mathbf{S}_1$ *along* $\mathbf{S}_2$.

i.e., the projector and its complement are the unique solution to the problem:

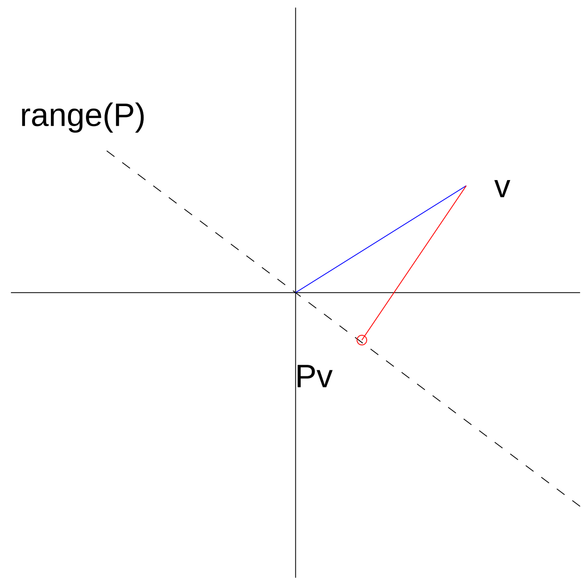
> Given any $\mathbf{v} \in \mathbf{S}$, find vectors $\mathbf{v}_1 \in \mathbf{S}_1$, $\mathbf{v}_2 \in \mathbf{S}_2$ such that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$

i.e., $\mathbf{v}_1 = \mathbf{P}\mathbf{v}$ and $\mathbf{v}_2 = (\mathbf{I} - \mathbf{P})\mathbf{v}$.

$\diamond$ ORTHOGONAL PROJECTORS

Definition 2. *An* orthogonal projector *is one that projects onto a subspace* $\mathbf{S}_1$ *along a space* $\mathbf{S}_2$ *where* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are orthogonal.*

WARNING: ORTHOGONAL PROJECTORS ARE NOT ORTHOGONAL MATRICES!



Geometry can only take us so far
$\rightarrow$ it becomes hard to visualize in more than 3D!

Fortunately, there is an algebraic test to see if a projector is orthogonal.

Theorem 1. *A projector* $\mathbf{P}$ *is orthogonal if and only if* $\mathbf{P} = \mathbf{P}^T$.

$\diamond$ PROJECTION WITH AN ORTHONORMAL BASIS

Let $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ be any set of orthonormal vectors in $\mathbb{R}^m$.

Let $\hat{\mathbf{Q}} \in \mathbb{R}^{m\times n}$ be the matrix of the vectors $\mathbf{q}_j$.

From Lecture 2, we know that any vector $\mathbf{v} \in \mathbb{R}^m$ can be decomposed into a component in the column space of $\hat{\mathbf{Q}}$ plus a component $\mathbf{r}$ in the orthogonal space:

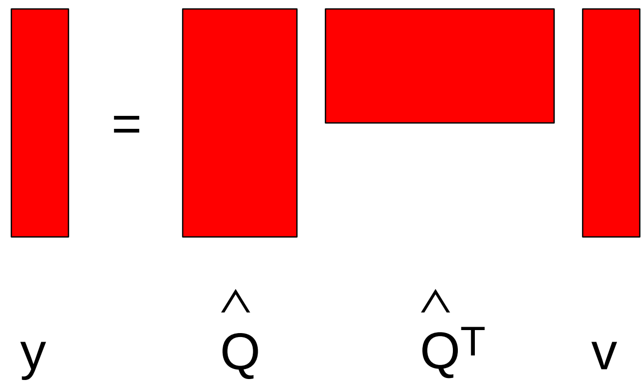
$$\mathbf{v} = \mathbf{r} + \sum_{i=1}^{n} (\mathbf{q}_i \mathbf{q}_i^T)\mathbf{v}$$

$$\therefore \qquad \mathbf{v} \mapsto \sum_{i=1}^{n} (\mathbf{q}_i \mathbf{q}_i^T)\mathbf{v}$$

$$\text{is an orthogonal projection onto range}(\hat{\mathbf{Q}})$$

In matrix form,

$$\mathbf{y} = \hat{\mathbf{Q}}\hat{\mathbf{Q}}^T \mathbf{v}$$



Note 4. *The complement of an orthogonal projector is also an orthogonal projector.*
The complement projector projects onto the space orthogonal to $\mathrm{range}(\hat{\mathbf{Q}})$*.*

An important special case of orthogonal projectors is the *rank-one orthogonal projector* that isolates the component in a single direction $\mathbf{q}$

$$\mathbf{P_q} = \mathbf{q}\mathbf{q}^T$$

Their complements are rank-$(m-1)$ orthogonal projectors that eliminate the component in direction $\mathbf{q}$

$$P_{\perp\mathbf{q}} = \mathbf{I} - \mathbf{q}\mathbf{q}^T$$

If $\mathbf{q}$ is not a unit vector, the analogous formulas are

$$\begin{aligned} \mathbf{P}_{\mathbf{a}} &= \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} \\ \mathbf{P}_{\perp\mathbf{a}} &= \mathbf{I} - \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} \end{aligned}$$

◇ PROJECTION WITH AN ARBITRARY BASIS

An orthogonal projector can be constructed beginning with an arbitrary (nonorthogonal) basis.

Let the subspace be spanned by the linearly independent vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be the matrix of the vectors a_j.

Let $\mathbf{v}$ be the vector to be projected, and let $\mathbf{y} \in \mathrm{range}(\mathbf{A})$ be the projected vector.

We know that

$$\mathbf{y} - \mathbf{v} \perp \mathrm{range}(\mathbf{A})$$

i.e., $\mathbf{a}_j^T(\mathbf{y} - \mathbf{v}) = 0$ for every j.

Since $\mathbf{y} \in \mathrm{range}(\mathbf{A})$, we can write $\mathbf{y} = \mathbf{A}\mathbf{x}$ for some $\mathbf{x}$ and hence

$$\begin{array}{ll} & \mathbf{a}_j^T(\mathbf{A}\mathbf{x} - \mathbf{v}) = \mathbf{0} \\ \text{or} & \mathbf{A}^T(\mathbf{A}\mathbf{x} - \mathbf{v}) = \mathbf{0} \\ \text{or} & (\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{v} \end{array}$$

Because $\mathbf{A}$ has full rank, $(\mathbf{A}^T\mathbf{A})$ is nonsingular, so

$$\begin{aligned} \mathbf{x} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{v} \\ \mathbf{y} &= \mathbf{P}\mathbf{v} \end{aligned}$$

where

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

Note 5.

1. *This* $\mathbf{P}$ *is the multi-dimensional generalization of* $\mathbf{P}_{\mathbf{a}}$.

2. *If* $\mathbf{A} = \hat{\mathbf{Q}}$ *(the orthonormal case), then* $\mathbf{P} = \hat{\mathbf{Q}}\hat{\mathbf{Q}}^T$ *as before.*