Phys 427 - HW#7

1a) w/ $\underline{\mathcal{D}(\epsilon)\,d\epsilon} = 4 \cdot \frac{\frac{1}{8}4\pi k^2\,dk}{\pi^3/V} = \frac{2V}{\pi^2} k^2\,dk$ w/ $k = \sqrt{2m/\hbar^2}\,\sqrt{\epsilon}$

$dk = \sqrt{2m/\hbar^2}\,d\epsilon/2\sqrt{\epsilon}$

$$= \frac{2V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{d\epsilon}{2\sqrt{\epsilon}}\,\epsilon = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} d\epsilon\,\sqrt{\epsilon} = \mathcal{D}(\epsilon)\,d\epsilon$$

$$\Rightarrow \underline{N} = \int_0^\infty d\epsilon\,\mathcal{D}(\epsilon)\,f = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{d\epsilon\sqrt{\epsilon}}{\exp(\beta(\epsilon-\mu))} = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\exp(\beta\mu)\int_0^\infty \frac{d\epsilon\sqrt{\epsilon}}{\exp(\beta\epsilon)}$$

$$= \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\exp(\beta\mu)}{\beta^{3/2}}\int_0^\infty \frac{d(\beta\epsilon)(\beta\epsilon)^{1/2}}{\exp(\beta\epsilon)} = \frac{V}{\pi^2}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2}\exp(\beta\mu)\int_0^\infty \frac{d(\beta\epsilon)\,(\beta\epsilon)^{1/2}}{\exp(\beta\epsilon)}$$

[solve $\int_0^\infty \frac{dx\,x^{1/2}}{\exp(-x)}$ $y = \sqrt{x}$, $dy = \frac{1}{2}x^{-1/2}\,dx$ $\Rightarrow dx = 2x^{1/2}dy = 2y\,dy$, $\sqrt{x}\,dx = 2y^2\,dy$

$$\Rightarrow \int_0^\infty \frac{2dy\,y^2}{\exp(y^2)} = -2\,\partial_\alpha\int_0^\infty dy\,\exp(-y^2) = -2\,\partial_\alpha\left[\frac{\sqrt{\pi}}{\sqrt{\alpha}}\,\frac{1}{2}\right] = -\sqrt{\pi}\left[\frac{-1}{2}\,\alpha^{-3/2}\right]$$

w/ $\alpha = 1$, $= \sqrt{\pi}/2$]

$$N = \frac{V}{\pi^2}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2}\exp(\beta\mu)\left(\frac{\sqrt{\pi}}{2}\right) = \boxed{\frac{V\exp(\beta\mu)}{2}\left(\frac{2m}{\pi\beta\hbar^2}\right)^{3/2} = N}$$

b) $n_Q = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \Rightarrow N = V\exp(\beta\mu)\,\frac{2^{3/2}}{2}\cdot\frac{2^{3/2}}{2^{3/2}}\left(\frac{mkT}{\pi\hbar^2}\right)^{3/2} = 4V\exp(\beta\mu)\overbrace{\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}}^{n_Q}$

$$\Rightarrow \frac{N}{V} = n = 4\exp(\beta\mu)\,n_Q \Rightarrow \frac{n}{4n_Q} = \exp(\beta\mu) \Rightarrow \beta\mu = \ln\left(\frac{n}{4n_Q}\right) \Rightarrow \mu = \frac{-1}{\beta}\ln\left(\frac{4n_Q}{n}\right)$$

$\boxed{\therefore \mu(T) = -kT\ln\left(\frac{4n_Q}{n}\right)}$

compare to classical result of $\mu = -kT\ln(n_Q/n)$, there is an extra 4 multiplying the n_Q in the gravitons.

The extra 4 occurs because the quantum density is lower in a graviton gas because the states are closer ~~together~~ in space.

c) $$U = \int_0^\infty d\epsilon\,\mathcal{D}(\epsilon)\,f\,\epsilon = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\exp(\beta\mu)\int_0^\infty \frac{d\epsilon\,\epsilon^{3/2}}{\exp(\beta\epsilon)} = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\exp(\beta\mu)}{\beta^{5/2}}\int_0 \frac{d(\beta\epsilon)\,(\beta\epsilon)^{3/2}}{\exp(\beta\epsilon)}$$

[solve $\int_0^\infty \frac{dx\,x^{3/2}}{\exp(-x)}$ w/ $y = \sqrt{x}$, $dx = 2y\,dy$, $\Rightarrow \int_0^\infty \frac{2y^4\,dy}{\exp(y^2)} = 2\,\partial_\alpha^2\int_0^\infty dy\,\exp(-\alpha y^2)$

$$= 2\,\partial_\alpha^2\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right] = \sqrt{\pi}\,\partial_\alpha^2(\sqrt{\alpha})^{-1} = \sqrt{\pi}\left(\frac{-1}{2}\cdot\frac{-3}{2}\cdot\alpha^{-5/2}\right) \text{ w/ } \alpha = 1, \; = \frac{3\sqrt{\pi}}{4}$$]

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2c) $\therefore \mathcal{U} = \frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\exp(\beta\mu)}{\beta^{5/2}}\left(\frac{3\sqrt{\pi}}{4}\right) = \frac{3V\exp(\beta\mu)}{4\beta^{5/2}}\left(\frac{2m}{\beta\pi\hbar^2}\right)^{3/2}$

$= \frac{3}{2\beta}\left[\left(\frac{V\exp(\beta\mu)}{2}\left(\frac{2m}{\pi\beta\hbar^2}\right)^{3/2}\right]\right. = \boxed{\frac{3N}{2\beta} = \mathcal{U}}$

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2a) $D\,d\epsilon = 2 \cdot \dfrac{\frac{1}{4}2\pi k\,dk}{\pi^2/A}$ w/ $k = \sqrt{2m/\hbar^2}\,\sqrt{\epsilon}$

$dk = \sqrt{2m/\hbar^2}\,d\epsilon / 2\sqrt{\epsilon}$

$$D(\epsilon)\,d\epsilon = \frac{A}{\pi}\left(\frac{2m}{\hbar^2}\right)\sqrt{\epsilon}\,\frac{d\epsilon}{2\sqrt{\epsilon}} = \frac{A}{2\pi}\left(\frac{2m}{\hbar^2}\right)d\epsilon = \boxed{\frac{mA}{\pi\hbar^2}\,d\epsilon = D(\epsilon)\,d\epsilon}$$

b) ϵ_F @ $T=0$ $\Rightarrow N = \int_0^\infty d\epsilon\, D(\epsilon)\, f(\epsilon,0) \approx \int_0^\infty d\epsilon\, D(\epsilon) = \frac{mA}{\pi\hbar^2}\int_0^{\mu(0)} d\epsilon = \frac{mA}{\pi\hbar^2}\mu(0) = N$

$\Rightarrow \frac{N}{A}\frac{\pi\hbar^2}{m} = \mu(0)$ w/ $\epsilon_F = \mu(0)$, $\boxed{\epsilon_F = \frac{n\pi\hbar^2}{m}}$

c) $U(T=0) = \frac{mA}{\pi\hbar^2}\int_0^{\mu(0)} \epsilon\,d\epsilon = \frac{mA}{\pi\hbar^2}\left[\frac{1}{2}\epsilon^2\right]_0^{\mu(0)} = \frac{mA}{2\pi\hbar^2}\mu(0) \Rightarrow \boxed{U(T=0) = \frac{mA}{2\pi\hbar^2}\epsilon_F^2}$

d) $N = \int_0^\infty d\epsilon\, D(\epsilon)\, f(N,T) = \int_0^\infty d\epsilon\,\frac{mA}{\pi\hbar^2}\,\frac{1}{\exp(\beta(\epsilon-\mu)+1)} = \frac{mA}{\pi\hbar^2}\int_0^\infty \frac{d\epsilon}{\exp(\beta(\epsilon-\mu))+1}$

$= \frac{mA}{\pi\hbar^2}\left[-\frac{1}{\beta}\ln\left[\exp(-\beta(\epsilon-\mu))+1\right]\right]_0^\infty = \frac{mA}{\pi\hbar^2\beta}\left[0 + \ln\left(\exp(\beta\mu)+1\right)\right]$

$N = \frac{mA}{\pi\hbar^2\beta}\ln\left[\exp(\beta\mu)+1\right] \Rightarrow \frac{n\pi\hbar^2\beta}{m} = \ln\left[\exp(\beta\mu)+1\right] \Rightarrow \exp\left(\overbrace{\frac{n\pi\hbar^2\beta}{m}}^{=\epsilon_F\beta}\right) = \exp(\beta\mu)+1$

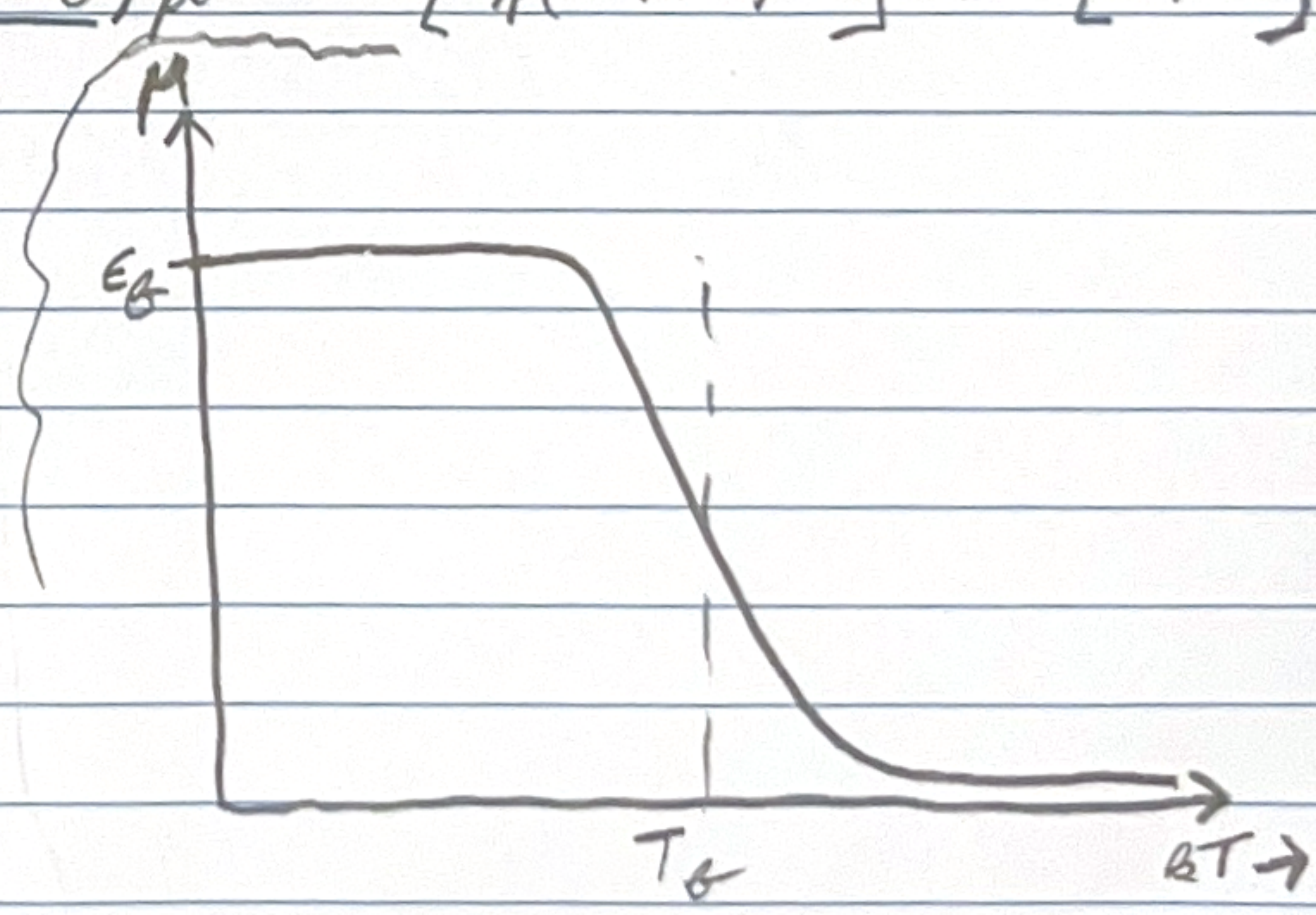
$\Rightarrow \boxed{\mu = \frac{1}{\beta}\ln\left[\exp(\beta\epsilon_F)-1\right]} = kT\ln\left[\exp\left(\frac{\epsilon_F}{kT}\right)-1\right]$

e) $\lim_{T\to 0}\mu = kT\ln\left[\exp(\epsilon_F/kT)-1\right] \approx kT\ln\left(1+\epsilon/kT-1\right) \approx kT\left[\epsilon/kT - 1\right] = \epsilon_F - \cancelto{0}{kT}$

$e^x \approx x+1$ for small x

$\boxed{\text{as } T\to 0,\ \mu = \epsilon_F}$ & when $kT \gg \epsilon_F$, $\mu = kT\ln\left[\exp(\epsilon_F/kT)-1\right] \approx kT\ln\left[\epsilon_F/kT\right]$

$\mu = kT\ln\left(\frac{n\pi\hbar^2}{mkT}\right)$ w/ n_Q in 2D $= kTm/\pi\hbar^2$



$\mu = kT\ln\left(\frac{n}{n_Q}\right) \Rightarrow \boxed{\mu = -kT\ln\left(\frac{n_Q}{n}\right) \text{ for } kT \gg \epsilon_F}$

the same as the classical limit