CONFIDENCE INTERVALS FOR A REGRESSION COEFFICIENT

Based on Stock and Watson, ch. 5

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THE t-TEST

THE SIMPLE LINEAR REGRESSION MODEL

- The **population** regression model is

$$Y_i = \beta_0 + \beta_1 X_i + u_i; \quad i = 1, \ldots, n$$

- β_0 and β_1 are population parameters; they are **fixed (non-random), but unknown**
- The data $(Y_i, X_i; i = 1, \ldots, n)$ is a **random sample** from the population of Y and X
- Assume the Least Squares assumptions for causal inference are satisfied

THE LARGE-SAMPLE SAMPLING DISTRIBUTION

Under the OLS assumptions for causal inference, in large samples, $\hat{\beta}_1$ is approximately distributed according to the Normal distribution:

$$\hat{\beta}_1 \overset{\text{approx}}{\sim} \mathcal{N}\left(\beta_1, \sigma^2_{\hat{\beta}_1}\right)$$

THE t-STATISTIC

$$H_0: \beta_1 = \beta_{1,0}; \quad H_1: \beta_1 \neq \beta_{1,0}$$

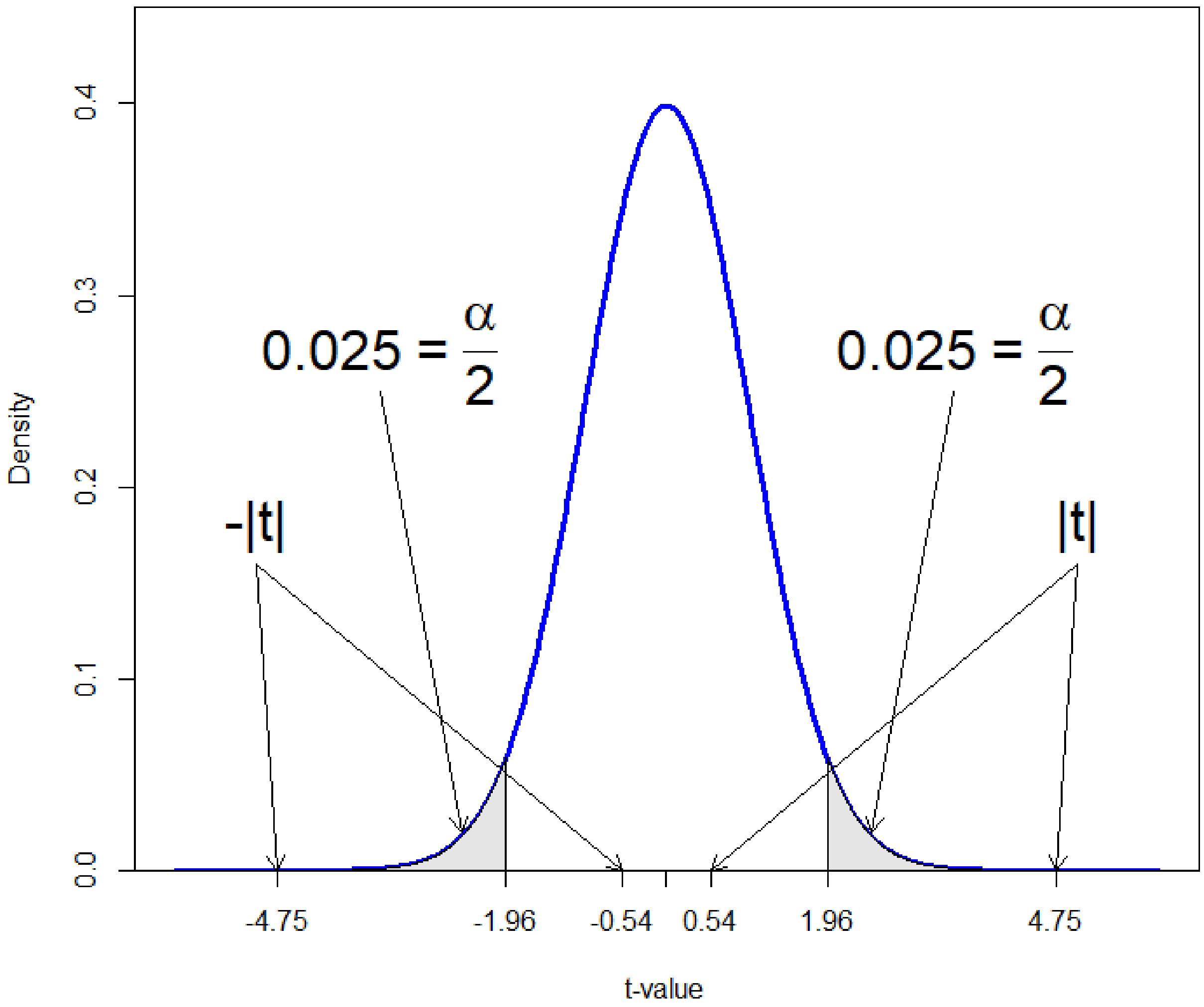
- When the Least Squares assumptions for causal inference holds and H_0 is true, then $\hat{\beta}_1 \overset{\text{approx}}{\sim} \mathcal{N}(\beta_{1,0}, \hat{\sigma}^2_{\hat{\beta}_1})$, and

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{\hat{\sigma}_{\hat{\beta}_1}} \overset{\text{approx}}{\sim} \mathcal{N}(0, 1)$$

  where $\hat{\sigma}_{\hat{\beta}_1}$ is a consistent (robust!) estimator of $\sigma_{\hat{\beta}_1}$.

- t-test of H_0 with significance level $\alpha = 0.05$: reject H_0 if $|t| > 1.96$; otherwise, do not reject H_0

SIGNIFICANCE LEVELS AND CRITICAL VALUES



TEST SCORES AND CLASS SIZES

$$Score_i = \beta_0 + \beta_1 STR_i + u_i; \quad i = 1, \ldots, n$$

$$H_0 : \beta_1 = 0; \quad H_1 : \beta_1 \neq 0$$

```
# Regression of Score on STR using CASchools dataframe
lm1 <- lm(Score ~ STR, data = CASchools) # Fitted model in lm1
# Regression output with heteroskedastic robust SEs
parameters(lm1, robust = TRUE, vcov_type = "HC1")
```

```
## Parameter   | Coefficient |    SE |             95% CI | t(418) |
## ------------------------------------------------------------------
## (Intercept) |      698.93 | 10.36 | [678.56, 719.31] |  67.44 | < .00
## STR         |       -2.28 |  0.52 | [ -3.30,  -1.26] |  -4.39 | < .00
```

CONFIDENCE INTERVALS

MANY, MANY t-TESTS

- Thought experiment: test all possible hypothesized values for β_1 with 5% significance level, record rejections and non-rejections
- Given $\hat{\beta}_1$ and $\hat{\sigma}_{\hat{\beta}_1}$, which hypothesized values $\beta_{1,0}$ for β_1 are not rejected?

$$\left| \frac{\hat{\beta}_1 - \beta_{1,0}}{\hat{\sigma}_{\hat{\beta}_1}} \right| < 1.96$$

95%-CONFIDENCE INTERVAL

- The outlined test procedure implies that we fail to reject $H_0 : \beta_1 = \beta_{1,0}$ for $\beta_{1,0}$ in the following interval:

$$\hat{\beta}_1 - 1.96 \times \hat{\sigma}_{\hat{\beta}_1} \leq \beta_{1,0} \leq \hat{\beta}_1 + 1.96 \times \hat{\sigma}_{\hat{\beta}_1}$$

- Estimate of range of β_1-values, called a **95%-confidence interval** for β_1 (abbreviated $CI_{\beta_1,0.95}$), all of which are consistent with the estimate $\hat{\beta}_1$

TEST SCORES AND CLASS SIZES

$$Score_i = \beta_0 + \beta_1 STR_i + u_i; \quad i = 1, \ldots, n$$

```r
# Regression output with heteroskedastic robust SEs
parameters(lm1, robust = TRUE, vcov_type = "HC1")
```

```
## Parameter   | Coefficient |    SE |             95% CI | t(418) |       p
## ------------------------------------------------------------------------
## (Intercept) |      698.93 | 10.36 | [678.56, 719.31] |  67.44 | < .001
## STR         |       -2.28 |  0.52 | [ -3.30,  -1.26] |  -4.39 | < .001
```

95% COVERAGE PROBABILITY

$$CI_{\beta_1,0.95} = \left[\hat{\beta}_1 - 1.96 \times \hat{\sigma}_{\hat{\beta}_1}, \hat{\beta}_1 + 1.96 \times \hat{\sigma}_{\hat{\beta}_1}\right]$$

- Random sampling implies that the **confidence interval limits are random variables**
- $CI_{\beta_1,0.95}$ is the set of β_1-values that are not rejected by a two-sided t-test with a 5% significance level
- $CI_{\beta_1,0.95}$ is also a interval that has a 95% **coverage probability** of containing the true value β_1

SUMMARY

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- A 95%-confidence interval is the range of β_1-values that are not rejected by a two-sided t-test with a 5% significance level

  Provides an estimate of a range of values for β_1 consistent with estimate $\hat{\beta}_1$
- A 95%-confidence interval is a random interval with a 95% **coverage probability**: will cover the true value β_1 in 95/100 random samples
- Confidence intervals are routinely reported by regression software including in R