Phys 427 - HW4

1) maxwell distribution. $E = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2+v_y^2+v_z^2)$

a) v_0, most probable value from $E = \frac{1}{2}mv_0^2 = kT \Rightarrow \boxed{v_0 = \sqrt{\frac{2kT}{m}}}$

when you see...
① fine spacing, so $\sum_0^\infty \approx \int_0^\infty dx$
② integrate over angle, $\int d\vec{x} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^\infty dx = 4\pi \int_0^\infty dx$
③ convert dV_x to dx, know $V = x^3 \Rightarrow dV = 3x^2 dx$

→ $\langle v \rangle = \sum_0^\infty v \exp(-E_i/kT)/Z$ w/ $\alpha = m/2kT$

• $\sum_0^\infty v \exp(-E_i/kT) \overset{①}{\approx} \int_0^\infty dV_E\, v \exp(-E/kT) = \int_0^\infty dV_v\, \vec{v} \exp(-\alpha \vec{v}^2) \overset{②}{=} 4\pi \int_0^\infty dV_v\, v \exp(-\alpha v^2)$

$\overset{③}{=} 12\pi \int_0^\infty dv\, v^3 \exp(-\alpha v^2) = 12\pi \left[-\partial_\alpha \int_0^\infty dv\, v \exp(-\alpha v^2)\right]$

$= 12\pi \left[-\partial_\alpha \left(-\frac{\exp(-\alpha v^2)}{2\alpha}\right)\Big|_0^\infty\right] = 6\pi\left[\partial_\alpha(-\alpha^{-1})\right] = 6\pi(+\alpha^{-2}) = \underline{6\pi\alpha^{-2}}$

• $Z = \sum_0^\infty \exp(-E_i/kT) \overset{①}{\approx} \int_0^\infty dV_E \exp(-E/kT) = \int_0^\infty d\vec{V}_v \exp(-\vec{v}^2\alpha) \overset{②}{=} 4\pi \int_0^\infty dV_v \exp(-v^2\alpha)$

$\overset{③}{=} 12\pi \int_0^\infty dv\, v^2 \exp(-v^2\alpha) = 12\pi\left[-\partial_\alpha \int_0^\infty dv \exp(-v^2\alpha)\right] = 6\pi\left[-\partial_\alpha\left(\sqrt{\frac{\pi}{\alpha}}\right)\right]$

$= 6\pi^{3/2}\left(\frac{1}{2}\alpha^{-3/2}\right) = \underline{3\pi^{3/2}\alpha^{-3/2}}$

$\Rightarrow \langle v \rangle = \frac{6\pi}{\alpha^2} \cdot \frac{\alpha^{3/2}}{3\pi^{3/2}} = \frac{2}{\sqrt{\alpha\pi}} = 2\sqrt{\frac{2kT}{m\pi}} = \boxed{\sqrt{\frac{8kT}{m\pi}} = \langle v \rangle}$

b) $\langle v^2 \rangle = \sum_0^\infty v^2 \exp(-E_i/kT)/Z$

• $\sum_0^\infty v^2 \exp(-E_i/kT) \overset{①}{\approx} \int_0^\infty dV_E\, v^2 \exp(-E/kT) = \int_0^\infty dV_v\, \vec{v}^2 \exp(-\vec{v}^2\alpha)$

$\overset{②}{=} 4\pi \int_0^\infty dV_v\, v^2 \exp(-v^2\alpha) \overset{③}{=} 12\pi \int_0^\infty dv\, v^4 \exp(-v^2\alpha) = 12\pi\left[\partial_\alpha^2 \int_0^\infty dv \exp(-v^2\alpha)\right]$

$= 12\pi\left[\partial_\alpha^2\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right]\right] = 6\pi^{3/2}\left(-\frac{1}{2}\cdot -\frac{3}{2}\cdot \alpha^{-5/2}\right) = \frac{9}{2}\pi^{3/2}\alpha^{-5/2}$

$\langle v^2 \rangle = \frac{9\pi^{3/2}}{2\alpha^{5/2}} \cdot \frac{\alpha^{3/2}}{3\pi^{3/2}} = \frac{3}{2\alpha} = \boxed{\frac{3kT}{m} = \langle v^2 \rangle}$

$U_1 = -\partial_\beta \ln Z_1$
$U_N = -\partial_\beta \ln Z_1^N = N(-\partial_\beta \ln(Z_1)) = NU_1$

w/ $U = \langle E \rangle = N\langle \frac{1}{2}mv^2 \rangle = \frac{Nm}{2}\langle v^2 \rangle = \frac{Nm}{2}\left(\frac{3kT}{m}\right) = \boxed{\frac{3}{2}NkT = U}$ ✓ answer agrees w/ equipartition ✓ (D=3)

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2a) $E = -\mu B\cos\theta \qquad U = -\partial_\beta \ln Z \qquad \alpha = B\beta$

w/ $Z = \sum_0^\infty \exp(-E\beta) = \int_0^\infty dV_E \exp(-E\beta) = \int_0^{2\pi} d\phi \int_0^\pi d\theta\, \mu^2 \sin\theta \exp(\mu\alpha\cos\theta)$

don't integrate over μ as only orientation affects energy, μ is fixed

$= 2\pi\mu^2 \int_0^\pi d\theta \sin\theta \exp(\mu\alpha\cos\theta) \quad$ w/ $\gamma = \cos\theta \quad d\theta = -\frac{d\gamma}{\sin\theta}$

$= 2\pi\mu^2 \int_1^{-1} -d\gamma \exp(\mu\alpha\gamma) = 2\pi\mu^2 \left[\exp(\mu\alpha\gamma)/\mu\alpha\right]_{-1}^{1} = \frac{2\pi\mu}{\alpha}\left[\exp(\mu\alpha) - \exp(\mu\alpha)\right]$

$= \frac{2\pi\mu}{\alpha}\left[\sinh(\mu\alpha)/2\right] = \underline{\frac{\pi\mu}{\alpha}\sinh(\mu\alpha) = Z}$

$\Rightarrow \underline{U} = -\partial_\beta \ln Z = -\partial_\beta\left[\ln(\pi\mu) - \ln(\alpha) + \ln(\sinh(\mu\alpha))\right]$

since $\alpha = B\beta$, $\partial_\beta \alpha = B$

$\Rightarrow U = -\left[0 - \frac{1}{\alpha}B + \frac{\cosh(\mu\alpha)}{\sinh(\mu\alpha)} \cdot \mu B\right] = -\left[\mu B\coth(\mu B\beta) - \frac{B}{B\beta}\right]$

$$\boxed{= -\mu B\left[\coth\left(\frac{\mu B}{k T}\right) - \frac{k T}{\mu B}\right] = U}$$

b) know $U = \langle E\rangle = -\mu B\langle\cos\theta\rangle \Rightarrow \langle\cos\theta\rangle = -U/\mu B$

$\Rightarrow \langle\cos\theta\rangle = \left(\coth\left(\frac{\mu B}{k T}\right) - \frac{k T}{\mu B}\right)$ w/ $\mu = 5e{-16}\ \mathrm{Am^2}$
$B = 50e{-6}\ \mathrm{T}$
$T = 300\ \mathrm{K}$
$k = 1.380649e{-23}\ \mathrm{J/K}$

$$\boxed{\begin{array}{l} \Rightarrow \langle\cos\theta\rangle = 0.83433 \\ \text{w/ } \theta = 0.58387\ \text{rad} \\ \quad = 33.45379^\circ \end{array}}$$

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3) low temp glassy material, some atom can jump b/w 2 states w/ ϵ

a) $Z = Z_1^N$, since you have $E = \epsilon$ or 0, $Z_1 = \exp(-\epsilon/kT) + \exp(-0/kT)$

$$Z_1 = \exp(-\epsilon/kT) + 1 \Rightarrow \boxed{Z = Z_1^N = \left(\exp(-\epsilon/kT) + 1\right)^N}$$

b) $F = -kT \ln Z = \boxed{-kTN \ln\left(\exp(-\epsilon/kT) + 1\right) = F}$

c) $S = -\partial_T F = kN\left[\ln\left(\exp(-\epsilon/kT) + 1\right) + T\dfrac{\exp(-\epsilon/kT)}{\exp(-\epsilon/kT) + 1} \cdot \dfrac{-\epsilon}{kT^2}\right]$

$$\boxed{S = Nk\left[\ln\left(\exp(-\epsilon/kT) + 1\right) + \frac{\epsilon \exp(-\epsilon/kT)}{kT\left(\exp(-\epsilon/kT) + 1\right)}\right]}$$

d) $\lim\limits_{\epsilon \to 0} S = Nk \ln\left(\exp(0) + 1\right) + 0) = \boxed{Nk \ln(2) = \lim\limits_{\epsilon \to 0} S}$

This is the same entropy as for the high temp limit of a paramagnet. This makes sense as decreasing ϵ is effectively the same as increasing T. The effect of both are more atoms are able to overcome ϵ. If $\epsilon \to 0$, this is basically saying any energy that is put into the system will be enough to saturate all available states to ~~the~~ max energy.