

# Chapter 1, Section 1.2

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## 1 Vector Spaces

A vector taught in previous courses may have defined one as a quantity with a magnitude and direction, commonly represented as an arrow. Here, we define what a vector is rigorously.

**Definition (Vector Space).** A **vector space**  $V$  over a field  $F$  is a set in which two operations, addition and scalar multiplication, are defined so that for each pair of elements  $x, y \in V$ , the sum  $x + y$  is also in  $V$ , and for any scalar  $c \in F$ , the product  $cx$  is also an element of  $V$ , such that the following axioms hold:

- (1) For all  $x, y \in V$ ,  $x + y = y + x$
- (2) For all  $x, y, z \in V$ ,  $(x + y) + z = x + (y + z)$
- (3) There exists the zero vector in  $V$ , denoted  $0$ , such that  $x + 0 = x$ ,  $\forall x \in V$
- (4) For each  $x \in V$ , there exists  $y \in V$  such that  $x + y = 0$
- (5) For each  $x$  in  $V$ ,  $1x = x$
- (6) For each scalar  $a, b \in F$  and each element  $x \in V$ ,  $(ab)x = a(bx)$
- (7) For each scalar  $a \in F$ , and each pair of elements  $x, y \in V$ ,  $a(x + y) = ax + ay$
- (8) For each pair of scalars  $a, b \in F$  and each element  $x \in V$ ,  $(a + b)x = ax + bx$

★ Elements of  $F$  are scalars, elements of  $V$  are vectors. In most cases, the vector space is over the field  $\mathbb{R}$  or  $\mathbb{C}$ .

### Vector Space Examples

- **N-tuples:**  $(a_1, a_2, a_3, \dots, a_n)$ ,  $a_n \in F$ . The set of n-tuples with  $n$  entries from  $F$  is the set  $F^n$ .
- **Matrices:** An  $m \times n$  matrix with entries  $a_{ij}$ , where  $m$  is the row and  $n$  is the column.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Functions:** Let  $S$  be a nonempty set and  $F$  be any field. Denote  $F(S, F)$  as the set of all functions that map from  $S$  to  $F$ .
- **Polynomials:** Denote  $P(F)$  as the set of all polynomials with coefficients from  $F$ .

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Let the zero polynomial to have degree -1

★ These are all vector spaces because they are sets that satisfy the addition and scalar multiplication operations.

**Theorem** (Cancellation Law for Vector Addition). *If  $x, y, z \in V$ , such that  $x + z = y + z$ , then  $x = y$ .*

*Proof.* By (4), there exists a vector  $v$  such that  $z + v = 0$ .

Thus,

$$x + 0 = x \quad (3) = x + (z + v) = (x + z) + v \quad (2) = (y + z) + v = y + (z + v) \quad (2) = y + 0 = y \quad (3)$$

□

**Corollary.** *The vector 0 in (3) is unique.*

*Proof.* Consider  $0'$ .

$$0 = 0 + 0' \quad (3) = 0'$$

□