THE UNIVERSITY OF TEXAS AT SAN ANTONIO

EXPLORATION OF BLACK-SCHOLES WITH NUMERICAL APPLICATION TO FINANCIAL ENGINEERING

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ABSTRACT

This paper on the Black Scholes model is written to be accessible to readers with a background in basic calculus. Within this paper is a derivation of the equation, a proof showing that the equation is the diffusion equation with a change of variable, a derivation of the analytic solution, and detailed discussion of the background and significance of the model. Finally, a numerical scheme is used to solve the PDE and this is compared with the analytic solution.

CHAPTER 1

BACKGROUND INFORMATION

1.1 Financial Glossary

Contract means a legally-binding obligation between two parties, namely the investor and a counterparty.

Counterparty means the other party in a financial transaction.

Market inefficiency means that the price of anything is too high or too low. Investors seek to profit from market inefficiencies.

Exercise price (aka strike price) is the predetermined price the holder of an option may buy or sell for.

Maturity date (aka expiration date) is last day in which the option may be exercised. For European options, this is the only day that the option may be exercised. American options allow the holder to exercise their option at any time before the option expires.

Derivatives are financial contracts whose value depends on something else (the underlying asset), such as the market price of a stock.

Options are a category of derivatives that give the owner the option but not the obligation to buy or sell the underlying asset at the predetermined strike price.

Call options allow the holder to buy the underlying asset. For example, A buys a call option from B for $2, and the call option contract allows for A to force B to sell a stock for $50 in one week. If after a week the stock price has gone up to $55, A exercises the call option to force B to sell the stock for $50 and A now has a stock valued at $55. Were the stock price to instead be below the $50 strike price at the time of maturity, the call option that A bought would be worthless.

Put options allow the holder to sell the underlying asset. Following the previous example, A purchasing a put option from B allows for A to force B to buy the stock from A. Now, A benefits from the stock price going down because if the market price of the stock is below the strike price at the maturity date, A can profit from purchasing the stock from the market and using the put option to force B to buy it at a higher price. The put option is worthless if the market price is higher than the strike price at the maturity date.

Arbitrage is the opportunity to profit from simultaneously buying an asset at a lower price and selling it at a higher price. Arbitrage can also mean "risk free profit with zero upfront investment".[1]

Short selling is the practice of borrowing an asset or option from a lender and immediately selling. This is profitable if the market price of the borrowed item falls soon after, allowing the investor to buy back the borrowed item from the market at the lower price and return it to the lender.

Transaction costs refer to costs incurred for carrying out a transaction.

Dividend means amounts paid out for owning a stock. For example, a company may pay shareholders 50 cents per share owned every three months.

Hedging is a terms that describes a strategy of protecting against risks. A basic example would be purchasing a product in addition to insurance on the product. The insurance is a derivative product that hedges against risk, meaning if the product value falls, the insurance contract will reimburse the investor for the lost value.

In the money (ITM), **at the money** (ATM), and **out of the money** (OTM) are terms referring to whether an option has value. A call option is ITM if the market price is above its exercise price, and a put option is ITM if the market prices is below its exercise price.

Interest rate refers to the rate of interest an investor may earn by holding cash (i.e. govt bonds) instead of an asset or option.

Volatility quantifies the frequency and amount of price changes of an asset.

1.2 Historical Significance

In 1973, Fischer Black from the University of Chicago and Myron Scholes from MIT published *The Pricing of Options and Corporate Liabilities* to the *Journal of Political Economy*.[2] That same year, Robert Merton, also from MIT, expanded upon their work with his article *Theory of Rational Option Pricing* published in *The Bell Journal of Economics and Management Science*.[3] These papers introduced the "Black-Scholes theory of option pricing" to the financial world. This model "was the first widely used mathematical method to calculate the theoretical value of an option contract, using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration, and expected volatility."[4] This work earned them the Nobel Memorial Prize in Economic Sciences in 1997. Being considered as the first of its kind, the Black-Scholes equation inspired decades of further work in financial engineering.

The assumptions of the model can be found within its philosophical foundations, specifically the efficient market hypothesis (EMH) that was "[d]eveloped independently by Paul A. Samuelson and Eugene F. Fama in the 1960s".[5] Thereby, the assumptions and formulation of the Black Scholes equation treats the market as if pricing is already optimal. Specifically, the equation is more of an analytic exploration of the mathematical properties of an ideal market. Therefore, the model represents a thought experiment endowed with mathematical properties rather than a practical trading strategy.

Despite the impracticality of the model's assumptions, it has an elegant closed-form solution and its derivation gives insightful intuition into market pricing. This is why it is still widely taught today in courses involving financial engineering and probability. Understanding the derivation and properties of the model allows for numerical schemes that solve more realistic versions of the equation, such as ones incorporating concepts including but not limited to the "volatility smile", dividends, changing interest rates, jumps, transaction costs, and American options.

CHAPTER 2

THE BLACK SCHOLES MODEL

2.1 Equation And Assumptions

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

For the maturity date T and exercise price K, $V(0,t) = 0 \ \forall t$ and

$$V(S,T) = \begin{cases} S - K, & \text{if } x \geq K \\ 0, & \text{if } x < K \end{cases}$$

Variables

- $V = V(S,t)$ is the value of an option.
- $r \in \mathbb{R}^+$ is the short-term interest rate, i.e. 5% interest means $r = 0.05$.
- $S \in \mathbb{R}^+$ is the current price of the underlying stock.
- $\sigma \in \mathbb{R}^+$ is the volatility of the geometric Brownian motion of S.
- $t \in \mathbb{R}^+$ is time.

Assumptions

(i) The asset price S follows geometric Brownian motion.

(ii) r are σ are known constants.

(iii) There are no transaction costs.

(iv) No dividends are paid.

(v) No arbitrage opportunities exist.

(vi) Trading is continuous in time.

(vii) Short selling is possible.

(viii) Any real number of the asset may be traded.

2.2 Heuristic Derivation

Investors are considering a hedged investment involving purchasing a stock using the funds received from short selling some amount of call options for the same stock. Short selling call options means borrowing such options and selling them immediately so that the investors have cash on hand to buy stocks but owe call options to a counterparty. The value of call options goes up if the stock price goes up and down if the stock price goes down, so the value of the call option debt will go up if the stock price goes down and down if the stock price goes up. This allows for the investors to hedge against risks.

Following the original work published by Black and Scholes in 1973, the amount of call options the investors borrow to short sell is determined by what creates a zero-risk situation. This means that whether the stock price goes up or down, the equity of the investment portfolio remains the same. In order to have a zero-risk portfolio, the change in portfolio value over Δt is set as $0 = \beta \cdot \Delta S - \alpha \cdot \Delta V$ where S is the stock price, β is the amount of stock purchased, V is the call option price, and α is the amount of call options short sold. For simplicity, we divide the equation by β and consider $\frac{\alpha}{\beta}$ as the number of call options to short sell per stock purchased to instead have $0 = \Delta S - \frac{\alpha}{\beta} \cdot \Delta V$. We can easily compute $\frac{\alpha}{\beta}$ with

$$0 = \Delta S - \frac{\Delta V}{\Delta V}\Delta S = \Delta S - \left[\frac{\Delta V}{\Delta S}\right]^{-1} \Delta V$$

By short selling $\frac{\alpha}{\beta} = \left[\frac{\Delta V}{\Delta S}\right]^{-1} \approx \left[\frac{\partial V}{\partial S}\right]^{-1}$ call options for some known volatility $\frac{\partial V}{\partial S}$ at the current prices, the risk associated with price changes in the stock is negated at time t.

We now use Taylor series to expand ΔV and then approximate by removing higher-derivative terms, cross-derivative terms, and terms where the Δt term is taken to a power higher than 1.

$$\Delta V(S,t) = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2 + \frac{\partial^2 V}{\partial t \partial S}\Delta S \Delta t + ... \approx ...$$

$$... \approx \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2$$

ΔS is modelled as $\Delta S = \mu S \Delta t + \sigma S \Delta W$ (see Appendix 2: Geometric Brownian Motion) where W is a Brownian motion process and σ is the known constant (over Δt) volatility of the stock price S. The multiple of the stochastic process W by σS means that the stock is equally likely to go up or down by 1% of its present value rather than by \$1. Here, 1% and \$1 are just examples. Now, we compute $(\Delta S)^2$ as

$$(\Delta S)^2 = (\mu S \Delta t + \sigma S \Delta W)^2 = \mu^2 S^2 (\Delta t)^2 + \mu(\Delta t)\sigma S^2 \Delta W + \sigma^2 S^2 (\Delta W)^2$$

Exercise 2.17 in Øksendal (1998)[6] gives the result that $(\Delta W)^2 \to \Delta t$, so we treat ΔW as $t^{1/2}$ so that $\Delta t \Delta W \approx (\Delta t)^{3/2}$. Because $(\Delta t)^x \to 0$ for $x > 1$ compared to Δt, we zero out the first two terms of the above result to be left with $(\Delta S)^2 \approx \sigma^2 S^2 (\Delta W)^2 \approx \sigma^2 S^2 \Delta t$. The equation now becomes

$$\Delta V(S,t) \approx \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \Delta t$$

Then the change in equity of the portfolio over Δt is

$$0 = \Delta S - \Delta V \left[\frac{\partial V}{\partial S}\right]^{-1} \approx \Delta S - \left(\frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \Delta t\right)\left[\frac{\partial V}{\partial S}\right]^{-1} = ...$$

$$... = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\Delta t \left[\frac{\partial V}{\partial S}\right]^{-1}$$

Because the portfolio is self-funding, $0 = S - \left[\frac{\partial V}{\partial S}\right]^{-1} V$. The assumption that the market perfectly prices the call options allows for the purchase and short sell of exactly $\left[\frac{\partial V}{\partial S}\right]^{-1}$ call options at price V in order to purchase a share of stock at price S. We can already equate

$\Delta S - \left[\frac{\partial V}{\partial S}\right]^{-1} \Delta V = 0 = \left(S - \left[\frac{\partial V}{\partial S}\right]^{-1} V\right) r\Delta t$, but heuristically this is justified because of the assumption of perfect pricing within the market. The equity value of the portfolio, $S - \left[\frac{\partial V}{\partial S}\right]^{-1} V$, could instead be invested in "cash" (i.e. govt bonds) at a constant short-term interest rate r. The assumption of perfect market valuation means that the return on this cash investment will equal the return on the hedged portfolio. Therefore, from the equation

$$\Delta S - \Delta V \left[\frac{\partial V}{\partial S}\right]^{-1} = \left(S - \left[\frac{\partial V}{\partial S}\right]^{-1} V\right) r\Delta t$$

we get

$$-\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) \Delta t \left[\frac{\partial V}{\partial S}\right]^{-1} \approx \left(S - V\left[\frac{\partial V}{\partial S}\right]^{-1}\right) r\Delta t$$

We replace $\approx$ with $=$, multiply both sides by $\left[\frac{\partial V}{\partial S}\right] \frac{1}{\Delta t}$, and distribute to get

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = Sr\frac{\partial V}{\partial S} - Vr$$

and we rearrange to reach

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

This is the classic Black Scholes equation.

2.3 Derivation to The Diffusion Equation

Recall that the Black Scholes equation is $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$.

We'll use the following variables to transform the equation into $u_\tau = u_{xx}$.

$x = x(S) = \ln \frac{S}{K}$

$\tau = \tau(t) = \frac{\sigma^2}{2}(T-t) \Leftrightarrow t = T - \frac{2\tau}{\sigma^2}$

We compute the partial derivatives as follows.

$\frac{\partial}{\partial S}V = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}V_x$

$\frac{\partial^2}{\partial S^2}V = \frac{\partial}{\partial S}\left(\frac{\partial}{\partial x}V\right)\cdot\left(\frac{1}{S}\right) + \left(\frac{\partial}{\partial x}V\right)\cdot\left(\frac{\partial}{\partial S}\frac{1}{S}\right) = ...$

$... = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial S}V\right)\cdot\left(\frac{1}{S}\right) + (V_x)\cdot\left(-\frac{1}{S^2}\right) = ...$

$... = \frac{\partial}{\partial x}\left(\frac{1}{S}V_x\right)\cdot\left(\frac{1}{S}\right) - \frac{1}{S^2}V_x = \frac{1}{S^2}V_{xx} - \frac{1}{S^2}V_x = \frac{1}{S^2}\left(V_{xx} - V_x\right)$

We note that $S^2 V_{SS} = V_{xx} - V_x$

$V_t = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = V_\tau \cdot \left(-\frac{\sigma^2}{2}\right)$

We now rewrite the original equation as $\left(-\frac{\sigma^2}{2}\right)V_\tau + \frac{1}{2}\sigma^2\left(V_{xx} - V_x\right) + rV_x - rV = 0$

We now define the following variables.

$V(S,t) = u(x,\tau)e^{ax+b\tau}$

$a = \frac{1}{2} - \frac{r}{\sigma^2} \Rightarrow a^2 = r^2\sigma^{-4} - r\sigma^{-2} + \frac{1}{4}$

$b = 2\sigma^{-2}\left(\frac{\sigma^2}{2}a^2 - \frac{\sigma^2}{2}a + ar - r\right) = -\left(r^2\sigma^{-4} + r\sigma^{-2} + \frac{1}{4}\right)$

Now we compute partial derivatives using these variables.

$V_\tau = (u_\tau + bu)e^{ax+b\tau}$

$V_x = (u_x + au)e^{ax+b\tau}$

$V_{xx} = (u_{xx} + au_x + au_x + a^2u)e^{ax+b\tau} = (u_{xx} + 2au_x + a^2u)e^{ax+b\tau}$

Using $e^{ax+b\tau} > 0$, we rewrite the equation as the following.

$\left(-\frac{\sigma^2}{2}\right)(u_\tau + bu) + \left(\frac{\sigma^2}{2}\right)(u_{xx} + 2au_x + a^2u - (u_x + au)) + r(u_x + ua) - ru = 0$

Rearranging, $\left(\frac{\sigma^2}{2}\right)(u_{xx} - u_\tau) + u_x\left(\sigma^2 a - \frac{\sigma^2}{2} + r\right) + u\left(-\frac{\sigma^2}{2}b + \frac{\sigma^2}{2}a^2 - \frac{\sigma^2}{2}a + ar - r\right) = 0$

The definitions of a and b, along with the fact that $\sigma > 0$, reduce the equation to $u_\tau = u_{xx}$

2.4 Black Scholes Solution From The Diffusion Equation

In this section, the previous derivation of the diffusion equation is used to solve the Black Scholes differential equation.

1. Black Scholes Equation With Conditions

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

$$V(S,T) = \max\{S-K,0\} \text{ and } V(0,t) = 0$$

The first condition is transformed with $V(S,t) = u(x,\tau)e^{ax+b\tau}$ and $S = Ke^x$ together with the observation that at $t = T$, $\tau(t) = \frac{\sigma^2}{2}(T-T) = 0$ to reach $u(x,0) = Ke^{-ax}(e^x - 1)H(x)$ where $H(x)$ is the Heaviside function defined to be 1 where $x \geq 1$ and 0 otherwise. Steps:
$V(S,T) = \max\{S-K,0\} \Leftrightarrow u(x,0)e^{ax+0} = \max\{Ke^x - K, 0\}$
$u(x,0) = Ke^{-ax}\max\{e^x - 1, 0\} = Ke^{-ax}(e^x-1)H(x)$ because $e^x - 1 \leq 0$ where $x \leq 0$
The second condition becomes the limit

$$\lim_{x\to-\infty}\left(u(x,\tau)e^{ax+b\tau}\right) = 0$$

2. Derivation of The Black Scholes Solution

Now, we will use the initial conditions and a known solution of the heat equation (proven in the appendix). This proof closely follows the work of Dr. Rouah.[7]

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x-\xi,\tau)u_0(\xi)\, d\xi$$

where

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}$$

Applying the first condition $u_0 = u(x,0) = Ke^{-ax}(e^x - 1)H(x)$ and noting this is 0 where $x \leq 0$, we get

$$u(x,\tau) = \int_0^\infty G(x-\xi,\tau)Ke^{-a\xi}(e^\xi - 1)\,d\xi$$

Because the Gaussian G vanishes at large $|x|$ and the initial condition is zero for $x \leq 0$, the second condition is satisfied.

$$u(x,\tau) = \int_0^\infty \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x-\xi)^2}{4\tau}} Ke^{-a\xi}(e^\xi - 1)\,d\xi$$

$$u(x,t) = \frac{K}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(x-\xi)^2}{4\tau}} (e^{\xi(1-a)} - e^{-a\xi})d\xi$$

Now we make the change of variable $z = \frac{\xi - x}{\sqrt{2\tau}}$ so that $\xi = \sqrt{2\tau}z + x$ and $d\xi = \sqrt{2\tau}dz$.

$$u(x,t) = \frac{K}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^\infty e^{-\frac{1}{2}z^2}(e^{(\sqrt{2\tau}z+x)(1-a)} - e^{-a(\sqrt{2\tau}z+x)})dz$$

$$u(x,t) = \frac{K}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^\infty e^{-\frac{1}{2}z^2} e^{(1-a)(\sqrt{2\tau}z+x)}dz - \frac{K}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^\infty e^{-\frac{1}{2}z^2} e^{-a(\sqrt{2\tau}z+x)}dz$$
$$= I_1 - I_2$$

We complete the square in the exponential of the first integral I_1 as

$$-\frac{1}{2}z^2 + (1-a)\sqrt{2\tau}z + (1-a)x = -\frac{1}{2}\left(z - (1-a)\sqrt{2\tau}\right)^2 + (1-a)x + (1-a)^2\tau$$

The first integral, I_1, becomes

$$I_1 = e^{(1-a)x+(1-a)^2\tau} \frac{K}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^\infty e^{-\frac{1}{2}(z-(1-a)\sqrt{2\tau})^2}dz$$

We now make the transformation $y = z - (1-a)\sqrt{2\tau}$ so that the integral I_1 becomes

$$I_1 = e^{(1-a)x+(1-a)^2\tau} \frac{K}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}-(1-a)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}y^2} dy$$

We transform the integral by replacing y with $-y$ to get

$$I_1 = e^{(1-a)x+(1-a)^2\tau} \frac{K}{\sqrt{2\pi}} (-1) \int_{x/\sqrt{2\tau}+(1-a)\sqrt{2\tau}}^{-\infty} e^{-\frac{1}{2}y^2} dy$$

And we exchange the limits of integration to get

$$= e^{(1-a)x+(1-a)^2\tau} K\Phi\left(\frac{x}{\sqrt{2\tau}} + (1-a)\sqrt{2\tau}\right)$$

Note that $\Phi(d) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-y^2/2} dy$ is the cumulative distribution function (CDF) of the standard normal distribution.

We compute the input of Φ by recalling $x = \ln \frac{S}{K}$, $\tau = \frac{\sigma^2}{2}(T-t)$, and $a = \frac{1}{2} - \frac{r}{\sigma^2}$

$$\frac{x}{\sqrt{2\tau}} + (1-a)\sqrt{2\tau} = \frac{x}{\sqrt{2\tau}} + \frac{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)(2\tau)}{\sqrt{2\tau}} = \frac{\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1$$

The second integral, I_2, is the same except with $-a$ replacing (1-a) to reach

$$I_2 = e^{(-a)x+(-a)^2\tau} K\Phi\left(\frac{x}{\sqrt{2\tau}} + (-a)\sqrt{2\tau}\right)$$

The input of Φ in I_2 is

$$\frac{x}{\sqrt{2\tau}} + (-a)\sqrt{2\tau} = \frac{x}{\sqrt{2\tau}} + \frac{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)(2\tau)}{\sqrt{2\tau}} = \frac{\ln \frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} = d_2$$

We now have

$$V(S,t) = e^{ax+b\tau} u(x,\tau) = e^{ax+b\tau}(I_1 - I_2) =$$

$$e^{ax+b\tau}\left(e^{(1-a)x+(1-a)^2\tau}K\Phi(d_1)\right)-e^{ax+b\tau}\left(e^{(-a)x+(-a)^2\tau}K\Phi(d_2)\right)$$

Using definitions for a and b given in the section Derivation to The Diffusion Equation, we simply the exponential of the first term to e^x as follows.

$ax+b\tau+(1-a)x+(1-a)^2\tau$

$ax+b\tau+x-ax+(1-2a+a^2)\tau$

$x+b\tau+(1-2a+a^2)\tau$

$x+b\tau+\left(1-2\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)+\left(r^2\sigma^{-4}-r\sigma^{-2}+\frac{1}{4}\right)\right)\tau$

$x+b\tau+\left(r^2\sigma^{-4}+r\sigma^{-2}+\frac{1}{4}\right)\tau$

$x-\left(r^2\sigma^{-4}+r\sigma^{-2}+\frac{1}{4}\right)\tau+\left(r^2\sigma^{-4}+r\sigma^{-2}+\frac{1}{4}\right)\tau=x$

We use the result $b=1-2a+a^2$ to simplify the exponent of the second term as follows.

$ax+b\tau-ax+a^2\tau$

$b\tau+a^2\tau+\tau-2a\tau+(-\tau+2a\tau)$

$b\tau+(1-2a+a^2)\tau+(-\tau+2a\tau)$

$-\tau+2a\tau$

$\left(-1+2\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)\right)\left(\frac{\sigma^2}{2}(T-t)\right)$

$\left(-2\frac{r}{\sigma^2}\right)\left(\frac{\sigma^2}{2}(T-t)\right)$

$-r(T-t)$

Because $x=\ln\frac{S}{K}$, we have

$$V(S,t)=S\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1=\frac{\ln\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2=d_1-\sigma\sqrt{T-t}=\frac{\ln\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

2.5 Zero Volatility Model

This section will explore what happens to the solution of the Black Scholes equation when $\sigma \to 0$. Clearly, the upper limit of each of the CDFs will approach ∞, causing the definite integrals to evaluate to 1. This leaves us with

$$V(S,t) = S(t) - Ke^{-r(T-t)}$$

Decrypting this equation involves analysing the future value of an investment given the possibility of a fixed interest rate r on cash held. If $r = 0$, there is no value to holding cash and thereby no relative disbenefit to swapping cash with an option or stock. Unsurprisingly, the equation would then become $V = S - K$, where the profit of a call option is simply the unchanging value of the stock minus the strike price.

Reintroducing a nonzero interest rate, we note that the disbenefit to purchasing the call option for price V is the future value of that cash amount, $e^{r(T-t)}V$. At the maturity date T, the call option will allow for the purchase of a stock at price K, and the stock price will be $S(T)$. The equivalent cash value of $S(T)$ at time t is $S(t)$ where $e^{r(T-t)}S(t) = S(T)$. Therefore, the equation is written as its equivalent (multiplied by $e^{r(T-t)}$) to get

$$e^{r(T-t)}V(S,t) = e^{r(T-t)}S(t) - K$$

Because K, r, and $T - t$ are positive, $-Ke^{-r(T-t)}$ is monotone decreasing. At $t = T$, it reaches the value K. This all means that the cash amount K at the future time T is worth a smaller amount of cash at time t because of the existence of the alternative strategy that is simply receiving interest on cash. Given the fact that the call option allows for purchase of the underlying asset at price K at time T, the burden of that price is merely $-Ke^{-r(T-t)}$ at time t, meaning that the future cost K is discounted according to the interest rate r.

CHAPTER 3

NUMERICAL APPLICATIONS

Contained in this chapter is a simple numerical scheme of forward Euler to solve the Black Scholes PDE. The solution is compared with the known analytic solution. For a more extensive numerical scheme involving nonconstant volatility, please refer to Dr. Hull's paper *The Pricing of Options on Assets with Stochastic Volatilities*.[8]

We first discretize the PDE with forward-in-time (t) and central-in-space (S) discretization schemes as follows. Note the notation indices are like $V_{\text{space}}^{\text{time}}$.

$$\frac{V_i^{j+1} - \Delta f V_i^j}{\Delta t} = rV_i^j - rS_i^j \frac{V_{i+1}^j - V_{i-1}^j}{2\Delta S} - \frac{1}{2}\sigma^2 \left(S_i^j\right)^2 \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\Delta S^2}$$

Note that $\Delta f = (1 + r\Delta t)^{-1}$ is the discount factor accounting for interest over Δt. Now, we know the value of $V(S,t)$ at the maturity date T and at $S = 0$. This numerical scheme (see Appendix 3: Numerical Python Code) produces

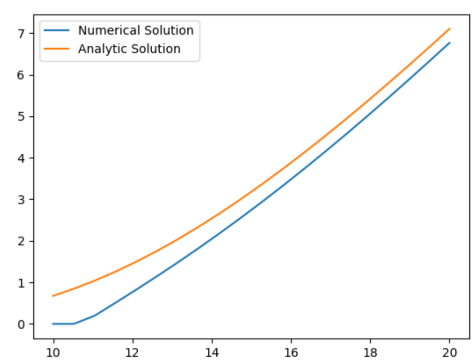

Figure 3.1: Solutions

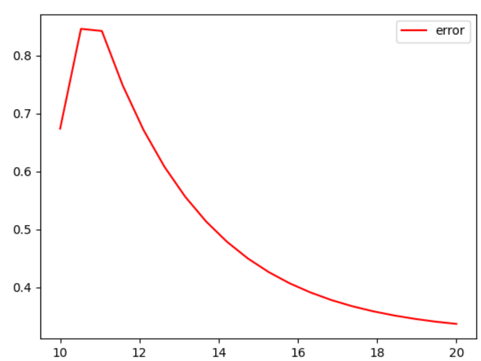

Figure 3.2: Error

Appendix A: Geometric Brownian Motion

This appendix was added to make clarifying comments on geometric Brownian motion (GBM) to explain how the stock price is modeled and how this leads to the solution of the Black Scholes PDE involving a normal distribution with somewhat cryptic integral limits. We model the change in stock price S as the stochastic process

$$\Delta S = (\mu \Delta t + \sigma \Delta W)\, S$$

The assumptions of the Black Scholes model force $\mu = r$ to achieve a risk-neutral model.[9] The volatility σ is assumed to be a known constant. W is a Brownian motion process, meaning that it takes a random path over time (stochastic process) and that W satisfies

Definition: Brownian motion process

A stochastic process $[X(t), t \geq 0]$ is said to be a *Brownian motion process* if:

(i) $W(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has stationary independent increments

(iii) for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $c^2 t$.

The above definition is given in the book *Stochastic Processes* (1996).[10] The first property means that W starts at 0, and the second means that at each point W moves randomly over Δt regardless of any prior behavior. The third and final property is most interesting because it involved the normal distribution. Basically, if a large amount of Brownian motion processes were taken and the point reached at time t was plotted for each, the result would be a Gaussian, meaning bell-shaped like the normal distribution.

The most interesting property of the representation of ΔS is that the multiplication of S with ΔW models the stock price as lognormally distributed. The solution to the Black Scholes PDE is given as a CDF of a normal distribution, and thus the cryptic limits of integration by the lognormal distribution of S.

Appendix B: Solution to The Heat Equation

This is a proof that the convolution of the heat kernel with the initial condition $u_0(x)$ satisfies the heat equation $u_\tau = u_{xx}$ with the initial condition $u(x,0) = u_0(x)$. In other words, we are proving that the solution is

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x-\xi,\tau)u_0(\xi)\,d\xi$$

where the heat kernel $G(x,\tau)$ is

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}$$

1. Computation of The Time Derivative u_τ

The time derivative u_τ is

$$u_\tau(x,\tau) = \frac{\partial}{\partial\tau}\left(\int_{-\infty}^{\infty} G(x-\xi,\tau)u_0(\xi)\,d\xi\right)$$

Since $u_0(\xi)$ is independent of τ, we have:

$$u_\tau(x,\tau) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\tau}G(x-\xi,\tau)u_0(\xi)\,d\xi$$

The time derivative of the heat kernel $G(x,\tau)$ is:

$$G_\tau(x,\tau) = \frac{\partial}{\partial\tau}\left(\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)$$

Using the product rule:

$$G_\tau(x,\tau) = \frac{\partial}{\partial\tau}\left(\frac{1}{\sqrt{4\pi\tau}}\right)e^{-\frac{x^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}}\frac{\partial}{\partial\tau}\left(e^{-\frac{x^2}{4\tau}}\right)$$

Calculating each term separately:

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\sqrt{4\pi\tau}}\right) = -\frac{1}{2}\frac{1}{\sqrt{4\pi}}\tau^{-\frac{3}{2}} = -\frac{1}{2\tau}\frac{1}{\sqrt{4\pi\tau}}$$

$$\frac{\partial}{\partial \tau}\left(e^{-\frac{x^2}{4\tau}}\right) = e^{-\frac{x^2}{4\tau}}\left(\frac{x^2}{4\tau^2}\right)$$

Combining these results:

$$G_\tau(x,\tau) = -\frac{1}{2\tau}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\frac{x^2}{4\tau^2} = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\left(-\frac{1}{2\tau} + \frac{x^2}{4\tau^2}\right)$$

2. Computation of The Spacial Derivative u_{xx}

The second spatial derivative u_{xx} is:

$$u_{xx}(x,\tau) = \frac{\partial^2}{\partial x^2}\left(\int_{-\infty}^{\infty} G(x-\xi,\tau)u_0(\xi)\,d\xi\right)$$

Since $u_0(\xi)$ is independent of x, we have:

$$u_{xx}(x,\tau) = \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}G(x-\xi,\tau)u_0(\xi)\,d\xi$$

The second spatial derivative of the heat kernel $G(x,\tau)$ is:

$$G_{xx}(x,\tau) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)$$

First, compute the first derivative:

$$G_x(x,\tau) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right) = \frac{1}{\sqrt{4\pi\tau}}\left(-\frac{x}{2\tau}\right)e^{-\frac{x^2}{4\tau}} = -\frac{x}{2\tau}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}$$

Then, compute the second derivative using the product rule:

$$G_{xx}(x,\tau) = \frac{\partial}{\partial x}\left(-\frac{x}{2\tau}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right) = \left(-\frac{1}{2\tau}\right)\left(\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right) + \left(-\frac{x}{2\tau}\right)\left(-\frac{x}{2\tau}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)$$

Simplifying:

$$G_{xx}(x,\tau) = \left(-\frac{1}{2\tau} + \frac{x^2}{4\tau^2}\right)\left(\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)$$

3. Implication that $u_{xx} = u_\tau$

Recall that

$$G_\tau(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}\left(-\frac{1}{2\tau} + \frac{x^2}{4\tau^2}\right)$$

It is thus clear that $G_{xx} = G_\tau$, so

$$u_\tau(x,\tau) = \int_{-\infty}^{\infty} G_\tau(x-\xi,\tau)u_0(\xi)\,d\xi = \int_{-\infty}^{\infty} G_{xx}(x-\xi,\tau)u_0(\xi)\,d\xi = u_{xx}(x,\tau)$$

and thus

$$u_{xx} = u_\tau$$

Appendix C: Numerical Python Code

```python
from numpy import log, exp, sqrt, linspace, searchsorted, array
from scipy.stats import norm
import matplotlib.pyplot as plt


def call_option(S, K, T, r, sig):
    if S == 0: return 0
    d1 = (log(S/K) + (r + 0.5 * sig**2) * T) / (sig * sqrt(T))
    d2 = d1 - sig * sqrt(T)
    Nd1 = norm.cdf(d1)
    Nd2 = norm.cdf(d2)
    return S * Nd1 - K * exp(-r*T) * Nd2


def solve_pde(S, K, T, r, sig, m = 5000):
    n = max(min(round(6 * S + 1), 240), 120)
    N = 1.4*S  # Set grid as 0 to 140% of S
    dS = N / n
    dt = T / m
    df = 1 / (1 + dt * r)
    S_grid = linspace(0, N, n)
    V_grid = [max(S - K, 0) for S in S_grid]
    V_temp = V_grid.copy()
    t = T
    while (t > dt + 1e-6):
        t -= dt
```

```
        for i in range(1, n - 1):
            D2 = (S_grid[i] * sig) ** 2 * (V_grid[i + 1] - 2 * V_grid[i] + V_grid[i - 1]) / (2 * dS ** 2)
            V_temp[i] = df * V_grid[i] + dt * (D2 + r * S_grid[i] * (V_grid[i + 1] - V_grid[i - 1]) / (2 * dS))
        V_grid[1:n - 1] = V_temp[1:n - 1]
    idx = searchsorted(S_grid, S) - 1
    s1, s2 = S_grid[idx], S_grid[idx + 1]
    v1, v2 = V_grid[idx], V_grid[idx + 1]
    weight1 = (s2 - S) / (s2 - s1)
    weight2 = (S - s1) / (s2 - s1)
    return weight1 * v1 + weight2 * v2


r = 0.05  # Interest rate
K = 15    # Strike price
sig = 0.3   # Volatility
T = 2     # Time till maturity
N = K+5     # Maximum stock price
n = 20   # Spacial partitions

S_grid = linspace(K-5, N, n)  # Stock prices
V_temp = linspace(K-5, N, n)  # Option prices
V_grid = [max(S - K, 0) for S in S_grid]

for i in range(1, n):
    V_grid[i] = solve_pde(S_grid[i], K, T, r, sig)

```

```python
O = array([call_option(S, K, T, r, sig) for S in S_grid])
plt.plot(S_grid, V_grid, label="Numerical Solution")
plt.plot(S_grid, O, label="Analytic Solution")
plt.legend()
plt.savefig("solutions.png")

plt.close()
plt.plot(S_grid, abs(O - V_grid), label="error", color='r')
plt.legend()
plt.savefig("error.png")
```

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