Chapter 1, Section 1.2

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1 Vector Spaces

A vector taught in previous courses may have defined one as a quantity with a magnitude and direction, commonly represented as an arrow. Here, we define what a vector is rigorously.

Definition 1.1 (Vector Space). *A* **vector space** *V over a field F is a set in which two operations, addition and scalar multiplication, are defined so that for each pair of elements $x, y \in V$, the sum $x + y$ is also in V, and for any scalar $c \in F$, the product cx is also an element of V, such that the following axioms hold:*

(1) For all $x, y \in V, x + y = y + x$

(2) For all $x, y, z \in V, (x + y) + z = x + (y + z)$

(3) There exists the zero vector in V, denoted 0, such that $x + 0 = x, \forall x \in V$

(4) For each $x \in V, \exists y \in V$ such that $x + y = 0$

(5) For each x in V, $1x = x$

(6) For each scalar $a, b \in F$ and each element $x \in V$, $(ab)x = a(bx)$

(7) For each element $a \in F$, and each pair of elements $x, y \in V$, $a(x + y) = ax + ay$

(8) For each pair of scalars $a, b \in F$ and each element $x \in V, (a + b)x = ax + bx$

These are the axioms that build the definition of a vector space.
Elements of F are scalars, elements of V are vectors. Most cases, the vector space is over the field $\mathbb{R}$ or $\mathbb{C}$