E9 285 Biomedical imaging-Inverse problems

Chapter 2.b
Representative reconstruction methods for fluorescence microscopy

Iterative constrained Tikhonov Miller method (ICTM)

[1] H.T.M. van der Voort and K.C. Strasters, “Restoration of confocal images for quantitative image analysis,” *J. Microsc., vol. 178, no. 2, pp. 165–181, 1995.*

[2] G.M.P. van der Kempen, L.J. van Vliet, P.J. Verveer, and H.T.M. van der Voort, “A quantitative comparison of image restoration methods for confocal microscopy,” *J. Microsc., vol. 185, no. 3, pp. 354–365, 1997.*

ICTM

1) $\{L_1, L_2\}$: first derivative filters

2) Implements steepest descent and conjugate gradient algorithm where the gradient is clipped for negative values at each step.

Recap: Tikhonov filtering

$$x_{opt}(\mathbf{r}) = \underset{x(\mathbf{r})}{\arg\min} \underbrace{\left[\underbrace{\sum_{\mathbf{r}' \in [1:N]^D} ((h * x)(\mathbf{r}') - D(\mathbf{r}'))^2}_{J_e(x)} + \lambda J_R(x) \right]}_{J_T(x)}$$

$$J_R(x) = \sum_{\mathbf{r}'} \left(\sum_{j=1}^{N_F} \big((L_j * x)(\mathbf{r}')\big)^2 \right)$$

First consider the data term:

$$J_e(x) = \sum_{\mathbf{r}'} \left((h * x)(\mathbf{r}')\right)^2 + \sum_{\mathbf{r}'} D^2(\mathbf{r}') - 2\sum_{\mathbf{r}'} (h * x)(\mathbf{r}') D(\mathbf{r}')$$

Consider the individual terms:

$$\sum_{\mathbf{r}'} \left((h * x)(\mathbf{r}')\right)^2 = \langle h * x, h * x \rangle = \langle x, h^T * h * x \rangle, \quad \text{where} \quad h^T(\mathbf{r}) = h(-\mathbf{r})$$

$$\sum_{\mathbf{r}'} (h * x)(\mathbf{r}') D(\mathbf{r}') = \langle h * x, D \rangle = \langle x, h^T * D \rangle$$

Hence we get

$$J_e(x) = \langle x, h^T * h * x \rangle - 2\langle x, h^T * D \rangle + \langle D, D \rangle$$

$$J_R(x) = \sum_{\mathbf{r}'} \left(\sum_{j=1}^{N_F} \left((L_j * x)(\mathbf{r}')\right)^2 \right) = \sum_{j=1}^{N_F} \langle x, L_j^T * L_j * x \rangle$$

Total cost: $J_e(x) + \lambda J_R(x)$

$$J_T(x) = \langle x, M * x \rangle - 2\langle x, b \rangle + c$$

$$M = h^T * h + \lambda \sum_{j=1}^{N_F} L_j^T * L_j$$

$$b = h^T * D$$

$$c: \quad \langle D, D \rangle$$

$$J_T(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x} - 2\mathbf{x}^T \mathbf{b} + c$$

$\mathbf{x}$: scanned vector of the image x

$\mathbf{b}$: scanned vector of the image b

$\mathbf{Q}$: convolution matrix of the filter M with respect to the scan order of $\mathbf{x}$

The gradient:

$$\nabla J_T(\mathbf{x}) = \mathbf{Qx} - \mathbf{b}$$

$$\nabla J_T(x) = M * x - h^T * D$$

Tikhonov solution:

$$\nabla J_T(x) = M * x - h^T * D = 0$$

$$M(\mathbf{\Omega}) x_{opt}(\mathbf{\Omega}) - \tilde{H}(\mathbf{\Omega}) D(\mathbf{\Omega}) = 0$$

$$x_{opt}(\mathbf{\Omega}) = \frac{\tilde{H}(\mathbf{\Omega}) D(\mathbf{\Omega})}{M(\mathbf{\Omega})}$$

The steepest descent method

Intialization:

Given $\mathbf{x}_0$

$$\mathbf{d}_0 = -\mathbf{g}_0 = -(\mathbf{Q}\mathbf{x}_0 - \mathbf{b}).$$

Iteration:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

$$\alpha_k == -\frac{\mathbf{g}_k^{\mathbf{T}} \mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}} \mathbf{Q} \mathbf{d}_k}$$

$$\mathbf{d}_{k+1} = -\mathbf{g}_{k+1}$$

where $\mathbf{g}_k = \mathbf{Q}\mathbf{x}_k - \mathbf{b}$.

Intialization: $x_0(\mathbf{r}) = F^{-1}\left[\frac{\tilde{H}(\boldsymbol{\Omega}) D(\boldsymbol{\Omega})}{M(\boldsymbol{\Omega})} \right]$

$$d_0 = -g_0 = -\nabla J_T(x) = -\left(M * x_0 - h^T * D \right)$$

Iteration:

$$x_{k+1} = x_k + \alpha_k d_k$$

$$\alpha_k = -\frac{\langle g_k, d_k \rangle}{\langle d_k, M * d_k \rangle}$$

$$d_{k+1} = -g_{k+1}$$

where

$$g_k = -\left(M * x_0 - h^T * D \right)$$

ICTM

$$\text{Intialization: } x_0(\mathbf{r}) = \text{ClipNeg}\left(F^{-1}\left[\frac{\tilde{H}(\mathbf{\Omega})D(\mathbf{\Omega})}{M(\mathbf{\Omega})} \right] \right)$$

$$d_0 = -g_0 = -\nabla J_T(x) =$$

$$\text{ClipNeg}\left(-\left(M * x_0 - h^T * D\right)\right)$$

Iteration:

$$x_{k+1} = x_k + \alpha_k d_k$$

$$\alpha_k = -\frac{\langle g_k, d_k \rangle}{\langle d_k, M * d_k \rangle}$$

$$d_{k+1} = \text{ClipNeg}\left(-g_{k+1}\right)$$

where

$$g_k = -\left(M * x_0 - h^T * D\right)$$

The conjugate gradient method:

Given an initial guess for $\mathbf{x}_0$ for the minimum of the quadratic form $f(\mathbf{x}) = (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x}$, let $\mathbf{d}_0 = -\mathbf{g}_0 = -(\mathbf{Q}\mathbf{x}_0 - \mathbf{b})$.

Then the iteration is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad \alpha_k = -\frac{\mathbf{g}_k^{\mathbf{T}}\mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k}$$

$$\mathbf{d}_{k+1} = -\mathbf{g}_{k+1} + \beta_k \mathbf{d}_k, \quad \beta_k = \frac{\mathbf{g}_{k+1}^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k}{\mathbf{d}_k^{\mathbf{T}}\mathbf{Q}\mathbf{d}_k},$$

where $\mathbf{g}_k = \mathbf{Q}\mathbf{x}_k - \mathbf{b}$.

ICTM-CG

Intialization: $x_0(\mathbf{r}) = \text{ClipNeg}\left(F^{-1}\left[\frac{\tilde{H}(\boldsymbol{\Omega})D(\boldsymbol{\Omega})}{M(\boldsymbol{\Omega})} \right] \right)$

$$d_0 = -g_0 = -\nabla J_T(x) =$$

$$\text{ClipNeg}\left(-\left(M * x_0 - h^T * D\right)\right)$$

Iteration:

$$x_{k+1} = x_k + \alpha_k d_k, \quad \alpha_k = -\frac{\langle g_k, d_k \rangle}{\langle d_k, M * d_k \rangle}$$

$$\tilde{d}_{k+1} = \text{ClipNeg}\left(-g_{k+1}\right)$$

$$d_{k+1} = \tilde{d}_{k+1} + \beta_k d_k, \beta_k = \frac{\langle \tilde{d}_{k+1}, M * d_k \rangle}{\langle d_k, M * d_k \rangle}$$

where

$$g_k = -\left(M * x_k - h^T * D\right)$$

Assignment 3:

Write a program to compute $\nabla J_T(x) = M * x - h^T * D$.

Assignment 4:

Write a program for ICTM

Assignment 5:

Write a program for ICTM-CG

Quadratic data fitting with L1 regularization

S. Lefkimmiatis, A. Bourquard, M. Unser, "Hessian-Based Regularization for 3-D Microscopy Image Restoration", Proceedings of the Ninth IEEE International Symposium on Biomedical Imaging: From Nano to Macro (ISBI'12), Barcelona, Spain, May 2-5, 2012, pp. 1731-1734.

The minimization problem

$$x_{opt}(\mathbf{r}) = \underset{x(\mathbf{r})}{\arg\min} \underbrace{\left[\underbrace{\sum_{\mathbf{r}' \in [1:N]^D} ((h * x)(\mathbf{r}') - D(\mathbf{r}'))^2}_{J_e(x)} + \lambda J_R(x) \right]}_{J_T(x)}$$

$$J_R(x) = \sum_{\mathbf{r}'} \sqrt{\sum_{j=1}^{N_F} \left((L_j * x)(\mathbf{r}') \right)^2}$$

The regularization functional

Consider the following:

$$\bar{R}_{\bar{x}}^2(\mathbf{r}) = \left(\frac{\partial^2}{\partial r_2^2}\bar{x}(\mathbf{r})\right)^2 + \left(\frac{\partial^2}{\partial r_1^2}\bar{x}(\mathbf{r})\right)^2 + \left(\frac{\partial^2}{\partial r_3^2}\bar{x}(\mathbf{r})\right)^2$$

$$2\left(\frac{\partial^2}{\partial r_1 \partial r_3}\bar{x}(\mathbf{r})\right)^2 + 2\left(\frac{\partial^2}{\partial r_1 \partial r_2}\bar{x}(\mathbf{r})\right)^2 + 2\left(\frac{\partial^2}{\partial r_2 \partial r_3}\bar{x}(\mathbf{r})\right)^2,$$

where $\bar{x}(\mathbf{r})$ is a continuous domain function.

If $\bar{x}_1$ and $\bar{x}_2$ are images such that $\bar{x}_1(\mathbf{r}) = \bar{x}_2(\Theta\mathbf{r})$, where Θ is a 3×3 rotation matrix, then we have

$$\bar{R}_{\bar{x}_1}(\mathbf{r}) = \bar{R}_{\bar{x}_2}(\Theta\mathbf{r})$$

Now consider

$$R_x^2(\mathbf{r}) = \sum_{j=1}^{N_F} \Big((L_j * x)(\mathbf{r}') \Big)^2 .$$

Question: what are the filters that make R_x be an approximation $\overline{R}_{\overline{x}}(\mathbf{r})$

$$\mathbf{r} = (l, m, n) \quad \text{(compare it with } \mathbf{r} = (r_1, r_2, r_3) \text{ for continuous domain)}$$

$$\delta = \frac{xy \text{ step size}}{z \text{ step size}}$$

$$L_1(\mathbf{r}) * x(\mathbf{r}) = x(l-1, m, n) - 2x(l, m, n) + x(l+1, m, n)$$

$$L_2(\mathbf{r}) * x(\mathbf{r}) = x(l, m-1, n) - 2x(l, m, n) + x(l, m+1, n)$$

$$L_3(\mathbf{r}) * x(\mathbf{r}) = \delta^2 \left(x(l, m, n-1) - 2x(l, m, n) + x(l, m, n+1) \right)$$

$$L_4(\mathbf{r}) * x(\mathbf{r}) = \sqrt{2} \left[x(l, m, n) - x(l-1, m, n) - x(l, m-1, n) + x(l-1, m-1, n) \right]$$

$$L_5(\mathbf{r}) * x(\mathbf{r}) = \sqrt{2}\delta \left[x(l, m, n) - x(l, m-1, n) - x(l, m, n-1) + x(l, m-1, n-1) \right]$$

$$L_6(\mathbf{r}) * x(\mathbf{r}) = \sqrt{2}\delta \left[x(l, m, n) - x(l-1, m, n) - x(l, m, n-1) + x(l-1, m, n-1) \right],$$

Comparison:

$$\bar{R}^2_{\bar{x}}(\mathbf{r}) = \left(\frac{\partial^2}{\partial r_2^2} \bar{x}(\mathbf{r}) \right)^2 + \left(\frac{\partial^2}{\partial r_1^2} \bar{x}(\mathbf{r}) \right)^2 + \left(\frac{\partial^2}{\partial r_3^2} \bar{x}(\mathbf{r}) \right)^2$$

$$2\left(\frac{\partial^2}{\partial r_1 \partial r_3} \bar{x}(\mathbf{r}) \right)^2 + 2\left(\frac{\partial^2}{\partial r_1 \partial r_2} \bar{x}(\mathbf{r}) \right)^2 + 2\left(\frac{\partial^2}{\partial r_2 \partial r_3} \bar{x}(\mathbf{r}) \right)^2,$$

The majorization-minimization method

The mojorization relation: for any non-negative funtion, g, we have

$$\sqrt{g(y)} \le 0.5\sqrt{g(z)} + 0.5\frac{g(y)}{\sqrt{g(z)}} \quad \text{for every y and z}$$

satisfying $g(y) \ge 0$ and $g(z) \ge 0$.

Given two images x and x', substitute $g(\bullet) = (\bullet)^2$ and y=$R_x(\mathbf{r})$, $z = R_{x'}(\mathbf{r})$:

$$R_x(\mathbf{r}) \le 0.5 R_{x'}(\mathbf{r}) + 0.5\frac{R_x^2(\mathbf{r})}{R_{x'}(\mathbf{r})}$$

$$\Rightarrow \underbrace{\sum_{\mathbf{r}} R_x(\mathbf{r})}_{J_R(x)} \le \underbrace{0.5\sum_{\mathbf{r}} R_{x'}(\mathbf{r}) + 0.5\sum_{\mathbf{r}} \frac{R_x^2(\mathbf{r})}{R_{x'}(\mathbf{r})}}_{Q'_R(x,x')}$$

$$\underbrace{\sum_{\mathbf{r}} R_x(\mathbf{r})}_{J_R(x)} \le \underbrace{0.5\sum_{\mathbf{r}} R_{x'}(\mathbf{r}) + 0.5\sum_{\mathbf{r}} \frac{R_x^2(\mathbf{r})}{R_{x'}(\mathbf{r})}}_{Q_R'(x,x')}$$

$Q_R'(x,x')$ is quadratic in x.

Given $x^{(t)}$, define

$$x^{(t+1)} = \underset{x}{\arg\min} \; J_e(x) + \lambda Q_R'(x, x^{(t)}).$$

Then

$$J_e(x^{(t+1)}) + \lambda \mathrm{J}_R(x^{(t+1)}) \leq$$

$$J_e(x^{(t+1)}) + \lambda Q'(x^{(t+1)}, x^{(t)}) \leq J_e(x^{(t)}) + \lambda Q'(x^{(t)}, x^{(t)})$$

$$= J_e(x^{(t)}) + \lambda \mathrm{J}_R(x^{(t)})$$

$$Q_R'(x, x^{(t)}) = 0.5 \sum_{\mathbf{r}} R_{x^{(t)}}(\mathbf{r}) + 0.5 \underbrace{\sum_{\mathbf{r}} \frac{R_x^2(\mathbf{r})}{R_{x^{(t)}}(\mathbf{r})}}_{Q_R(x, x^{(t)})}$$

The iteration:

$$x^{(t+1)} = \underset{x}{\arg\min} \underbrace{\left[\sum_{\mathbf{r} \in [1:N]^D} ((h * x)(\mathbf{r}) - D(\mathbf{r}))^2 + 0.5\lambda Q_R(x, x^{(t)}) \right]}_{J_{x^{(t)}}(x)}$$

$$Q_R(x, x^{(t)}) = \sum_{\mathbf{r}} \left[\frac{\sum_{j=1}^{N_F} \left((L_j * x)(\mathbf{r}) \right)^2}{\sqrt{\sum_{j=1}^{N_F} \left((L_j * x^{(t)})(\mathbf{r}) \right)^2}} \right]$$

$$Q_R(x,x^{(t)}) = \sum_{\mathbf{r}} \left[\frac{\sum_{j=1}^{N_F} \left((L_j * x)(\mathbf{r}) \right)^2}{\sqrt{\sum_{j=1}^{N_F} \left((L_j * x^{(t)})(\mathbf{r}) \right)^2}} \right]$$

Define $W_{x^{(t)}}(\mathbf{r}) = \dfrac{1}{\sqrt{\sum_{j=1}^{N_F} \left((L_j * x^{(t)})(\mathbf{r}) \right)^2}}$.

Then $\quad Q_R(x,x^{(t)}) = \sum_{j=1}^{N_F} \left\langle L_j * x, W_{x^{(t)}}(L_j * x) \right\rangle$

$$= \sum_{j=1}^{N_F} \left\langle x, L_j^T * \left(W_{x^{(t)}}(L_j * x) \right) \right\rangle$$

Relating to the standard quadratic form

$$J_{x^{(t)}}(x) = \left\langle x, A_{x^{(t)}} x \right\rangle - 2\langle x, b\rangle + c$$

$$A_t x = h^T * h * x + \lambda \sum_{j=1}^{N_F} L_j^T * \left(W_{x^{(t)}} \left(L_j * x \right) \right)$$

$$b = h^T * D$$

$$c = \langle D, D \rangle$$

$$J_{x^{(t)}}(\mathbf{x}) = \mathbf{x}^T \mathbf{Q}_t \mathbf{x} - 2\mathbf{b}^T \mathbf{x} + c$$

$\mathbf{x}$: scanned vector of x

$\mathbf{b}$: scanned vector of b

$\mathbf{Q}_t$: matrix equivalent of A_t with respect to the scan order

$D_{x^{(t)}}$: Diagonal approximation of $A_{x^{(t)}}$.

Define

$$\tilde{A}_{x^{(t)}} x = D^{-1/2}\left(h^T * h * \left(D_{x^{(t)}}^{-1/2} x\right)\right) + \lambda \sum_{j=1}^{N_F} D^{-1/2}\left(L_j^T * \left(W_{x^{(t)}}\left(L_j * \left(D_{x^{(t)}}^{-1/2} x\right)\right)\right)\right)$$

Minimization of $J_{x^{(t)}}(x)$

Intialization: $x_0 = x^{(t)}$

Compute $\tilde{A}_{x^{(t)}}$ from $A_{x^{(t)}}$

$$d_0 = -g_0 = -\nabla J_{x^{(t)}}(x_0) = -\left(\tilde{A}_{x^{(t)}} x_0 - h^T * D\right)$$

Iteration:

$$x_{k+1} = x_k + \alpha_k d_k, \quad \alpha_k = -\frac{\left\langle g_k, d_k \right\rangle}{\left\langle d_k, \tilde{A}_{x^{(t)}} d_k \right\rangle}$$

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \quad \beta_k = \frac{\left\langle d_{k+1}, \tilde{A}_{x^{(t)}} d_k \right\rangle}{\left\langle d_k, \tilde{A}_{x^{(t)}} d_k \right\rangle}$$

where $g_k = -\left(\tilde{A}_{x^{(t)}} x_k - h^T * D\right)$

What is $D_{x^{(t)}}$?

$\mathbf{Q}_{x^{(t)}}$: matrix equivalent of $A_{x^{(t)}}$ with respect to the scan order

Let $\mathbf{d}_{x^{(t)}}$ be the diagonal of $\mathbf{Q}_{x^{(t)}}$.

Then $D_{x^{(t)}}$ obtained from reassembling $\mathbf{d}_{x^{(t)}}$.

Direct definition:

$$D_{x^{(t)}}(\mathbf{r}') = B_{x^{(t)}}(\mathbf{r}'), \text{ where } B_{x^{(t)}}(\mathbf{r}) = A_{x^{(t)}}\delta(\mathbf{r} - \mathbf{r}')$$

Assignment 6:

Derive a formula to compute $D_{x^{(t)}}(\mathbf{r})$ from $W_{x^{(t)}}(\mathbf{r})$, h, and derivative filters $L_1, \ldots\ldots, L_6$.