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Problem 1. Multi-Firm LP Alliance

Read the following statement and answer the following five questions.
Consider an linear programming games where a finite set K of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of each firm k in K entails choosing an n-dimension vector $x \geqslant 0$ of activity levels that maximize the firm's profit

$$c^T x$$

subject to the constraint that its consumption Ax of resources minorizes its resource vector $b^{\{k\}} \in R^m$, that is,

$$Ax \leqslant b^{\{k\}}.$$

Here A is the so called resource/product consumption matrix for firm k.
An alliance is a subset of the firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-dimension vector $x \geqslant 0$ that maximizes the profit $c^T x$ that S earns subject to its resource constraint

$$Ax \leqslant b^S,$$

where

$$b^S = \sum_{k \in S} b^{\{k\}},$$

that is, the alliance will utilize all their members' resources.
Let V^S be the resulting maximum profit of alliance S:

$$\begin{aligned} V^S := \ \max \quad & c^T x \\ s.t. \quad & Ax \leqslant b^S, \ x \geqslant 0. \end{aligned}$$

The grand alliance is the set $S = K$ of all firms.
Core is the set of payment vector $z = (z_1, \ldots, z_{|K|})$ to each firm such that

$$\sum_{k \in K} z_k = V^K$$

and

$$\sum_{k \in S} z_k \geqslant V^S, \ \forall S \subset K.$$

that is, the payment to any possible alliance S is not worse than the maximum profit of its own production.

a) Show that the core is a convex set.
For any $z^1 = (z_1^1, z_2^1, \ldots, z_{|K|}^1)$, $z^2 = (z_1^2, z_2^2, \ldots, z_{|K|}^2)$ $\in$ core, and any $\lambda \in [0, 1]$, let z denote $\lambda z^1 + (1-\lambda) z^2$,

then we have

$$\sum_{k\in K} z_k = \sum_{k\in K}\left[\lambda z_k^1 + (1-\lambda) z_k^2\right] = \lambda\sum_{k\in K} z_k^1 + (1-\lambda)\sum_{k\in K} z_k^2 = \lambda V^K + (1-\lambda)V^K = V^K$$
$$\sum_{k\in S} z_k = \sum_{k\in S}\left[\lambda z_k^1 + (1-\lambda) z_k^2\right] = \lambda\sum_{k\in S} z_k^1 + (1-\lambda)\sum_{k\in S} z_k^2 \geqslant \lambda V^S + (1-\lambda)V^S = V^S,\ \forall S\subset K$$

By definition, that is, the core is a convex set.

b) Write out the dual of the grand alliance problem.
The grand alliance problem:

$$\begin{aligned} \max \quad & c^T x \\ s.t. \quad & Ax \leqslant b^K,\ x\geqslant 0 \end{aligned}$$

The dual problem:

$$\begin{aligned} \min \quad & (b^K)^T y \\ s.t. \quad & A^T y \geqslant c,\ y\geqslant 0 \end{aligned}$$

c) Prove that for each optimal dual price vector y^* for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices, $b^{\{k\}}y^*$ for $k=1,\dots,|K|$, yields a profit allocation in the core.
Let y^* denote the optimal solution of the dual problem in b). For any subset $S\in K$, let y_S^* denote the optimal solution of the corresponding dual problem. Then we have

$$(b^K)^T y^* = V^K,\ (b^S)^T y_S^* = V^S$$

Write $z_k = (b^{\{k\}})^T y^*$, and we obtain

$$\sum_{k\in K} z_k = \sum_{k\in K}(b^{\{k\}})^T y^* = (b^K)^T y^* = V^K$$
$$\sum_{k\in S} z_k = \sum_{k\in S}(b^{\{k\}})^T y^* = (b^S)^T y^* \geqslant (b^S)^T y_S^* = V^S,\ \forall S\subset K$$

By definition, that is, $(b^{\{k\}})^T y^*,\ k=1,\dots,|K|$ yields a profit allocation in the core.

d) Construct a sample example where a core payment is not necessarily drawn from the dual price vector of the grand alliance problem.
Write $c=(1,\dots,1)^T$, $b^{\{k\}}=(0,\dots,0,1,0,\dots,0)^T$ where the k^{th} component is 1, and

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

Then for any $S \subsetneqq K$, we have $V^S=0$, Thus, any allocation that satisfies $\sum_{k\in K} z_k = V^K,\ z_k\geqslant 0$ forms a core payment, and it is not necessarily drawn from the dual price vector of the grand alliance problem.

e) Consider the following Multi-Firm LP Alliance Problem and answer the two questions:
There are 3 firms A, B, C in the problem. The common profit margin vector c is given by $(1;2;4)$. The resource consumption rate matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

There are 3 available resources during production. The resource vector for company A, B, C are given separately by $(1;2;3), (2;3;1)$ and $(3;2;1)$.

e1) What is the profit of the grand alliance? Explain why the alliance is preferred in this problem.
Solve the grand alliance problem:

$$\begin{aligned} \max \quad & x_1 + 2x_2 + 4x_3 \\ s.t. \quad & x_1 + x_2 \leqslant 6 \\ & x_1 + x_3 \leqslant 7 \\ & x_2 + x_3 \leqslant 5 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

We obtain:

$$x^* = (x_1^*, x_2^*, x_3^*)^T = (2, 0, 5)^T, \ c^T x^* = 22$$

When $Ax \leqslant b^{\{A\}}, \ x \geqslant 0$, we have

$$V^{\{A\}} = c^T x^* = x_1^* + 2x_2^* + 4x_3^* \leqslant 2x_1^* + 2x_2^* + 4x_3^* \leqslant 10 \leqslant 22 = V^{\{A,B,C\}}$$

When $Ax \leqslant b^{\{B\}}, \ x \geqslant 0$, we have

$$V^{\{B\}} = c^T x^* = x_1^* + 2x_2^* + 4x_3^* \leqslant x_1^* + 5x_2^* + 4x_3^* \leqslant 6 \leqslant 22 = V^{\{A,B,C\}}$$

When $Ax \leqslant b^{\{C\}}, \ x \geqslant 0$, we have

$$V^{\{C\}} = c^T x^* = x_1^* + 2x_2^* + 4x_3^* \leqslant x_1^* + 3x_2^* + 4x_3^* \leqslant 5 \leqslant 22 = V^{\{A,B,C\}}$$

Thus, the alliance is preferred in this problem.

e2) What is the core of the problem? Solve it by checking the dual problem. And verify the profit allocation derived from the core is desirable.

Solve the dual problem:

$$\begin{aligned}
\min \quad & 6y_1 + 7y_2 + 5y_3 \\
s.t. \quad & y_1 + y_2 \geqslant 1 \\
& y_1 + y_3 \geqslant 2 \\
& y_2 + y_3 \geqslant 4 \\
& y_1, y_2, y_3 \geqslant 0
\end{aligned}$$

We obtain:

$$y^* = (y_1^*, y_2^*, y_3^*)^T = (2, 0, 5)^T,\ (b^{\{A,B,C\}})^T y^* = 22$$

Using the results in c) and e1), we have the profit allocation $z_k = (b^{\{k\}})^T y^*,\ k \in A, B, C$, and we can verify it by the following facts:

$$\begin{aligned}
& \sum_{k\in K} z_k = z_A + z_B + z_C = 22 = c^T x^* = (b^{\{A,B,C\}})^T y^* \\
& z_A = (b^{\{A\}})^T y^* = 11 \geqslant 10 \geqslant V^{\{A\}} \\
& z_B = (b^{\{B\}})^T y^* = 6 \geqslant 6 \geqslant V^{\{B\}} \\
& z_C = (b^{\{C\}})^T y^* = 5 \geqslant 5 \geqslant V^{\{C\}} \\
& \sum_{k\in S} z_k \geqslant \sum_{k\in S} V^k \geqslant V^S,\ \forall S \subset A, B, C
\end{aligned}$$

Problem 2. Arbitrage-Free Strike Price of Put Options

Consider a market that involves a stock A, which has a price of \$1 per share today. The price of A tomorrow is a random variable S, which can take two values \$2 or \$0.5. The market also has a "put option" P, which is a financial product that gives the owner the right to sell one share of A at a strike price K tomorrow. Each share of P has a price of \$0.1 today and has a payoff $\max\{K - S, 0\}$ tomorrow. We use the notation $(K - S)^+ = \max\{K - S, 0\}$ throughout this problem. We want to determine the optimal amounts of stocks θ_A and put options θ_P to purchase, given by the linear program below

$$\begin{aligned}
\min \quad & \theta_A + 0.1\theta_P \\
s.t. \quad & 2\theta_A + (K-2)^+\theta_P \geqslant 0 \\
& 0.5\theta_A + (K-0.5)^+\theta_P \geqslant 0 \\
& \theta_A, \theta_P\ free
\end{aligned}$$

A market has an *arbitrage opportunity* if there is a way to earn a strictly positive payoff by buying and selling assets in the market today, with nonnegative payoffs in the future (in other words, guaranteed free money, regardless of which outcome happens). A market with no arbitrage opportunity is called *arbitrage free*.

a) Suppose $K = 0.5$ (Note $(K-2)^+ = 0$ and $(K-0.5)^+ = 0$) and answer the questions below (the simplex method should not be necessary).

a1) Will you buy any positive amount of put options with this striking price?

Will not. Since the pay off of P is 0 in this case, and buying positive amount of P might lead to a positive cost.

a2) Give a feasible solution (θ_A, θ_P) to the above linear program with **negative** objective function value. Argue that the problem is unbounded, and explain why this corresponds to an arbitrage opportunity in the market.

Let $\theta_A = 0,\ \theta_P = -10$, then the value of the objective function is -1.
Let $\theta_A = 0,\ \theta_P \to -\infty$, then $theta_A + 0.1\theta_P \to -\infty$, which implies that the problem is unbounded.
This corresponds to an arbitrage opportunity cause if I buy 0 share of stock A and negative share of stock P, then I will earn positive pay off today and \$0 pay off tomorrow.

b) Suppose $K = 0.7$ (Note $(K-2)^+ = 0$ and $(K-0.5)^+ = 0.2$) and answer the questions below (the simplex method should not be necessary).

b1) Write the dual program.
When $K = 0.7$, the primal problem is

$$\begin{aligned} \min \quad & \theta_A + 0.1\theta_P \\ s.t. \quad & 2\theta_A \geqslant 0 \\ & 0.5\theta_A + 0.2\theta_P \geqslant 0 \end{aligned}$$

and the dual problem is

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & 0.2y_2 = 0.1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

b2) Is the dual feasible? What is the dual optimal solution?

The feasible region only contains $(y_1, y_2) = (0.375, 0.5)$, so the dual problem is feasible and the dual optimal solution is $(y_1^*, y_2^*) = (0.375, 0.5)$

b3) Argue that if the dual program is feasible then the market is arbitrage free (so there is no portfolio with negative cost today, with a nonnegative payoff tomorrow.)

When the dual problem is feasible, the primal problem is also feasible with the same optimal value which is 0 in this case. That is, min $\theta_A + 0.1\theta_P = 0$, which implies that there is no feasible solution of the primal problem with negative cost today and nonnegative payoff tomorrow.

c) Answer the questions below concerning the existence of an arbitrage opportunity in the market. You may use conclusions from previous questions a) and b).

c1) The case $K = 0.5$ shows that improper pricing of the put option leads to an arbitrage opportunity in the market. Find the range of K within which there is no arbitrage opportunity.

By the conclusion above, we know that if the dual is feasible, then there is no arbitrage opportunity. So we hope to find the range of K within which the dual is feasible.
The dual problem is

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & (K-2)^+ y_1 + (K-0.5)^+ y_2 = 0.1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

We already know that when $K \leqslant 0.5$, the dual is infeasible.
When $0.5 < K \leqslant 2$, the dual can be written as

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & (K-0.5) y_2 = 0.1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

the solution to the first two equations is

$$\begin{cases} y_1 = 0.5 - \frac{0.025}{K-0.5} \\ y_2 = \frac{0.1}{K-0.5} \end{cases}$$

Setting $y_1, y_2 \geqslant 0$, we obtain $0.55 \leqslant K \leqslant 2$.
When $K > 2$, the dual can be written as

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & (K-2) y_1 + (K-0.5) y_2 = 0.1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

the solution to the first two equations is

$$\begin{cases} y_1 = \frac{2K-1.1}{3K} \\ y_2 = \frac{1.1-0.5K}{0.75K} \end{cases}$$

Setting $y_1, y_2 \geqslant 0$, we obtain $2 < K \leqslant 2.2$.
Thus, the range of K within which the dual is feasible is $0.55 \leqslant K \leqslant 2.2$

c2) Suppose now a new financial product, called a *bond*, has been introduced into the market. Investing one dollar in a bond at the beginning of the period results in a constant payoff of r (a positive scalar) at the end of the period. We have the option to purchase or sell an amount θ_B of bonds; the resulting linear program describing our portfolio is now given by

$$\begin{aligned} \min \quad & \theta_A + 0.1\theta_P + \theta_B \\ s.t. \quad & 2\theta_A + (K-2)^+\theta_P + r\theta_B \geqslant 0 \\ & 0.5\theta_A + (K-0.5)^+\theta_P + r\theta_B \geqslant 0 \\ & \theta_A, \theta_P, \theta_B \ free \end{aligned}$$

With $K = 0.7$, find the value r that makes the market arbitrage-free. Then, provide a general expression $r(K)$ under which the market is arbitrage-free, assuming that K is within the range computed in the preceding item.

The dual problem is

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & (K-2)^+y_1 + (K-0.5)^+y_2 = 0.1 \\ & ry_1 + ry_2 = 1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

Thus, we still have the conclusion that if the dual is feasible, then there is no arbitrage opportunity. So we hope to find the range of K within which the dual is feasible.
When $K \leqslant 0.5$, it's easy to know that the dual is infeasible.
When $0.5 < K \leqslant 2$, the dual can be written as

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1 + 0.5y_2 = 1 \\ & (K-0.5)y_2 = 0.1 \\ & ry_1 + ry_2 = 1 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

the solution to the first two equations is

$$\begin{cases} y_1 = 0.5 - \frac{0.025}{K-0.5} \\ y_2 = \frac{0.1}{K-0.5} \end{cases}$$

Hence, $r(K) = 1/(y_1 + y_2) = (4K-2)/(2K-0.7)$

When $K>2$, the dual can be written as

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & 2y_1+0.5y_2=1 \\ & (K-2)y_1+(K-0.5)y_2=0.1 \\ & ry_1+ry_2=1 \\ & y_1,y_2 \geqslant 0 \end{aligned}$$

the solution to the first two equations is

$$\begin{cases} y_1=\frac{2K-1.1}{3K} \\ y_2=\frac{1.1-0.5K}{0.75K} \end{cases}$$

Hence, $r(K)=1/(y_1+y_2)=K/1.1$
Thus,

$$r(K)=\begin{cases} \frac{4K-2}{2K-0.7} & 0.5<K\leqslant 2 \\ \frac{K}{1.1} & K>2 \end{cases}$$

When $K=0.7$, $r(K)=8/7$

Problem 3. 生产销售计划

一奶制品加工厂用牛奶生产A1,A2两种普通奶制品，和B1，B2两种高级奶制品，B1,B1分别是由A1,A2深加工开发得到的。已知每1桶牛奶可以在甲类设备上用12小时加工成3公斤A1，或者在乙类设备上用8小时加工成4公斤A2. 深加工时，用2小时并花1.5元加工费，可将1公斤A1加工成0.8公斤B1，也可将1公斤A2加工成0.75公斤B2. 根据市场需求，生产的4种奶制品全部能售出，且每公斤A1,A2,B1,B2 获利分别为12元，8元，22元和16元.
现在加工厂每天能得到50桶牛奶的供应，每天正式工人总的劳动时间最多为480小时，并且乙类设备和深加工设备的加工能力没有限制，但甲类设备的数量相对较少，每天至多能加工100公斤A1.
试为该厂制订一个生产销售计划，使每天的净利润最大。

由题，一桶牛奶的获利为

$$\begin{aligned} & A1: 3\times 12=36 && A2: 4\times 8=32 \\ & B1: 3\times(0.8\times 22-1.5)=48.3 && B2: 4\times(0.75\times 16-1.5)=42 \end{aligned}$$

设用于生产A1,A2,B1,B2的牛奶桶数为x_1, x_2, x_3, x_4，则有如下优化问题

$$
\begin{aligned}
\max \quad & 36x_1 + 32x_2 + 48.3x_3 + 42x_4 \\
s.t. \quad & x_1 + x_2 + x_3 + x_4 \leqslant 50 \\
& 12x_1 + 8x_2 + 18x_3 + 16x_4 \leqslant 480 \\
& x_1 + x_3 \leqslant 100/3 \\
& x_1, x_2, x_3, x_4 \geqslant 0
\end{aligned}
$$

在Lingo中运行得到如下结果：

Global optimal solution found.	
Objective value:	1730.400
Infeasibilities:	0.000000
Total solver iterations:	2
Elapsed runtime seconds:	0.34
Model Class:	LP
Total variables:	4
Nonlinear variables:	0
Integer variables:	0
Total constraints:	4
Nonlinear constraints:	0
Total nonzeros:	14
Nonlinear nonzeros:	0

Variable	Value	Reduced Cost
X1	0.000000	2.520000
X2	42.00000	0.000000
X3	8.000000	0.000000
X4	0.000000	3.040000

Row	Slack or Surplus	Dual Price
1	1730.400	1.000000
2	0.000000	18.96000
3	0.000000	1.630000
4	25.33333	0.000000

1) 若投资15元可以增加供应1桶牛奶，应否作这项投资？

   由以上结果知对偶问题的最优解为$y^* = (18.96; 1.63; 0)$，故每增加1桶牛奶，收益增加$18.96 > 15$，从而应该作这项投资。

2) 若可以聘用临时工人以增加劳动时间，付给临时工人的工资最多是每小时几元？

   同样由对偶问题的最优解$y^* = (18.96; 1.63; 0)$知，付给临时工人的工资最多是每小时1.63元。

3) 如果B1，B2的获利经常有10%的波动，波动后是否需要制订新的生产销售计划？

Objective Coefficient Ranges:

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	36.00000	2.520000	INFINITY
X2	32.00000	16.30000	4.200000
X3	48.30000	23.70000	3.800000
X4	42.00000	3.040000	INFINITY

Righthand Side Ranges:

Row	Current RHS	Allowable Increase	Allowable Decrease
2	50.00000	10.00000	23.33333
3	480.0000	253.3333	80.00000
4	33.33333	INFINITY	25.33333

由以上Lingo运行结果知，B1的变动范围为$-3.8 \sim 23.7$，B2的变动范围为$-\infty \sim 3.04$

$$3.8 < 48.3 \times 10\% = 4.83$$
$$3.04 < 42 \times 10\% = 4.2$$

故需要制定新的生产销售计划。