School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

UMA007 : Numerical Analysis
Assignment 4
Direct Methods for Solving Linear Systems

1. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear system. Do not reorder the equations. (The exact solution to each system is $x_1 = -1,\ x_2 = 1,\ x_3 = 3$.)

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11. \end{aligned}$$

2. Using four-digit arithmetic operations, solve the following system of equations by Gaussian elimination with and without partial pivoting

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!

3. Use the Gaussian elimination algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary or not:

(a)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3. \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3. \end{aligned}$$

4. Use Gaussian elimination and three-digit chopping arithmetics to solve the following linear system, and compare the approximations with the actual solution $[0, 10, 1/7]^T$.

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\ 6.11x_1 - 14.2x_2 + 21x_3 &= -139. \end{aligned}$$

5. Repeat the above Exercise 4 using Gaussian elimination with scaled partial pivoting and three-digit rounding arithmetic.

6. Given the linear system

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + x_3 &= 2. \end{aligned}$$

(a) Find value(s) of α for which the system has no solution.

(b) Find value(s) of α for which the system has infinite number of solutions.

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(c) Assuming a unique solution exists for a given α, find the solution.

7. Modify the LU Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear systems.

(a)

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

(b)

$$\begin{aligned} 1.012x_1 - 2.132x_2 + 3.104x_3 &= 1.984, \\ -2.132x_1 + 4.096x_2 - 7.013x_3 &= -5.049, \\ 3.104x_1 - 7.013x_2 + 0.014x_3 &= 3.895. \end{aligned}$$

8. Show that the LU Factorization Algorithm requires

(a)

$$\frac{1}{3}n^3 - \frac{1}{3}n \quad \text{multiplications/divisions and} \quad \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \quad \text{additions/subtractions.}$$

(b) Show that solving $Ly = b$, where L is a lower-triangular matrix with $l_{ii} = 1$ for all i, requires

$$\frac{1}{2}n^2 - \frac{1}{2}n \quad \text{multiplications/divisions and} \quad \frac{1}{2}n^2 - \frac{1}{2}n \quad \text{additions/subtractions.}$$

(c) Show that solving $Ax = b$ by first factoring A into $A = LU$ and then solving $Ly = b$ and $Ux = y$ requires the same number of operations as the Gaussian Elimination Algorithm.