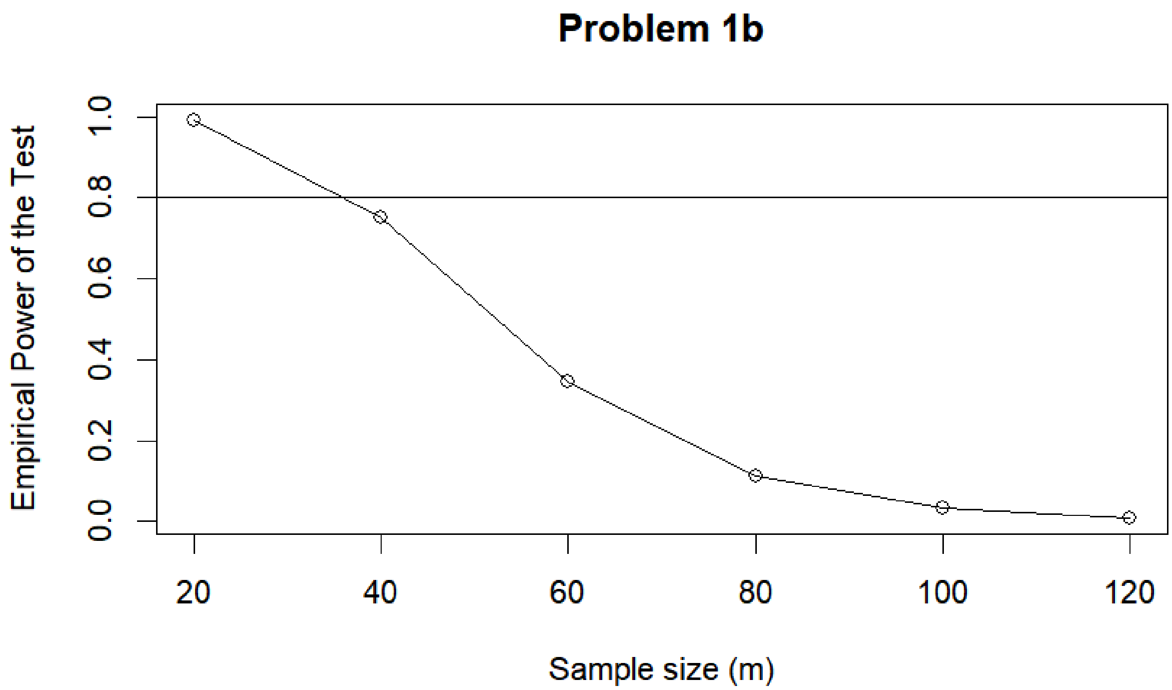
Problem 1b
Empirical Power of the Test
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
100
120
Sample size (m)

1a)

m	20	40	60	80	100	120
Power	0.9923	0.7520	0.3448	0.1125	0.0324	0.0085

Power = P(reject H_0 | H_1 true)

1c) Approximate sample size (m) of 35 is required to have a power of 0.8

2a)

$$\theta = \int_{-1}^{1} (1-x^2)^{-1/2}\,dx = \pi$$

Let $h(x) = \frac{1}{2}(x+1)$ be a change of coordinates function.

$$x = 2y - 1$$

$$dx = 2 \cdot dy$$

$$\theta = \int_0^1 2\left(1-(2y-1)^2\right)^{-1/2} dy$$

$$\int_0^1 2\left(1-(4y^2-4y+1)\right)^{-1/2} dy$$

$$\int_0^1 2\left(4y-4y^2\right)^{-1/2} dy$$

$$\int_0^1 \frac{2}{\sqrt{4y(1-y)}}\,dy$$

$$\int_0^1 \frac{1}{\sqrt{y(1-y)}}\,dy$$

2a)

$$\theta = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

make substitution $u = -x$ and $u = x$

$$-\int_{1}^{0} \frac{1}{\sqrt{1-u^2}}\,du + \int_{0}^{1} \frac{1}{\sqrt{1-u^2}}\,du$$

$$\int_{0}^{1} \frac{1}{\sqrt{1-u^2}}\,du + \int_{0}^{1} \frac{1}{\sqrt{1-u^2}}$$

$$\int_{0}^{1} \frac{2}{\sqrt{1-u^2}}\,du$$

2b) $\theta = 3.134102$

$$\frac{Var(\bar{Y})}{Var(\bar{T})}$$

2c) $U_i = \text{Uniform}(0,1)$ $\quad Y_i = g(U_i)$

$V_i = 1 - U_i$ $\quad W_i = g(V_i)$

$\bar{Y} = \Sigma \; Y_i/m \qquad \bar{W} = \Sigma \; W_i/m$

$T_i = (Y_i + W_i)/2 \qquad E[T] = \Sigma_{i=1}^{m} T_i/m$

$E[T] = \Sigma_{i=1}^{m} \frac{1}{2m}(Y_i + W_i)$

$E[T]^2 = \Sigma_{i=1}^{m} \frac{1}{4m^2}(Y_i + W_i)^2 \qquad E[T^2] = \Sigma_{i=1}^{m} T_i^2/m$

$E[T^2] = \Sigma_{i=1}^{m} \frac{1}{4m}(Y_i + W_i)^2$

$Var(T) = E[T^2] - E[T]^2$

$Var(T) = \frac{1}{4m}(Y_i + W_i)^2 - \frac{1}{4m^2}(Y_i + W_i)^2$

$Var(T) = \left(\frac{1}{4m} - \frac{1}{4m^2}\right) \Sigma_{i=1}^{m} (Y_i + W_i)^2$

$Var(\bar{Y}) = \Sigma_{i=1}^{m} (Y_i - \bar{Y})^2/m^2 \qquad \bar{T} = \Sigma_{i=1}^{m} T_i/m$

$Var(\bar{T}) = \Sigma_{i=1}^{m} (T_i - \bar{T})^2/m^2 \qquad T_i = (Y_i + W_i)/2$

$$Var(\bar{T}) = \sum_{i=1}^{m} \frac{1}{m^2}\left(\frac{1}{2}(Y_i + W_i) - \frac{1}{m}\Sigma_{i=1}^{m} T_i\right)^2$$

$$Var(\bar{T}) = \sum_{i=1}^{m} \frac{1}{m^2}\left[\frac{1}{2}(Y_i + W_i) - \frac{1}{m}\Sigma_{i=1}^{m} \frac{1}{2}(Y_i + W_i)\right]^2$$

$$Var(\bar{T}) = E[\bar{T}^2] - E[\bar{T}]^2$$

$E[\bar{T}] = \bar{T} = \Sigma_{i=1}^{m} T_i/m = \Sigma_{i=1}^{m} \frac{1}{2m}(Y_i + W_i)$

$E[\bar{T}^2] = \Sigma_{i=1}^{m} T_i^2/m = \Sigma_{i=1}^{m} \frac{1}{4m}(Y_i + W_i)^2$

$Var(\bar{T}) = \Sigma_{i=1}^{m} \frac{1}{4m}(Y_i + W_i)^2 - \Sigma_{i=1}^{m} \frac{1}{4m^2}(Y_i + W_i)^2$

$Var(\bar{T}) = \Sigma_{i=1}^{m} \left(\frac{1}{4m} - \frac{1}{4m^2}\right)(Y_i + W_i)^2$

$Var(\bar{T}) = \left(\frac{1}{m} - \frac{1}{m^2}\right) \cdot \Sigma_{i=1}^{m} T_i^2$

3a) Estimate $E[g(Y)]$ where $Y \sim N(\mu, 1)$ by samples of $\text{Laplace}(\mu, \alpha)$

Normal - $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x-\mu)^2)$

Laplace - $f(x) = \frac{1}{2\alpha} \exp(-\frac{1}{\alpha}|x-\mu|)$

$$E[g(Y)] = \int g(y) \cdot \phi(y) \, dy$$

$$E[g(Y)] = \int g(x) \cdot \frac{\phi(x)}{f(x)} f(x) \, dx$$

Change of coordinates because I can generate samples of $\text{Laplace}(\alpha, \mu)$ and not $\text{Normal}(\mu, 1)$

$$\int e^{-|x-\mu| + \frac{1}{2}(x-\mu)^2} \cdot \frac{\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x-\mu)^2)}{\frac{1}{2\alpha} \exp(-\frac{1}{\alpha}|x-\mu|)} f(x) \, dx$$

$$h(x) = \sqrt{\frac{2}{\pi}} \alpha \exp\left(\frac{1}{\alpha}|x-\mu| - |x-\mu|\right)$$

$$h(x) = \frac{2\alpha}{\sqrt{2\pi}} \exp\left(-(1-\frac{1}{\alpha})|x-\mu|\right)$$

3b) $E[g(Y)] = \int g(y) \cdot f(y) \; dy$

$$\int_{-\infty}^{\infty} \exp\left(-|y-\mu| + \tfrac{1}{2}(y-\mu)^2\right) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\tfrac{1}{2}(y-\mu)^2\right) dy$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-|y-\mu|\right) dy$$

Split integral into two parts around $y = \mu$

$$\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}} e^{y-\mu} dy + \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mu-y} dy$$

$z = y - \mu$ and $dz = dy$

$$\frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{z} dz + \int_{0}^{\infty} e^{-z} dz \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[e^{z} \Big|_{-\infty}^{0} - e^{-z} \Big|_{0}^{\infty} \right]$$

$$\frac{1}{\sqrt{2\pi}} [1 - -1]$$

$$\frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \approx 0.798$$

3c) If $X \sim Exp(\alpha^{-1})$ and $Y \sim Exp(\alpha^{-1})$ are independent, then $X - Y + \mu \sim Laplace(\mu, \alpha)$

estimate = 0.7943 theoretical = 0.798

These are similar to an acceptable degree of precision.

4a) $\theta = \int_S g(x)\,dx$
And $f(x)$ is a "simple" distribution we can generate observations from. Using importance sampling

$$\hat{\theta}^{IS} = \int_S \frac{g(x)}{f(x)} f(x)\,dx = E\left[\frac{g(x)}{f(x)}\right]$$

$$E\left[\frac{g(x)}{f(x)}\right] = \frac{1}{m}\sum_{i=1}^{m} \frac{g(x_i)}{f(x_i)}$$ Monte Carlo estimate for the sample mean

4b) $$Var(\hat{\theta}^{IS}) = E[(\hat{\theta}^{IS})^2] - E[\hat{\theta}^{IS}]^2$$

$$= E\left[\left(\frac{g(x_i)}{f(x_i)}\right)^2\right] - E\left[\frac{g(x_i)}{f(x_i)}\right]^2$$
is given to be finite

$|g(x_i)/f(x_i)| < c$ for all $i = 1 \dots m$
because $g(x)/f(x)$ is bounded
Upper bound estimate of
$E\left[\frac{g(x_i)}{f(x_i)}\right]^2 \leq c^2$ which is finite.

$Var(\hat{\theta}^{IS})$ is finite because both components of the expected value is finite.

5.1) 0.8969
5.2) 0.9354
5.3) 0.9433
5.4) 0.9502