Chapter 1, Section 1.2

Joseph Song

1 Vector Spaces

A vector taught in previous courses may have defined one as a quantity with a magnitude and direction, commonly represented as an arrow. Here, we define what a vector is rigorously.

Definition (Vector Space). A vector space V over a field F is a set in which two operations, addition and scalar multiplication, are defined so that for each pair of elements $x, y \in V$, the sum x + y is also in V, and for any scalar $c \in F$, the product cx is also an element of V, such that the following axioms hold:

- (1) For all $x, y \in V$, x + y = y + x
- (2) For all $x, y, z \in V$, (x + y) + z = x + (y + z)
- (3) There exists the zero vector in V, denoted 0, such that x + 0 = x, $\forall x \in V$
- (4) For each $x \in V$, there exists $y \in V$ such that x + y = 0
- (5) For each x in V, 1x = x
- (6) For each scalar $a, b \in F$ and each element $x \in V$, (ab)x = a(bx)
- (7) For each scalar $a \in F$, and each pair of elements $x, y \in V$, a(x+y) = ax + ay
- (8) For each pair of scalars $a, b \in F$ and each element $x \in V$, (a + b)x = ax + bx
- \star Elements of F are scalars, elements of V are vectors. In most cases, the vector space is over the field $\mathbb R$ or $\mathbb C$.

Vector Space Examples

- N-tuples: $(a_1, a_2, a_3, \ldots, a_n), a_n \in F$. The set of n-tuples with n entries from F is the set F^n .
- Matrices: An $m \times n$ matrix with entries a_{ij} , where m is the row and n is the column.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Functions: Let S be a nonempty set and F be any field. Denote F(S, F) as the set of all functions that map from S to F.
- Polynomials: Denote P(F) as the set of all polynomials with coefficients from F.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Let the zero polynomial to have degree -1

 \star These are all vector spaces because they are sets that satisfy the addition and scalar multiplication operations.

Theorem (Cancellation Law for Vector Addition). If $x, y, z \in V$, such that x + z = y + z, then x = y. Proof. By (4), there exists a vector v such that z + v = 0. Thus,

$$x + 0 = x$$
 (3) = $x + (z + v) = (x + z) + v$ (2) = $(y + z) + v = y + (z + v)$ (2) = $y + 0 = y$ (3)

Corollary. The vector 0 in (3) is unique.

Proof. Consider 0'.

$$0 = 0 + 0' \quad (3) = 0'$$