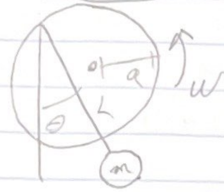
HW#11 - Joseph Specht

1) 

disk ~~fixed~~

a) Need to find position in terms of $\hat{x}$ & $\hat{y}$

If we take $\vec{r}$ to be $\vec{r}_p + \vec{r}_b$, we can find $\vec{r}_p$ in terms of $\hat{x}$ & $\hat{y}$ & $\vec{r}_b$ in terms of $\hat{x}$ & $\hat{y}$

$\vec{r}_p$: distance from center of disk to pendulum connection

$$\vec{r}_p = a\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right)$$ if we start @

$$\vec{r}_b = l\left(\sin(\theta)\hat{x} - \cos(\theta)\hat{y}\right)$$ ← distance from contact to mass

$$\therefore \vec{r} = \left[\left(a\cos(\omega t) + l\sin(\theta)\right)\hat{x} + \left(a\sin(\omega t) - l\cos(\theta)\right)\hat{y}\right]$$

now to find $\vec{v}$, we take $\vec{\omega}\times\vec{r}$, so

$$\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ r_{px} & r_{py} & 0 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \dot{\theta} \\ r_{bx} & r_{by} & 0 \end{vmatrix} \Rightarrow$$

$$\left(l\dot{\theta}\cos\theta - a\omega\sin(\omega t)\right)\hat{x} + \left(l\dot{\theta}\sin\theta + a\omega\cos(\omega t)\right)\hat{y}$$

now, to find ~~the~~ T, take $\frac{1}{2}m|v|^2$

$$|v|^2 = v\cdot v = \left(l^2\dot{\theta}^2\cos^2\theta - 2l\dot{\theta}a\omega\cos(\theta)\sin(\omega t) + a^2\omega^2\sin^2(\omega t) + l^2\dot{\theta}^2\sin^2\theta + 2l\dot{\theta}a\omega\sin(\theta)\cos(\omega t) + a^2\omega^2\cos^2(\omega t)\right)$$

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$$|v|^2 = l^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + a^2w^2(\sin^2(wt) + \cos^2(wt))$$
$$- 2l\dot{\theta}aw(\cos\theta\sin(wt) - \sin\theta\cos(wt))$$

but $(\cos^2\theta + \sin^2\theta)$ & $\sin^2(wt) + \cos^2(wt) = 1$, so

$$|v|^2 = l^2\dot{\theta}^2 + a^2w^2 + 2l\dot{\theta}aw(\sin(wt)\cos\theta - \cos(wt)\sin\theta)$$

but $\sin(wt)\cos\theta - \cos(wt)\sin\theta = \sin(wt - \theta)$

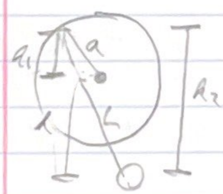
$|v|^2 = l^2\dot{\theta}^2 + a^2w^2 + 2l\dot{\theta}aw\sin(wt - \theta)$, but $-\sin(C) = \sin(-C)$, so

$$|v|^2 = l^2\dot{\theta}^2 + a^2w^2 + 2l\dot{\theta}aw\sin(\theta - wt)$$

we also know $T = \frac{1}{2}m|v|^2 \Rightarrow$ so

$$T = \frac{1}{2}m(l^2\dot{\theta}^2 + a^2w^2 + 2l\dot{\theta}aw\sin(\theta - wt))$$

now we need to find U to solve L, so we can say $U = 0$ @ the disk center, so we need h



$h = h_1 - h_2$ & $h_1 = a\sin(wt)$
$h_2 = l\cos(\theta)$

$\therefore h = a\sin(wt) - l\cos\theta$

$\therefore U = mgh = mg(a\sin(wt) - l\cos\theta)$

now we know $L = T - U$, which is

$$\boxed{L = \frac{1}{2}m(l^2\dot{\theta}^2 + a^2w^2 + 2l\dot{\theta}aw\sin(\theta - wt)) - mg(a\sin(wt) - l\cos\theta)}$$

which matches!

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b) To find eq of motion, apply E-L eq to $\mathcal{L}$, so need $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$, so need each side

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(m l \dot{\theta} a \omega \sin(\theta - \omega t) + m g l \cos\theta\right)$$

$$= m l \dot{\theta} a \omega \cos(\theta - \omega t) - m g l \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} m l^2 \dot{\theta}^2 + m l \dot{\theta} a \omega \sin(\theta - \omega t)\right)\right]$$

$$= \frac{d}{dt}\left(m l^2 \dot{\theta} + m l a \omega \sin(\theta - \omega t)\right)$$

$$= m l^2 \ddot{\theta} + m l a \omega (\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$\Rightarrow m l \dot{\theta} a \omega \cos(\theta - \omega t) - m g l \sin\theta = m l^2 \ddot{\theta} + m l a \omega (\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$\Rightarrow \dot{\theta} a \omega \cos(\theta - \omega t) - g \sin\theta = l \ddot{\theta} + a \omega (\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$- g \sin\theta = l \ddot{\theta} - a \omega^2 \cos(\theta - \omega t)$$

$$\Rightarrow \boxed{l \ddot{\theta} = a \omega^2 \cos(\theta - \omega t) + g \sin\theta}$$

c) if $\omega = 0$, eq of motion goes to

$l \ddot{\theta} = g \sin\theta$, which is just the eq of motion of a pendulum not rotating, which is what we have

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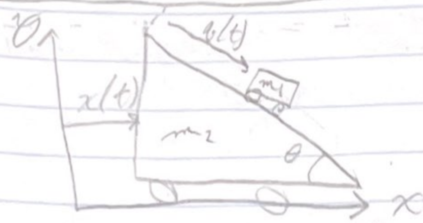
2) 

a) To find E-L eqs, we need L, which = T-U

T of m_1 → $T_1 = \frac{1}{2} m_1 v_1^2$ & $v_1 = \vec{\dot{x}} + \vec{\dot{q}} = \hat{x}(\dot{x} + \dot{q}\cos\theta) - \hat{y}(\dot{q}\sin\theta)$

T of m_2 → $T_2 = \frac{1}{2} m_2 v_2^2$ & $v_2 = \hat{x}\,\dot{x}$

$T_{Tot} = T_1 + T_2 = \frac{1}{2} m_1 |v_1|^2 + \frac{1}{2} m_2 |v_2|^2$

$$\Rightarrow |v_1|^2 = v_1 \cdot v_1 = \dot{x}^2 + 2\dot{x}\dot{q}\cos\theta + \overbrace{\dot{q}^2\cos^2\theta + \dot{q}^2\sin^2\theta}^{\dot{q}^2}$$

$|v_2|^2 = v_2 \cdot v_2 = \dot{x}^2$

$$\therefore T_{Tot} = \frac{1}{2} m_1 (\dot{x}^2 + \dot{q}^2 + 2\dot{x}\dot{q}\cos\theta) + \frac{1}{2} m_2 (\dot{x}^2)$$

If we say $U = 0$ when U is at its max, then

$$U = -mg\, q(t) \sin\theta$$

$$\therefore \mathcal{L} = \frac{1}{2} m_1 (\dot{x}^2 + \dot{q}^2 + 2\dot{x}\dot{q}\cos\theta) + \frac{1}{2} m_2 \dot{x}^2 + m_1 g q \sin\theta$$

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now to find EL equation of $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$ & $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$

$\frac{\partial L}{\partial x} = 0$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m_1 \dot{x} + m_1 \dot{q}_j \cos\theta + m_2 \dot{x}\right)$
$= m_1 \ddot{x} + m_1 \ddot{q} \cos\theta + m_2 \ddot{x}$

$\frac{\partial L}{\partial q_j} = m_1 g \sin\theta$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(m_1 \dot{q} + \dot{x}\cos\theta\right)$
$= m_1 \ddot{q} + \ddot{x} m_1 \cos\theta$

Setting each of these equal to their other half gives

$m_1 \ddot{x} + m_1 \ddot{q}_j \cos\theta + m_2 \ddot{x} = 0$
$\ddot{x}(m_1 + m_2) = -m_1 \ddot{q} \cos\theta \Rightarrow \ddot{x} = \frac{-m_1 \ddot{q} \cos\theta}{m_1 + m_2}$

$m_1 g \sin\theta = m_1 \ddot{q} + \ddot{x}\cos\theta$
$m_1 \ddot{q}_j = m_1 g \sin\theta - \ddot{x} m_1 \cos\theta \Rightarrow \ddot{q} = g\sin\theta - \ddot{x}\cos\theta$

Decoupling gives

$\ddot{x} = \frac{-m_1\cos\theta}{m_1+m_2}\left(g\sin\theta - \ddot{x}\cos\theta\right) = \ddot{x}\cos\theta \frac{m_1 \cos\theta}{m_1+m_2} - \frac{g\sin\theta\, m_1\cos\theta}{m_1+m_2}$

$\Rightarrow \ddot{x} - \ddot{x}\cos\theta^2 \frac{m_1}{m_1+m_2} = -g\sin\theta\cos\theta \frac{m_1}{m_1+m_2} \Rightarrow \ddot{x} = -g\sin\theta\cos\theta \frac{m_1}{m_1+m_2}\left(\frac{1}{1-\cos^2\theta \frac{m_1}{m_1+m_2}}\right)$

$\ddot{q} = g\sin\theta + \frac{m_1}{m_1+m_2}\ddot{q}\cos^2\theta \Rightarrow \ddot{q} - \ddot{q}\frac{m_1\cos^2\theta}{m_1+m_2} = g\sin\theta$

$\ddot{q}_j = g\sin\theta\left(\frac{1}{1-\frac{m_1}{m_1+m_2}\cos^2\theta}\right)$

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$$\boxed{\ddot{x} = -g\sin\theta\cos\theta \frac{m_1}{m_1+m_2}\left(\frac{1}{1-\cos^2\theta\frac{m_1}{m_1+m_2}}\right) \qquad \ddot{q} = g\sin\theta\left(\frac{1}{1-\cos^2\theta\frac{m_1}{m_1+m_2}}\right)}$$

b) if $\theta = 90°$ we expect freefall for $\ddot{x}$ & $\ddot{q} = 0$

$$\ddot{x} = g\sin(90)\cos(90)\frac{m_1}{m_1+m_2}\left(\frac{1}{1-\cos^2(90)\frac{m_1}{m_1+m_2}}\right)$$

$$= 0\,\frac{m_1}{m_1+m_2}\left(\frac{1}{1-0}\right) = 0 \quad , \quad \boxed{\ddot{x} = 0 \text{ like we expect}}$$

$$\ddot{q} = g\sin(90)\left(\frac{1}{1-\cos^2\theta\frac{m_1}{m_1+m_2}}\right) = -g(1)\left(\frac{1}{1-0}\right) = g$$

$\boxed{\ddot{q} = g \text{ which is what we expect}}$ $g = -9.81\ m/s^2$

c) if $m_2 \gg m_1$, we expect $\ddot{x} = 0$ b/c m_1 can't move m_2

$$\ddot{x} = -g\sin\theta\cos\theta\,(0)\left(\frac{1}{1-\cos^2\theta\cdot 0}\right) = 0 \qquad \text{if } \frac{m_1}{m_1+m_2} \approx 0$$

$\boxed{\ddot{x} = 0 \text{ like we expect}}$

$\ddot{q}$ we expect to be a block moving down a stationary ramp, so

$$\ddot{q} = g\sin\theta\left(\frac{1}{1-0}\right) = g\sin\theta$$

$\boxed{\ddot{q} = g\sin\theta \text{ which is what we expect}}$

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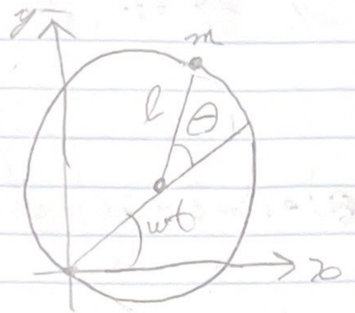
3)



a) To get this, we can get the LaGrangian from $L = T - U$
But $U = 0$ since we are on the ground / constant U
$\therefore \mathcal{L} = T$, so find T

$T = \frac{1}{2} m v^2$, but we can find $\vec{x}$ & $\vec{y}$ to get $\vec{v}$

$\vec{x} = l\cos(\omega t + \theta) + l\cos(\omega t)$

distance 2 center of circle

$\vec{y} = l\sin(\omega t + \theta) + l\sin(\omega t)$

distance from center to mass

$\vec{r} = l\left[\hat{x}\left(\cos(\omega t + \theta) + \cos(\omega t)\right) + \hat{y}\left(\sin(\omega t + \theta) + \sin(\omega t)\right)\right]$

$$\vec{v} = \frac{dr}{dt} = l\left[\hat{x}\left(-(\omega + \dot{\theta})\sin(\omega t + \theta) - \omega\sin(\omega t)\right) + \hat{y}\left((\omega + \dot{\theta})\cos(\omega t + \theta) + \omega\cos(\omega t)\right)\right]$$

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} = l^2\left[(\omega + \dot{\theta})^2\sin^2(\omega t + \theta) + 2(\omega + \dot{\theta})\omega\sin(\omega t + \theta)\sin(\omega t) + \omega^2\sin^2(\omega t) + (\omega + \dot{\theta})^2\cos^2(\omega t + \theta) + 2(\omega + \dot{\theta})\omega\cos(\omega t + \theta)\cos(\omega t) + \omega^2\cos^2(\omega t)\right]$$

this is equal to from wolfram

$$|\vec{v}|^2 = l^2\left[(\omega + \dot{\theta})^2 + \omega^2 + 2\omega(\dot{\theta} + \omega)\cos\theta\right]$$

$$L = T = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m l^2\left[(\omega + \dot{\theta})^2 + \omega^2 + 2\omega(\dot{\theta} + \omega)\cos\theta\right]$$

$$\boxed{L = \frac{1}{2} m l^2\left[(\omega + \dot{\theta})^2 + \omega^2\right] + m l^2 \omega(\dot{\theta} + \omega)\cos\theta}$$

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now we need to find EL Eq w/ $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)$

$$\frac{\partial L}{\partial \theta} = -m l^2 \omega(\dot\theta + \omega)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}\left(l^2 m(\dot\theta + \omega) + m l^2 \omega \cos\theta\right) = l^2 m \ddot\theta - m l^2 \omega \sin\theta\,\dot\theta$$

setting these equal we get

$$-m l^2 \omega(\dot\theta + \omega)\sin\theta = l^2 m\ddot\theta - m l^2 \omega \sin\theta\,\dot\theta$$
$$-m l^2 \omega\dot\theta \sin\theta - m l^2 \omega^2 \sin\theta = l^2 m\ddot\theta - m l^2 \omega\sin\theta\,\dot\theta$$

$$\Rightarrow -m l^2 \omega^2 \sin\theta = l^2 m \ddot\theta$$

Solving for $\ddot\theta$ we get...

$$\boxed{\ddot\theta = -\omega^2 \sin\theta}$$

b) Taylor expanding, $\sin\theta \approx \theta$ $\therefore$ $\ddot\theta + \omega^2\theta = 0$

w/ ~~this~~ we know the roots for this eq as

$\theta = \exp(\lambda t)$ $\Rightarrow$ $\lambda^2\theta + \omega^2\theta = 0$ $\Rightarrow \lambda^2 + \omega^2 = 0$

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega$$

$\boxed{\text{we know ~~the~~ frequency is } \omega \text{ \& } \theta_{eq} = n\pi,\ n \in \mathbb{Z}}$

we know this b/c @ $\theta = 0$, $\dot\theta = 0$ & no motion

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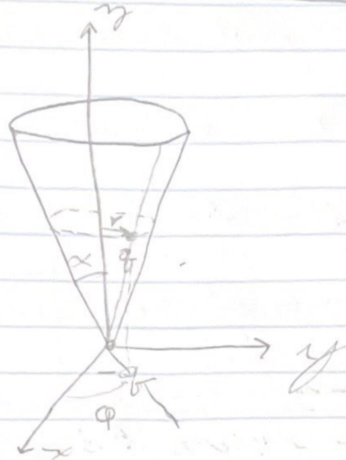
4)



a) There is 1 constraint

$$\tan(\theta) = \frac{r}{z} \quad // \quad \theta = \alpha$$

There are 2 Degrees of freedom r & z

b) To find $\mathcal{L}$, need U & T

U of 2 charges: $U = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{d}$

$\sin\alpha = \frac{r}{d} \Rightarrow d = \frac{r}{\sin\alpha}$ $\quad \therefore U = \frac{-1}{4\pi\epsilon_0} \frac{q^2 \sin\alpha}{r}$

now to find T we need $\vec{v}$ in cylindrical coords

$v = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$

$z = r / \tan\alpha \Rightarrow \dot{z} = \dot{r} / \tan\alpha$

$\vec{v} = \langle \dot{r}, r\dot{\phi}, \dot{r}/\tan\alpha \rangle \; [\hat{r}, \hat{\phi}, \hat{z}]$

$||\vec{v}||^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2/\tan^2\alpha = \dot{r}^2\left(1 + \frac{1}{\tan^2\alpha}\right) + r^2\dot{\phi}^2$

But $\frac{1}{\tan^2\alpha} = \cot^2\alpha$ & $1 + \cot^2\alpha = \csc^2\alpha$, so

$\vec{v}^2 = \dot{r}^2 \csc^2\alpha + r^2\dot{\phi}^2$

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now to find T we do $\frac{1}{2}m\vec{v}^2$, so

$$T = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\varphi}^2\right)$$

now plug in values for $L = T - U$

$$\boxed{L = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\varphi}^2\right) + \frac{q^2\sin\alpha}{4\pi\epsilon_0 r}}$$

c) φ is cyclic because L has no explicit φ dependence

d) $H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - \mathcal{L}$ where $\dot{q}_j = \dot{r}$ & $\dot{\varphi}$

$$\dot{r}\frac{\partial L}{\partial \dot{r}} = \dot{r}\left(m\dot{r}\csc^2\alpha\right) = m\dot{r}^2\csc^2\alpha \equiv A_1$$

$$\dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi}\left(mr^2\dot{\varphi}\right) = mr^2\dot{\varphi}^2 \equiv A_2$$

$$H = A_1 + A_2 - L$$

$$\boxed{H = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\varphi}^2\right) - \frac{q^2\sin\alpha}{4\pi\epsilon_0 r}}$$

Since no nonconservative forces act, we know H is the total energy, which is a conserved quantity

$l = m v_0 \times r$, but $v_0 \perp r$, so $l = m v_0 r$ c

$l = m r(r\dot{\varphi}) = mr^2\dot{\varphi}$, which is $\frac{\partial L}{\partial \dot{\varphi}}$ $\therefore$ $p_\varphi = l$

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e) To find the eq of motion, we apply EL eqs ($r, \dot{r}; t$)

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) \Rightarrow \frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - \frac{q^2 \sin\alpha}{4\pi\epsilon_0 r^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}\left(m\dot{r}\csc^2\alpha\right) = m\ddot{r}\csc^2\alpha$$

$$\boxed{m\ddot{r}\csc^2\alpha = mr\dot{\varphi}^2 - \frac{q^2\sin\alpha}{4\pi\epsilon_0 r^2}}$$

If $\frac{d}{dt}(\dot{\varphi}) = 0$, then we know $\ddot{r} = 0$ because to balance angular momentum, we would have to have $\ddot{\varphi} \neq 0$ to balance if $\ddot{r} \neq 0$, so $\ddot{r} = 0$

$$\therefore mr\dot{\varphi}^2 = \frac{q^2 \sin\alpha}{4\pi\epsilon_0 r^2} \Rightarrow 4\pi\epsilon_0 m r^3\dot{\varphi}^2 = q^2\sin\alpha$$

$$\Rightarrow r^3 = \frac{q^2\sin\alpha}{4\pi\epsilon_0 m\dot{\varphi}^2} \Rightarrow \boxed{r = \sqrt[3]{\frac{q^2\sin\alpha}{4\pi\epsilon_0 m\dot{\varphi}^2}}}$$

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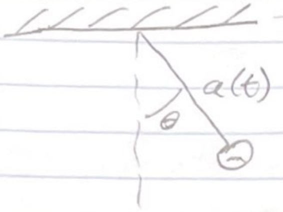
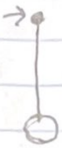
5)



$a(t)$ is length & can change

a) To find $L = T - U$, find T & U

If →  (top of pendulum) is $U = 0$, then

$$U = -mga\cos\theta$$

To find T, we need $\|\vec{v}\|^2$, so finding v

$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\quad r = a(t)$ $\quad \dot{r} = \dot{a}(t)$ $\quad \dot{\theta} = \dot{\theta}$

$$v = \langle \dot{a}, a\dot{\theta} \rangle \qquad \|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$$

$\vec{v}\cdot\vec{v} = \dot{a}^2 + a^2\dot{\theta}^2$

$$\boxed{L = \tfrac{1}{2}m\left(\dot{a}^2 + a^2\dot{\theta}^2\right) + mga\cos\theta}$$

b) $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$, so we find ~~this~~ below

$$\boxed{p_\theta = m a^2 \dot{\theta}}$$

c) This makes sense as the p_θ is the velocity, times the radius, times mass

$$\boxed{\begin{array}{l}\text{where } \vec{v}_\theta = a\dot{\theta}\text{, radius} = a\text{, \& mass} = m\text{, so} \\ \qquad p_\theta = m a^2\dot{\theta} \text{ makes sense}\end{array}}$$

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d) To find eq of motion, take E-L eq $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$

$\frac{\partial L}{\partial \theta} = -mga\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ma^2\dot{\theta}\right) = ma^2\ddot{\theta} + 2ma\dot{a}\dot{\theta}$

$\Rightarrow \quad -mga\sin\theta = ma^2\ddot{\theta} + 2ma\dot{a}\dot{\theta}$

$a^2\ddot{\theta} + 2a\dot{a} + ga\sin\theta = 0 \Rightarrow a\ddot{\theta} + 2\dot{a}\dot{\theta} + g\sin\theta = 0$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{2\dot{a}\dot{\theta}}{a} + \frac{g}{a}\sin\theta = 0}$$

e) $H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - \mathcal{L}$ w/ $\dot{q}_j = \dot{\theta}$ & $\dot{a}$

$\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}\left(ma^2\dot{\theta}\right) = ma^2\dot{\theta}^2$

$\dot{a}\frac{\partial L}{\partial \dot{a}} = \dot{a}\left(m\dot{a}\right) = m\dot{a}^2$

$$\boxed{h = \frac{1}{2}m\dot{a}^2 + \frac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta}$$

h is the total energy & energy is conserved even though g is applying a torque