Induction

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1 Induction

Let $S \subseteq \mathbb{N}$. If S contains 1, and for any element $n \in S$, $n+1 \in S$, then $S = \mathbb{N}$.

Proof by induction: show the base case ($n = 0, 1$). Then with the inductive hypothesis, the case where it holds for n, show it holds for $n+1$.

Example 1.1. *Let $y_1 = 6$, and define $y_{n+1} = (2y_n - 6)/3$. Prove $y_n > -6, \forall n \in \mathbb{N}$.*

Proof. First, we prove the base case, ($n = 1$). In this case, it is given. $y_1 = 6 > -6$. Then, with the inductive hypothesis, show it holds for $n+1$. The inductive hypothesis is we claim $y_n > -6$, because it holds for n. Using that, derive $y_{n+1} > -6$ as well to complete the proof.

With some scratch work:

$$\begin{aligned} & y_{n+1} = (2y_n - 6)/3 > -6 \\ \Rightarrow \quad & 2y_n - 6 > -18 \\ \Rightarrow \quad & 2y_n > -12 \\ \Rightarrow \quad & y_n > -6 \end{aligned}$$

We get $y_n > -6$ at the end, but we want to use that in the beginning. So our inductive step goes like this:

$$\begin{aligned} & y_n > -6 \\ \Rightarrow \quad & 2y_n > -12 \\ \Rightarrow \quad & 2y_n - 6 > -18 \\ \Rightarrow \quad & (2y_n - 6)/3 > -6 \\ \Rightarrow \quad & (2y_n - 6)/3 = y_{n+1} \\ \Rightarrow \quad & y_{n+1} > -6 \end{aligned}$$

$\square$

Use induction again to show the sequence is decreasing.

Proof. Want to show that $y_n > y_{n+1}$ for all n in the natural numbers.
The base case: $y_1 > y_2 \implies 6 > 2$.
Now with $y_n > y_{n+1}$, show $y_{n+1} > y_{n+2}$.

$$\begin{aligned} & y_n > y_{n+1} \\ \Rightarrow \quad & 2y_n > 2y_{n+1} \\ \Rightarrow \quad & 2y_n - 6 > 2y_{n+1} - 6 \\ \Rightarrow \quad & (2y_n - 6)/3 > (2y_{n+1} - 6)/3 \\ \Rightarrow \quad & y_{n+1} > y_{n+2} \end{aligned}$$

$\square$