HW#6 - Joseph Spacito

1)

$U(\theta=0)=0$, Energy E

a) E_{max} when 0 potential energy & all kinetic energy.
We know

$$T = \frac{1}{2}m\vec{v}^2 \quad \& \text{ in polar } T = \frac{1}{2}m(\dot{r}+r\dot{\theta})^2$$

but $\dot{r}=0$ c/c fixed pendulum, so

$$T = \frac{1}{2}mr^2\dot{\theta}^2 = E, \text{ plugging in to this problem}$$

$$E = \frac{1}{2}m\mathcal{L}^2\dot{\theta}^2 \Rightarrow \dot{\theta}^2 = \frac{2E}{\mathcal{L}^2 m} \Rightarrow \boxed{\dot{\theta} = \sqrt{\frac{2E}{\mathcal{L}^2 m}}}$$

b) θ_{max} is when $T=0$ or all energy is potential, so

$$E = mgh(1-\cos\theta), \text{ but for this problem}$$

$$E = mg\mathcal{L}(1-\cos\theta) \Rightarrow \frac{E-mg\mathcal{L}}{mg\mathcal{L}} = -\cos\theta \Rightarrow \frac{E}{mg\mathcal{L}} - 1 = -\cos\theta$$

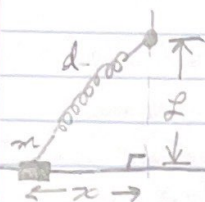$$\Rightarrow \cos\theta = 1 - E/mg\mathcal{L} \Rightarrow \boxed{\theta_{max} = a\cos\left(1 - \frac{E}{mg\mathcal{L}}\right)}$$

c) To swing in a circle, $\theta_{max} > \pi$, so

$$a\cos\left(1 - \frac{E}{mg\mathcal{L}}\right) > \pi \Rightarrow 1 - \frac{E}{mg\mathcal{L}} > -1 \Rightarrow \frac{E}{mg\mathcal{L}} > 2$$

$$\boxed{E > 2mg\mathcal{L}}$$

2)

$$L_0 = L/2$$

a) Force of spring $= -kx$ & eq length is $L_0 = L/2$, so this is the stretched length

$d = \sqrt{x^2 + L^2}$
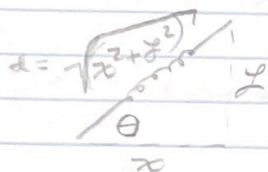
$$U = \frac{1}{2} k \left(d - \frac{L}{2} \right)^2$$

$$U = \frac{1}{2} k \left(\sqrt{x^2 + L^2} - \frac{L}{2} \right)^2$$

$$\boxed{U = \frac{1}{2} k \left(\sqrt{x^2 + L^2} - \frac{L}{2} \right)^2}$$

b) Using a derivative calculator, I found U' to be

$$U' = kx \frac{\left(\sqrt{\sqrt{x^2 + L^2}} - \frac{L}{2} \right)}{\sqrt{x^2 + L^2}} = kx \left(\frac{\sqrt{x^2 + L^2}}{\sqrt{x^2 + L^2}} - \frac{L}{2\sqrt{x^2 + L^2}} \right)$$

$$= kx \left(1 - \frac{L}{2\sqrt{x^2 + L^2}} \right) = kx - \frac{Lkx}{2\sqrt{x^2 + L^2}}, \text{ this is only } 0$$
when $x = 0$

$$\boxed{\therefore x_{min} = 0}$$

HW#6 - Joseph Szold

1) using an online calculator we get

$$u''(x) = k \frac{\left(2(x^2 + \ell^2)^{3/2} - \ell^3\right)}{2(x^2 + \ell^2)^{3/2}}$$

, but since we are expanding around $x_{min} = 0$, we plug in 0 for $u''(x)$

$$u''(x) = k \frac{\left(2(\ell^2)^{3/2} - \ell^3\right)}{2(\ell^2)^{3/2}}$$ & thus simplified gives

$$u''(0) = k/2 \quad , \quad \boxed{\therefore k_{eff} = k/2}$$

2) $$f = \frac{1}{2\pi} \omega = \frac{1}{2\pi} \sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi} \sqrt{\frac{k}{2m}}$$

$$\boxed{f = \frac{1}{2\pi} \sqrt{\frac{k}{2m}} = \sqrt{\frac{k}{8\pi^2 m}}}$$

3)

$$\tau = I\alpha$$

$$I = I_1 + I_2 = 2m(a)^2 + m(2a)^2 = 2ma^2 + 4ma^2$$
$$= 6ma^2$$

$$\tau = -g\sin\theta(m_1 r_1 + m_2 r_2)$$

$$= -g\sin\theta(2ma + m2a) = -g\sin\theta(4ma)$$

$$\tau = -4mag\sin\theta = I\alpha = 6ma^2\ddot{\theta}$$

$$-4mag\sin\theta = 6ma^2\ddot{\theta} \Rightarrow -4g\sin\theta = 6a\ddot{\theta}$$

Since θ is really small, $\sin\theta \approx \theta$, so

$$-4g\theta = 6a\ddot{\theta} \Rightarrow 6a\ddot{\theta} + 4g\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{4}{6a}g\theta = 0 \Rightarrow \ddot{\theta} + \frac{2g}{3a}\theta = 0 \qquad \text{guess } \theta = \exp(\lambda t)$$

so, $\ddot{\theta} = \lambda^2 \exp(\lambda t)$ & plugging them in gives

$$\lambda^2\exp(\lambda t) + \frac{2g}{3a}\exp(\lambda t) = 0, \text{ so } \lambda^2 + \frac{2g}{3a} = 0$$

$$\therefore \lambda = \pm i\sqrt{\frac{2g}{3a}} \quad \& \quad \omega = \sqrt{\frac{2g}{3a}}, \text{ but } T = \frac{2\pi}{\omega}, \text{ so}$$

$$\boxed{T = 2\pi\sqrt{\frac{3a}{2g}} = \sqrt{\frac{6\pi^2 a}{g}}}$$

HW#6 - Joseph Specht

4a) $x(t) = c_1 \cos(wt - \varphi_1)$

$c_1 = 5$, amplitude is $= 5$ & this is c_1

$\varphi_1 = \pi/3$, @ $x=0$, $c_1 \cos(-\varphi_1) = \frac{1}{2} c_1$ ∴ $\varphi_1 = -a\cos(1/2) = \pm \pi/3$, but we know the graph is almost @ a max, so it has to be positive (1st quadrant)

$w = 2$, @ $x = \pi/6$, we have a max, so $c_1 \cos(wt - \pi/3) = c_1$
∴ $\cos(wt = \pi/3) = 1$, so $wt - \pi/3 = 0$, so $wt = \pi/3$
∴ $w = \pi/3t = \pi/3 \cdot \frac{6}{\pi} = 2$

$$x(t) = 5\cos(2t - \pi/3)$$

b) sin lags $-\pi/2$ behind cos, so $\varphi_2 = \varphi_1 - \frac{\pi}{2} = \frac{\pi}{3} - \frac{\pi}{2} = \frac{-\pi}{6}$

$$\therefore x(t) = 5\sin(2t + \pi/6)$$

c) we know $x(t) = A\cos(2t) + B\sin(wt)$ & $c_1 = \sqrt{A^2 + B^2}$

also $\tan(\varphi) = \frac{B}{A} = \tan(\frac{\pi}{3}) = \sqrt{3}$, so $B = \sqrt{3} A$

Plugging this in we get $c_1 = \sqrt{A^2 + 3A^2} = \sqrt{4A^2} = 2A$, so $A = \frac{c_1}{2}$

& we know $B = \sqrt{3} \frac{c}{2}$, so $A = \frac{5}{2}$ & $B = \frac{5}{2}\sqrt{3}$

$$\therefore x(t) = \frac{5}{2}\cos(2t) + \frac{5}{2}\sqrt{3}\sin(2t)$$

d) If x takes the form $O_1 \exp(iwt) + O_2 \exp(-iwt)$, then using euler's identity, we get

$$x = O_1 \cos(wt) + O_1 i \sin(wt) + O_2 \cos(wt) - O_2 i \cos(wt)$$

& grouping these terms gives,

$$x = (O_1 + O_2)\cos(wt) + (O_1 i - O_2 i)\sin(wt)$$

but we know from part (the constants in front of each, so we know,

$$\frac{5}{2} = O_1 + O_2 \qquad \& \qquad \frac{5}{2}\sqrt{3} = O_1 i - O_2 i$$

$$O_1 = \frac{5}{2} - O_2 \Rightarrow \quad \frac{5}{2}\sqrt{3} = \frac{5}{2}i - O_2 i - O_2 i \Rightarrow \frac{5}{2}\sqrt{3} - \frac{5}{2}i = -2O_2 i$$

$$\Rightarrow O_2 = \frac{5}{4} + \frac{5}{4}\sqrt{3}i \quad \text{since} \quad i = \frac{1}{i} = \sqrt{-1} = \frac{1}{\sqrt{-1}}, \text{ so } O_1 \text{ is}$$

$$O_1 = \frac{5}{2} - \left(\frac{5}{4} + \frac{5}{4}\sqrt{3}i\right) = \frac{5}{4} - \frac{5}{4}\sqrt{3}i$$

$$\boxed{\therefore x(t) = \left(\frac{5}{4} - \frac{5i\sqrt{3}}{4}\right)\exp(2it) + \left(\frac{5}{4} + \frac{5i\sqrt{3}}{4}\right)\exp(-2it)}$$

$$\omega_d = \sqrt{1-\zeta^2}\, \omega_n$$

HW#6 — Joseph Specht

$$\zeta = \frac{c}{2m\omega_n}$$

(5)

$$F_{drag} = -cv$$

a) $T = cl\alpha \qquad cl = L^2 m$

$cl\alpha = T_{drag} + T_g = L^2 m\alpha = L(F_{drag}) + L(mg\sin\theta)$

$L^2 m\alpha = -Lc(\dot{r} + r\dot{\theta}) - L\,mg\sin\theta$, and $\dot{r} = 0$, so

$L^2 m\ddot{\theta} = -L(cL\dot{\theta}) - L\,mg\sin\theta \Rightarrow mL\ddot{\theta} = -cL\dot{\theta} - mg\sin\theta$

$$\boxed{mL\ddot{\theta} + cL\dot{\theta} + mg\sin\theta = 0}$$

b) if $\theta \ll 0$, $\sin\theta \approx \theta$, so $(\sin\theta = \theta + \dots)$

$$mL\ddot{\theta} + cL\dot{\theta} + mg\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\theta = 0$$

guessing $\theta = \exp(\lambda t)$, we get

$\lambda^2 \exp(\lambda t) + \frac{c}{m}\lambda \exp(\lambda t) + \frac{g}{L}\exp(\lambda t) = 0$, so $\lambda^2 + \frac{c}{m}\lambda + \frac{g}{L} = 0$

use quadratic formula to get λ, so

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{g}{L}}}{2} \qquad \& \text{ using } \zeta = \frac{c}{2m\omega_n}, \text{ so}$$

$$\lambda = \frac{2\omega_n\left(-\zeta \pm \sqrt{4\omega_n^2 \zeta^2 - 4\omega_n^2\left(\frac{g}{L}\right)\frac{1}{\omega_n^2}}\right)}{2}$$

This is equal to ... (next page)

HW #6 - Joseph Specht

$$\lambda = \frac{2\,\omega_n \left(-\zeta \pm 2\,\omega_n \sqrt{\zeta^2 - \left(\frac{g}{L}\right)\frac{1}{\omega_n^2}}\right)}{2}$$ however, for pendulum
$$\omega_n = \sqrt{\frac{g}{L}}, \text{ so}$$

$$\frac{g}{L} \cdot \frac{1}{\omega_n^2} = \frac{g}{L} \cdot \frac{L}{g} = 1, \text{ so} \quad \lambda = \omega_n \left(-\zeta \pm \omega_n \sqrt{\zeta^2 - 1}\right)$$

$$\Rightarrow \lambda = -\zeta\,\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\,\omega_n \pm \omega_n\,i\sqrt{1-\zeta^2} = -\zeta\,\omega_n \pm i\,\omega_d, \text{ so}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}, \text{ but} \quad \zeta = \frac{c}{2m\omega_n}, \text{ so}$$

$$\omega_d = \omega_n\sqrt{1 - \frac{c^2}{4m^2\omega_n^2}} = \sqrt{\omega_n^2 - \frac{c^2\omega_n^2}{4m^2\omega_n^2}} = \sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}$$

$$\boxed{\omega_d = \sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}}$$

d) $\theta(t=0) = \theta_0$ & $\dot{\theta}(t=0) = \Omega$ find $\theta(t)$

we know $\theta(t) = \exp(-\zeta\omega_n t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$

but $\theta(0) = A\cos(0) = A = \theta_0$ $\therefore A = \theta_0$

$$\dot{\theta}(t) = \exp(-\zeta\omega_n t)\left[-A\omega_d\sin(\omega_d) + B\omega_d\cos(\omega_d t)\right] - \zeta\omega_n\exp(-\zeta\omega_n t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$$

$$\dot{\theta}(0) = B\omega_d\cos(0) - \zeta A\cos(0) = B\omega_d - \zeta A = \Omega$$

$$\Rightarrow B = \frac{\Omega + \zeta A}{\omega_d} = \frac{\Omega + \zeta\theta_0}{\omega_d}$$

$$\omega_d = \sqrt{\frac{g}{l} - \frac{c^2}{4m^2}} \qquad\qquad \omega_d = \sqrt{1 - \zeta^2}\,\omega_n$$

$$\zeta = \frac{c}{2m\omega_n} \qquad \omega_n = \sqrt{g/l}$$

HW#6 - Joseph Specht

Plugging everything in, we get,

$$\theta(t) = \exp(-\zeta\omega_n t)\left[\theta_0 \cos(\omega_d t) + \left(\frac{\Omega + \zeta\theta_0}{\omega_d}\right)\sin(\omega_d t)\right]$$

now plugging in $\omega_n = \sqrt{g/l}$, $\zeta = \frac{c\sqrt{l}}{2m\sqrt{g}}$, &

$\omega_d = \sqrt{\frac{g}{l} - \frac{c^2}{4m^2}}$, we get

$$\theta(t) = \exp\left(\frac{-c\,t}{2m}\right)\left[\theta_0 \cos\left(\sqrt{\frac{g}{l} - \frac{c^2}{4m^2}}\;t\right) + \right.$$

$$\left(\frac{\Omega + \frac{c\theta_0}{2m}\sqrt{\frac{g}{l}}}{\sqrt{\frac{g}{l} - \frac{c^2}{4m^2}}}\right)\sin\left(t\sqrt{\frac{g}{l} - \frac{c^2}{4m^2}}\right)$$