

Chapter 1, Section 1.4

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1 Linear Combinations

Definition (Linear Combination). *Let S be a nonempty subset of the vector space V . A vector $v \in V$ is a **linear combination** of vectors in S if there exists a finite numbers of vectors $u_1, u_2, \dots, u_n \in S$ and coefficient scalars $c_1, c_2, \dots, c_n \in \mathbb{F}$ such that $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$.*

Notice how the zero vector exists in any V , so the zero vector is a linear combination of the vectors in any S .

Example 1.1 (Polynomials).

Is $2x^3 - 2x^2 + 12x - 6$ a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$?

There must exists scalars a and b such that:

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

Distributing:

$$2x^3 - 2x^2 + 12x - 6 = ax^3 - 2ax^2 - 5ax - 3a + (3bx^3 - 5bx^2 - 4bx - 9b)$$

Setting the coefficients in a system of equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= 2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6 \end{aligned}$$

Solve for a and b :

$$\begin{array}{r} 2a + 6b = 4 \\ -2a - 5b = 2 \\ \hline b = 2 \end{array}$$

$$a + 3(2) = 2 \implies a = -4$$

Plug into the other equations:

$$-2(-4) - 5(2) = 8 - 10 = -2 \neq 2$$

We have reached a contradiction, so it is NOT a linear combination

Example 1.2 (Vectors).

Is $(2,1,9)$ a linear combination of $(1,2,0)$ and $(0,-1,3)$?

$$\begin{aligned} a &= 2 \\ 2a - b &= 1 \\ 3b &= 9 \end{aligned}$$

$a = 2$ and $b = 3$, so we verify:

$$2(2) - 1(3) = 1$$

So it is a linear combination.

2 Span

Definition. Let S be a nonempty subset of the vector space V . The **span** of S , denoted $\text{span}(S)$, is the set of all linear combinations of the vectors in S . Define $\text{span}(\emptyset) = 0$.

Theorem. The span of any subset of V , S , is a subspace of V . Additionally,