

# Chapter 1, Section 1.3

Joseph Song

## 1 Subspaces

**Definition** (Subspace). *A subset  $W$  of a vector space  $V$  over a field  $F$  is a **subspace** if  $W$  is a vector space over  $F$  with the operations defined on  $V$ . The subspace  $\{0\}$  is called the **zero subspace** of  $V$ .*

The following is the theorem to prove if a subset is a subspace of  $V$ :

**Theorem.** *Let  $W$  be a subset of the vector space  $V$ . Then  $W$  is a subspace of  $V$  if and only if the following hold:*

- (1)  $0 \in W$
- (2)  $x + y \in W$ , for  $x, y \in W$
- (3)  $cx \in W$ , for  $c \in \mathbb{F}$  and  $x \in W$

**Example 1.1** (Polynomials).

For  $n \in \mathbb{N}$ , let  $P_n(F)$  be the set containing all the polynomials in  $P(F)$  having degree less than or equal to  $n$ .

To show that it is a subspace, we use the theorem and see if the three condition holds. The zero polynomial has degree  $-1 \leq n, \forall n \in \mathbb{N}$ . The sum of two polynomials with degree less than or equal to  $n$  still has the degree less than or equal to  $n$ , along with scalar multiplication.

**Example 1.2** (Functions).

Let  $C(R)$  denote the set of all continuous real-valued functions defined on  $\mathbb{R}$ . Claim  $C(R)$  is a subspace of  $F(R, R)$ .

The zero function of  $F(R, R)$  is just  $f(t) = 0, \forall t \in \mathbb{R}$ . But since constant functions are continuous, it is also an element of  $C(R)$ . Also notice that the sum of two continuous functions is continuous, along with the scalar multiplication of a continuous function, meaning they also belong in  $C(R)$ . So  $C(R)$  is closed under addition and scalar multiplication, hence is a subspace of  $F(R, R)$ .