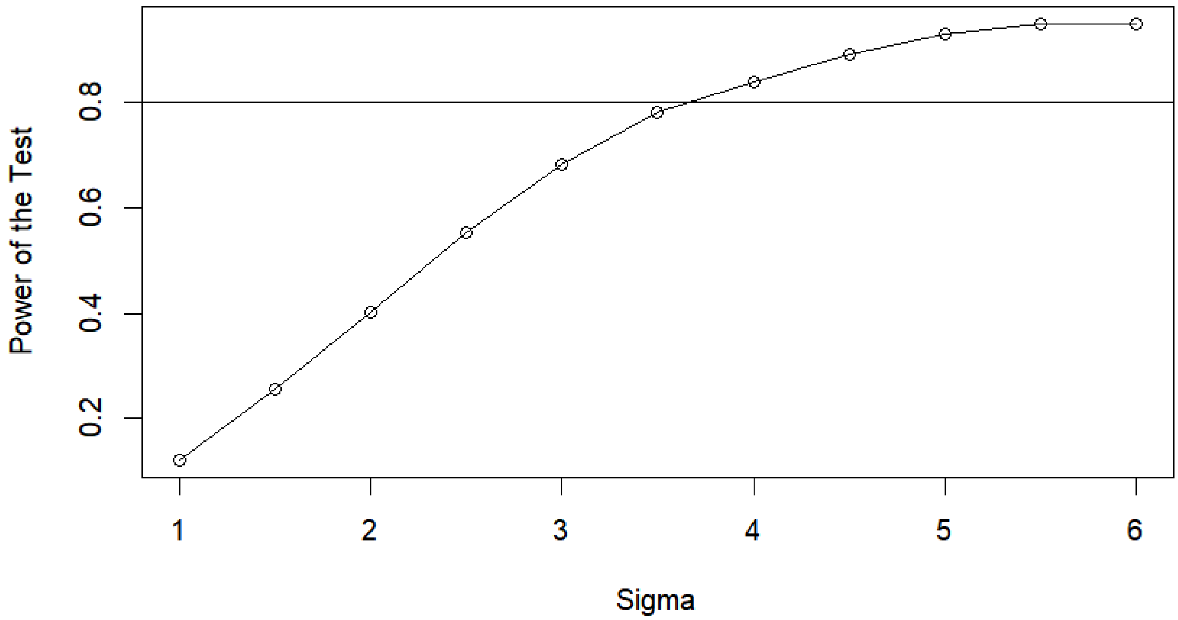
1a) Please reference R code.

1b)




1c) By eye, it appears that a sigma = 3.2 would achieve a power of the test with 0.8

# HW 5

1a) $\alpha = 0.05$ $\quad \sigma^2 = r$ $\quad$ Gama$(r, 1)$ $\quad m = 10$

$H_0: \sigma_0^2 = 1$ $\quad H_1: \sigma_0^2 > 1$ $\quad$ Power = P(Ho true | H, true)

2a) $F(x) = 1 - e^{-x/\theta}$ $\quad x > 0$

$p = 1 - e^{-x/\theta}$

$1 - p = e^{-x/\theta}$

$x = -\theta \ln(1-p) = -\theta \cdot \ln(0.75) = Q_1$

2b) Given a sample first quantile estimate $\hat{Q}_1$

$\hat{Q}_1 = -\theta \ln(0.75)$ $\qquad Var(\hat{\theta}) = Var(-\hat{Q}_1 / \ln(0.75))$

$\theta = -\hat{Q}_1 / \ln(0.75)$ $\qquad Var(\hat{\theta}) = Var(\hat{Q}_1) / \ln(0.75)$

2c) $\boxed{B \times m}$

① Find a Monte Carlo estimate for $\hat{Q}_1^{(b)}$ Using quantile(data, probs = 0.25)

② Transform estimates of $\hat{Q}_1$ into $\hat{\theta}$ using $\hat{\theta} = -\hat{Q}_1 / \ln(0.75)$

③ Compute $Var(\hat{\theta}_{0.25}) = \frac{1}{B} \sum_{i=1}^{B} (\hat{\theta} - \bar{\hat{\theta}})^2$

2d) $Var(\hat{\theta}_{0.25}) = 4.382$

2e) $Var(\hat{\theta}_{0.25}) = Var(\hat{x}_p) / \ln(0.75)^2$

$Var(\hat{x}_p) = \frac{p(1-p)}{m f(x_p)^2}$ $\qquad f(x) = \theta^{-1} e^{-x/\theta}$ $\quad x > 0$

Using part A, $\hat{x}_p = -\theta \cdot \ln(0.75)$

$f(\hat{x}_p) = \theta^{-1} e^{\ln(0.75)} = 1/4$

$Var(\hat{\theta}_{0.25}) = \frac{0.25(1-0.25)}{m \cdot (\frac{3}{4}\theta^{-1})^2} = \frac{3/16}{1/16 \cdot m \cdot \theta^{-2}} = \frac{\theta^2}{3m}$

$\frac{1}{4}$

$= (3/m) \div \ln(0.75)^2$

$\approx 3.625$

2f) Theoretical $= (\theta/m) \div \ln(1-p) = 3.625$
Expirimental $= 4.382$

The difference between the two variances is because of the low number of samples ($m=10$). When I run my R code with larger $m$, the experimental variance approaches the theoretical.

2g) Let $X_i \sim Exp(\theta)$
$E[X] = \theta \qquad Var[X] = \theta^2$
$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$

$$\begin{aligned}
bias(\hat{\theta}_{mean}) &= E[\bar{X}] - \theta \\
&= E\left[\tfrac{1}{m}\textstyle\sum_{i=1}^{m} X_i\right] - \theta \\
&= \tfrac{1}{m}\textstyle\sum_{i=1}^{m} E[X_i] - \theta \qquad \text{using } E[X_i] = \theta \\
&= \tfrac{1}{m}\textstyle\sum_{i=1}^{m} \theta - \theta \qquad \text{when } X_i \sim Exp(\theta) \\
bias(\hat{\theta}_{mean}) &= \theta - \theta = 0
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\theta}_{mean}) &= Var(\bar{X}) \\
&= Var\left(\tfrac{1}{m}\textstyle\sum_{i=1}^{m} X_i\right) \\
&= \tfrac{1}{m^2}\textstyle\sum_{i=1}^{m} Var(X_i) \qquad \text{using } Var(X_i) = \theta^2 \\
&= \tfrac{1}{m^2}\textstyle\sum_{i=1}^{m} \theta^2 \qquad \text{when } X_i \sim Exp(\theta) \\
&= \theta^2/m \\
Var(\hat{\theta}_{mean}) &= 9/10 = 0.9
\end{aligned}$$

$\hat{\theta}_{mean}$ is an unbiased estimator for $\theta$ with variance $= 0.9$ when $\theta = 3$ and $m = 10$

2h) $\hat{\theta}_{mean}$ is a better estimator than $\hat{\theta}_{0.25}$ for $\hat{\theta}$ because it has a smaller variance.

# HW 5

3) Non-parametric Bootstrapping: $[49.14, 55.72]$
Semi-Parametric Bootstrapping: $[49.40, 55.53]$
Studentized Bootstrapping: $[49.26, 55.78]$

4a) Jackknife is leave one out resampling.

$$\text{bias}(\hat{\theta}) = E[\hat{\theta}] - E[\theta]$$
$$= E[\bar{x}_j] - \bar{x}$$

expected sample mean of a jackknife resample is

$$\bar{X}_j = \frac{1}{m-1} \Sigma_{i=1}^{m} X_i \quad \text{where } i \neq j$$
$$\bar{X}_j = \frac{1}{m-1} \Sigma_{i=1}^{m} X_i - X_j$$
$$\bar{x}_j = \frac{1}{m-1}(m\bar{x} - x_j)$$

$$E[\bar{x}_j] = \frac{1}{m} \Sigma_{j=1}^{m} \frac{1}{m-1}(m\bar{x} - x_j)$$
$$= \frac{1}{m} \cdot \frac{1}{m-1} \Sigma_{j=1}^{m} m\bar{x} - x_j$$
$$= \frac{1}{m} \cdot \frac{1}{m-1} \left[ \Sigma_{j=1}^{m} m\bar{x} - \Sigma_{j=1}^{m} x_j \right]$$
$$= \frac{1}{m} \cdot \frac{1}{m-1} \left[ m^2 \bar{x} - m\bar{x} \right]$$
$$= \frac{1}{m} \cdot \frac{1}{m-1} \left[ m\bar{x}(m-1) \right]$$
$$E[\bar{x}_j] = \bar{x}$$

The expected sample mean from a jackknife resample is $\bar{x}$.

$$\text{bias}(\hat{\theta}) = (m-1)[E(\bar{x}_j) - \bar{x}]$$
$$\text{bias}(\hat{\theta}) = 0$$

4b) $\text{Var}(\hat{\theta}_j) = \frac{1}{m} \sum_{j=1}^{m} (\bar{J}_j - \bar{J})^2$

The variance of the $j^{th}$ jackknife resample.

$\bar{J}_j$ is the average of the $j^{th}$ resample

$\bar{J}$ is the average across all resamples

$\bar{J}_j = \frac{1}{m-1}(m\bar{x} - x_j)$ [from part a]

$\bar{J} = \bar{x}$ [from part a]

$$\begin{aligned}
\text{Var}(\hat{\theta}_j) &= \frac{m-1}{m} \sum_{j=1}^{m} \left[\frac{1}{m-1}(m\bar{x} - x_j) - \bar{x}\right]^2 \\
&= \frac{m-1}{m} \sum_{j=1}^{m} \left[\frac{1}{m-1}(m\bar{x} - x_j - (m-1)\bar{x})\right]^2 \\
&= \frac{m-1}{m} \sum_{j=1}^{m} \left[\frac{1}{m-1}(\bar{x} - x_j)\right]^2 \\
&= \frac{m-1}{m} \cdot \frac{1}{(m-1)^2} \sum_{j=1}^{m} (\bar{x} - x_j)^2 \\
&= \frac{1}{m} \cdot \frac{1}{m-1} \sum_{j=1}^{m} (x_j - \bar{x})^2 \\
&= \frac{1}{m} \cdot s^2
\end{aligned}$$