Least Upper Bound - Supremum

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1 Axiom of Completeness

Axiom of Completeness - Every nonempty set of real numbers that is bounded above has a least upper bound.

Definition (Upper bound). *Set $A \subseteq R$ is bounded above if there exists a real number b such that it is greater than all the values in A. ($\exists\ b \in \mathbb{R} : a \leq b, \forall\ a \in A$)*

Definition (Least Upper Bound - Supremum). *The least upper bound is an element s in A, such that:*

1. *s is an upper bound*
2. *For any other upper bound, b, $s \leq b$*

This means that out of all the bounds, s is the smallest. The supremum is denoted as $s = \sup(A)$.

Example 1.1. *For a set A and a number $c \in \mathbb{R}$, define*

$$c + A = \{c + a : a \in A\}$$

and show $\sup(c + A) = c + \sup A$.

Proof. Let s be the least upper bound of $\sup(A)$. By definition, this means that $a \leq s, \forall\ a \in A$. Addting $c \implies c + a \leq c + s$. Notice that the $c + a$ for all a is just the set $c + A$, so $c + s \implies c + supA$ is an upper bound. For (2), let b be another arbitrary upper bound for c + A, meaning $b \geq c+a$. $b \geq c+a \implies b-c \geq a$, so $b - c$ is an upper bound for set A. Since we assumed s is the least upper bound, $s \leq b - c \implies c + s \leq b$, but b is an upper bound for c + A, therefore $c + s$ is the least upper bound. □

Example 1.2. *Given two nonempty, bounded above sets , A and B, define $A+B = \{a+b : a \in A$ and $b \in B\}$. Prove* $\sup(A + B) = \sup A + \sup B$.

Proof. Let $s = \sup A$ and $t = \sup B$. We first show that $s + t$ is an upper bound for $A + B$. $a \leq s, \forall\ a \in A$, and $b \leq t, \forall\ b \in B$. This implies that $a + b \leq s + t$. Let u be an arbitrary bound for $A + B$. $a + b \leq u \implies b \leq u - a$, and since t is the least upper bound, this implies $t \leq u - a$. Same logic can be applied to get $a \leq u - b \implies s \leq u - b$. $s + t \leq 2u - a - b$, where $2u - a - b$ is an upper bound for A + B, therefore $s + t$ is the least upper bound. □