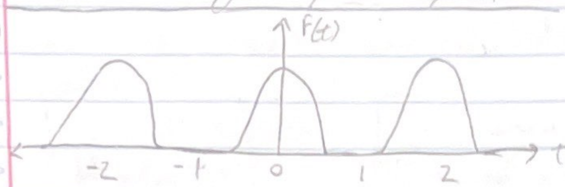
HW#8 - Joseph Specht

1)



$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

a) $T = 2\,s$ $\quad \omega = \frac{2\pi}{T} \quad \therefore \omega = \frac{2\pi}{2} = \pi$

$$\boxed{T = 2\,s \qquad \omega = \pi \ \text{rad/s}}$$

b) a_0 will not vanish b/c the average value ~~of~~ $F(t)$ is not 0

a_n will not be 0 as this is an even function

b_n will be 0 as this is an even function & b_n is the coefficient on the odd periodic basis vectors

c) $m\ddot{x} + c\dot{x} + kx = F(t)$ from L#16 slide 19

$$x_p(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n G(n\omega)\cos(n\omega t - \varphi(n\omega))$$

$$G(n\omega) = \frac{1}{k}\left[\left(1-\left(\frac{n\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{n\omega}{\omega_n}\right)\right)^2\right]^{(-1/2)}$$

$$\varphi(n\omega) = \operatorname{atan}\left(\frac{2\zeta\left(\frac{n\omega}{\omega_n}\right)}{1-\left(\frac{n\omega}{\omega_n}\right)^2}\right)$$

$$a_0 = \frac{2}{T}\int_T F(t)\,dt \qquad a_n = \frac{2}{T}\int_T F(t)\cos(n\omega t)\,dt$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2m\omega_n}$$

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d) We know average position is $\boxed{\bar{x} = \frac{a_0}{2k}}$ as this is the value of x after a long time

2) $F_{rec} = |F_0 \sin(\omega t)|$

a) graphed & $\boxed{T = \frac{\pi}{\omega}}$ w/ $\omega = \sqrt{k/m}$

b) $F_{rec} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$

This function is even, so $b_n = 0$, so

$F_{rec} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$, so need to find a_0 & a_n, so

$$a_0 = \frac{2}{T}\int_0^{\frac{\pi}{\omega}} F(t)\,dt = \frac{4}{T}\int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} F(t)\,dt = \frac{4}{T}\int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} |F_0 \sin(\omega t)|\,dt$$

but $\sin(\omega t) \geq 0$ here so we can remove abs, so

$$a_0 = \frac{4F_0}{T}\int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} \sin(\omega t)\,dt = \frac{4F_0}{T\omega}\left[-\cos(\omega t)\right]_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}}$$

all @ the end

$$= \frac{4F_0}{T\omega}\left(-\cos(\pi) + \cos\left(\frac{\pi}{2}\right)\right) = \frac{4F_0}{T\omega}\left(-(-1) + 0\right) = \frac{4F_0}{T\omega} = \boxed{\frac{4F_0}{\pi} = a_0}$$

now solving for a_n, we know

$$a_n = \frac{2}{T}\int_T F(t)\cos(n\omega_2 t)\,dt$$

$$a_n = \frac{2}{T}\int_0^{\frac{\pi}{\omega}} |F_0 \sin(\omega t)|\cos(n\omega_2 t)\,dt$$

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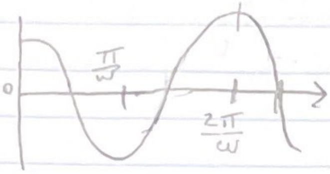
To find ω_2, we graph $\cos(n\omega t)$ $\boxed{\Omega = 2\omega}$



$T_2 = 2T_1$, so to get an entire period in a single period, Ω needs to be 2ω & since we know the period is 2x as long, the frequency is 1/2 as much & this means there are 2 periods of the Force in 1 of the cos, so we need it to go 2x as fast, so

$$a_n = \frac{2}{T}\int_0^{\frac{\pi}{\omega}} |F_0 \sin(\omega t)| \cos(2n\omega t)\, dt$$, but at this interval, $\sin > 0$, so

$$a_n = \frac{2}{T}\int_0^{\frac{\pi}{\omega}} F_0 \sin(\omega t)\cos(2n\omega t)\, dt$$ applying an identity gives

$$a_n = \frac{2F_0}{T}\int_0^{\frac{\pi}{\omega}} \frac{1}{2}\left(\sin[(2n+1)\omega t] - \sin[(2n-1)\omega t]\right) dt$$

$$a_n = \frac{F_0}{T}\left(\int_0^{\frac{\pi}{\omega}} \sin[(2n+1)\omega t]\, dt - \int_0^{\pi/\omega} \sin[(2n-1)\omega t]\, dt\right)$$

$$= \frac{F_0}{T}\left[\frac{-\cos[(2n+1)\omega t]}{(2n+1)\omega}\Bigg|_0^{\frac{\pi}{\omega}} + \frac{\cos[(2n-1)\omega t]}{(2n-1)\omega}\Bigg|_0^{\frac{\pi}{\omega}}\right]$$

$$= \frac{F_0}{T}\left(\frac{-\cos(\text{odd}\,\mathbb{Z}\cdot\pi) + \cos(0)}{(2n+1)\omega} + \frac{\cos(\text{odd}\,\mathbb{Z}\cdot\pi) - \cos(0)}{(2n-1)\omega}\right)$$

$$= \frac{F_0\omega}{\pi\omega}\left(\frac{-(-1)+1}{2n+1} + \frac{(-1)-1}{2n-1}\right) = \frac{F_0}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right)$$ getting common denominators

$$= \frac{2F_0}{\pi}\left(\frac{(2n-1)}{(2n+1)(2n-1)} - \frac{(2n+1)}{(2n-1)(2n+1)}\right) = \frac{2F_0}{\pi}\left(\frac{-2}{4n^2-1}\right) = \frac{-4F_0}{\pi(4n^2-1)} = a_n$$

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Now we know every coefficient, so in

$$F_{rec} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t)$$

$$\boxed{a_0 = \frac{4F_0}{\pi} \qquad \Omega = 2\omega \qquad b_n = 0 \qquad a_n = \frac{-4F_0}{\pi(4n^2-1)}}$$

c) To find the particular solution, use form

$$x_p = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n\, G(2n\omega) \cos(2n\omega t - \phi(2n\omega))$$

now we need to find $G(2n\omega)$ & $\phi(2n\omega)$, so

$$G(2n\omega) = \frac{1}{k}\left[\left(1-\left(\frac{2n\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2n\omega}{\omega_n}\right)\right)^2\right]^{(-1/2)}$$

but we know $\omega = \omega_n$ & $\zeta = 0$ b/c it is undamped, so

$$G(2n\omega) = \frac{1}{k}\left[\left(1-\left(\frac{2n\omega_n}{\omega_n}\right)^2\right)^2 + 0\right]^{(-1/2)} = \frac{1}{k}\left[1-(2n)^2\right]^{(-1/2)}$$

$$G(2n\omega) = \frac{1}{k}\frac{1}{\sqrt{(1-4n^2)^2}} = \boxed{\frac{1}{k}\frac{1}{|1-4n^2|} = G(2n\omega)}$$ ← we'll assemble @ the end

Finding $\phi(2n\omega)$

$$\phi(2n\omega) = \operatorname{atan}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$ but $\zeta = 0$, so $\boxed{\phi(2n\omega) = \operatorname{atan}(0) = 0}$

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now plugging into the expression for $x_p(t)$

$$x_p = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n B(2n\omega)\cos(2n\omega t - \varphi(2n\omega))$$ we get

$$x_p = \frac{2F_0}{k\pi} + \sum_{n=1}^{\infty} \frac{-4F_0}{\pi(4n^2-1)} \frac{1}{k|1-4n^2|} \cos(2n\omega t - 0)$$

$$x_p = \frac{2F_0}{k\pi}\left(1 + \sum_{n=1}^{\infty} \frac{-2\cos(2n\omega t)}{(4n^2-1)|1-4n^2|}\right)$$

but we know $n \geq 1$, so we can rewrite

$|1-4n^2|$ as $|(-1)(4n^2-1)| = (|-1|)(|4n^2-1|) = |4n^2-1|$ but $n\geq 1$, so

$(4n^2-1) > 0$, so we can disregard the absolute value, so in this case

$|1-4n^2| = (4n^2-1)$ which we can combine to

$$\boxed{x_p = \frac{2F_0}{k\pi}\left(1 + \sum_{n=1}^{\infty} \frac{-2\cos(2n\omega t)}{(4n^2-1)^2}\right)}$$

d) now finding F_{rec} these n's we have

$$F_{rec} = \frac{2F_0}{\pi} + \sum_{n=1}^{\infty} \frac{-4F_0}{(4n^2-1)} \cos(2n\omega t) = \frac{2F_0}{\pi}\left(1 + \sum_{n=1}^{\infty} \frac{-2\cos(2n\omega t)}{(4n^2-1)}\right)$$

now we need to plug in the desired n values

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amplitude @ $n=1$ term $= \max\left(\dfrac{-4F_0 \cos(2\omega t)}{\pi(4-1)}\right) = \dfrac{4F_0}{3\pi} = A_1$

amplitude @ $n=2$ term $= \max\left(\dfrac{-4F_0 \cos(4\omega t)}{\pi(16-1)}\right) = \dfrac{4F_0}{15\pi} = A_2$

amplitude @ $n=3$ term $= \max\left(\dfrac{-4F_0 \cos(6\omega t)}{\pi(36-1)}\right) = \dfrac{4F_0}{35\pi} = A_3$

now we can find the ratios $\left|\dfrac{A_2}{A_1}\right|$ & $\left|\dfrac{A_3}{A_1}\right|$

$\left|\dfrac{A_2}{A_1}\right| = \dfrac{4F_0}{15\pi} \cdot \dfrac{3\pi}{4F_0} = \dfrac{1}{5}$ $\qquad$ $\left|\dfrac{A_3}{A_1}\right| = \dfrac{4F_0}{35\pi} \cdot \dfrac{3\pi}{4F_0} = \dfrac{3}{35}$

We see $A_1 > A_2 > A_3$ & this is because $n=1$ is the lowest integer multiple of the resonance frequency available. We also know the largest response happens when $n\omega_n$ is as small as possible, which is what happened here

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3) $T = 1s$ $F(t) = 2F_0 t$ for $-.5s < t < .5s$ & then periodic

a) $\Omega = \frac{2\pi}{T} = 2\pi$, just use the formula relating T & w

$a_0 = 0$, because the average value is 0 of the function & this is centered @ F=0

$a_n = 0$, because this is an odd function, so the even part cancels out

$$b_n = \frac{2}{T}\int_T F(t)\sin(n\omega t)\,dt = \frac{2}{T}\int_T 2F_0 t \sin(n\omega t)$$

$$= \frac{4F_0}{T}\int_{-.5}^{.5} t\sin(n\omega t)\,dt = \frac{4F_0}{1}\left(\frac{\sin(n\omega t) - n\omega t\cos(n\omega t)}{n^2\omega^2}\Bigg|_{-.5}^{.5}\right)$$

but we know $\omega = 2\pi$, so

$$b_n = 4F_0\left(\frac{\sin(2\pi n t) - 2\pi n t\cos(2\pi n t)}{4\pi^2 n^2}\Bigg|_{-.5}^{.5}\right)$$

$$= 4F_0\left(\frac{\sin(n\pi) + \sin(-n\pi) - n\pi\cos(\pi n) - \pi n\overbrace{\cos(-\pi n)}^{=\cos(\pi n)}}{4\pi^2 n^2}\right)$$

but we know $\sin(n\pi) = 0$, so

$$b_n = \frac{F_0}{\pi^2 n^2}\left(0 + 0 - 2n\pi\cos(\pi n)\right) = \frac{-2F_0\overbrace{\cos(\pi n)}^{\text{either } -1 \text{ or } 1}}{n\pi} = \boxed{\frac{-2F_0(-1)^n}{\pi n} = b_n}$$

$$\boxed{\Omega = 2\pi \text{ rad/s} \qquad a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{-2F_0(-1)^n}{\pi n}}$$

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a) find $x_p(t)$. know it follows form below

$$x_p = \sum_{n=1}^{\infty} b_n\, g(n\omega) \sin(n\omega t - \varphi(n\omega)) \quad w/ \quad a_0 = a_n = 0$$

now we need to find g & φ, so we just plug in w/ $\omega = 2\pi$

$$\boxed{\begin{aligned} x_p &= \sum_{n=1}^{\infty} \frac{-2F_0(-1)^n}{\pi n}\, g(2\pi n) \sin(2\pi n t - \varphi(2\pi n)) \\ g(2\pi n) &= \frac{1}{k}\left[\left(1-\left(\frac{2\pi}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2\pi}{\omega_n}\right)\right)^2\right]^{(-1/2)} \\ \varphi(2\pi n) &= \operatorname{atan}\left(\frac{2\zeta\left(\frac{2\pi}{\omega_n}\right)}{1-\left(\frac{2\pi}{\omega_n}\right)^2}\right) \qquad \zeta = \frac{c}{2m\omega_n} \end{aligned}}$$

c) resonance is $\omega_d = \omega_n\sqrt{1-2\zeta^2}$ where ω_n is

$$\omega_n = \sqrt{k/m} = \sqrt{200/.6} = \sqrt{\frac{1000}{3}}$$ & to find ζ we do

$$\zeta = \frac{c}{2m\omega_n} \quad \& \quad 2\zeta^2 = \frac{2c^2}{2m^2}\cdot\frac{3}{1000} = \frac{3c^2}{2{,}000\, m^2}$$ & plugging in values we get

$$2\zeta^2 = \frac{3 * .5^2}{2000 * .6^2} = \frac{1}{960}$$

$$\omega_d = \sqrt{\frac{1000}{3}}\sqrt{1-\frac{1}{960}} = \sqrt{\frac{1000}{3}}\sqrt{\frac{959}{960}} = \sqrt{\frac{23{,}975}{72}}$$

$$\boxed{\omega_d = \sqrt{\frac{23{,}975}{72}}\ \text{rad/s} \approx 18.2479\ \text{rad/s}}$$

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1) I used a computer program to complete these calculations. I will also attach it below.

n	max $\lvert F \rvert$ [N]	Ω in $\sin(\Omega t - \varphi(2\pi n))$ [rad/s]
1	.00361 F_0	6.28
2	.00302 F_0	12.57
3	.01309 F_0	18.85
4	.00089 F_0	25.13
5	.00032 F_0	31.42

The amplitude of $n=3$ was the largest & this makes sense as the frequency of oscillations Ω is closest to the resonance frequency $\omega \approx 18.2479$ rad/s

$$18.85 \text{ rad/s} = \Omega_3 \approx \omega_d = 18.2479 \text{ rad/s}$$

These values are closer for $n=3$ than any other n

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4a) $\int_0^\infty \delta(2t^{1/3} - 54)\exp(2t)\,dt$

let $u = 2t^{1/3} \Rightarrow du = \frac{2}{3}t^{-2/3}\,dt \Rightarrow dt = \frac{3}{2}t^{2/3}\,du$

$\Rightarrow \frac{3}{2}\int_0^\infty \delta(u-54)\exp(2t)\,t^{2/3}\,du$, since $u = 2t^{1/3}$ $t = \left(\frac{u}{2}\right)^3$

$= \frac{3}{2}\int_0^\infty \delta(u-54)\exp\left(2\left(\frac{u}{2}\right)^3\right)\left(\left(\frac{u}{2}\right)^3\right)^{2/3}du = \frac{3}{2}\int_0^\infty \delta(u-54)\exp\left(2\left(\frac{u}{2}\right)^3\right)\left(\frac{u}{2}\right)^2 du$

we know $\int_0^\infty \delta(u-54)\,du = 1$ if $u = 54$ & 0 if else, so we can say $f(x) = \exp\left(2\left(\frac{u}{2}\right)^3\right)\left(\frac{u}{2}\right)^2$, so we know the integral is simply $f(54)$

if $f(x) = \exp\left(2\left(\frac{u}{2}\right)^3\right)\left(\frac{u}{2}\right)^2$, then the integral evaluates to

$$f(54) = \exp\left(2\left(\frac{54}{2}\right)^3\right)\left(\frac{54}{2}\right)^2 = \exp(2(27)^3)(27)^2$$

$$\boxed{\int_0^\infty \delta(2t^{1/3}-54)\exp(2t)\,dt = \exp(2\cdot 27^3)(27)^2 \approx 1.992\,e17{,}099}$$

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e) $\int_{-\infty}^{\infty} \left(\frac{d^2}{dt^2}(\delta(t-4)\right) \sin\left(\frac{t^2}{4}\right) dt$; $\int u\,dv = uv - \int v\,du$

$u = \sin\left(\frac{t^2}{4}\right)$ $\quad v = \frac{d}{dt}\delta(t-4)$

$du = \frac{t}{2}\cos\left(\frac{t^2}{4}\right)$ $\quad dv = \frac{d^2}{dt^2}\delta(t-4)$

$= \underbrace{\frac{d}{dt}(\delta(t-4))\sin\left(\frac{t^2}{4}\right)\Big|_{-\infty}^{\infty}}_{=} - \int_{-\infty}^{\infty}\frac{d}{dt}(\delta(t-4))\frac{t}{2}\cos\left(\frac{t^2}{4}\right)$

0 b/c $\delta(t-4) = 0$ @ $\pm\infty$ & the derivative of $\delta(t-4) = 0$ too

$= -\int_{-\infty}^{\infty}\frac{d}{dt}(\delta(t-4))\frac{t}{2}\cos\left(\frac{t^2}{4}\right) dt$

$u = \frac{t}{2}\cos\left(\frac{t^2}{4}\right)$ $\quad v = \delta(t-4)$

$du = \frac{\cos\left(\frac{t^2}{4}\right)}{2} - \frac{t^2\sin(t^2/4)}{4}$ $\quad dv = \frac{d}{dt}(\delta(t-4))$

$= \underbrace{-\delta(t-4)\frac{t}{2}\cos\left(\frac{t^2}{4}\right)}_{=0\text{; same reason as before}} + \int_{-\infty}^{\infty}\delta(t-4)\,du(t)\,dt$

This expression is only non-zero when $t=4$, so the answer is $du(4)$

$$\boxed{\int_{-\infty}^{\infty}\frac{d^2}{dt^2}(\delta(t-4))\sin\left(\frac{t^2}{4}\right)dt = \frac{\cos(4)}{2} - 4\sin(4)}$$

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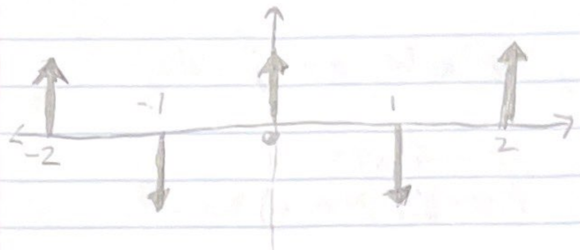
5)



$$F(t) = \sum_{d=-\infty}^{\infty} (-1)^d \delta(t-d), \quad d \in \mathbb{Z}$$

a) even function w/ average value of $F(t)=0$,
we know $a_0 = 0 = b_n$

$$a_n = \frac{2}{T}\int_T F(t)\cos(n\omega t), \quad T = 2 \qquad \therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \boxed{\pi = \omega}$$

We can rewrite $F(t)$ as another function that follows the same values of $(-1)^d$, so we have

$$F(t) = \sum_{d=-\infty}^{\infty} \cos(n\pi t)\,\delta(t-d) \quad d \in \mathbb{Z}$$

$$\therefore \quad a_n = \frac{2}{2}\int_{-.5}^{1.5} \cos(d\pi t)\,\delta(t-d)\cos(n\pi t)\,dt$$

We use these bounds to not cut a δ in half

$$a_n = \int_{-.5}^{1.5} \cos(d\pi t)\cos(n\pi t)\,\delta(t-d)\,dt$$, but $\delta(t-d) \neq 0$ @ $d = 0$ & 1, so

$$a_n = \cos(0)\cos(0) + \cos(d\pi)\cos(n\pi) = 1 + \cos(d\pi)\cos(n\pi)$$

but d is either 0 or 1 w/ values 1 & -1 @ these times, so

$$a_n = 1 + (-1)^n$$

HW#2 - Joseph Specht

$$\boxed{\begin{array}{c} \omega = \pi \qquad a_0 = 0 \qquad a_n = 1 + (-1)^n \quad b_n = 0 \\ F(t) = \sum_{n=1}^{\infty} (1 + (-1)^n) \cos(n\pi t) \end{array}}$$

b) guess for x_p is

$$x_p = \sum_{n=1}^{\infty} (1+(-1)^n)\, g(n\pi) \cos(n\pi t - \varphi(n\pi))$$

$$\zeta = \frac{c}{2m\omega_n} \qquad \omega_n = \sqrt{k/m} \qquad \varphi(n\pi) = \operatorname{atan}\left(\frac{2\zeta\left(\frac{n\pi}{\omega_n}\right)}{1-\left(\frac{n\pi}{\omega_n}\right)^2}\right)$$

$$g(n\pi) = \frac{1}{k}\left[\left(1-\left(\frac{n\pi}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{n\pi}{\omega_n}\right)\right)^2\right]^{(-1/2)}$$

$n\omega = \omega_n\sqrt{1-2\zeta^2}$

c) we know it resonates when we have

$$n\omega = \omega_n\sqrt{1-2\zeta^2}$$

$$n\omega = \omega_n\sqrt{1 - \frac{2c^2}{4m^2\omega_n^2}} = \omega_n\sqrt{1-\frac{c^2}{2mk^2}}$$

$$\boxed{\therefore\ \omega = \sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{2mk^2}}\left(\frac{1}{n}\right) \text{ where } \frac{n}{2} \in \mathbb{Z}}$$

we need an even n because when n is odd, $a_n = 0$, so we have no amp, so not resonant