Phys 427 - HW6

1a) Know $\mathcal{Z} = \sum_N \lambda^N \frac{(Vn_a)^N}{N!} = \sum_N \frac{(\lambda V n_a)^N}{N!}$

Know $\sum_{n=0}^{N} \frac{x^n}{n!} \approx \exp x \Rightarrow \boxed{\mathcal{Z} \approx \exp(\lambda V n_a)}$

b) $P = \frac{\sum_\epsilon \exp(-\beta(\epsilon - \mu N))}{\mathcal{Z}} = \frac{\exp(\mu N \beta) Z_1}{\exp(-\lambda n_a V)} = \frac{\lambda^N (n_a V)^N}{N!} \cdot \frac{1}{\exp(-\lambda n_a V)}$

$$\boxed{P = \frac{(\lambda n_a V)^N}{N!} \cdot \frac{1}{\exp(-\lambda n_a V)}}$$

c) w/ $\langle N \rangle = \frac{1}{\beta} \partial_\mu \ln \mathcal{Z} = \frac{V n_a}{\beta} \beta \lambda = \underline{V n_a \lambda = \langle N \rangle}$

w/ $\frac{1}{Z\beta^2} \partial_\mu^2 Z = \frac{1}{Z\beta^2} \partial_\mu^2 \left[\exp(\lambda n_a V)\right] = \frac{1}{Z\beta^2} \frac{\partial}{\partial\mu}\left[\exp(\lambda n_a V) n_a V \exp(\mu\beta) \beta\right]$

$= \frac{1}{Z\beta} \partial_\mu \left[Z n_a V \lambda\right] = \frac{1}{Z\beta}\left[Z(n_a V \lambda)^2_\beta + Z n_a V \lambda \beta\right]$

$= \frac{Z\beta}{Z\beta}\left[(n_a V\lambda)^2 + n_a V \lambda\right] = \underline{(n_a V \lambda)^2 + n_a V \lambda = \langle N^2 \rangle}$

$\therefore \sigma^2 = |\langle N^2 \rangle - \langle N \rangle^2| = |(n_a V \lambda)^2 - (n_a V \lambda)^2 - n_a V \lambda| = \underline{n_a V \lambda = \sigma^2}$

$\Rightarrow \frac{\sigma}{\langle N \rangle} = \frac{(n_a V \lambda)^{1/2}}{n_a V \lambda} = \frac{1}{(n_a V \lambda)^{1/2}} \Rightarrow \boxed{\therefore \frac{\sigma}{\langle N \rangle} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{n_a V \lambda}} = \frac{1}{\sqrt{\langle N \rangle}}}$

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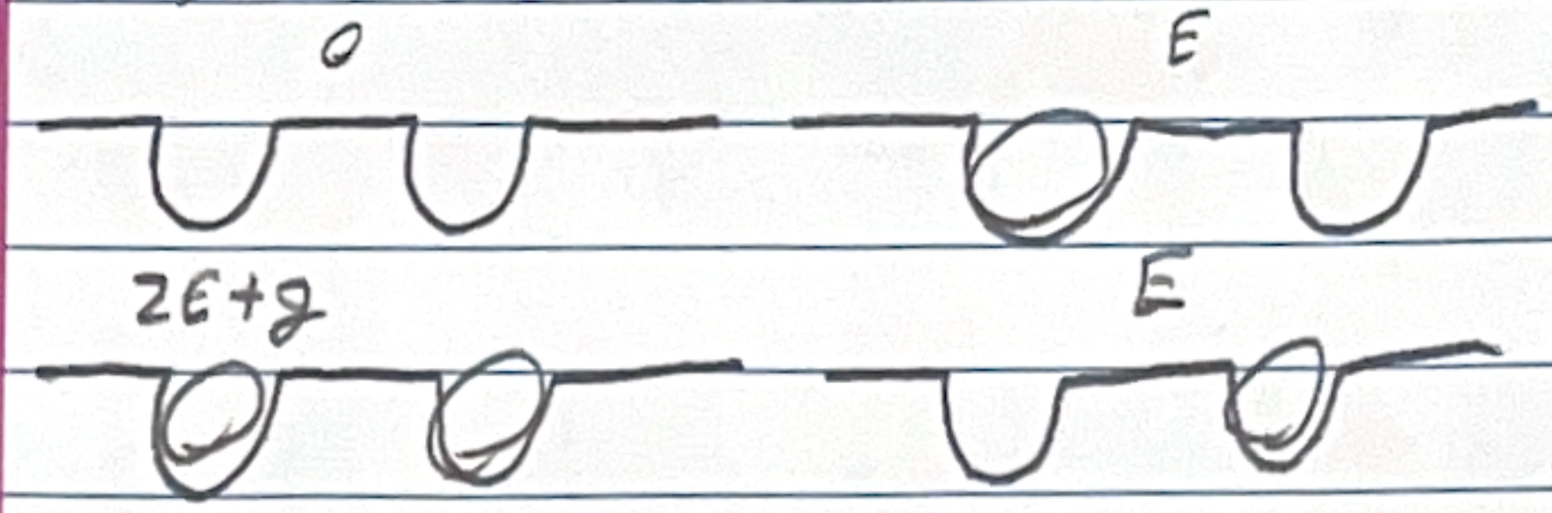
2a)



$$\mathcal{Z} = 1 + 2\exp(-\beta(E-\mu)) + \exp(-\beta(2E+g-2\mu))$$

$$\langle N \rangle = \frac{1}{\beta}\partial_\mu \ln \mathcal{Z} = \frac{1}{\beta}\left[\frac{1}{\mathcal{Z}} \cdot 2\beta\exp(-\beta(E-\mu)) + 2\beta\exp(-\beta(2E+g-2\mu))\right]$$

$$\langle N \rangle = \frac{2\exp(-\beta(E-\mu)) + 2\exp(-\beta(2E+g-2\mu))}{1+2\exp(-\beta(E-\mu)) + \exp(-\beta(2E+g-2\mu))}$$

know $\mu = \overset{\text{set to } 0}{\mu^\circ} + \frac{1}{\beta}\ln\left(\frac{p\beta}{n_Q}\right) \Rightarrow \exp(\beta\mu) = \exp\left(\beta\frac{1}{\beta}\ln\left(\frac{p\beta}{n_Q}\right)\right) = \frac{p\beta}{n_Q}$

also know $p_0 = \frac{n_Q}{\beta}\exp(E\beta) \Rightarrow \frac{\beta}{n_Q} = p_0\exp(E\beta) \therefore \underline{\exp(\beta\mu) = \frac{p}{p_0}\exp(E\beta)}$

$$\Rightarrow \langle N \rangle = \frac{2\exp(\beta\mu)\exp(-\beta E) + 2\exp(\beta\mu)^2\exp(-\beta E)^2\exp(-\beta g)}{1+2\exp(\beta\mu)\exp(-\beta E) + \exp(\beta\mu)^2\exp(-\beta E)^2\exp(-\beta g)}$$

w/ $\exp(\beta\mu)\exp(-\beta E) = \left(\frac{p}{p_0}\exp(E\beta)\right)\exp(-E\beta) = \frac{p}{p_0} = p_r$

$$\exp(\beta\mu)^2\exp(-\beta E)^2 = \left(\frac{p}{p_0}\exp(E\beta)\right)^2\exp(-E\beta)^2 = \left(\frac{p}{p_0}\right)^2 = p_r^2$$

$$\Rightarrow \langle N \rangle = \frac{2p_r + 2p_r^2\exp(-\beta g)}{1+2p_r+\cancel{2}p_r^2\exp(-\beta g)}$$

$$\boxed{\therefore f = \frac{\langle N \rangle}{2} = \frac{p_r + p_r^2\exp(-\beta g)}{1+2p_r+p_r^2\exp(-\beta g)}} \qquad p_r = \frac{p}{p_0}$$

b) $\lim_{g\to 0} f = \frac{p_r + p_r^2}{1+2p_r+p_r^2} = \frac{p_r(1+p_r)}{(1+p_r)^2} = \boxed{\frac{p_r}{1+p_r}}$

$f = \frac{p_r}{1+p_r}$ is the same as myoglobin because 2 sites w/ no cooperation is the same as 2 one-site molecules w/ the same parameters.

w/o cooperation, a 2-site molecule functions as 2 1-site molecules!

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c) $f = \frac{1}{2} = \frac{p_r + p_r^2 \exp(-\beta J)}{1 + 2p_r + p_r^2 \exp(-\beta J)} \Rightarrow 2p_r + 2p_r^2 \exp(-\beta J) = 1 + 2p_r + p_r^2 \exp(-\beta J)$

$$\Rightarrow p_r^2 \exp(-\beta J) = 1 \Rightarrow \frac{p}{p_o} = \exp\left(\frac{\beta J}{2}\right) \Rightarrow \boxed{p_{1/2} = p_o \exp\left(\frac{\beta J}{2}\right)}$$

if $J = 0$, $p_{1/2} = p_o$ or the pressures of the external environment & the O_2 are equal. w/o cooperation, this makes sense because equal pressures are the result of mechanical equilibrium.

if $J > 0$, $p_{1/2} > p_o$ or cooperation occurs. This makes sense as the cooperation overcomes the desire for mechanical equilibrium. Instead of equal mechanical potentials, the sum of O_2 mechanical & chemical is equal to environment mechanical. The pressure of the O_2 can be higher because the J now counteracts the chemical gradient.

if $J < 0$, $p_{1/2} < p_o$ or negative cooperation occurs. Same logic, but inverted from $J > 0$. The J makes each bound O_2 a "source" of chemical potential.

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3)

a) $F = \mathcal{U} - TS = Vu - \frac{4}{3}\alpha V T^4 = V\alpha T^4 - \frac{4}{3}\alpha V T^4 = \boxed{-\frac{1}{3}\alpha V T^4 = F}$

$$p = -\frac{\partial F}{\partial V}\bigg|_{u,N} = \boxed{+\frac{1}{3}\alpha T^4 = p}$$

$$\mathcal{G} = F + pV = -\frac{1}{3}\alpha V T^4 + \frac{1}{3}\alpha T^4 V = \boxed{0 = \mathcal{G}}$$

The expression for $\mathcal{G}$ implies $\underline{\mu = 0}$

b) $\langle N \rangle = \int_0^\infty d\epsilon\, \mathcal{D}(\epsilon)\, f(\epsilon, T) = \int_0^\infty d\epsilon \frac{V\epsilon^2}{\pi^2 c^3 \hbar^3} \cdot \frac{1}{\exp(\beta(\epsilon - \mu)) - 1}$

$$= \frac{V}{\pi^2 \hbar^3 c^3} \int_0^\infty d\epsilon \frac{\epsilon^2}{\exp(\beta\epsilon) - 1} = \frac{V}{\pi^2 \hbar^3 c^3 \beta^2} \int_0^\infty d\epsilon \frac{\beta^2 \epsilon^2}{\exp(\beta\epsilon) - 1} \qquad x = \beta\epsilon, \quad dx = \beta d\epsilon$$

$$= \frac{V}{\pi^2 \hbar^3 c^3 \beta^3} \left(\Gamma(3)\, \zeta(3) \right) = \frac{V}{\pi^2 \hbar^3 c^3 \beta^3} (2!)(1.202)$$

$$\boxed{\langle N \rangle = \frac{(2.402) V}{\pi^2 \hbar^3 c^3 \beta^3}}$$

$\therefore \langle N \rangle \propto T^2$, so the number of photons by a lot at higher temps

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4a)

find $\sigma(\omega)\,d\omega$ for 1D, 2D

1D: $\sigma(\omega)\,d\omega = \dfrac{\text{length of line in } k \text{ space w/ radius } k}{\text{length in } k \text{ space per state}}$

polarization

$$= \frac{dk}{2} \Big/ \frac{\pi}{L} \cdot 2 \quad \text{w/ } k = \frac{\omega}{c} \quad dk = \frac{d\omega}{c}$$

$$\boxed{= \frac{d\omega\, L}{c\pi} = \sigma(\omega)\,d\omega \quad \text{for 1D}}$$

2D: $\sigma(\omega)\,d\omega = \dfrac{\text{area " "}}{\text{area " "}} = \dfrac{2\pi k\,dk}{4} \Big/ \dfrac{\pi^2}{L^2} \cdot 2 = \dfrac{k\,dk\,L^2}{\pi} = \boxed{\dfrac{\omega\,d\omega\,L^2}{c^2\pi} = \sigma(\omega)\,d\omega \text{, 2D}}$

b) $u = \sum_N \dfrac{\hbar\omega}{\exp(\beta\hbar\omega_n)-1} \approx \displaystyle\int_0^\infty d\omega\,\sigma(\omega)\frac{\hbar\omega}{\exp(\beta\hbar\omega)-1}$ w/ $x = \beta\hbar\omega$, $dx = \beta\hbar\,d\omega \Rightarrow d\omega = \dfrac{dx}{\beta\hbar}$

1D) $u = \displaystyle\int_0^\infty \frac{d\omega\, L}{c\pi}\frac{\hbar\omega}{\exp(\beta\hbar\omega)-1} = \frac{L}{\pi c\beta}\int_0^\infty d\omega\frac{\hbar\omega\beta}{\exp(\beta\hbar\omega)-1} = \frac{L}{\pi c\beta^2\hbar}\int_0^\infty dx\frac{x}{\exp x - 1}$

$$= \frac{L}{\pi c\beta^2\hbar}\left(\Gamma(2)\zeta(2)\right) = \frac{L}{\pi c\beta^2\hbar}(1)\frac{\pi^2}{6} = \underline{\frac{\pi L k^2 T^2}{c\hbar 6}}$$

2D) $u = \displaystyle\int_0^\infty d\omega \frac{L^2\omega}{c^2\pi}\frac{\hbar\omega}{\exp(\beta\hbar\omega)-1} = \frac{L^2}{\pi c^2\beta^2\hbar}\int_0^\infty \frac{\hbar^2\omega^2\beta^2\,d\omega}{\exp(\beta\hbar\omega)-1} = \frac{L^2}{\pi c^2\beta^3\hbar^2}\int_0^\infty \frac{dx\,x^2}{\exp(x)-1}$

$$= \frac{L^2}{\pi c^2\beta^3\hbar^2}\left(\Gamma(3)\zeta(3)\right) = \frac{L^2}{\pi c^2\beta^3\hbar^2}(2)\zeta(3) = \underline{\frac{2L^2\zeta(3)}{\pi c^2\beta^3\hbar^2}}$$

1D) $u = \dfrac{\pi k^2}{6c\hbar} L T^2$

2D) $u = \dfrac{2k^3\zeta(3)}{\pi c^2\hbar^2} L^2 T^3$

$$\boxed{\begin{aligned} \therefore\ a_1 &= \frac{\pi k^2}{6c\hbar} \\ \therefore\ a_2 &= \frac{2k^3\zeta(3)}{\pi c^2\hbar^2} \end{aligned}}$$

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5) w/ $m\frac{d^2r_j}{dt^2} = K(r_{j+1} - 2r_j + r_{j-1})$ & $r_j \propto exp(ikja - i\omega t)$

$\Rightarrow m(i\omega)^2 exp(ikja - i\omega t) = K[exp(ik(j+1)a - \omega t)$
$- 2 exp(ikja - \omega t)$
$+ exp(ik(j-1)a - \omega t)]$

$\Rightarrow -m\omega^2 = K\, exp(ika) - 2 + exp(-ika) = 2K[\cosh(ika) - 1]$

w/ $\cosh(ika) = \cos(ka)$

$$\boxed{\Rightarrow \omega^2 = \frac{2K}{m}[1 - \cos(ka)]}$$

w/ Taylor series, $\cos(ka) \approx 1 - \frac{(ka)^2}{2}$ when $k \ll \frac{2\pi}{a}$

$\Rightarrow \omega^2 \approx \frac{2K}{m}\left[1 - 1 + \frac{k^2a^2}{2}\right] = \frac{Kk^2a^2}{m}$

w/ $c_s = \frac{a^2K}{m}$ $\boxed{\therefore \omega = \sqrt{\frac{Ka^2}{m}}\, k = c_s k}$