# Chapter 1, Section 1.4

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## 1 Linear Combinations

**Definition** (Linear Combination)**.** *Let $S$ be a nonempty subset of the vector space $V$. A vector $v \in V$ is a* **linear combination** *of vectors in $S$ if there exists a finite numbers of vectors $u_1, u_2, \ldots, u_n \in S$ and coefficient scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.*

Notice how the zero vector exists in any $V$, so the zero vector is a linear combination of the vectors in any $S$.

**Example 1.1** (Polynomials)**.**

Is $2x^3 - 2x^2 + 12x - 6$ a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$?
There must exists scalars $a$ and $b$ such that:

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

Distributing:

$$2x^3 - 2x^2 + 12x - 6 = ax^3 - 2ax^2 - 5ax - 3a + (3bx^3 - 5bx^2 - 4bx - 9b)$$

Setting the coefficients in a system of equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= 2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6 \end{aligned}$$

Solve for $a$ and $b$:

$$\begin{array}{c} 2a + 6b = 4 \\ -2a - 5b = 2 \\ \hline b = 2 \end{array}$$

$$a + 3(2) = 2 \implies a = -4$$

Plug into the other equations:

$$-2(-4) - 5(2) = 8 - 10 = -2 \neq 2$$

We have reached a contradiction, so it is NOT a linear combination

**Example 1.2** (Vectors)**.**

Is (2,1,9) a linear combination of (1,2,0) and (0,-1,3)?

$$\begin{aligned} a &= 2 \\ 2a - b &= 1 \\ 3b &= 9 \end{aligned}$$

$a = 2$ and $b = 3$, so we verify:

$$2(2) - 1(3) = 1$$

So it is a linear combination.

## 2 Span

**Definition.** *Let $S$ be a nonempty subset of the vector space $V$. The* **span** *of $S$, denoted span$(S)$, is the set of all linear combinations of the vectors in $S$. Define span$(\emptyset) = 0$.*

**Theorem.** *The span of any subset of $V, S$, is a subspace of $V$. Additionally, i think i coooked*