Joseph Specht
jspecht3

HW #4

2.38)

Splitting in half, we know one hemisphere has $q/2$ charge, so...

$$\vec{E}_{Top} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cdot\frac{1}{2} = \frac{q\hat{r}}{8\pi\epsilon_0 r^2}$$

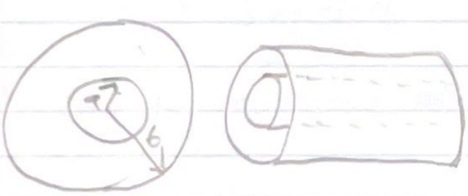
$\Rightarrow d\vec{F} = \vec{E}_{top}\,dq = \vec{E}_{top}\,\sigma\,dA \qquad \sigma = \frac{q}{4\pi r^2} \qquad dA = r^2\sin\theta\,d\theta\,d\phi$

$\Rightarrow F = \int dF = \int \frac{q\hat{r}}{8\pi\epsilon_0 r^2}\cdot\frac{q}{4\pi r^2}\cdot r^2\sin\theta\,d\theta\,d\phi = \int\frac{q^2\hat{z}}{32\pi^2\epsilon_0 r^2}\sin\theta\cos\theta\,d\theta\,d\phi$

$\Rightarrow F = \frac{q^2}{32\pi^2\epsilon_0 r^2}\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{q^2}{16\pi\epsilon_0 r^2}\left[\frac{1}{2}\sin^2(\theta)\right]_0^{\pi/2}$

$\Rightarrow F = \frac{q^2}{32\pi\epsilon_0 r^2}\left[1-0\right] = \boxed{\frac{q^2}{32\pi\epsilon_0 r^2} = F}$

2.39)



$C \equiv \frac{q}{V}$ Using Gauss's Law gives

$\int E\cdot dA = \frac{q_{enc}}{\epsilon_0}$, but $\vec{E}\perp d\vec{A}$, so

$\Rightarrow \vec{E}A = \frac{q_{enc}}{\epsilon_0}$ where $A = 2\pi r l$, so $E = \frac{q\hat{r}}{2\pi\epsilon_0 r l}$

moving on to Voltage

$V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b \frac{q\hat{r}}{2\pi\epsilon_0 r l}\cdot dr\hat{r} = -\frac{q}{2\pi\epsilon_0 l}\int_a^b\frac{dr}{r}$

$V = -\frac{q}{2\pi\epsilon_0 l}\ln\left(\frac{b}{a}\right) \Rightarrow C = q\cdot\frac{-2\pi\epsilon_0 l}{q\ln\left(\frac{b}{a}\right)}$

$\Rightarrow \boxed{C = -2\pi\epsilon_0 l / \ln\left(\frac{b}{a}\right)}$

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3.1) Know voltage forms the smoothest path from boundary layers, & if it is a sphere in question, the voltage at the center will be the average from equidistant boundaries, so $V_{ave} = V_{center}$. ← no charge enclosed

Due to superposition, we know if there's a charge in the sphere, its voltage adds linearly. The voltage just looks like the sum of $V_{average}$ + V from a point charge Q, so

$$\boxed{V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}}$$

3.4) Say we have $\vec{E}_1$ & $\vec{E}_2$ such that

$$\vec{\nabla}\vec{E}_1 = \frac{\rho(\vec{r})}{\epsilon_0} \quad , \quad \vec{\nabla}\vec{E}_2 = \frac{\rho(\vec{r})}{\epsilon_0}$$

& $E_3 \equiv E_1 - E_2$, then we can say $\vec{\nabla}\cdot\vec{E}_3 = \left(\frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0}\right) = 0$

We also know V @ any point won't change based on the method we use, so

$$V_1\big|_b = V = V_2\big|_b \qquad \therefore V_3 = V_1 - V_2 = 0$$

Product Rule trick

$$\nabla \cdot V_3 E_3 = V_3(\underbrace{\nabla E_3}_{0}) + E_3(\overbrace{\nabla V_3}^{-E_3}) = -(E_3)^2$$

Divergence Theorem

$$\int_V -(E_3)^2\, d\tau = \oint_S V_3 \vec{E}_3 \cdot d\vec{A}$$

, but for this to be true, either V_3 or $E_{3\perp} = 0$ because $E = -\nabla V$, so they are not possibly equal unless 0

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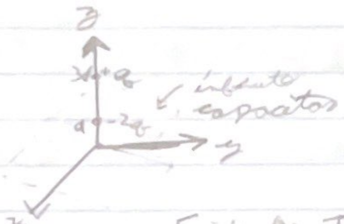
HW#4 - cont

This shows that V_1 has to $= V_2$ because E_3 can only satisfy such an equation if they are.

3.6) 

Generate image charges of $+2q$ @ $-d$ & $-q$ @ $-3d$.

Finding the E field @ 3d due to the real $-2q$ & the image $-q$ & $+2q$ gives

$$\vec{E} = \frac{\hat{z}}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d^2)} - \frac{q}{(6d^2)}\right] \quad \& \quad q\vec{E} = \vec{F}\text{, so}$$

$$\vec{F} = \frac{q^2\hat{z}}{4\pi\epsilon_0}\left[\frac{-2}{4d^2} + \frac{2}{16d^2} - \frac{1}{36d^2}\right] = \frac{q^2\hat{z}}{4\pi\epsilon_0 d^2}\left[\frac{-72}{144} + \frac{18}{144} - \frac{4}{144}\right] = \frac{-q^2\hat{z}}{4\pi\epsilon_0 d^2}\left[\frac{58}{144}\right]$$

$$\Rightarrow \boxed{\vec{F} = \frac{-q^2\hat{z}}{4\pi\epsilon_0 d^2}\left(\frac{29}{72}\right) = \frac{-29}{288}\,\frac{q^2\hat{z}}{\pi\epsilon_0 d^2}}$$

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3.7) From law of cosines, we can get r & r' from figure 3.13

a) $$r = \sqrt{r^2 + a^2 - 2ra\cos\theta} \quad \& \quad r' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

we also know from eq 3.15 & 3.16 that

$$q' = -\frac{R}{a} q \quad \& \quad b = \frac{R^2}{a}, \text{ so}$$

$$\frac{q'}{r'} = \frac{-Rq}{a} \frac{1}{\sqrt{r^2 + (R^2/a)^2 - 2rR^2/a\cos\theta}} = \frac{-q}{\sqrt{\frac{r^2a^2}{R^2} + R^2 - 2ra\cos\theta}} \quad \&$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\left[\left(r^2 + a^2 - 2ra\cos\theta\right)^{-1/2} - \left(\frac{r^2a^2}{R^2} + R^2 - 2ra\cos\theta\right)^{-1/2}\right]}$$

b) Know $\vec{E} = \frac{\sigma}{\epsilon_0}$ & $-\nabla V = E$, but it is a sphere, so E is only radially outward

$$\therefore -\frac{\partial V}{\partial r} = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial r}$$

↓ derivative calculator

$$\left.\frac{\partial V}{\partial r}\right|_R = \frac{q}{8\pi\epsilon_0}\left[\left(r^2 + a^2 - 2ra\cos\theta\right)^{-3/2}(2r - 2a\cos\theta) - \left(R^2 + r^2a^2/R^2 - 2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r - 2a\cos\theta\right)\right]$$

$$\left.\frac{\partial V}{\partial r}\right|_R = \frac{q}{4\pi\epsilon_0}\left[\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}(R - a\cos\theta) - \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R} - a\cos\theta\right)\right]$$

$$\left.\frac{\partial V}{\partial r}\right|_R = \frac{q}{4\pi\epsilon_0}\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left[R - \cancel{a\cos\theta} - \frac{a^2}{R} + \cancel{a\cos\theta}\right]$$

$$\boxed{\sigma = \frac{q}{4\pi}\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left[R - \frac{a^2}{R}\right]}$$

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HW#4 - cont

Finding total charge.

$Q = \int \sigma \, dA$ w/ $dA = R^2 \sin\theta \, d\theta \, d\varphi$

$\Rightarrow \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \; R^2 \sin\theta \left(\frac{q}{4\pi} \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2} \left[R - \frac{a^2}{R}\right] \right)$

$\Rightarrow \left(R - \frac{a^2}{R}\right) \frac{qR^2}{2} \int_0^{\pi} \sin\theta \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}$

↳ integral calculator: $Q = \frac{qR}{2}\left(R^2 - a^2\right)\left[\frac{-1}{Ra}\left(\frac{1}{\sqrt{R^2+a^2+2Ra}} - \frac{1}{\sqrt{R^2+a^2-2Ra}}\right)\right]$

$Q = \frac{q}{2a}\left(a^2 - R^2\right)\left(\frac{1}{\sqrt{(a+R)^2}} - \frac{1}{\sqrt{(a-R)^2}}\right)$ ← R − a doesn't work later b/c a > R since outside

$Q = \frac{q}{2a}\left(a^2 - R^2\right)\left(\frac{1}{a+R} - \frac{1}{a-R}\right) = \frac{q}{2a}\left(\frac{(a-R)\cancel{(a+R)}}{\cancel{(a+R)}} - \frac{\cancel{(a-R)}(a+R)}{\cancel{(a-R)}}\right)$

$Q = \frac{q}{2a}\left(a - R - a - R\right) = -\frac{qR}{a}$ $\boxed{Q = -\frac{qR}{a}}$

c) know $W = \int_{-\infty}^{a} F\,dx$, now need to find force

Force can be found using image charge of $-q$ @ distance b away from a

$F = \frac{-q^2}{4\pi\epsilon_0}(a-b)^{-2}$ $\quad (a-b)^2 = a^2 - 2ab + b^2 = \left(a^2 - R^2 + \frac{R^4}{a^2}\right)$

$F = \frac{-q^2}{4\pi\epsilon_0} \frac{1}{a^2 - R^2 + \frac{R^4}{a^2}}$ $\quad \Rightarrow W = \int_{-\infty}^{a} F\,dx \overset{\text{integral calc}}{=} \frac{-q^2 R}{8\pi\epsilon_0}\left[\frac{1}{x^2 - R^2}\right]_{\infty}^{a}$

$\Rightarrow W = -\frac{q^2 R}{8\pi\epsilon_0}\left(\frac{1}{a^2 - R^2} + 0\right) \Rightarrow \boxed{W = -\frac{q^2 R}{8\pi\epsilon_0}\left(\frac{1}{a^2 - R^2}\right)}$