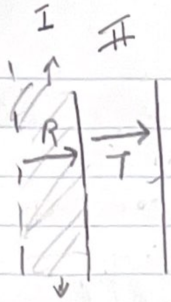
HW #6

1)



Infinite in $\hat{z}$
homogenous in $\hat{\theta}$

a) ϕ_1: $-D\nabla^2\phi_1 + \Sigma_a\phi_1 = \frac{1}{k}\nu\Sigma_f\phi_1 \Rightarrow \nabla^2\phi_1 + \underbrace{\frac{\frac{1}{k}\nu\Sigma_f - \Sigma_a}{D_1}}_{B^2}\phi_1 = 0$

$\Rightarrow \nabla^2\phi_1 + B^2\phi_1 = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi_1}{\partial r}\right) + B^2\phi_1 = 0$

Solutions are Bessel functions: $\phi_1 = C_1J_0(Br) + \cancel{C_2} Y_0(Br)$

ϕ_2: $-D\nabla^2\phi_2 + \Sigma_a\phi_2 = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi_2}{\partial r}\right) - \frac{1}{L_2^2}\phi_2 = 0$

Solutions are modified bessel functions: $\phi_2 = C_3I_0\left(\frac{r}{L_2}\right) + C_4K_0\left(\frac{r}{L_2}\right)$

BC) i) finite flux: $\phi_1(r=0) \neq \infty$
ii) flux continuity: $\phi_1(r=R) = \phi_2(r=R)$
iii) current continuity: $J_1(r=R) = J_2(r=R)$, $-D_1\frac{\partial\phi_1}{\partial r} = -D_2\frac{\partial\phi_2}{\partial r}$
iv) extrapolated length: $\phi_2(r=\tilde{T}) = 0$, $\tilde{T} = R+T+2D_2$

i) $\phi_1 = C_1J_0(0) + C_2Y_0(0) = 0 - C_2(\infty) = 0$ $\therefore C_2 = 0$

ii) $C_1J_0(BR) = C_3I_0\left(\frac{R}{L_2}\right) + C_4K_0\left(\frac{R}{L_2}\right)$

iii) $-D_1\left(-C_1BJ_1(BR)\right) = -D_2\left(\frac{C_3}{L_2}I_1\left(\frac{R}{L_2}\right) - \frac{C_4}{L_2}K_1\left(\frac{R}{L_2}\right)\right)$

↳ $C_1D_1BJ_1(BR) = \frac{D_2}{L_2}\left(C_4K_1\left(\frac{R}{L_2}\right) - C_3I_1\left(\frac{R}{L_2}\right)\right)$

iv) $C_3I_0\left(\frac{\tilde{T}}{L_2}\right) + C_4K_0\left(\frac{\tilde{T}}{L_2}\right) = 0$

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solve the system of equations

iv) $\Rightarrow$ $C_3\, Il_0(\bar{r}/L_2) + C_4 K_0(\bar{r}/L_2) = 0 \Rightarrow C_4 = -\dfrac{C_3\, Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)}$

ii) $\Rightarrow$ $C_1 J_0(BR) = C_3\, Il_0(R/L_2) - C_3 \dfrac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} K_0(R/L_2)$

$$C_3 = C_1 J_0(BR) \Big/ \left[Il_0(R/L_2) - \frac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} K_0(R/L_2) \right]$$

iii) $\Rightarrow$ $C_1 \sigma_1 B J_1(BR) = \frac{\sigma_2}{L_2}\left[C_4 K_1(R/L_2) - C_3\, Il_1(R/L_2) \right]$

$$\frac{L_2 C_1 \sigma_1 B}{\sigma_2} J_1(BR) + C_3\, Il_1(R/L_2) = C_4 K_1(R/L_2)$$

$$\frac{\frac{L_2 C_1 \sigma_1 B}{\sigma_2} J_1(BR) + C_3\, Il_1(R/L_2)}{C_4 K_1(R/L_2)} = 1$$

implicit cancelation of C_1

$$\boxed{\frac{\frac{L_2 \sigma_1}{\sigma_2} B J_1(BR) + \dfrac{J_0(BR)\, Il_1(R/L_2)}{\left[Il_0(R/L_2) - \frac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} K_0(R/L_2)\right]}}{\left[\frac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} J_0(BR) \Big/ \left(\frac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} K_0(R/L_2) - Il_0(R/L_2)\right)\right]} = 1}$$

CC

$\phi_1 = C_1 J_0(Br)$ $\qquad \phi_2 = C_3\, Il_0(r/L_2) + C_4 K_0(r/L_2)$

$$C_3 = C_1 J_0(BR) \Big/ \left[Il_0(R/L_2) - \frac{Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)} K_0(R/L_2) \right]$$

$$C_4 = -\frac{C_3\, Il_0(\bar{r}/L_2)}{K_0(\bar{r}/L_2)}$$

$\frac{11}{0}$

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b) $B_g^2 = \left(\frac{2.405}{R}\right)^2 = \frac{\frac{1}{k}\nu\Sigma_f - \Sigma_{a_1}}{D_1} = B^2$ solve for k

$\Rightarrow D_1\left(\frac{2.405}{R}\right)^2 + \Sigma_{a_1} = \frac{1}{k}\nu\Sigma_f \Rightarrow k = \frac{\nu\Sigma_f}{D_1\left(\frac{2.405}{R}\right)^2 + \Sigma_{a_1}}$

plug in $\Sigma_{a_1} = 0.066$ 1/cm, $D_1 = 2$ cm, $\Sigma_f = 0.02805$ 1/cm, $R = 30$ cm, $\nu = 2.4$

$$\boxed{k = 0.85374}$$

c) Ermmm? Increase ↑ relative to bare

d) Plug in known quantities into the criticality condition & solving for T.

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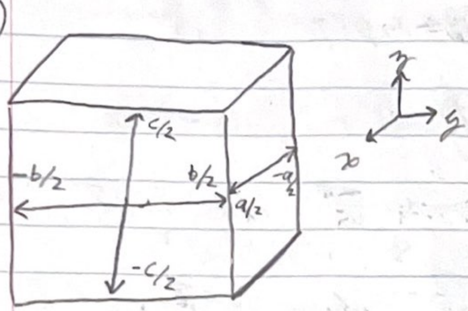
2)



$-\frac{a}{2} < x < \frac{a}{2}$

$-\frac{b}{2} < y < \frac{b}{2}$

$-\frac{c}{2} < z < \frac{c}{2}$

a) $\frac{1}{v}\frac{\partial\phi}{\partial t} - D\nabla^2\phi + \Sigma_a\phi = \nu\Sigma_f\phi \Rightarrow \frac{\partial\phi}{\partial t} - Dv\,\nabla^2\phi + v(\Sigma_a - \nu\Sigma_f)\phi = 0$

w/ $\phi = X(x)Y(y)Z(z)T(t)$, $\frac{\partial\phi}{\partial t} = XYZ\frac{\partial T}{\partial t}$, $\frac{\partial\phi}{\partial x} = YZT\frac{\partial X}{\partial x}$, $\frac{\partial\phi}{\partial y} = XZT\frac{\partial Y}{\partial y}$, $\frac{\partial\phi}{\partial z} = XYT\frac{\partial Z}{\partial z}$

$\frac{1}{\phi}$ $\Rightarrow \underbrace{\frac{1}{T}\frac{\partial T}{\partial t}}_{-\lambda} \underbrace{- Dv\left[\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right] + v(\Sigma_a - \nu\Sigma_f)}_{\lambda} = 0$

T: $\frac{1}{T}\frac{\partial T}{\partial t} = -\lambda \Rightarrow \frac{dT}{dt} = -\lambda T$ ~~say~~ $\underline{T = T_0\exp(-\lambda t)}$

λ: $-Dv\left[\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right] - v(\nu\Sigma_f - \Sigma_a) = \lambda$

$+ \quad - \quad +$

$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{(\nu\Sigma_f - \Sigma_a + \lambda/v)}{D} = 0$, $B_n^2 = \frac{\nu\Sigma_f - \Sigma_a + \lambda/v}{D}$

$\Rightarrow \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\mu^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{-\gamma^2} + B_n^2 = 0$

X: $\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \Rightarrow \frac{d^2X}{dx^2} + \alpha^2 X = 0$, $X = A\cos(\alpha x) + C\sin(\alpha x)$

Y: $\frac{1}{Y}\frac{d^2Y}{dy^2} = -\mu^2 \Rightarrow \frac{d^2Y}{dy^2} + \mu^2 Y = 0$, $Y = E\cos(\mu y) + F\sin(\mu y)$

Z: $\frac{1}{Z}\frac{d^2Z}{dz^2} = -\gamma^2 \Rightarrow \frac{d^2Z}{dz^2} + \gamma^2 Z = 0$, $Z = G\cos(\gamma z) + H\sin(\gamma z)$

know from symmetry BC:

$J_x(0,y,z) = 0 \Rightarrow -D\frac{dX}{dx}(0) = 0 \Rightarrow X = A\cos(\alpha x)$

$J_y(x,0,z) = 0 \Rightarrow -D\frac{dY}{dy}(0) = 0 \Rightarrow Y = E\cos(\mu y)$

$J_z(x,y,0) = 0 \Rightarrow -D\frac{dZ}{dz}(0) = 0 \Rightarrow Z = G\cos(\gamma z)$

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Region is symmetric, so work w/ $0 < x, y, z < \frac{\tilde{a}}{2}, \frac{\tilde{b}}{2}, \frac{\tilde{c}}{2}$

Also know from extrapolation length BC,

$$
\begin{cases}
X\left(\frac{\tilde{a}}{2}\right) = A\cos\left(\tilde{a}\,\alpha\right) = 0 \quad \therefore \tilde{a}\alpha = \frac{n\pi}{2} \Rightarrow \alpha_n = \frac{n\pi}{2\tilde{a}} \quad \text{where } n \in \mathbb{Z},\ \frac{n}{2} \notin \mathbb{Z} \\
Y\left(\frac{\tilde{b}}{2}\right) = E\cos\left(\tilde{b}\,\mu\right) = 0 \quad \therefore \tilde{b}\mu = \frac{n\pi}{2} \Rightarrow \mu_n = \frac{n\pi}{2\tilde{b}} \quad \text{where } n \in \mathbb{Z},\ \frac{n}{2} \notin \mathbb{Z} \\
Z\left(\frac{\tilde{c}}{2}\right) = G\cos\left(\tilde{c}\,\gamma\right) = 0 \quad \therefore \tilde{c}\gamma = \frac{n\pi}{2} \Rightarrow \gamma_n = \frac{n\pi}{2\tilde{c}} \quad \text{where } n \in \mathbb{Z},\ \frac{n}{2} \notin \mathbb{Z}
\end{cases}
$$

$$\therefore \phi = XYZT = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} C_n \cos(\alpha x)\cos(\mu y)\cos(\gamma z)\exp(-\lambda_n t)$$

Know index n is the same for all X, Y, Z, T as the criticality condition imposes $\alpha_n^2 + \mu_n^2 + \gamma_n^2 = \lambda_n$ & this will not be generally true unless $n_\alpha = n_\mu = n_\gamma = n_\lambda$

$$\boxed{\phi(x,y,z,t) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} C_n \cos\left(\frac{n\pi x}{2\tilde{a}}\right)\cos\left(\frac{n\pi y}{2\tilde{b}}\right)\cos\left(\frac{n\pi z}{2\tilde{c}}\right)\exp(-\lambda_n t)}$$

b) Know $B^2 = \frac{\nu\Sigma_f - \Sigma_a + \lambda_1/v}{D} \Rightarrow v(B^2 D + \Sigma_a - \nu\Sigma_f) = \lambda_1$

$\Rightarrow v\Sigma_a(B^2L^2 + 1 - k_\infty) = \lambda_1 \Rightarrow \Sigma_a v(B^2L^2+1)\left(1 - \frac{k_\infty}{1+L^2B^2}\right) = \lambda_1$

$\Rightarrow \Sigma_a v(B^2L^2+1)(1 - k) = \lambda_1 \quad \& \quad \ell = \frac{1}{\Sigma_a v(B^2L^2+1)}$

$$\boxed{\lambda_1 = \frac{1-k}{\ell}} \quad ✓$$

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c) fundamental mode when $n=1$

$$\phi(x,y,z,t) = C_1 \cos\left(\frac{\pi x}{2\tilde{a}}\right)\cos\left(\frac{\pi y}{2\tilde{b}}\right)\cos\left(\frac{\pi z}{2\tilde{c}}\right)\exp\left[\overbrace{\left(\frac{B-1}{l}\right)}^{\lambda_1} t\right]$$

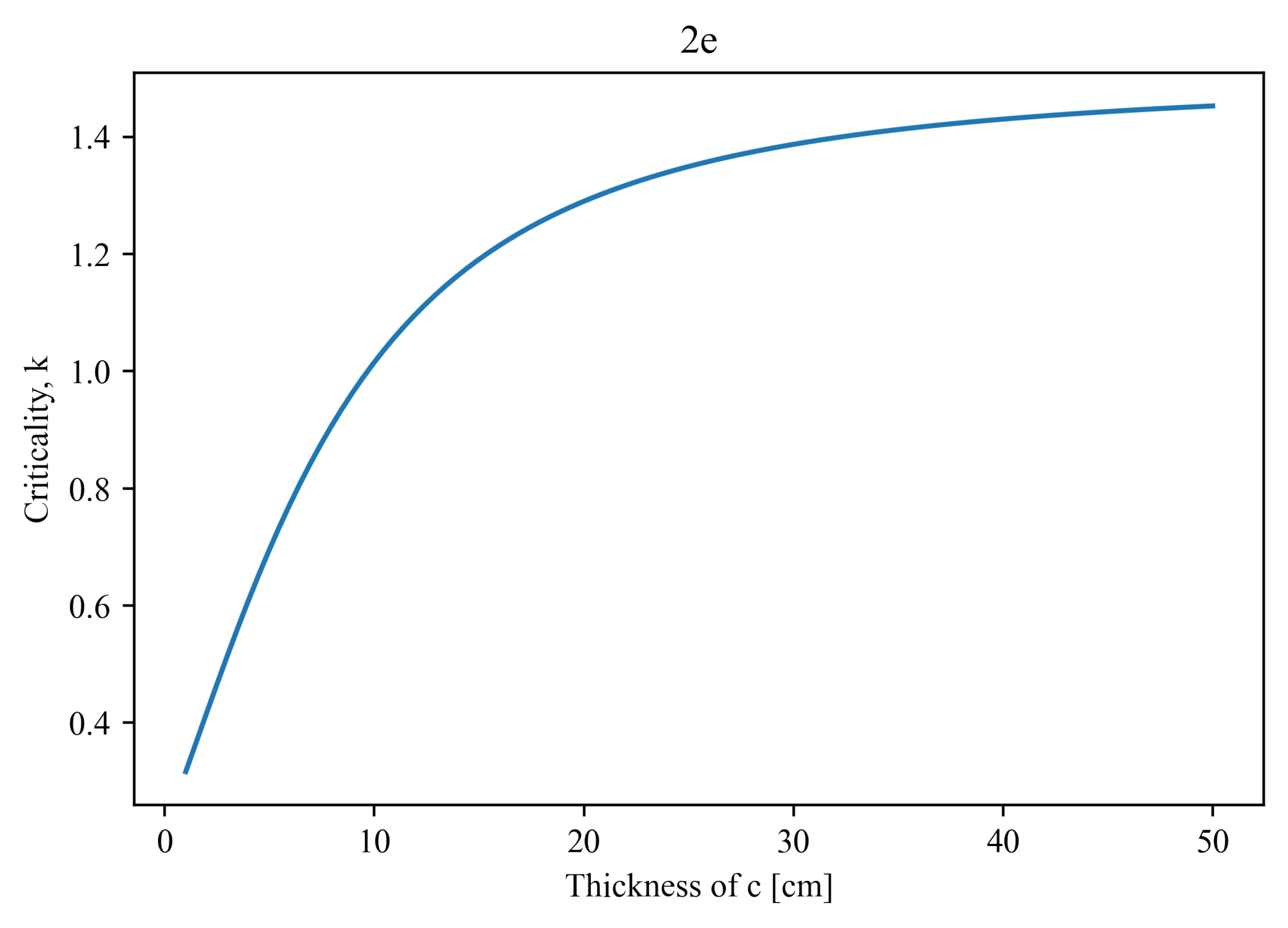
d) $B_n^2 = \alpha^2 + \mu^2 + \alpha^2 \Rightarrow$

$$\boxed{\frac{\nu\Sigma_f - \Sigma_a + \lambda_n/v}{D} = \left(\frac{n\pi}{2\tilde{a}}\right)^2 + \left(\frac{n\pi}{2\tilde{b}}\right)^2 + \left(\frac{n\pi}{2\tilde{c}}\right)^2 \quad n = \text{odd } \mathbb{Z}}$$

e) say $\lambda=0,\ n=1$

$$\frac{\frac{1}{k}\nu\Sigma_f - \Sigma_a}{D} = \left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2$$

plotted in python


2e
Criticality, k
1.4
1.2
1.0
0.8
0.6
0.4
0
10
20
30
40
50
Thickness of c [cm]

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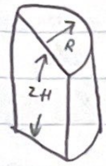
3)  $0<\phi<\pi$, radius R, heigh $2H$ $(-H<z<H)$

a) $-D\nabla^2\phi + \Sigma_a\phi = \frac{1}{k}\nu\Sigma_f\phi \Rightarrow \nabla^2\phi + \overbrace{\frac{\frac{1}{k}\nu\Sigma_f - \Sigma_a}{D}}^{B^2}\phi = 0 \Rightarrow \nabla^2\phi + B^2 = 0$

say $\phi = R(r)\Theta(\phi)Z(z)$ & divide by ϕ

$\Rightarrow \underbrace{\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}\underbrace{\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\phi^2}}_{-\mu^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{-\gamma^2} + B^2 = 0$

Z: $\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\gamma^2 \Rightarrow \frac{\partial^2 Z}{\partial z^2} + \gamma^2 Z = 0 \Rightarrow Z = A\sin(\gamma z) + C\cos(\gamma z)$

β^2 Θ: $\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\phi^2} = -\mu^2 \Rightarrow \frac{\partial^2\Theta}{\partial\phi^2} + \mu^2\Theta = 0 \Rightarrow \Theta = D\sin(\mu\Theta) + E\cos(\mu\theta)$

$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\mu^2}{r^2} = -\beta^2 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\underset{\lambda^2}{\beta^2} - \underset{\frac{n^2}{r^2}}{\frac{\mu^2}{r^2}}\right)\underset{u}{R} = 0$

$R = FJ_\mu(\beta r) + \cancel{G\,Y_\mu(\beta r)}$

BC

i) $\Theta(0)=0 \Rightarrow 0 = D\sin(0) + E\cos(0) \Rightarrow E=0$

ii) $\Theta(\pi)=0 \Rightarrow 0 = D\sin(\pi\mu) \Rightarrow \pi\mu = \pi n \Rightarrow \mu = n$, odd, but 1 for fundamental
$\mu = 1$

iii) $-J_z = 0 \Rightarrow -D\frac{\partial Z}{\partial z} = 0 \Rightarrow 0 = A\gamma\cos(0) + C\gamma\sin(0) \Rightarrow C=0$

iv) $Z(H)=0 \Rightarrow 0 = C\cos(\gamma H) \Rightarrow \gamma H = \frac{n\pi}{2} \Rightarrow \gamma = \frac{n\pi}{2H}$, @ $n=1$ $\gamma = \frac{\pi}{2H}$

v) $R(0)\neq\infty \Rightarrow \infty \neq FJ_1(0) + G\,Y_1(0) \Rightarrow G=0$

vi) $R(\tilde{R}) = 0 \Rightarrow 0 = FJ_1(\beta R)$ $\quad \beta\tilde{R} = 3.8317 \Rightarrow \beta = \frac{3.8317}{\tilde{R}}$

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$\phi = R\Theta Z = \boxed{C_1 J_1\left(\frac{3.8317}{\tilde{R}} r\right) \sin(\theta) \cos\left(\frac{z\pi}{2H}\right) = \phi(r,\phi,z)}$

$$B^2 = \beta^2 + \mu^2 = \left(\frac{3.8317}{\tilde{R}}\right)^2 + \left(\frac{\pi}{2H}\right)^2$$

$$\Rightarrow \boxed{\frac{\frac{1}{k}\nu\Sigma_f - \Sigma_a}{D} = \left(\frac{3.8317}{\tilde{R}}\right)^2 + \left(\frac{\pi}{2H}\right)^2}$$

b) $D\left[\left(\frac{3.8317}{\tilde{R}}\right)^2 + \left(\frac{\pi}{2H}\right)^2\right] + \Sigma_a = \frac{1}{k}\nu\Sigma_f$

$$\Rightarrow k = \frac{\nu\Sigma_f}{D\left[\left(\frac{3.8317}{\tilde{R}}\right)^2 + \left(\frac{\pi}{2H}\right)^2\right] + \Sigma_a} = \boxed{1.06235 = k}$$

$D = 1\text{ cm}$, $\nu = 2.4\ n^0/\text{fission}$, $\Sigma_f = 0.026\text{ cm}^{-1}$, $\Sigma_a = 0.035\text{ cm}^{-1}$
$R = 25\text{ cm}$ $H = 100\text{ cm}$

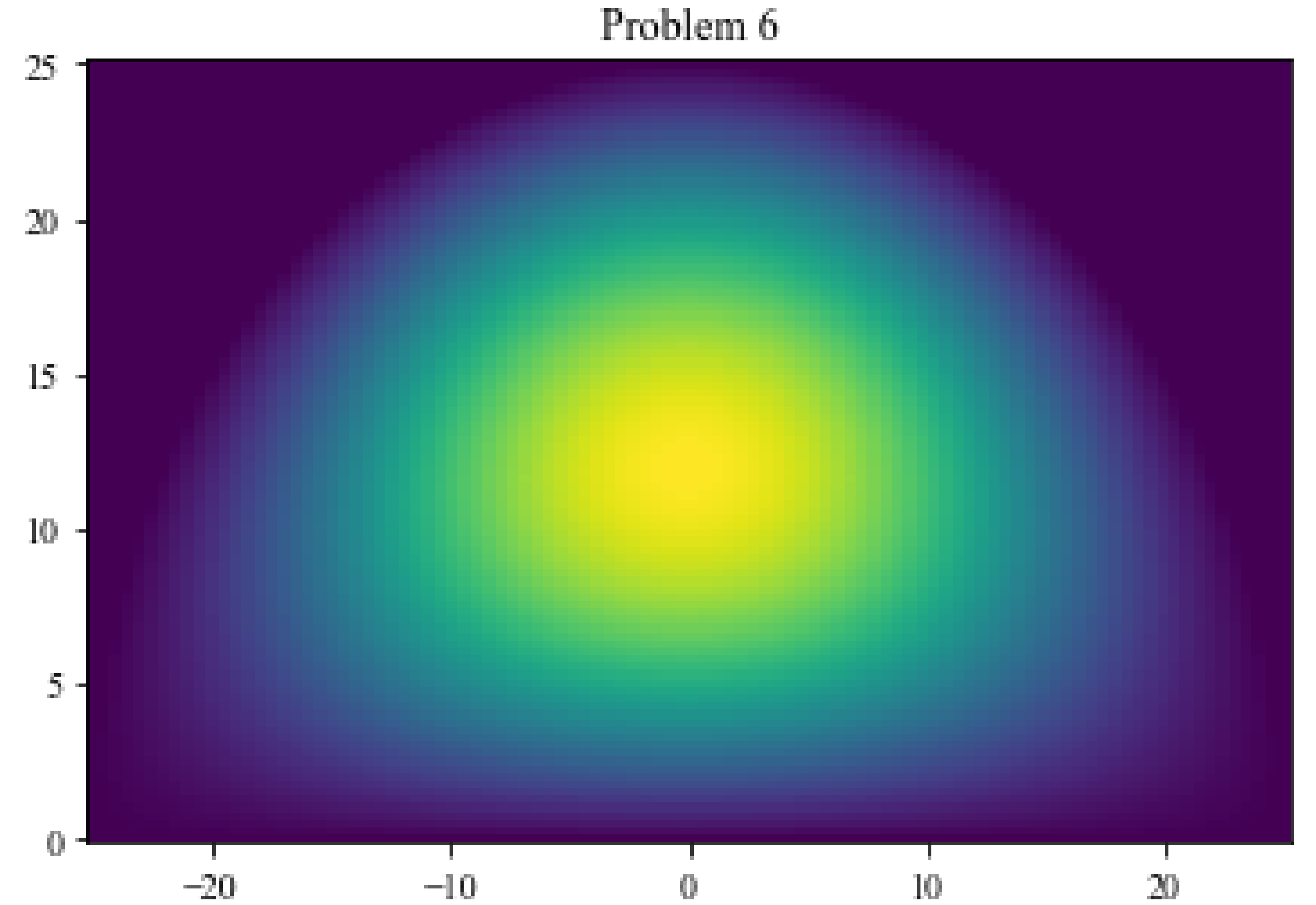
$\boxed{\text{k of half cylinder is smaller than k for } k \text{ of a whole cylinder}}$

c) plot next

Part a) addendum

a) For this distribution, we obtain incongruent plots, so in code (c), substitute $\sin(\theta)$ for $\cos(\theta)$, which is the same as φ ranging from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ & not $[0, \pi]$

thus $\phi = C_1 J_1\left(\frac{3.8317}{\tilde{R}} r\right) \cos(\theta) \cos\left(\frac{z\pi}{2H}\right) = \phi(r,\phi,z)$


Problem 6
25
20
15
10
5
0
−20
−10
0
10
20

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4a) True, the flux is not geometrically attenuated, so the flux in vacuum retains intensity incident of slab seperation distance

c) False, the neutron flux of each shere scales proportionaly to r^2, which is not linear.

d) False, k will stay the same as k is not a function of external sources.

e) False, k will stay the same as k is not a function of density as $k = \frac{\nu \Sigma_f}{\Sigma_a} P_{NL} = \frac{\nu \sigma_f}{\sigma_a} \frac{N}{N} \cdot P_{NL} = \frac{\nu \sigma_f}{\sigma_a} P_{NL}$

& ν, σ_f, σ_a are material properties & $P_{NL} = 1$ for an infinite medium.