1a) $X \sim Exp(\theta = 2)$ transform this into a random observation from the standard uniform.

Use Inverse transform method: $X = F^{-1}(u)$
Solve for u: $F(x) = u$

pdf of exponential $f(x) = \theta e^{-\theta x}$
cdf of exponential $F(x) = 1 - e^{-\theta x}$

$u \sim U(0,1) = 1 - e^{-2x}$ where $x \sim Exp(\theta = 2)$

1b) Beta distribution pdf $= \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha + \beta)}$

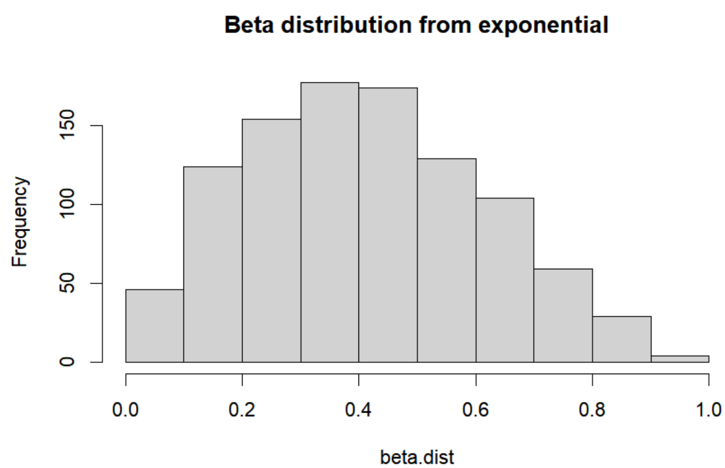
where $\Gamma(x)$ is the gamma distribution.
Knowing $\{X_1, \dots, X_n\}$ are samples from the exponential distribution, $\sum_{i=1}^{n} X_i \sim Gamma(n, \theta^{-1})$

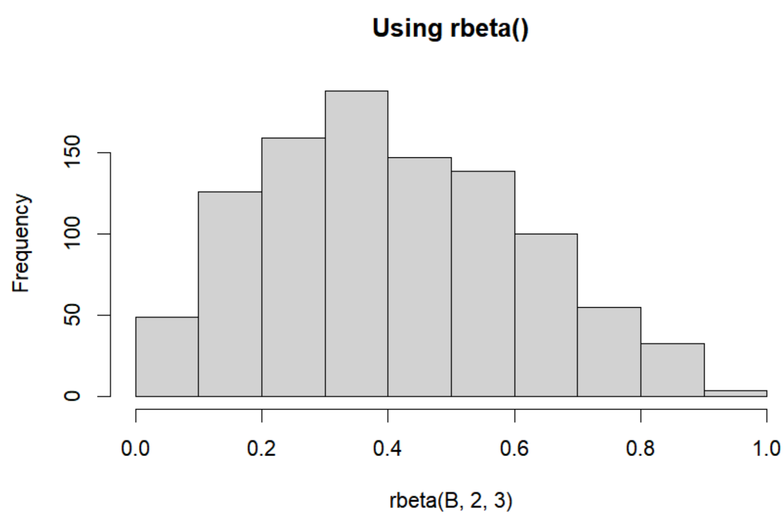
① Generate n observations from $Exp(\theta)$
② Use the first α samples to compute
$\sum_{i=1}^{\alpha} X_i \sim Gamma(\alpha, \theta^{-1})$
③ Use samples $(n-\beta, n)$ to compute $Gamma(\beta)$
$\sum_{i=\alpha+1}^{n} X_i \sim Gamma(\beta, \theta^{-1})$
④ Finally, compute the Beta distrn. as:

$$Beta(\alpha, \beta) = \frac{Gamma(\alpha)}{Gamma(\alpha) + Gamma(\beta)}$$

Problem 1c

```
n = 1000
alpha = 2
beta = 3
x = rexp(n*(alpha + beta))
x = matrix(x, nrow = n)
gamma.alpha = rowSums(x[, 1:alpha])
gamma.beta = rowSums(x[, (alpha+1):(alpha+beta)])
gamma.all = rowSums(x)
beta.dist = gamma.alpha * gamma.beta / gamma.all
hist(beta.dist, main="Beta distr. from exponential")
hist(rbeta(n, 2, 3), main="Using rbeta()")
```





2a) $E[I(x \le c)]$

$$I(x \le c) = \begin{cases} 1 & x \le c \\ 0 & x > c \end{cases}$$

$$E[g(x)] = \sum_{\text{all } x} g(x) P(X = x)$$

$$E[I(x \le c)] = 1 \cdot P(x \le c) + 0 \cdot P(x > c)$$
$$E[I(x \le c)] = P(x \le c)$$

2b) $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx$

$$E[g(x)] = \int_{-\infty}^{c} 1 \cdot f(x) \, dx + \int_{c}^{\infty} 0 \cdot f(x) \, dx$$

$$E[g(x)] = \int_{-\infty}^{c} f(x) \, dx$$

$$E[g(x)] = P(x \le c)$$

3a) $\theta = \int_0^1 x^x \, dx$ Estimate θ using Monte Carlo estimation

```
m = 100,000
x = runif (m)
y = x ^ x
theta = mean (y)
```

$\theta = 0.7840$

3b) $\theta = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} = (1+1) \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{1}{1+1} \, dx$

```
m = 100,000
x = runif(m, min = -1, max = 1)
y = 1 / sqrt(1 - x ^ 2)
y.bar = mean (y)
theta = (1 + 1) · y.bar
```

$\theta = 3.135$

3c) $[1, \infty) \to [0, 1]$

$$h(x) = 1 - e^{1-x}$$
$$u = 1 - e^{1-x}$$
$$1 - x = \log(1-u)$$
$$x = 1 - \log(1-u)$$

$$dx = \frac{d}{du}\left(-\log(1-u)\right)$$
$$dx = \frac{1}{1-u}$$

$$\int_1^\infty -\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\, dx$$

$$\int_0^1 \left(\frac{-1}{1-\log(1-u)} + \frac{1}{\lfloor 1-\log(1-u) \rfloor} \right) \frac{1}{1-u}\, du$$

```
m = 100,000
x = runif(m)
y = 1-log(1-x)
z = [1/y + 1/floor(y)] · 1/(1-x)
theta = mean(z)
```

$\theta = 0.547$

Problem 3a

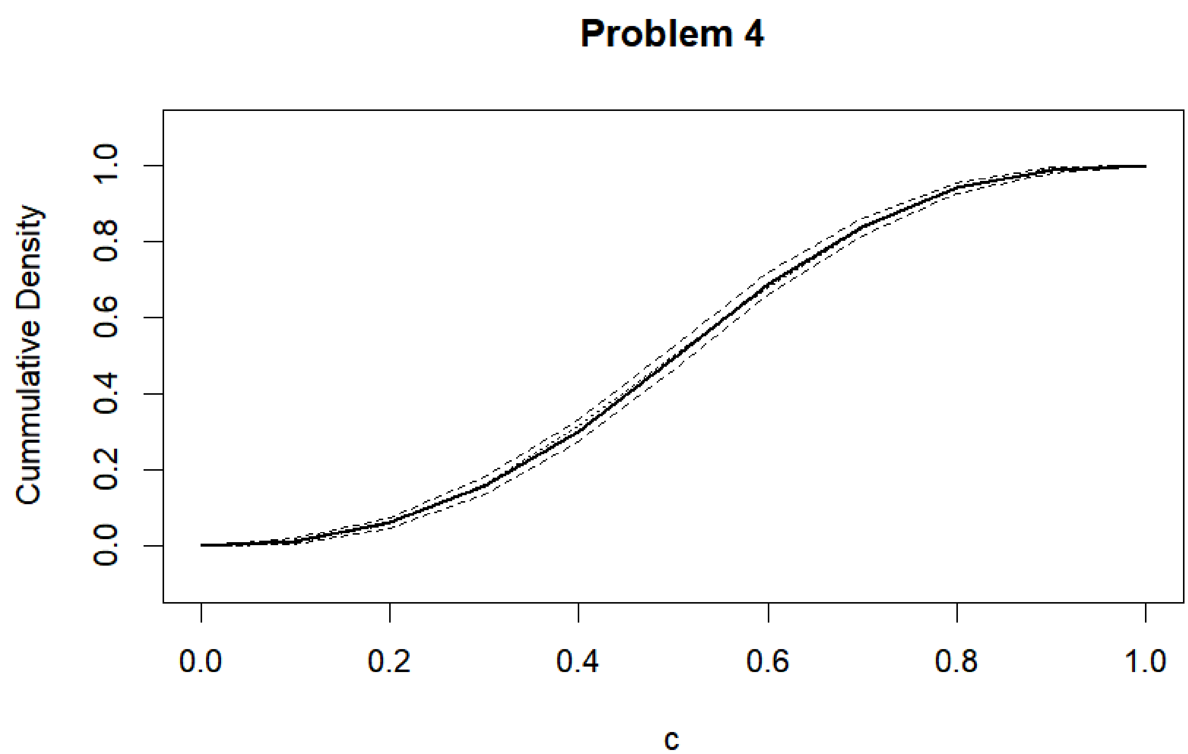
```
m = 100000
x = runif(m)
y = x^x
theta = mean(y)
```

Problem 3b

```
m = 100000
a = -1
b = 1
x = runif(m, min=a, max=b)
y = 1 / sqrt(1 - x^2)
y.bar = mean(y)
theta = (b - a) * y.bar
```

Problem 3c

```
m = 100000
x = runif(m)
y = 1 - log(1-x)
z = (-1/y + 1/floor(y)) * 1/(1-x)
theta = mean(z)
```


Problem 4
Cummulative Density
0.0
0.2
0.4
0.6
0.8
1.0
c

5a)

γ_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
r	100	44.4	25	16	11.1	8.16	6.25	4.94

using $\gamma_1 = 2/\sqrt{r}$

$r = 4/\gamma_1^2$

5d) The maximum skewness tolerated when the empirical Type I error rate is at most 0.025 would be (approximately) $\gamma_1 = 0.725$ with $r = 7.61$

Problem 5b

```
library(EnvStats)
B = 10000
m = 30
errors = c()
for (r in r.sequence) {
  data = rgamma(B*m, shape=r, scale=1)
  data = matrix(data, nrow = B)
  tb <- apply(data, 1, function(x){varTest(x, alternative = "greater",
                                          sigma.squared = r)$statistic})
  empirical.error <- mean(tb > qchisq(alpha, df=m-1, lower.tail=FALSE))
  errors = c(errors, empirical.error)
}
print(errors)
```