DS-288 NUMERICAL METHODS
Assignment–1 [1]

`Due date: 31, August, 2017 (18:00 Hrs IST)`
Instructor: Prof. Sashikumaar Ganesan.

1. Let p be a fixed point of the function g. Suppose $g \in C[a,b]$ and g' exists in (a,b), show that if $|g'(x)| > 1$, for all $x \in (a,b)$, then the fixed point sequence $p_n = g(p_{n-1})$ will fail to converge for any choice of p_0, except if $p_0 = p$. [`2 points`]

2. Develop the functional form for a cubically convergent fixed point iteration function $g(p_n)$ to solve the problem $f(x) = 0$ by writing

$$g(x) = x - \phi(x)f(x) - \psi(x)f^2(x)$$

   and determining $\phi(x)$ and $\psi(x)$. Specify the asymptotic order of convergence (α) and write the asymptotic error constant (λ). Write all expressions in terms of $f(x)$ and its derivatives and *simplify* your answers.
   Hint: Extend the approach we used in class to derive Newton's method. [`4 points`]

3. Consider the recurrence relation $a_{n+2} = 10a_{n+1} - 9a_n$, where $a_1 = a_2 = 1.24$. If we use infinite arithmetic, the solution for the recurrence is $a_n = 1.24$, for all $n \in \mathbb{Z}^+$. Calculate (upto 16 significant digits) $a_3, a_4, ..., a_{20}$ using finite arithmetic and obtain the relative errors (w.r.t. infinite arithmetic) as well. Repeat the same calculations for $a_1 = a_2 = 1.25$. Explain your observations.
   Note: Use matlab for calculations. [`3 points`]

4. Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.
   Note: Use only theorems discussed in the class to prove. [`3 points`]

5. (a) Find the fourth Taylor polynomial $P_4(x)$ for the function $f(x) = xe^{x^2}$ about $x_0 = 0$ and determine an upper bound for $|f(x) - P_4(x)|$, for $0 \leq x \leq 0.4$. [`3 points`]
   (b) Approximate $\int_0^{0.4} f(x)dx$ using $\int_0^{0.4} P_4(x)dx$ and determine a bound for the accuracy of the approximation. [`3 points`]

6. The following four methods are proposed to compute $7^{1/5}$. Report the number of iterations to converge and rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$. Use stopping criterion as $|p_n - p_{n-1}| \leq 10^{-8}$. [`3 points`]

   (a) $p_n = p_{n-1}\left(1 + \frac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$

   (b) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{p_{n-1}^2}$

   (c) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

   (d) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$

   Note: Use matlab for calculations.

[1] Posted on: `August 23, 2017`.

7. Using Newton's method, Secant method and modified Newton's method, find the solution of $f(x) = 0$ for the functions listed. For the Newton methods start with an initial guess of $p_0 = 0$, while for Secant method start with $p_0 = 0$ and $p_1 = 1$. Iterate until you reach a relative tolerance of 10^{-8} between successive iterates. Report the root found and the number of iterations needed for each method.

   (a) $f(x) = x + e^{-x^2} \cos x$

   (b) $f(x) = \left(x + e^{-x^2} \cos x\right)^2$.

   Comment on the observed convergence rates in these cases. Does your results agree with the analysis we did in class ? [`6 points`]
   Note: Use matlab for calculations.

8. Show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.
   Hint: Use Intermediate Value theorem and Rolle's theorem. [`3 points`]