# E9 285 Biomedical imaging-Inverse problems

## Chapter 2.b (5)
## Information for first lab exam

# The minimization problem

$$x_{opt}(\mathbf{r}) = \underset{x(\mathbf{r})}{\arg\min} \underbrace{\left[ \underbrace{\sum_{\mathbf{r}' \in [1:N]^D} ((h * x)(\mathbf{r}') - D(\mathbf{r}'))^2}_{J_e(x)} + \lambda J_R(x) \right]}_{J_T(x)}$$

$$J_R(x) = \sum_{\mathbf{r}'} \sqrt{\sum_{j=1}^{N_F} \left( (L_j * x)(\mathbf{r}') \right)^2}$$

$$\mathbf{r} = (l, m, n) \quad \text{(compare it with } \mathbf{r} = (r_1, r_2, r_3) \text{ for continuous domain)}$$

$$\delta = \frac{xy \text{ step size}}{z \text{ step size}}$$

$$L_1(\mathbf{r}) * x(\mathbf{r}) = x(l, m, n) - x(l-1, m, n)$$

$$L_2(\mathbf{r}) * x(\mathbf{r}) = x(l, m, n) - x(l, m-1, n)$$

$$L_3(\mathbf{r}) * x(\mathbf{r}) = \delta(x(l, m, n) - x(l, m, n-1))$$

# The iteration:

$$x^{(t+1)} = \underset{x}{\arg\min} \underbrace{\left[ \sum_{\mathbf{r}\in[1:N]^D} ((h * x)(\mathbf{r}) - D(\mathbf{r}))^2 + 0.5\lambda Q_R(x, x^{(t)}) \right]}_{J_{x^{(t)}}(x)}$$

$$Q_R(x, x^{(t)}) = \sum_{\mathbf{r}} \left[ \frac{\sum_{j=1}^{N_F} \left( (L_j * x)(\mathbf{r}) \right)^2}{\sqrt{\sum_{j=1}^{N_F} \left( (L_j * x^{(t)})(\mathbf{r}) \right)^2}} \right]$$

$$Q_R(x, x^{(t)}) = \sum_{\mathbf{r}} \left[ \frac{\sum_{j=1}^{N_F} \left( (L_j * x)(\mathbf{r}) \right)^2}{\sqrt{\sum_{j=1}^{N_F} \left( (L_j * x^{(t)})(\mathbf{r}) \right)^2}} \right]$$

Define $W_{x^{(t)}}(\mathbf{r}) = \dfrac{1}{\sqrt{\epsilon + \sum_{j=1}^{N_F} \left( (L_j * x^{(t)})(\mathbf{r}) \right)^2}}$.

Then $\quad Q_R(x, x^{(t)}) = \sum_{j=1}^{N_F} \left\langle L_j * x, W_{x^{(t)}}(L_j * x) \right\rangle$

$$= \sum_{j=1}^{N_F} \left\langle x, L_j^T * \left( W_{x^{(t)}}(L_j * x) \right) \right\rangle$$

# Relating to the standard quadratic form

$$J_{x^{(t)}}(x) = \left\langle x, A_{x^{(t)}} x \right\rangle - 2\langle x, b \rangle + c$$

$$A_{x^{(t)}} x = \; h^T * h * x + \lambda \sum_{j=1}^{N_F} L_j^T * \left( W_{x^{(t)}} \left( L_j * x \right) \right)$$

$$b = \; h^T * D$$

$$c = \langle D, D \rangle$$

# Minimization of $J_{x^{(t)}}(x)$

Intialization: $x_0 = x^{(t)}$

$$d_0 = -g_0 = -\nabla J_{x^{(t)}}(x_0) = -\left(A_{x^{(t)}} x_0 - \left(h^T * D\right)\right)$$

Iteration:

$$x_{k+1} = x_k + \alpha_k d_k, \quad \alpha_k = -\frac{\left\langle g_k, d_k \right\rangle}{\left\langle d_k, A_{x^{(t)}} d_k \right\rangle}$$

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \quad \beta_k = \frac{\left\langle g_{k+1}, A_{x^{(t)}} d_k \right\rangle}{\left\langle d_k, A_{x^{(t)}} d_k \right\rangle}$$

where $g_k = \left(A_{x^{(t)}} x_k - \left(h^T * D\right)\right)$