Chapter 1, Section 1.2

Joseph Song

1 Vector Spaces

A vector taught in previous courses may have defined one as a quantity with a magnitude and direction, commonly represented as an arrow. Here, we define what a vector is rigorously.

Definition (Vector Space). *A* ***vector space*** *V over a field F is a set in which two operations, addition and scalar multiplication, are defined so that for each pair of elements $x, y \in V$, the sum $x + y$ is also in V, and for any scalar $c \in F$, the product cx is also an element of V, such that the following axioms hold:*

(1) For all $x, y \in V$, $x + y = y + x$

(2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$

(3) There exists the zero vector in V, denoted 0, such that $x + 0 = x$, $\forall x \in V$

(4) For each $x \in V$, there exists $y \in V$ such that $x + y = 0$

(5) For each x in V, $1x = x$

(6) For each scalar $a, b \in F$ and each element $x \in V$, $(ab)x = a(bx)$

(7) For each scalar $a \in F$, and each pair of elements $x, y \in V$, $a(x + y) = ax + ay$

(8) For each pair of scalars $a, b \in F$ and each element $x \in V$, $(a + b)x = ax + bx$

$\star$ Elements of F are scalars, elements of V are vectors. In most cases, the vector space is over the field $\mathbb{R}$ or $\mathbb{C}$.

Vector Space Examples

- **N-tuples**: $(a_1, a_2, a_3, \ldots, a_n)$, $a_n \in F$. The set of n-tuples with n entries from F is the set F^n.
- **Matrices**: An $m \times n$ matrix with entries a_{ij}, where m is the row and n is the column.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Functions**: Let S be a nonempty set and F be any field. Denote $F(S, F)$ as the set of all functions that map from S to F.
- **Polynomials**: Denote $P(F)$ as the set of all polynomials with coefficients from F.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  Let the zero polynomial to have degree -1

$\star$ These are all vector spaces because they are sets that satisfy the addition and scalar multiplication operations.

Theorem (Cancellation Law for Vector Addition)**.** *If $x, y, z \in V$, such that $x + z = y + z$, then $x = y$.*

Proof. By (4), there exists a vector v such that $z + v = 0$.
Thus,

$$x + 0 = x \quad (3) = x + (z + v) = (x + z) + v \quad (2) = (y + z) + v = y + (z + v) \quad (2) = y + 0 = y \quad (3)$$

$\square$

Corollary. *The vector 0 in (3) is unique.*

Proof. Consider $0'$.

$$0 = 0 + 0' \quad (3) = 0'$$

$\square$