HW #7 - Joseph Specht

1) $m\ddot{x} + kx = F_0 \exp(-\lambda t)$

a) $\boxed{\omega_n = \sqrt{k/m}}$

b) $m\ddot{x} + kx = F_0 \exp(-\lambda t) \Rightarrow m\lambda^2 \vartheta \exp(-\lambda t) - k\vartheta \exp(-\lambda t) = F_0 \exp(-\lambda t)$

if $x(t) = \vartheta \exp(-\lambda t)$, canceling out $\exp(-\lambda t)$ gives

$m\lambda^2 \vartheta + k\vartheta = F_0 \Rightarrow \vartheta(m\lambda^2 + k) = F_0$

$$\boxed{\vartheta = \frac{F_0}{m\lambda^2 + k}} \qquad x_p = \frac{F_0 \exp(-\lambda t)}{m\lambda^2 + k}$$

c) $m\ddot{x} + kx = F_0 \exp(-\lambda t)$

$x_h = A\cos(\omega_n t) + B\sin(\omega_n t)$
$\dot{x}_h = -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t)$
$\ddot{x}_h = -A\omega_n^2 \cos(\omega_n t) - B\omega_n^2 \sin(\omega_n t)$

$x_p = \vartheta \exp(-\lambda t)$
$\dot{x}_p = -\vartheta\lambda \exp(-\lambda t)$
$\ddot{x}_p = \vartheta\lambda^2 \exp(-\lambda t)$

$x = A\cos(\omega_n t) + B\sin(\omega_n t) + \vartheta \exp(-\lambda t)$
$\ddot{x} = -A\omega_n^2 \cos(\omega_n t) - B\omega_n^2 \sin(\omega_n t) + \vartheta\lambda^2 \exp(-\lambda t)$

Plugging this in ~~to~~ original expression

$m(-A\omega_n^2 \cos(\omega_n t) - B\omega_n^2 \sin(\omega_n t) + \vartheta\lambda^2 \exp(-\lambda t)) +$
$k(A\cos(\omega_n t) + B\sin(\omega_n t) + \vartheta \exp(-\lambda t)) = F_0 \exp(-\lambda t)$

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This can be written as

$m(-A\omega_n^2\cos(\omega_n t) - B\omega_n^2\sin(\omega_n t)) + k(A\cos(\omega t) + B\sin(\omega_n t)) +$
$m(D\lambda^2 \exp(-\lambda t)) + k(D\exp(-\lambda t)) = F_0\exp(-\lambda t)$

This is the same as $\mathcal{L}(x_p + x_h) = F_0\exp(-\lambda t)$ & this can be written as $\mathcal{L}(x_p + x_h) = 0 + F_0\exp(-\lambda t)$

where $\mathcal{L}(x) = m\ddot{x} + kx$

this can be seen as the superposition of solution mode & this sum is equal to the original force

d) $x_0 = x(t=0) = 0$ & $\dot{x}_0 = \dot{x}(t=0) = 0$

Using homogeneous case, we get

$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + D\exp(-\lambda t)$
$x(0) = A\cos(0) + B\sin(0) + D\exp(0)$
$x(0) = A + D = 0 \Rightarrow A = -D$

$\dot{x}(t) = -A\omega_n\sin(\omega_n t) + B\omega_n\cos(\omega_n t) + D\lambda^2\exp(-\lambda t)$
$\dot{x}(0) = -A\omega_n\sin(0) + B\omega_n\cos(0) - D\lambda\exp(0)$
$\dot{x}(0) = B\omega_n + D\lambda = 0 \Rightarrow B = D\lambda/\omega_n$

This gives

$$x = x_h + x_p = -D\cos(\omega_n t) + \frac{D\lambda}{\omega_n}\sin(\omega_n t) + D\exp(-\lambda t)$$

$$x = D\left(-\cos(\omega_n t) + \frac{\lambda}{\omega_n}\sin(\omega_n t) + \exp(-\lambda t)\right)$$

w/ $D = \dfrac{F_0}{m\lambda^2 + k}$

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$$\boxed{x = \frac{F_0}{m\lambda^2 - k}\left[-\cos\left(t\sqrt{\frac{k}{m}}\right) + \lambda\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right) + \exp(-\lambda t)\right]}$$

e) $\lim_{t\to\infty}(x(t)) = D\left[-\cos(\infty) + \frac{\lambda}{\omega_n}\sin(\infty) + \exp(\infty)\right]$

$\lim_{\varphi\to\infty}(\sin(\varphi) \text{ or } \cos(\varphi))$ is meaningless as they oscillate periodically, so they just follow cos & sin rules

$\lim_{\varphi\to\infty}(\exp(-\varphi)) = 0$, so adding these limits together we get

as time $\gg \frac{1}{\lambda}$, $x(t) = \frac{F_0}{m\lambda^2 - k}\left[-\cos\left(t\sqrt{\frac{k}{m}}\right) + \lambda\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right)\right]$

This simply acts as a steady state oscillator, so this makes sense as there is no drag so all the energy the force put in, has to stay there & cannot dissipate, so it is left as a steady state oscillator

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2) $F = F_0 \sin(\omega t)$

$c = 6\ kg/s$; $k = 160\ N/m$; $m = 10\ kg$; $F_0 = 2N$; $\omega = 4\ Hz$

a) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160\ N/m}{10\ m}} = 4\ rad/s$

$\frac{\omega}{\omega_n} = \frac{4}{4} = 1$ $\boxed{\frac{\omega}{\omega_n} = 1}$

b) Driving force $F = F_0 \sin(\omega t)$ is equal to $F_0 \cos(\omega t - \frac{\pi}{2})$

now we can use $x_p(t) = F_0\, g(\omega) \cos(\omega t - \frac{\pi}{2} - \varphi(\omega))$

Solving for g

$$g = \frac{1}{k} \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$ but $\frac{\omega}{\omega_n} = 1$, so

$$g = \frac{1}{k} \frac{1}{\sqrt{(1-1)^{2})^2 + (2\zeta)^2}} = \frac{1}{k\sqrt{(2\zeta)^2}} = \frac{1}{2k\zeta}$$, but $\zeta = \frac{c}{2m\omega_n}$

$$g = \frac{2m\omega_n}{2kc} = \frac{\omega_n}{\omega_n^2 c} = \frac{1}{\omega_n c} = \frac{1}{4 \cdot 6} = \boxed{\frac{1}{24} = g}$$

Solving for φ

$\varphi = \operatorname{atan}\left(\frac{2\zeta\, \omega \omega_n}{(\omega_n^2 - \omega^2)}\right)$ but $\omega_n^2 - \omega^2 = 0$, so $\varphi = \operatorname{atan}(\infty)$

so $\boxed{\varphi = \frac{\pi}{2}}$

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$$\boxed{x_p = \frac{F_0}{24}\cos(\omega t - \pi)}$$

c) $\boxed{\zeta = \frac{3}{40} < 1, \quad \therefore \text{ system is underdamped}}$ $\quad \zeta = \frac{6}{2 \cdot 10 \cdot 4} = \frac{3}{40}$

d) $x_h(t) = \exp(-\omega_n \zeta t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$

$\omega_n = \sqrt{k/m} = 4 \text{ rad/s} \qquad \zeta = \frac{c}{2m\omega_n} = \frac{6}{2 \cdot 10 \cdot 4} = \frac{6}{80} = \frac{3}{40}$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-\frac{9}{1600}} = 4\sqrt{\frac{1591}{1600}} = \frac{4}{4}\sqrt{\frac{1591}{10}} = \sqrt{\frac{1591}{10}}$

$x_h = \exp\left(-\frac{3}{10}t\right)\left[A\cos\left(\sqrt{\frac{1591}{10}}t\right) + B\sin\left(\sqrt{\frac{1591}{10}}t\right)\right]$

$\therefore x = \exp\left(-\frac{3}{10}t\right)\left[A\cos\left(\sqrt{\frac{1591}{10}}t\right) + B\sin\left(\sqrt{\frac{1591}{10}}t\right)\right] + \frac{8}{24}\cos(4t - \pi)$

Finding $x(0)$,..

$x(0) = \exp(0)\left[A\cos(0) + B\sin(0)\right] + \frac{1}{3}\cos(-\pi)$

$2 = A - \frac{1}{3} \Rightarrow A = \frac{7}{3}$ 2

$\dot{x}$ from a derivative calculator is

$\dot{x} = \frac{\exp\left(-\frac{3}{10}t\right)}{3 \cdot 10^{3/2}}\left[\left(9\sqrt{10}B + 30\sqrt{1591}A\right)\sin\left(\sqrt{\frac{1591}{10}}t\right) + \left(9\sqrt{10}A - 30\sqrt{1591}B\right)\right.$ ✱

$\left.\cos\left(\sqrt{\frac{1591}{10}}t\right)\right] - 4 \cdot 10^{3/2}\exp\left(\frac{3t}{10}\right)\sin(4t)$

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$$\dot{x}(0) = 0 = \frac{\exp(0)}{3\cdot 10^{3/2}}\left[(\ldots)\sin(0) + (9\sqrt{10}A - 30\sqrt{1591}B)\cos(0) - (\ldots)\sin(0)\right]$$

$$\Rightarrow 0 = \frac{1}{3\cdot 10^{3/2}}\left[9\sqrt{10}A - 30\sqrt{1591}B\right] \Rightarrow 9\sqrt{10}A - 30\sqrt{1591}B$$

$$\therefore 9\sqrt{10}A = 30\sqrt{1591}B \Rightarrow B = \frac{9\sqrt{10}A}{30\sqrt{1591}} = \frac{9\cdot\sqrt{10}\cdot 7}{30\sqrt{1591}\cdot 3} = \frac{21\sqrt{10}}{30\sqrt{1591}}$$

$$= \frac{7}{10}\frac{\sqrt{10}}{\sqrt{1591}} = \boxed{\frac{7}{\sqrt{15{,}910}} = B}$$

Plugging in we get

$$\boxed{x = \exp\left(-\frac{3}{10}t\right)\left[\frac{7}{3}\cos\left(4t\right) + \frac{7}{\sqrt{1591}}\sin\left(4t\right)\right] + \frac{1}{3}\cos\left(4t - \pi\right)}$$

e) $\boxed{\text{conditions satisfied } \checkmark,\ @\ t=0,\ x=0\ \&\ \text{max, so } v=0}$

f) the $\frac{1}{e}$ period for x_h is $t = \frac{10}{3}s = 3.\bar{3}s$

The homogeneous solution decays in about 11 seconds, so the decay time & these are the same order of magnitude

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3) $F(t) = F_0 \exp(-\alpha t) \cos(\omega t + \theta)$

Since we want to write F in terms of only a real part of a function

$$F(t) = \mathrm{Re}\left[F_0 \exp(-\alpha t) \exp(i(\omega t + \theta))\right] = \mathrm{Re}\left[F_0 \exp((-\alpha + i\omega)t + i\theta)\right]$$

but $-\alpha + i\omega = \gamma$, so

$$F = \mathrm{Re}\left[F_0 \exp(\gamma t + i\theta)\right]$$

using the given x guess, we get

$$x_p = \mathrm{Re}\left[A \exp(\gamma t + i\theta)\right]$$

$\dot{x}_p = A\gamma \exp(\gamma t + i\theta)$ Plugging these in we get

$\ddot{x}_p = A\gamma^2 \exp(\gamma t + i\theta)$

$$F_0 \exp(\gamma t + i\theta) = A \exp(\gamma t + i\theta)\left[\gamma^2 m + c\gamma + k\right]$$

$$\Rightarrow F_0 = A\left[\gamma^2 m + c\gamma + k\right] \Rightarrow \boxed{A = F_0 / (\gamma^2 m + c\gamma + k)}$$

$$x_p = \mathrm{Re}\left[\frac{F_0}{\gamma^2 m + c\gamma + k} \exp(\gamma t + i\theta)\right]$$

Expanding γ to find real part

$$x_p = \mathrm{Re}\left[\frac{F_0 \exp((-\alpha + i\omega)t + i\theta)}{(-\alpha + i\omega)^2 m + c(-\alpha + i\omega) + k}\right] = \mathrm{Re}\left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{(\alpha^2 - 2\alpha i\omega - \omega^2)m + c(-\alpha + i\omega) + k}\right]$$

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$$x_p = \text{Re}\left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{\alpha^2 m - 2\alpha i \omega m - \omega^2 m - c\alpha + ci\omega + k}\right]$$

Rearrange

$$x_p = \text{Re}\left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{\alpha^2 - \omega^2 m - c\alpha + k - 2\alpha i \omega m + ci\omega}\right]$$ mult by $\bar{z}$ of denominator

$$x_p = \text{Re}\left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{\alpha^2 - \omega^2 m - c\alpha + k - 2\alpha i \omega m + ci\omega} \times \frac{\alpha^2 - \omega^2 m - c\alpha + k + 2\alpha i \omega m - ci\omega}{\alpha^2 - \omega^2 m - c\alpha + k + 2\alpha i \omega m - ci\omega}\right]$$

$G = \alpha^2 - \omega^2 m - c\alpha + k$ $\qquad F = -\alpha t + i\omega t + i\theta$

$H = -2\alpha\omega m + c\omega$

$$\therefore x_p = \text{Re}\left[\frac{F_0 \exp(F)}{G + Hi} \times \frac{G - Hi}{G - Hi}\right] = \frac{F_0 \exp(F)(G - Hi)}{G^2 + H^2}$$

Expanding $\exp(F)$...

$$\exp(F) = \exp(-\alpha t + i\omega t + i\theta) = \exp(-\alpha t + (\omega t + \theta)i)$$

$$= \exp(-\alpha t)\exp((\omega t + \theta)i) = \exp(-\alpha t)(\cos(\omega t + \theta) + i\sin(\omega t + \theta))$$

Plugging that in we get

$$x_p = \text{Re}\left[\frac{F_0 \exp(-\alpha t)(\cos(\omega t + \theta) + i\sin(\omega t + \theta))(G - Hi)}{G^2 + H^2}\right]$$

Taking the real

$$x_p = \frac{F_0 \exp(-\alpha t)}{G^2 + H^2}\,\text{Re}\left[(\cos(\omega t + \theta) + i\sin(\omega t + \theta))(G - Hi)\right]$$

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$$x_p = \frac{F_0 \exp(-\alpha t)}{G^2 + H^2} \,\mathrm{Re}\left[\cos(\omega t+\theta)\, G - Hi\cos(\omega t+\theta) + Gi\sin(\omega t+\theta) + H\sin(\omega t+\theta)\right]$$

now we can take the Real part of this

$$x_p = \frac{F_0 \exp(-\alpha t)}{G^2 + H^2} \left(\cos(\omega t+\theta)\, G + H\sin(\omega t+\theta)\right)$$

Plugging everything in we get

$$\boxed{x_p = \frac{F_0 \exp(-\alpha t)\begin{bmatrix}(\alpha^2 - \omega^2 m - c\alpha + k)\cos(\omega t+\theta) \\ (-2\alpha\omega m + c\omega)\sin(\omega t+\theta)\end{bmatrix}}{(\alpha^2 - \omega^2 m - c\alpha + k)^2 + (-2\alpha\omega m + c\omega)^2}}$$

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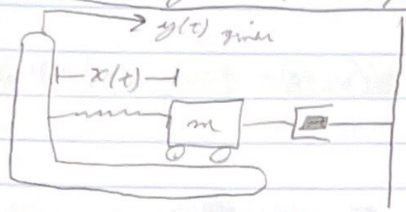
4)



$F_d = -cv$

$y = A_0 \cos(\omega t)$

a) $F = ma = -cv - kx$

$$m\frac{d^2}{dt^2}(x+y) = -c\frac{d}{dt}(x+y) - kx$$

We use $(x+y)$ for a & v because the relative position depends on both.

$$m\ddot{x} + m\ddot{y} = -c\dot{x} - c\dot{y} - kx \Rightarrow m\ddot{x} + c\dot{x} + kx = -m\ddot{y} - c\dot{y}$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = -m\ddot{y} - c\dot{y}}$$

This follows the form of the eq, so...

$$\boxed{M_{eff} = m \qquad C_{eff} = c \qquad K_{eff} = k \qquad D = -m \qquad E = -c}$$

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c) • $y(t)=0$, if this happens, eq of motion is

$$m\ddot{x} + c\dot{x} + kx = 0$$

This makes sense as this is the general eq of damped harmonic oscillators

• positive C_{eff} & K_{eff}. These are both positive as $C_{eff} = c > 0$ & $K_{eff} = k > 0$

• Dimensional analysis

$$\left[\frac{kg\, m}{s^2}\right] + \left[\frac{kg}{s}\frac{m}{s}\right] + \left[\frac{N}{m} = \frac{kg\, \cancel{m}}{s^2 \cancel{m}} \cdot m\right] = \left[\frac{kg\, m}{s^2}\right] - \left[\frac{kg}{s}\frac{m}{s}\right]$$

All of these units are equal to N as they should be

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c) obtain $x_p(t)$ for $y(t) = A_0 \cos(\omega t)$

find $\mathcal{G}(\omega)$ & $\varphi(\omega)$

$$\mathcal{G}(\omega) = \frac{1}{k} \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\varphi(\omega) = \operatorname{atan}\left[\frac{2\zeta\omega\omega_n}{(\omega_n^2-\omega^2)}\right]$$

Now we need to find ω_n, ω, ζ to get these solutions

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \underset{\text{given}}{\omega = \omega} \qquad \zeta = \frac{c}{2m\omega_n}$$

because this spans the (ωt) trig space & we need this b/c the force is in the form of $(\cos(\omega t \ldots) + \sin(\omega t \ldots))$

We know $\mathcal{G}$, φ, & ω, so

$$\mathcal{G}(\omega) = \frac{1}{k} \frac{1}{\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4\zeta^2\omega^2}{\omega_n^2}\right]^{1/2}} \Rightarrow \frac{4\zeta\omega^2}{\omega_n^2} = \frac{4c^2\omega^2}{4m^2\omega_n^2} = \frac{c^2\omega^2}{k^2}$$

$$\therefore \mathcal{G}(\omega) = \frac{1}{k}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{c\omega}{k}\right)^2\right]^{(-1/2)}$$

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simplifying $\varphi(\omega)$ gives...

$$\varphi(\omega) = \text{atan}\left[\frac{2\zeta\,\omega\omega_n}{(\omega_n^2-\omega^2)}\right] = \text{atan}\left[\frac{c\,\omega\omega_n}{m\omega_n(\omega_n^2-\omega^2)}\right] = \text{atan}\left[\frac{c\omega}{m(\omega_n^2-\omega^2)}\right]$$

$$= \text{atan}\left[\frac{c\omega}{m\left(\frac{k}{m}-\omega^2\right)}\right] = \text{atan}\left[\frac{c\omega}{k-m\omega^2}\right]$$

a guess for x_p, so we can assume a superposition of forces $-m\ddot{y}$ & $-c\dot{y}$, so we get

$$F_1 = -c(-A_0\omega\sin(\omega t)) = A_0\omega c\sin(\omega t)$$
$$F_2 = -m(-A_0\omega^2\cos(\omega t)) = A_0\omega^2 c\cos(\omega t)$$

now we can say the guesses are

$$x_{p1} = \mathcal{G}F_1 = \mathcal{G}A_0\omega c\sin(\omega t+\varphi)$$
$$x_{p2} = \mathcal{G}F_2 = \mathcal{G}A_0\omega^2 m\cos(\omega t+\varphi)$$

knowing $x_p = x_{p1}+x_{p2}$, we can write the final solution...

$$x_p = \mathcal{G}A_0\omega\left[\omega m\cos(\omega t+\varphi) + c\sin(\omega t+\varphi)\right]$$

where $$\mathcal{G} = \frac{1}{k}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{c\omega}{k}\right)^2\right]^{(-1/2)}$$

& $$\varphi = \text{atan}\left[\frac{c\omega}{k-m\omega^2}\right]$$