E9 285 Biomedical imaging-Inverse problems

Course organization

Part I:

Foundations (this lecture)

Part II:

a) Image formation in fluorescence microscopy

b) Representation reconstruction (deconvolution) methods for fluorescence microscopy

Part III:

a) Image formation in tomography

b) Representation reconstruction methods for for tomography

Part IV:

a) Image formation in MRI

b) Representation reconstruction methods for for MRI

Part V (student presentations):

a) Other derivative based regularization approaches

b) Other iterative methods

c) Algorithm for wavelet regularization

Part VI:

a) Introduction to compressed sensing

b) Compressed sensing for MRI

Grading

1) Assignments (both derivation and MATLAB simulation): 25 marks
2) Project (implementation of a journal paper): 25 marks
3) Presentations (for part V): 10 marks
4) Final written exam: 40 marks

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Lecture 1
Foundations

General form of the inverse problem

The image notation:

$x(\mathbf{r}) \leftrightarrow \mathbf{x}$

$x(\mathbf{r})$: 2D or 3D image with index $\mathbf{r} = (l, m)$ or (l, m, n)

$\mathbf{x}$: pixels/voxels of the array column vector

The measurement model:

$m_i = \mathbf{v}_i^T \mathbf{x}$

$s_i = \Re(m_i)$

$\mathbf{x}$: required unknown vector (scanned image)

$\mathbf{v}_i$: measuring vector

m_i: ideal measurement

s_i: actual measurement

$\Re$: random process with m_i as its mean
with likelihood $\mathrm{L}(m_i, s_i)$

The inversion framework: Penalized maximum-likelihood approach

$$\mathbf{x}_{opt} = \underset{\mathbf{x}}{\arg\min} \left[-\sum_{i=1}^{M} \log L(\mathbf{v}_i^T \mathbf{x}; s_i) + \lambda J(x) \right]$$

$J(x)$: penalty function/functional

Two types of likelihood functions:

Gaussian

$$L(\mu_i; s_i) = e^{-(s_i - \mu_i)^2/\sigma^2}$$

Poissons

$$L(\mu_i; s_i) = \frac{\mu_i^{s_i} e^{-\mu_i}}{s_i!}$$

P-MLE for Gaussian:

$$\mathbf{x}_{opt} = \underset{\mathbf{x}}{\arg\min} \left[\sum_{i=1}^{M} \frac{1}{\sigma^2} (\mathbf{v}_i^T \mathbf{x} - s_i)^2 + \lambda J(x) \right]$$

P-MLE for Poissons:

$$\mathbf{x}_{opt} = \underset{\mathbf{x}}{\arg\min} \left[\sum_{i=1}^{M} \left(\mathbf{v}_i^T \mathbf{x} - s_i \log(\mathbf{v}_i^T \mathbf{x}) \right) + \lambda J(x) \right]$$

General form of $J(x)$:

$$J(x) = \sum_{\mathbf{r}} \varphi\left(\sum_{j=1}^{N_F} \left(L_j(\mathbf{r}) * x(\mathbf{r}) \right)^2 \right)$$

$L_j(\mathbf{r}), j = 1,..,N_F$: derivative filters

φ: convex function

Alternate form:

$$J(\mathbf{x}) = \sum_{\mathbf{r}} \varphi\left(\sum_{j=1}^{N_F} \left(\mathbf{u}_{j,\mathbf{r}}^T \mathbf{x} \right)^2 \right)$$

Introduction to multivariable quadratic functions

The quadratic form

$$f(\mathbf{x}) = 0.5\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathrm{c}$$

$\mathbf{x}$: $n \times 1$ vector of optimization variable

$\mathbf{b}$: $n \times 1$ constant vector

$\mathbf{Q}$: $n \times n$ symmetric matrix

c: scalar constant

What if Q is not symmetric ?

$$f(\mathbf{x}) = 0.5\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathrm{c}$$

Let $\mathbf{Q}' = \frac{1}{2}\left(\mathbf{Q} + \mathbf{Q}^T\right)$

$$f'(\mathbf{x}) = 0.5\mathbf{x}^T\mathbf{Q}'\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathrm{c}$$

Are $f(\mathbf{x})$ and $f'(\mathbf{x})$ related ?

$$f(\mathbf{x}) = 0.5\mathbf{x}^T \mathbf{Q}\mathbf{x} - \mathbf{b}^T \mathbf{x} + c$$

$$f'(\mathbf{x}) = 0.5\mathbf{x}^T \mathbf{Q}'\mathbf{x} - \mathbf{b}^T \mathbf{x} + c$$

$$\mathbf{Q}' = \frac{1}{2}\left(\mathbf{Q} + \mathbf{Q}^T\right)$$

$f(\mathbf{x})$ and $f'(\mathbf{x})$ are identical.

Hence there is no loss of generality in assuming

$\mathbf{Q}$ to be symmetric.

Properties of symmetric matrices

(1) Eigen values are real
(2) Eigen vectors can be assumed to be real
(3) Eigen vectors are orthogonal
(4) Is diagonalizable

Properties of symmetric matrices

(1) An $n \times n$ symmetric matrix, $\mathbf{Q}$, is called positive definite if $\mathbf{v}^T\mathbf{Q}\mathbf{v} > \mathbf{0}$, for every $\mathbf{v} \in \mathbb{R}^n$

(2) Since $\mathbf{Q}$ is symmetric, we have $\mathbf{Q} = \mathbf{U}\Lambda\mathbf{U}^T$.
Also any $\mathbf{v}$ can be expressed as $\mathbf{v} = \mathbf{U}\mathbf{s}$, for some $\mathbf{s}$.

(1) and (2) imply
An $n \times n$ symmetric matrix, $\mathbf{Q}$, is called positive definite if and only if $\mathbf{s}^T\Lambda\mathbf{s} > \mathbf{0}$, for every $\mathbf{s} \in \mathbb{R}^n$

which in turn implies
an $n \times n$ symmetric matrix, $\mathbf{Q}$, is called positive definite if and only if all eigen values are postive.

Gradient and Hessian of quadratic form

$$f(\mathbf{x}) = 0.5\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathrm{c}$$

Gradient:

$$\nabla\left(f(\mathbf{x})\right) = \mathbf{Q}\mathbf{x} - \mathbf{b}$$

Hessian:

$$\mathbf{F}(\mathbf{x}) = \mathbf{Q}$$

Minimization of quadratic forms

(1) First order necessay condition:

$\nabla\left(f(\mathbf{x}^*)\right) = \mathbf{Q}\mathbf{x}^* - \mathbf{b} = 0$ is solvable.

$\Rightarrow \mathbf{Q}\mathbf{x}^* = \mathbf{b}$ is solvable.

(2a) Second order necessary condition

the Hessian, $\mathbf{F}(\mathbf{x}) = \mathbf{Q}$ is postive semi-definite.

(2b) Second order sufficient condition:

$\mathbf{Q}$ is postive definite.

The method of conjugate gradients for quadratic functions

The steepest descent method

Given an initial guess for $\mathbf{x}_0$ for the minimum of the quadratic form $f(\mathbf{x}) = (1/2)\mathbf{x}^T \mathbf{Q}\mathbf{x} - \mathbf{b}^T \mathbf{x}$, let $\mathbf{d}_0 = -\mathbf{g}_0 = -(\mathbf{Q}\mathbf{x}_0 - \mathbf{b})$.

Then the iteration is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

$$\alpha_k == -\frac{\mathbf{g}_k^{\mathbf{T}} \mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}} \mathbf{Q} \mathbf{d}_k}$$

$$\mathbf{d}_{k+1} = -\mathbf{g}_{k+1}$$

where $\mathbf{g}_k = \mathbf{Q}\mathbf{x}_k - \mathbf{b}$.

Consecutive directions are orthogonal, i.e., $\mathbf{d}_{k+1}^T \mathbf{d}_k = 0$

The conjugate gradient method:

Given an initial guess for $\mathbf{x}_0$ for the minimum of the quadratic form $f(\mathbf{x}) = (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x}$, let $\mathbf{d}_0 = -\mathbf{g}_0 = -(\mathbf{Q}\mathbf{x}_0 - \mathbf{b})$.

Then the iteration is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad \alpha_k = -\frac{\mathbf{g}_k^{\mathbf{T}}\mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k} \Rightarrow \mathbf{g}_{k+1}^T\mathbf{d}_k = 0$$

$$\mathbf{d}_{k+1} = -\mathbf{g}_{k+1} + \beta_k \mathbf{d}_k, \quad \beta_k = \frac{\mathbf{g}_{k+1}^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k} \Rightarrow \mathbf{d}_{k+1}^T\mathbf{Q}\mathbf{d}_k = 0$$

where $\mathbf{g}_k = \mathbf{Q}\mathbf{x}_k - \mathbf{b}$.

Consecutive directions are Q-orthogonal

The conjugate gradient theorem:

The conjugate gradient algorithm, at any step k satisfies the following:

(a) $[\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_k] = [\mathbf{g}_0, \mathbf{Q}\mathbf{g}_0, \ldots, \mathbf{Q}^k \mathbf{g}_0]$

(b) $[\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_k] = [\mathbf{g}_0, \mathbf{Q}\mathbf{g}_0, \ldots, \mathbf{Q}^k \mathbf{g}_0]$

(c) $\mathbf{d}_k^T \mathbf{Q} \mathbf{d}_i = 0$ for all $i < k$

(d) $\alpha_k = \mathbf{g}_k \mathbf{g}_k / \mathbf{d}_k^T \mathbf{Q} \mathbf{d}_k$

(e) $\beta_k = \mathbf{g}_{k+1}^T \mathbf{g}_{k+1} / \mathbf{g}_k^T \mathbf{g}_k$

Optimality of conjugate gradient method

Define

$$E(\mathbf{x}_k) = \left(\mathbf{x}_k - \mathbf{x}^*\right)^T \mathbf{Q}\left(\mathbf{x}_k - \mathbf{x}^*\right), \text{ where } \mathbf{x}^* \text{ is the optimal solution.}$$

Then

$$E(\mathbf{x}_{k+1}) \leq \max_{\lambda} \left[1 + \lambda P_k(\lambda)\right]^2 E(\mathbf{x}_0),$$

for any polynomial P_k of order k.