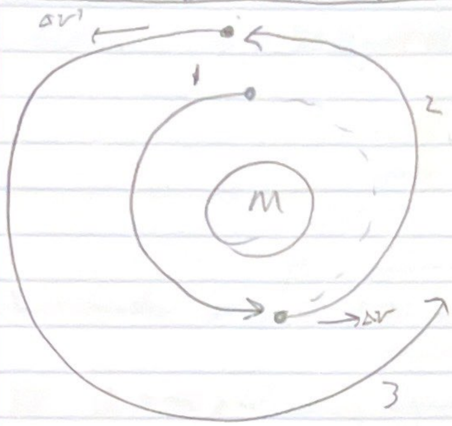
HW#5 – Joseph Specht

1)




$v_0 = \sqrt{GM/R}$

$v_f = \sqrt{GM/R'}$

a) We know @ $r_p$ & $r_a$, $r \perp v$, so $l = rv$ & in this transition orbit, $r_p = R$ & $r_a = R'$

using conservation of energy, we know

$e_T = \frac{v_1^2}{2} - \frac{GM}{R} = \frac{v_2^2}{2} - \frac{GM}{R'}$, so $\frac{v_1^2}{2} - \frac{v_2^2}{2} = \frac{GM}{R} - \frac{GM}{R'}$

$\Rightarrow \frac{1}{2}(v_1^2 - v_2^2) = GM\left(\frac{1}{R} - \frac{1}{R'}\right)$, but $l = rv$, so

$v_1 = \frac{l}{R}$ & $v_2 = \frac{l}{R'}$, so

$\frac{1}{2}\left(\frac{l^2}{R^2} - \frac{l^2}{R'^2}\right) = GM\left(\frac{1}{R} - \frac{1}{R'}\right) \Rightarrow \frac{l^2}{2}\underbrace{\left(\frac{1}{R^2} - \frac{1}{R'^2}\right)}_{\text{difference of 2 squares}} = GM\left(\frac{1}{R} - \frac{1}{R'}\right)$

we can break up this to get

$\frac{l^2}{2}\left(\frac{1}{R} + \frac{1}{R'}\right)\left(\frac{1}{R} - \frac{1}{R'}\right) = GM\left(\frac{1}{R} - \frac{1}{R'}\right) \Rightarrow \frac{l^2}{2} = \frac{GM}{\frac{1}{R} + \frac{1}{R'}}$

but $\frac{1}{R} + \frac{1}{R'} = \frac{R'}{RR'} + \frac{R}{RR'} = \frac{R'+R}{RR'}$ & $\frac{1}{\text{this}}$ is equal to

$\frac{RR'}{R'+R}$, so $l^2 = \frac{2GMRR'}{R'+R}$ $\boxed{\therefore l_T = \sqrt{\frac{2GMRR'}{R'+R}}}$

# HW#5 - graph sketch

Solving for $e_T$ let $a := R' + R$

$$e_T = \frac{\ell^2}{2R^2} - \frac{GM}{R} = \frac{2GMRR'}{2R^2(R'+R)} - \frac{GM}{R} = \frac{GMR'}{Ra} - \frac{GMa}{Ra}$$

$$\Rightarrow GM\left(\frac{R'-a}{Ra}\right) = GM\left(\frac{R'-R'-R}{R(R'+R)}\right) = GM\left(\frac{-R}{R(R'+R)}\right) = \frac{-GM}{R'+R}$$

$$\boxed{e_T = \frac{-GM}{R'+R}}$$

2) Solving for $\Delta v$ $R_a = R_p$ for a circle

$$e_T = e_0 + \frac{\Delta v^2}{2} \Rightarrow \frac{-GM}{R'+R} = -\frac{GM}{2R} + \frac{\Delta v^2}{2} \Rightarrow \Delta v^2 = \frac{GM}{R} - \frac{2GM}{R'+R}$$

$$\Rightarrow \Delta v^2 = GM\left(\frac{R'+R}{R(R'+R)} - \frac{2R}{R(R'+R)}\right) = GM\left(\frac{R'-R}{(R+R')R}\right)$$

$$\boxed{\therefore \Delta v = \sqrt{\frac{GM(R'-R)}{R(R+R')}}}$$

Solving for $\Delta v'$

$$e_f = e_T + \frac{(\Delta v')^2}{2} \Rightarrow \frac{-GM}{2R'} = -\frac{GM}{R+R'} + \frac{(\Delta v')^2}{2} \Rightarrow (\Delta v')^2 = \frac{2GM}{R+R'} - \frac{GM}{R'}$$

$$\Rightarrow (\Delta v')^2 = GM\left(\frac{2R'}{(R+R')R'} - \frac{(R+R')}{R'(R+R')}\right)$$

$$= GM\left(\frac{R'-R}{R'(R+R')}\right)$$

$$\boxed{\therefore \Delta v' = \sqrt{\frac{GM(R'-R)}{R'(R+R')}}}$$

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2a) $v_p > v_a$. This is $\boxed{\text{true}}$ since, $l_p = l_a = r_p v_p = r_a v_a$ since they are colinear

$\therefore v_p = \frac{r_a}{r_p} v_a$ since $r_a > r_p$, $\frac{r_a}{r_p} > 1$ $\therefore v_p > v_a$

$v_a r_a = v_p r_p$. This is $\boxed{\text{true}}$ because at these points, $\vec{v} \perp \vec{r}$ $\therefore l = vr\sin(90°)$ & $l = vr$. We know angular momentum is conserved @ all points, so they have to be equal.

$l_a > l_{midway}$. This is $\boxed{\text{false}}$ because $l$ is the same @ all points in an orbit.

$e < 0$. This is $\boxed{\text{true}}$ because the minimum energy to escape an orbit is $e = 0$; $\therefore$ if something is in orbit, $e < 0$.

$v_p > v_{midway}$. This is $\boxed{\text{true}}$ because, by Kepler's 2nd law, the satellite sweeps out an equal area each $dt$, so if $r$ is @ its lowest ($r_p$), $v$ has to be highest

$v_a r_a > v_{midway} r_{midway}$. This is $\boxed{\text{false}}$. Since $l$ is the same @ all points, we know the $l$ are equal & can set up the expression $l_m = l_a = r_m v_m \sin(\theta) = r_a v_a \sin(90)$. $\theta \neq 90°$ We know that $r_a \perp v_a$, but $r_m \not\perp v_m$ $\therefore \sin\theta < 1$ & $v_m r_m > v_a r_a$ to conserve $l$.

$e_a = e_p$. This is $\boxed{\text{true}}$ because there are no non-conservative forces acting on the satellite.

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$\frac{v_a^2}{r_a} = \frac{GM}{r_a^2}$. This is $\boxed{\text{false}}$ because $v_a^2/r_a$ is only equal to $GM/r_a^2$ in a circular orbit because $F/m = GM/r^2$ in circular orbits.

e) In a circle we know $F = \frac{mv^2}{R} = \frac{GMm}{R^2}$, we can use this to solve for $v$

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow \frac{v^2}{R} = \frac{GM}{R^2} \Rightarrow v^2 = \frac{GM}{R} \Rightarrow \boxed{v = \sqrt{\frac{GM}{R}}}$$

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3a) Since $l = v b \cos\theta$, & an infinite distance away, $\cos\theta \approx 1$


$v$ is effectively $\perp$ to $b$, so

$$\boxed{l = v_0 b}$$

b) @ $r = \infty$, there is only $T$, so

$$\boxed{e = \frac{v_0^2}{2}}$$

c) since $e > 0$, we know the orbit is $\boxed{\text{hyperbolic}}$

d) the speed $v'$ is $\boxed{v_0}$ because only conservative forces act & there is enough energy to escape orbit.

e) Since $l$ is conserved $v_0 b = v_\theta b'$, but $v_0 = v_\theta$, so

$$\boxed{b = b'}$$

f) $r(\varphi) = \frac{\alpha}{1 + \epsilon\cos(\varphi)}$, but @ $r_{min}$ $r = \frac{\alpha}{1+\epsilon}$

$\alpha = \frac{v_0^2 b^2}{GM}$ & $\epsilon = \sqrt{1 + \frac{v_0^4 b^2}{G^2 M^2}}$ w/ $v_0 = \sqrt{\frac{GM}{b}}$

$\epsilon = \sqrt{1 + \frac{b^2 G^2 M^2}{b^2 G^2 M^2}} = \sqrt{1+1} = \sqrt{2}$ & $\alpha = \frac{b^2 GM}{b GM} = b$

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plugging this in for $r$ we get

$$\boxed{r = \frac{b}{1+\sqrt{2}}}$$

2) $\varphi$ gives an infinite radius when the denominator of $r = 0$, so

$$r = \frac{b}{1+\epsilon\cos\varphi}, \text{ so we need}$$

$\epsilon\cos\varphi = -1$, so

$$\varphi = a\cos\left(\frac{-1}{\sqrt{2}}\right)$$

this $\varphi$ is satisfies this equation @

$(-\infty)$ $(\infty)$

$135°$ & $225°$

to get $\gamma$ we do $225° - 135° = 90°$

$$\boxed{\therefore \gamma = 90°}$$

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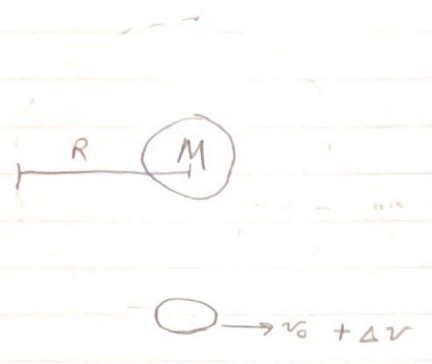
4)




$x v_0$ = escape velocity

$x > 1$

a) Energy has to = 0 to escape an orbit, so

$$0 = \frac{(x v_0)^2}{2} - \frac{GM}{R} \Rightarrow x^2 v_0^2 = \frac{2GM}{R}$$, but we

know $v_0 = \sqrt{\frac{GM}{R}}$, so $\frac{x^2 GM}{R} = \frac{2GM}{R}$, so $x^2 = 2$

$$\boxed{\therefore x = \sqrt{2}}$$

c) The energy is = 0 @ minimum escape velocity, so

$\epsilon = \sqrt{1} = 1$ $\boxed{\therefore \text{shape is parabolic}}$

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c) $\boxed{r_{perigee} = R}$

$\ell = R\beta v_0$ , $e = \frac{v^2}{2} - \frac{GM}{R} = \frac{\beta^2 v_0^2}{2} - \frac{GM}{R}$

$v = \beta v_0$

w/ $v_0 = \sqrt{\frac{GM}{R}}$ , so $e = \frac{\beta^2 GM}{2R} - \frac{GM}{R} = \frac{GM}{R}\left(\frac{\beta^2}{2} - 1\right)$

Solving for $\epsilon$,

$$\epsilon^2 = 1 + \frac{2e\ell^2}{G^2M^2} = 1 + 2\left(\frac{GM}{R}\left(\frac{\beta^2}{2} - 1\right)\right)\frac{R^2\beta^2 GM}{R} \cdot \frac{1}{G^2M^2}$$

$$\epsilon^2 = 1 + 2\left(\frac{\beta^2}{2} - 1\right)\beta^2 = 1 + \beta^4 - 2\beta^2 = \beta^4 - 2\beta^2 + 1$$

$$\epsilon^2 = (\beta^2 - 1)^2 \Rightarrow \underline{\epsilon = \beta^2 - 1}$$

now plugging into $r(\phi)$, we get $(\cos\phi) = -1$ b/c maxed to get $r_a$

$$r = \frac{\ell^2}{GM}\left(\frac{1}{1 - (\beta^2 - 1)}\right) = \frac{\ell^2}{GM}\left(\frac{1}{2 - \beta^2}\right) = \frac{R^2\beta^2 GM}{R\, GM}\left(\frac{1}{2 - \beta^2}\right)$$

$$= \frac{\beta^2 R}{2 - \beta^2}$$

$$\boxed{\begin{aligned} r_{perigee} &= R \\ r_{apogee} &= \frac{\beta^2 R}{2 - \beta^2} \end{aligned}}$$