Comp Stats HW 2

1a) $f_x(x) = x^{-1}$ Domain $e^{-1} < x < 1$

$E(x) = \int x f(x)\,dx = \int_{e^{-1}}^{1} x \cdot x^{-1}\,dx = \int_{e^{-1}}^{1} 1\,dx$

$= x\Big|_{e^{-1}}^{1} = 1 - e^{-1} \approx 0.6321$

$Var(x) = E(x^2) - [E(x)]^2$

$[E(x)]^2 = (1-e^{-1})^2 \approx 0.3996$

$E(x^2) = \int_{e^{-1}}^{1} x^2 x^{-1}\,dx = \int_{e^{-1}}^{1} x\,dx$

$= \frac{1}{2}x^2\Big|_{e^{-1}}^{1} = \frac{1}{2} - \frac{1}{2}e^{-2} \approx 0.3161$

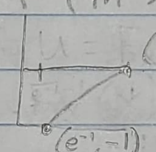
$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{1}{2}(1-e^{-2}) - (1-e^{-1})^2$

≈ 0.03276

1b) $f_x(x) = x^{-1}, \; e^{-1} < x < 1$

$F_x(x) = \ln(x), \; e^{-1} < x < 1$



$f^{-1}(\ln x) = e^{x} \;\{-1 < x < 0\}$

$X = u - 1$

$F_x^{-1}(u) = e^{u-1}$

1c) The inverse transform method could be applied. First, generate a random observation $u \sim U(0,1)$ then calculate $x = F_x^{-1}(u) = e^{u-1}$ which will be a random observation of $f_x(x)$.

1e) mean = 0.635

Variance = 0.0330

2a) $Exp(\theta)$ has pdf $\frac{1}{\theta}e^{-x/\theta}$ $x > 0, \theta > 0$

$V = \theta \log(1/U)$

$F_V(v) = P(\theta \log(\frac{1}{U}) \le v)$

$= P(-\theta \log(U) \le v)$

$= P(\log(U) \ge -v/\theta)$

$= P(U \ge \exp(-v/\theta))$

$= 1 - P(U < \exp(-v/\theta))$

$F_V(v) = 1 - \exp(-v/\theta)$

2b) $F_V(v) = 1 - \exp(-v/\theta)$

find $F_V^{-1}(v)$

$$x = 1 - \exp(-y/\theta)$$
$$\ln(1-x) = -y/\theta$$
$$-\theta \ln(1-x) = y$$
$$F_V^{-1}(v) = -\theta \ln(1-v)$$

2d) Waiting times b/w i and $i+1$ events are independent. The number of events in an hour is Poisson dist$^{\underline{n}}$ with $\lambda = 5$. The waiting time until the first event is exponential dist$^{\underline{n}}$ with $\theta = 1/5$

Poisson(λ) equals the number of samples from exp($1/\lambda$) which sum to ≤ 1.

```
cummulative_sum = 0
count = 0
while TRUE {
  e = rexp(1/λ)
  cummulative_sum += e
  if cummulative_sum <= 1
     count ++
  else
     break
  end
end
```

① Generate observation from Exponential ($1/\lambda$)
② Count the # of observations which occur in 1 unit of time.

count is a random observation from the Poisson distribution.

2f) See Appendix

3a) $f(x) = \frac{\alpha}{2} e^{-\alpha|x-\mu|}$ $\quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \alpha > 0$

$P(X \le x) = \int_{-\infty}^{x} \frac{\alpha}{2} e^{-\alpha|x-\mu|} \, dx$ with $x < \mu$

$= \int_{-\infty}^{x} \frac{\alpha}{2} e^{\alpha(x-\mu)} \, dx$

U substitution $\quad u = \alpha(x-\mu)$

$du = \alpha$

$= \int \frac{1}{2} e^{u} \, du$

$\frac{1}{2} e^{u}$

$P(X \le x) = \frac{1}{2} e^{\alpha(x-\mu)} \Big|_{-\infty}^{x} = \frac{1}{2} e^{\alpha(x-\mu)}$ when $x < \mu$

$P(X \le x) = P(X \le \mu) + P(\mu \le X \le x)$ when $x > \mu$

$X < \mu$ so $|x-\mu| = -(x-\mu)$ $\qquad x > \mu$ so $|x-\mu| = (x-\mu)$

$P(X \le x) = \int_{-\infty}^{\mu} \frac{\alpha}{2} e^{\alpha(x-\mu)} \, dx + \int_{\mu}^{x} \frac{\alpha}{2} e^{-\alpha(x-\mu)} \, dx$

$u = \alpha(x-\mu) \quad + \quad u = -\alpha(x-\mu)$

$du = -\alpha \quad + \quad du = -\alpha$

$\int \frac{1}{2} e^{u} \, du + \int -\frac{1}{2} e^{u} \, du$

$\frac{1}{2} e^{u} + -\frac{1}{2} e^{u}$

$\frac{1}{2} e^{\alpha(x-\mu)} \Big|_{-\infty}^{\mu} + -\frac{1}{2} e^{-\alpha(x-\mu)} \Big|_{\mu}^{x}$

$\frac{1}{2} e^{0} + \frac{1}{2} e^{-\infty} + -\frac{1}{2} e^{-\alpha(x-\mu)} + \frac{1}{2} e^{0}$

$\frac{1}{2} + 0 + -\frac{1}{2} e^{-\alpha(x-\mu)} + \frac{1}{2}$

$P(X \le x) = 1 - \frac{1}{2} e^{-\alpha(x-\mu)}$

3b) Laplace cdf from wikipedia is:

$$F(x) = \begin{cases} \frac{1}{2}\exp(\alpha(x-\mu)) & \text{if } x < \mu \\ 1 - \frac{1}{2}\exp(-\alpha(x-\mu)) & \text{if } x \ge \mu \end{cases}$$

$F^{-1}(x)$ $\quad x = \frac{1}{2}\exp(\alpha(y-\mu)) \qquad x = 1 - \frac{1}{2}\exp(-\alpha(y-\mu))$

$\ln 2x = \alpha(y-\mu) \qquad \ln(2-2x) = -\alpha(y-\mu)$

$\frac{1}{\alpha}\ln 2x + \mu = y \qquad -\frac{1}{\alpha}\ln(2-2x) + \mu = y$

$$F^{-1}(x) = \begin{cases} \mu + \frac{1}{\alpha}\ln(2x) & \text{if } x < \frac{1}{2} \\ \mu - \frac{1}{\alpha}\ln(2-2x) & \text{if } x \ge \frac{1}{2} \end{cases}$$

4a) laplace pdf $= \frac{\alpha}{2} e^{-\alpha|x-\mu|} = g(t)$

normal pdf $= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = f(t)$

$$\frac{f(t)}{g(t)} = \sqrt{\frac{2}{\pi\sigma^2}} \cdot \alpha^{-1} \frac{\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}{\exp(-\alpha|x-\mu|)}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha \cdot \sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + \alpha|x-\mu|\right)$$

normal $\sim N(0,1)$ $\mu = 0$ and $\sigma^2 = 1$

laplace $\sim L(0,\alpha)$ $\mu = 0$

$$\frac{f(t)}{g(t)} = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \exp\left(-\frac{x^2}{2} + \alpha x\right) = \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha x - x^2/2}$$

Show: $\sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha x - \frac{x^2}{2}} \leq \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha^2/2}$

$$\alpha x - \tfrac{1}{2}x^2 \leq \tfrac{1}{2}\alpha^2$$

$$-\alpha^2 + 2\alpha x - x^2 \leq 0$$

$$(-\alpha + x)(\alpha - x) \leq 0$$

$$(x-\alpha)(\alpha - x) \leq 0$$

Exactly one of $(x-\alpha)$ or $(\alpha - x)$ is negative at all times. The inequality holds true for all x and α.

Proven: $\frac{f(t)}{g(t)} \leq \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha^2/2}$

4b) Find minimum of $\sqrt{2/\pi}\,\alpha^{-1} e^{\alpha^2/2}$

Derivative! $\frac{d}{d\alpha}\left(\sqrt{\frac{2}{\pi}}\,\alpha^{-1} e^{\alpha^2/2}\right) = 0$

$$\sqrt{\frac{2}{\pi}}\left[-\alpha^{-2} e^{\alpha^2/2} + \alpha^{-1}\cdot\alpha\cdot e^{\alpha^2/2}\right] = 0$$

$$-\alpha^{-2} e^{\alpha^2/2} + e^{\alpha^2/2} = 0$$

$$1 - \alpha^{-2} = 0$$

$$1 = \alpha^{-2}$$

$$\alpha = 1$$

To see if $\alpha = 1$ is a minimum or maximum I need to evaluate at nearby values.

$f(\alpha = 0.95) = 1.319$

$f(\alpha = 1) = 1.315$ $\quad\alpha = 1$ is a minimum

$f(\alpha = 1.05) = 1.319$

4d) Theoretical probability of success

$P(\text{accept}) = \frac{1}{c} = \sqrt{\pi/2e} \approx 0.760$

Experimental probability of success [reference line 85 of R code] ≈ 0.7535

These values are close together, verifying the validity that the $P(\text{acceptance}) = 1/c = \sqrt{\pi/2e}$

4e) Let $X \sim N(0,1)$ then I can transform X as: $Y = \sigma X + \mu$ to generate a random variable $Y \sim N(\mu, \sigma^2)$

$\sigma^2 = Var(Y) = Var(\sigma X + \mu)$

$\sigma^2 = \sigma^2\, Var(X)$

$\sigma^2 = \sigma^2(1)$

✓

$\mu = E(Y) = E(\sigma X + \mu)$

$= \sigma E(X) + \mu$

$= \sigma(0) + \mu$

$\mu = \mu$

✓

EXTRA CREDIT

Prove $\Theta \cdot \log\left(\Sigma_{i=1}^{\infty} U^k\right) \sim Exp(\Theta)$ where $U \sim \text{uniform}(0,1)$

$\Sigma_{i=0}^{\infty} U^k = U^0 + U^1 + \ldots$

Apply the geometric series formula.

$$S_n = a\frac{1-r^n}{1-r} = u\frac{1-u^\infty}{1-u} = \frac{1}{1-u}$$

$\Theta \log\left(\frac{u}{1-u}\right) \sim Exp(\Theta)$

Use inverse transform method to generate a random variable of Exp(θ)

Exp(θ) has pdf $f(x) = \theta e^{-\theta x}$

and cdf $F(x) = 1 - e^{-\theta x}$

$$F^{-1}(x): \quad x = 1 - e^{-\theta y}$$

$$1 - x = e^{-\theta y}$$

$$\frac{1}{1-x} = e^{\theta y}$$

$$\frac{1}{\theta}\log\left(\frac{1}{1-x}\right) = y$$

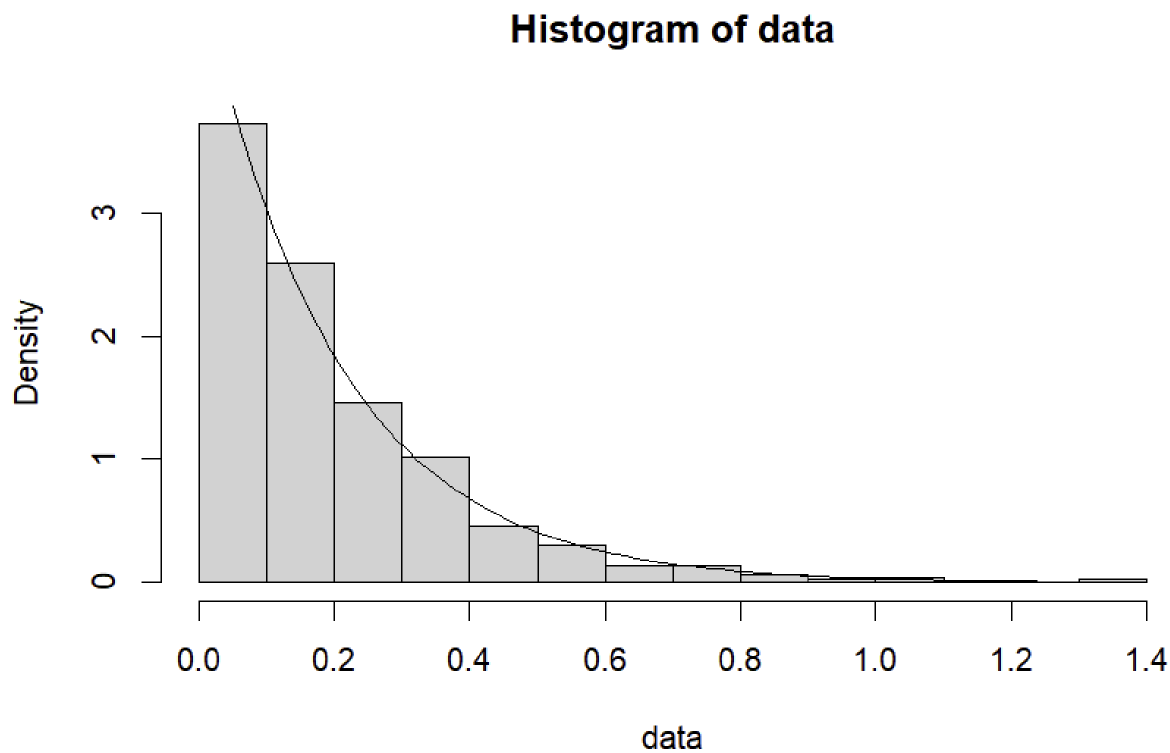
I could multiply Exp(θ) by any positive constant and get another exponential distribution.

$$F^{-1}(u) = \frac{1}{\theta}\log\left(\frac{1}{1-u}\right) \sim \theta\log\left(\frac{1}{1-u}\right)$$

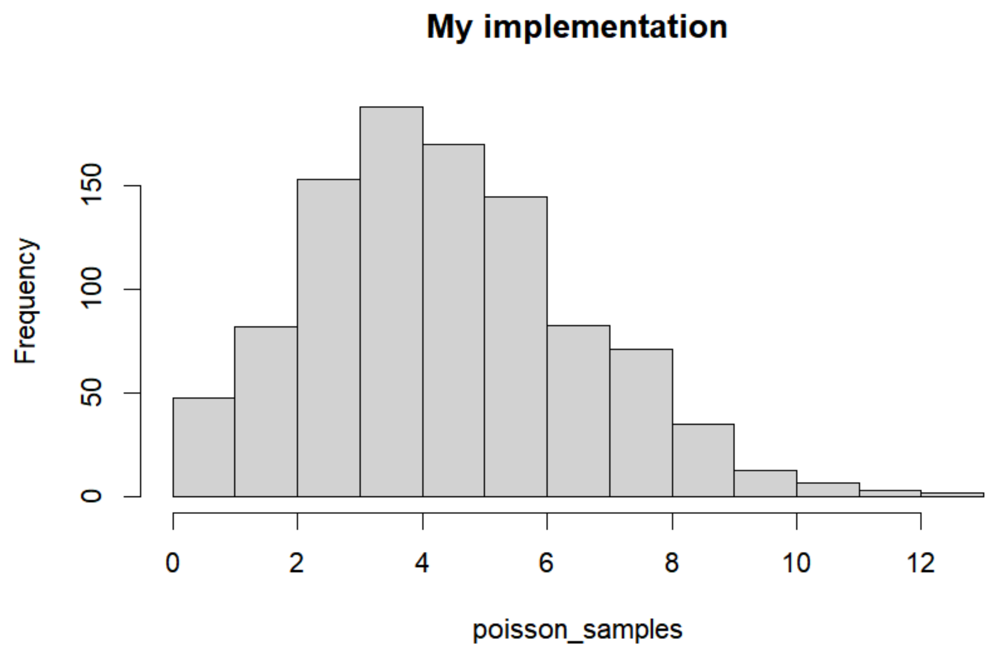
Therefore, $\theta \cdot \log\left(\Sigma_{i=0}^{k} U^k\right) \sim Exp(\theta)$

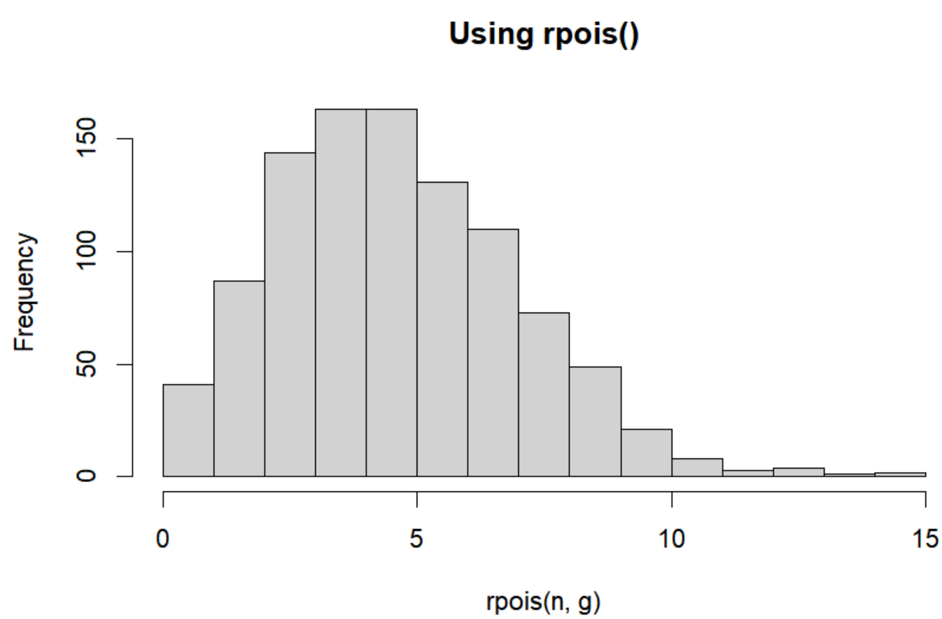
Appendix to HW 2

2c)



2f)



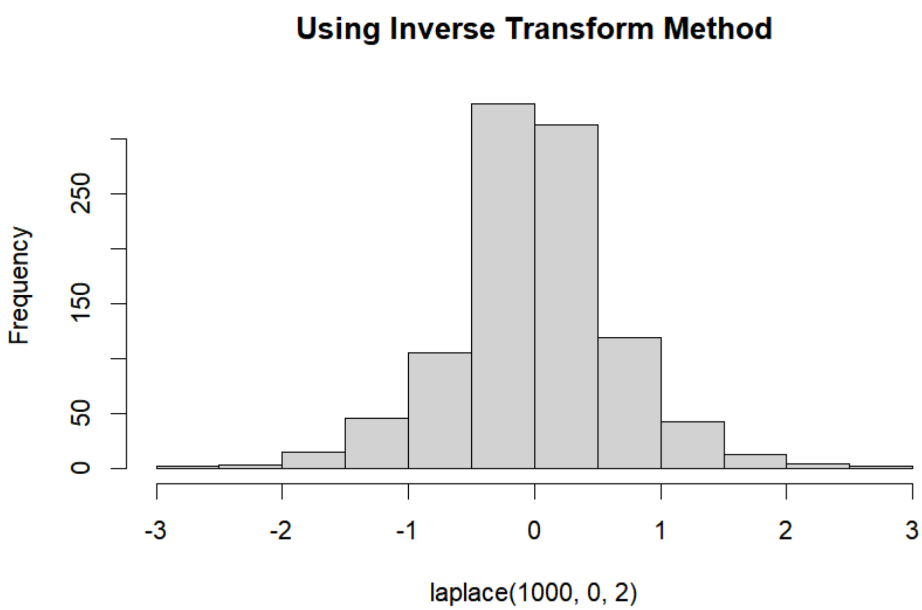


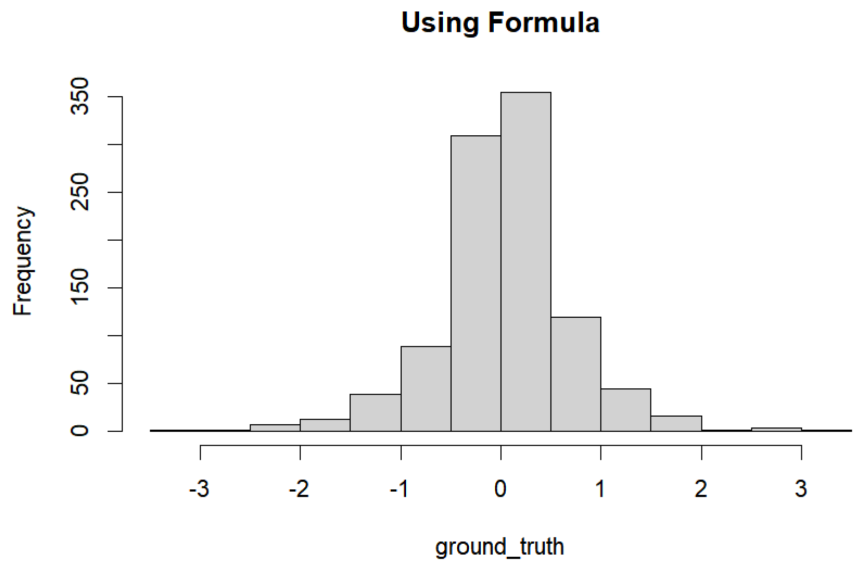
The histograms look to be about the same shape. I would believe they came from the same Poisson distribution. Both distributions have the same center and skew to the right.

3b)

```
laplace <- function(n, mu, alpha) {
  u = runif(1000)
  f1 = mu + 1/alpha * logb(2 * u[u<0.5])
  f2 = mu - 1/alpha * logb(2 - 2 * u[u>=0.5])
  return(c(f1, f2))
}
hist(laplace(1000, 0, 2))
```

3d)





Both distributions are centered at zero and approximately symmetric. They both reach ± 3 on the x-axis. I would believe they are observations from the same distribution.