Chapter 1, Section 1.4

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1 Linear Combinations

Definition (Linear Combination)**.** *Let S be a nonempty subset of the vector space V. A vector $v \in V$ is a* **linear combination** *of vectors in S if there exists a finite numbers of vectors $u_1, u_2, \ldots, u_n \in S$ and coefficient scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.*

Notice how the zero vector exists in any V, so the zero vector is a linear combination of the vectors in any S.

Example 1.1 (Polynomials)**.**

Is $2x^3 - 2x^2 + 12x - 6$ a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$?
There must exists scalars a and b such that:

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

Distributing:

$$2x^3 - 2x^2 + 12x - 6 = ax^3 - 2ax^2 - 5ax - 3a + (3bx^3 - 5bx^2 - 4bx - 9b)$$

Setting the coefficients in a system of equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= 2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6 \end{aligned}$$

Solve for a and b:

$$\begin{array}{c} 2a + 6b = 4 \\ -2a - 5b = 2 \\ \hline b = 2 \end{array}$$

$$a + 3(2) = 2 \implies a = -4$$

Plug into the other equations:

$$-2(-4) - 5(2) = 8 - 10 = -2 \neq 2$$

We have reached a contradiction, so it is NOT a linear combination

Example 1.2 (Vectors)**.**

Is (2,1,9) a linear combination of (1,2,0) and (0,-1,3)?

$$\begin{aligned} a &= 2 \\ 2a - b &= 1 \\ 3b &= 9 \end{aligned}$$

$a = 2$ and $b = 3$, so we verify:

$$2(2) - 1(3) = 1$$

So it is a linear combination.

2 Span

Definition. *Let S be a nonempty subset of the vector space V. The* **span** *of S, denoted span(S), is the set of all linear combinations of the vectors in S. Define span$(\emptyset) = 0$.*

Theorem. *The span of any subset S of V, is a subspace of V. Additionally, any subspace of V that contains S must also contain span(S).*

Proof. Let $z \in \text{span}(S)$. Then $0z = 0$. Let $x, y \in \text{span}(S)$. That means there exists $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in S$ such that $x = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ for scalars $a_n, b_n \in \mathbb{F}$.
Then $x + y = a_1u_1 + a_2, u_2 + \cdots + a_nu_n + b_1v_1 + b_2v_2 + \cdots + b_nv_n$ is an element of span(S).
For any $c \in \mathbb{F}$, $cx = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_n)u_n$ which is an element of span(S)

Let W be a subspace. Let $w \in \text{span}(S)$. Then $w = \sum a_nu_n$ for $u_n \in S$. But since $S \in W$, $u_n \in W \implies w \in W$, so span$(S) \subseteq W$.

$\square$

Definition. *Subset S spans V if* span$(S) = V$

Prove that span$(\{x\}) = \{ax : a \in \mathbb{F}\}$ for any vector x.

Proof. Want to show $\forall v \in \text{span}(\{x\}), v \in \{ax : a \in \mathbb{F}\}$. $\forall v \in \text{span}(\{x\}), \exists c \in \mathbb{F}$ such that $v = cx$, which is the definition of span$(\{x\})$

$\square$