

# Chapter 1, Section 1.4

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## 1 Linear Combinations

**Definition** (Linear Combination). *Let  $S$  be a nonempty subset of the vector space  $V$ . A vector  $v \in V$  is a **linear combination** of vectors in  $S$  if there exists a finite numbers of vectors  $u_1, u_2, \dots, u_n \in S$  and coefficient scalars  $c_1, c_2, \dots, c_n \in \mathbb{F}$  such that  $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$ .*

Notice how the zero vector exists in any  $V$ , so the zero vector is a linear combination of the vectors in any  $S$ .

**Example 1.1** (Polynomials).

Is  $2x^3 - 2x^2 + 12x - 6$  a linear combination of  $x^3 - 2x^2 - 5x - 3$  and  $3x^3 - 5x^2 - 4x - 9$ ?

There must exists scalars  $a$  and  $b$  such that:

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

Distributing:

$$2x^3 - 2x^2 + 12x - 6 = ax^3 - 2ax^2 - 5ax - 3a + (3bx^3 - 5bx^2 - 4bx - 9b)$$

Setting the coefficients in a system of equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= 2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6 \end{aligned}$$

Solve for  $a$  and  $b$ :

$$\begin{array}{r} 2a + 6b = 4 \\ -2a - 5b = 2 \\ \hline b = 2 \end{array}$$

$$a + 3(2) = 2 \implies a = -4$$

Plug into the other equations:

$$-2(-4) - 5(2) = 8 - 10 = -2 \neq 2$$

We have reached a contradiction, so it is NOT a linear combination

**Example 1.2** (Vectors).

Is  $(2,1,9)$  a linear combination of  $(1,2,0)$  and  $(0,-1,3)$ ?

$$\begin{aligned} a &= 2 \\ 2a - b &= 1 \\ 3b &= 9 \end{aligned}$$

$a = 2$  and  $b = 3$ , so we verify:

$$2(2) - 1(3) = 1$$

So it is a linear combination.

## 2 Span

**Definition.** Let  $S$  be a nonempty subset of the vector space  $V$ . The **span** of  $S$ , denoted  $\text{span}(S)$ , is the set of all linear combinations of the vectors in  $S$ . Define  $\text{span}(\emptyset) = 0$ .

**Theorem.** The span of any subset of  $V$ ,  $S$ , is a subspace of  $V$ . Additionally,