

Phys 427 - PS1

- 1) Bernoulli, $p \neq 1/2$. No molecules in V_0 . N molecules in sub-volume V . Equally distributed in space. Prob molecule in V is V/V_0 .

a) $\langle N \rangle = N_0 \frac{V}{V_0} = N_0 p$

b) $\frac{\sigma_N^2}{\langle N \rangle^2} = \frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle^2} = \frac{\langle N^2 \rangle - 2\langle N \rangle^2 + \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle N^2 \rangle}{\langle N \rangle^2} - 1$

know $\langle N \rangle = N_0 \frac{V}{V_0}$, need $\langle N^2 \rangle$

start w/ log derivative: $P \frac{\partial}{\partial P} (P^N) = P(N P^{N-1}) = NP$

\Rightarrow know $(P+2)^{N_0} = \sum_{N=0}^{N_0} \binom{N_0}{N} P^N 2^{N_0-N} \Rightarrow P \frac{\partial}{\partial P} (\text{LHS}) = P \frac{\partial}{\partial P} (\text{RHS})$

\Rightarrow apply log derivative: $P N_0 (P+2)^{N_0-1} = \sum_{N=0}^{N_0} \binom{N_0}{N} N P^N 2^{N_0-N} = \sum_{N=0}^{N_0} \binom{N_0}{N} N P^N 2^{N_0-N}$

\Rightarrow again: $P \frac{\partial}{\partial P} (P N_0 (P+2)^{N_0-1}) = P N_0 \left(\frac{\partial}{\partial P} (P(P+2)^{N_0-1}) \right)$

$= P N_0 ((P+2)^{N_0-1} + P(N_0-1)(P+2)^{N_0-2}) = \sum_{N=0}^{N_0} \binom{N_0}{N} N^2 P^N 2^{N_0-N}$

$\Rightarrow \text{RHS} = \langle N^2 \rangle$ & w/ $P+2=1: \langle N^2 \rangle = P N_0 (1 + P(N_0-1)) = P N_0 (1 + P N_0 - P) = 1 + P N_0^2 - P$

$= P N_0 + P^2 N_0^2 - P^2 N_0 = \langle N^2 \rangle$

\Rightarrow as $\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2: \sigma_N^2 = P N_0 + P^2 N_0^2 - P^2 N_0 - N_0^2 P^2 = P N_0 - P^2 N_0$

$\therefore \frac{\sigma_N^2}{\langle N \rangle^2} = \frac{P N_0 - P^2 N_0}{N_0^2 P^2} = \frac{1-P}{N_0 P} = \frac{1}{N_0} \left(\frac{1-P}{P} \right) = \frac{1}{N_0} \left(\frac{1-1}{1} \right)$ w/ $P = \frac{V}{V_0}$

$$\boxed{\frac{\sigma_N^2}{\langle N \rangle^2} = \frac{1}{N_0} \left(\frac{V_0}{V} - 1 \right)}$$

c) if $V \ll V_0$, $\left(\frac{V_0}{V} - 1 \right) = \left(\frac{V_0}{V} - \frac{V}{V} \right) = \left(\frac{V_0 - V}{V} \right) \approx \frac{V_0}{V} \Rightarrow \frac{\sigma_N^2}{\langle N \rangle^2} = \frac{V_0}{N_0 V}$

d) variance should be 0 if $V \rightarrow V_0$

$\Rightarrow \frac{1}{N_0} \left(\frac{V_0}{V_0} - 1 \right) = \frac{1}{N_0} (1-1) = 0 \checkmark$ guess agrees w/ b)

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2) Poisson is defined w/ $1 \ll N$ & $p \ll 1$ w/ $Np = \epsilon$, $n \ll N$

show $P_p(n, N) \approx \frac{a^n}{n!} \exp(-a)$ w/ $a = Np = \langle n \rangle$

start w/ $P_p(n, N) = \binom{N}{n} p^n 2^{N-n} = \frac{N!}{(N-n)! n!} p^n 2^{N-n}$

\Rightarrow w/ Sterling approx: $\ln \left(\frac{N!}{(N-n)!} \right) = \ln(N!) - \ln((N-n)!)$

$= N \ln(N) - N - (N-n) \ln(N-n) + (N-n) = N \ln(N) - (N-n) \ln(N-n) - n$

$=$ w/ $\ln(N-n) = \ln \left(N \left(1 - \frac{n}{N} \right) \right) = \ln(N) + \ln \left(1 - \frac{n}{N} \right) \approx \ln(N) - \frac{n}{N} \approx \ln(N)$
 \nearrow as $n \ll N$

$\Rightarrow N \ln(N) - (N-n) \ln(N) - n = N \ln(N) - N \ln(N) + n \ln(N) - n = n \ln(N) - n = \ln(N^n) - n$

\Rightarrow exp both sides: $\frac{N!}{(N-n)!} = \exp(\ln(N^n) - n) = N^n \exp(-n)$

as $N \rightarrow \infty$, $Np = \langle n \rangle \rightarrow \epsilon$ w/ $1 \ll n \ll N$ & $\exp(-n) \approx 1$ $\therefore \frac{N!}{(N-n)!} = N^n$

$\Rightarrow P_p \approx \frac{N^n p^n}{n!} 2^{N-n} = \frac{(Np)^n}{n!} 2^{N-n} = \frac{a^n}{n!} \left(1 - \frac{a}{N} \right)^{N-n}$

know: $\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n} \right)^{bn} = \exp(ab)$ & $\left(1 - \frac{a}{N} \right)^{N-n} \approx \left(1 - \frac{a}{N} \right)^N \approx \exp(-a)$

\therefore as $N \rightarrow \infty$ $P(n, N) \approx \frac{a^n}{n!} \exp(-a)$

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3) λ long, so rare event. $5 \text{ rpm} = \langle n \rangle = \lambda$. Use Poisson distribution

a) $P_p(0, N) = \frac{(5)^0 \exp(-5)}{0!} = \boxed{0.67379\%}$

b) $P_p(10, N) = \frac{(5)^{10} \exp(-5)}{10!} = \boxed{1.81328\%}$

c) See attached

4) cannot assume $n \gg N$, $n \ll N$. only $1 \ll n \ll N$. ignore zero-point energy

$$a) \Omega = \frac{(N+n-1)!}{n!(N-1)!} \Rightarrow \ln \Omega = \ln((N+n-1)!) - \ln(n!) - \ln((N-1)!)$$

starting approx: $\ln \Omega \approx (N+n-1) \ln(N+n-1) - (N+n-1) - n \ln(n) + n - (N-1) \ln(N-1) + (N-1)$

$$\Rightarrow \ln \Omega \approx (N+n-1) \ln(N+n-1) - n \ln(n) - (N-1) \ln(N-1)$$

$(x-a) \approx x$ if $x \gg a$

\Rightarrow apply same reasoning as 22) $\therefore \ln \Omega \approx (N+n) \ln(N+n) - n \ln(n) - N \ln(N)$

$$\Rightarrow \ln \Omega = N \ln(N+n) + n \ln(N+n) - n \ln(n) - N \ln(N)$$

$$= N \ln\left(\frac{N+n}{N}\right) + n \ln\left(\frac{N+n}{n}\right) = N \ln\left(1 + \frac{n}{N}\right) + n \ln\left(1 + \frac{N}{n}\right)$$

$$w/n = \frac{u}{kT} \Rightarrow \boxed{\ln \Omega = 0 = N \ln\left(1 + \frac{u}{kT N}\right) + \frac{u}{kT} \ln\left(1 + \frac{N kT}{u}\right)}$$

b) $\frac{1}{T} = k \frac{\partial \sigma}{\partial u} \Rightarrow \frac{\partial \sigma}{\partial u} = \underbrace{1 \cdot \frac{N^2 kT}{1 + u/kT N}}_{\textcircled{1} = \text{first term}} + \frac{1}{kT} \ln\left(1 + \frac{N kT}{u}\right) \dots$

$\dots + \frac{u}{kT} \left[\frac{1}{1 + \frac{N kT}{u}} \cdot \left(-\frac{N kT}{u^2} \right) \right] \} \textcircled{3} = \text{third term}$

$u/\textcircled{3} = -\frac{N}{u} \left(\frac{1}{1 + \frac{N kT}{u}} \right) = -\frac{N}{u + N kT} \quad \& \quad \textcircled{1} = \frac{N}{N kT + u} \quad \therefore \textcircled{1} + \textcircled{3} = 0$

$$\Rightarrow \frac{\partial \sigma}{\partial u} = \frac{1}{kT} \ln\left(1 + \frac{N kT}{u}\right) \Rightarrow \frac{kT}{u} = \ln\left(1 + \frac{N kT}{u}\right)$$

\Rightarrow exp both sides: $\exp\left(\frac{kT}{u}\right) = 1 + \frac{N kT}{u}$

$$\Rightarrow \frac{N kT}{u} = \exp\left(\frac{kT}{u}\right) - 1 \Rightarrow \boxed{u = \frac{N kT}{\exp\left(\frac{kT}{u}\right) - 1}}$$

4c) say $\hbar\omega/kT = x$ $\hbar\omega = kTx$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{N kTx}{\exp(x) - 1} \Rightarrow \text{L'Hospital} \Rightarrow \lim_{x \rightarrow 0} \frac{NkT}{\exp(x)} = NkT$$

u / N as # of oscillators, u/N = energy per oscillator

$$\Rightarrow u/N = \frac{NkT}{N} = \boxed{kT = u/N} \quad \text{for the high temp limit}$$

d) u / $\hbar\omega/kT = x$ $\hbar\omega = kTx$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{N kTx}{\exp(x) - 1} \Rightarrow \text{L'Hospital} \Rightarrow \lim_{x \rightarrow \infty} \frac{NkT}{\exp(x)} = 0$$

$$\Rightarrow \boxed{\frac{u}{N} = 0} \quad \text{for the low temp limit, no oscillator has energy}$$