jspecht3

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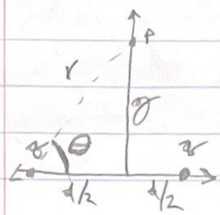
Phys 435 - Homework #1

2.2a)



By symmetry, know there is no x component & $\vec{E}$ is only a function of z when centered b/w charges

If we define r as the distance b/w the source & p, we get... $r=\sqrt{x^2+z^2}$

$\vec{E} = \underbrace{\frac{1}{4\pi\epsilon_0}}_{K} \cdot \frac{q}{r^2} \cdot \underbrace{2}_{\text{symmetry}} \cdot \hat{r}$, but we want the z component, so multiply by $\sin\theta$

$\sin\theta$ in this situation is...

$\sin\theta = \frac{z}{r}$, so $\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2} \cdot 2 \cdot z/r \cdot \hat{z}$

$$\vec{E} = \frac{2zq}{4\pi\epsilon_0 r^3} = \frac{2zq}{4\pi\epsilon_0} \cdot 1/(x^2+z^2)^{3/2}\,\hat{z} = \frac{2zq}{4\pi\epsilon_0} \cdot 1/\left(\left(\frac{d}{2}\right)^2+z^2\right)^{3/2}\hat{z}$$

$$\boxed{\vec{E} = \frac{2q}{4\pi\epsilon_0} \cdot \frac{z\hat{z}}{\left[\left(\frac{d}{2}\right)^2+z^2\right]^{3/2}}}$$

Follows inverse square law & when $z=0$, $E=0$, so makes sense

__Limit__

$$\lim_{z\gg d}\left(\frac{z\hat{z}}{\left[\left(\frac{d}{2}\right)^2+z^2\right]^{3/2}}\right) = \frac{z\hat{z}}{z^3} = z^{-2}\hat{z},$$ which lines up w/ what we'd expect from a point charge

$$\boxed{\vec{E} = \frac{2q}{4\pi\epsilon_0} \cdot z^{-2}\hat{z}}$$

E field of a particle w/ charge of 2q

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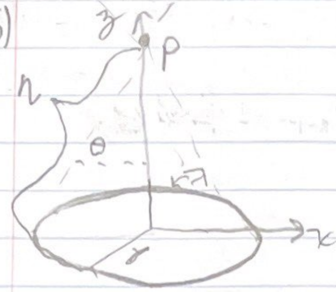
HW #1 cont

2.5)



Due to radial symmetry we have no horizontal components and only an E in $\hat{z}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}^2} \overbrace{\sin\theta\, \hat{z}}^{\text{instead of } \hat{\mathscr{r}}}$$

We know dq is equal to $\lambda d\theta$, if we revolve around $\hat{z}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda d\theta}{\mathscr{r}^2} \cdot \frac{z\hat{z}}{\mathscr{r}} = \frac{\lambda z}{4\pi\epsilon_0} \cdot \frac{1}{\mathscr{r}^3} \int_0^{2\pi} d\theta \hat{z} = \frac{\lambda z}{4\pi\epsilon_0} \cdot \frac{1}{\mathscr{r}^3} \cdot 2\pi \cdot \hat{z}$$

$\vec{E}(z) = \frac{\lambda z}{2\epsilon_0} \mathscr{r}^{-3} \hat{z}$ & $\mathscr{r} = \sqrt{r^2 + z^2}$, so

$$\boxed{\vec{E}(z) = \frac{\lambda z}{2\epsilon_0} \cdot \frac{1}{(r^2+z^2)^{3/2}}}$$

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HW#1 cont

2.6) This problem has a very similar setup to 2.5, but requires a double integral. However, this system still exhibits radial symmetry, so need to multiply by $\sin\theta$ again

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\eta^2}\sin\theta\,\hat{z} = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\eta^2}\cdot\frac{z}{\eta}\hat{z}$$

dq is different in cylindrical coordinates ~~ds~~ we are integrating over area

$\therefore dq = \sigma\, dA$, but $dA = r\,dr\,d\theta$ in polar coords

$$\therefore \vec{E} = \frac{z}{4\pi\epsilon_0}\iint^{D}\frac{1}{\eta^3}\cdot\sigma r\,dr\,d\theta\hat{z} = \frac{z\hat{z}}{4\pi\epsilon_0}\iint^{D}\frac{\sigma r\,dr\,d\theta}{(r^2+z^2)^{3/2}}$$
let $u = r^2+z^2$, $\frac{du}{dr} = 2r$ $\therefore dr = \frac{du}{2r}$

$$\Rightarrow \frac{z\hat{z}\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\theta\int_{r=0}^{r=R}\frac{r}{u^{3/2}}\cdot\frac{du}{2r} = \frac{2\pi z\hat{z}\sigma}{4\pi\epsilon_0}\int_{r=0}^{r=R}\frac{1}{2}\cdot\frac{du}{u^{3/2}}$$

$$\Rightarrow \frac{z\hat{z}\sigma}{2\epsilon_0}\int_{z^2}^{R^2+z^2}\frac{1}{2}u^{-3/2}\,du = \frac{z\hat{z}\sigma}{2\epsilon_0}\left[-u^{-1/2}\right]_{z^2}^{R^2+z^2} = \frac{z\hat{z}\sigma}{2\epsilon_0}\left[u^{-1/2}\right]_{R^2+z^2}^{z^2}$$

$$\Rightarrow \frac{z\hat{z}\sigma}{2\epsilon_0}\left[z^{-1}-(R^2+z^2)^{-1/2}\right] = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right]\hat{z} = \frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}$$

☆ all 2.6 answers here

$$\boxed{\therefore \vec{E}(z) = \frac{\sigma\hat{z}}{2\epsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]}$$

$\lim_{R\to\infty}\vec{E} = \frac{\sigma\hat{z}}{2\epsilon_0}$, which is E for an infinite plate so math checks out

for $z \gg R$ ... $(1+R)^{\alpha} \approx 1+\alpha R$

2.6 cont)

$(1+\epsilon)^{\alpha} \approx 1+\alpha\epsilon$

for $z \gg R$, $\vec{E} = \frac{\sigma \hat{z}}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]$

$\vec{E} = \frac{\sigma z \hat{z}}{2\epsilon_0}\left[\frac{1}{z} - (R^2+z^2)^{-1/2}\right]$ Taylor approx $(R^2+z^2)^{-1/2}$

$(R^2+z^2)^{-1/2} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1-\frac{R^2}{2z^3}\right)$

$\Rightarrow \vec{E} = \frac{\sigma z \hat{z}}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{z} + \frac{R^2}{2z^3}\right) = \frac{\sigma z \hat{z}}{2\epsilon_0}\left(\frac{R^2}{2z^3}\right)$

$\therefore \vec{E} = \frac{\sigma R^2 \hat{z}}{4\epsilon_0 z^2}$

same as point charge!

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HW#1 - cont

2.9) $E = kr^3\hat{r}$ in spherical coords where k is constant

a) $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, but E_θ & $E_\varphi = 0$, so...

$\Rightarrow \frac{\rho}{\epsilon_0} = \frac{1}{r^2} \left[\frac{\partial(r^2 E_r)}{\partial r} + 0 + 0\right]$

$\Rightarrow \frac{\partial(r^2 E_r)}{\partial r} = \frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \frac{\partial}{\partial r}(kr^5) = 5kr^4$

$\Rightarrow \frac{\rho}{\epsilon_0} = \frac{1}{r^2} \cdot 5kr^4 = 5kr^2$ $\boxed{\therefore \rho = 5k\epsilon_0 r^2}$

b) First way: Gauss's Law

$\frac{Q_T}{\epsilon_0} = \int \vec{E} \cdot d\vec{A}$, but $E \perp A$ $\therefore \int \vec{E} \cdot d\vec{A} = EA$

$\Rightarrow A =$ area of a sphere @ R away $= 4\pi R^2$

$\Rightarrow E =$ scalar component of $E_r = kR^3$

$\therefore \frac{Q_T}{\epsilon_0} = 4\pi R^2 \cdot kR^3 = 4\pi k R^5$

$$\boxed{\therefore Q_T = 4\pi\epsilon_0 k R^5}$$

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HW#1 - cont

Second way: integrating dq

$$Q_T = \int_{\mathcal{V}} dq \qquad dq = \rho \, d\mathcal{V}$$

However, $d\mathcal{V}$ can be written as the product of the boundary & the rate of motion of that boundary. In this case, the size of the boundary = the surface area of a sphere w/ radius r & the rate of change is dr.

$$\therefore Q_T = \int_{\mathcal{V}} \rho \, 4\pi r^2 \cdot dr = \int_0^R (5k\epsilon_0 r^2)(4\pi r^2 dr)$$

$$\Rightarrow Q_T = 4\pi\epsilon_0 k \int_0^R 5r^4 \, dr = 4\pi\epsilon_0 k \left[R^5\right]$$

$$\boxed{\therefore Q_T = 4\pi\epsilon_0 k R^5}$$

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HW #1 - cont

2.11) Gauss's Law inside & outside a spherical shell or radius R & surface density σ. Find E

Inside:

$\frac{Q_T}{\epsilon_0} = \int \vec{E} \cdot d\vec{A}$, but $Q_T = 0$, so

$\int \vec{E} \cdot d\vec{A} = 0$, but $\vec{E} \perp d\vec{A}$ & $d\vec{A} \neq 0$ $\boxed{\therefore \vec{E} = 0}$

Outside:

$\frac{Q_T}{\epsilon_0} = \int \vec{E} \cdot d\vec{A}$, but E is constant, so...

$\Rightarrow \frac{Q_T}{\epsilon_0} = \vec{E} \cancel{\int} d\vec{A} = EA$ where A = SA of sphere @ r, so

$\Rightarrow \frac{Q_T}{\epsilon_0} = E \cdot 4\pi r^2$ & Q_T = SA of sphere @ $R \cdot \sigma$

$\Rightarrow E \cdot 4\pi r^2 = \frac{4\pi R^2 \sigma}{\epsilon_0}$

$$\boxed{\therefore E = \frac{R^2 \sigma}{r^2 \epsilon_0}}$$

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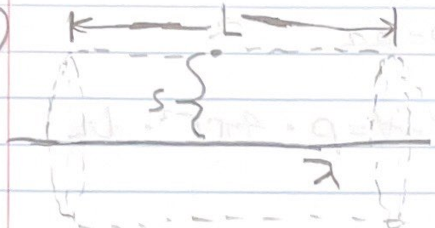
HW #1 - cont

2.13)



$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{inc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$ E is constant, so

$\Rightarrow E \oint dA = EA$ A = SA of cylinder w/ radius s & length L

$\Rightarrow E(2\pi s L) = \frac{\lambda L}{\epsilon_0} \Rightarrow E = \frac{\lambda L}{2\pi s L \epsilon_0} = \frac{\lambda}{2\pi s \epsilon_0}$

$$\boxed{\therefore \vec{E} = \frac{\lambda}{2\pi s \epsilon_0}}$$

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HW#1 - cont

2.14) inside a sphere where $\rho = kr$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\hat{r}, \text{ but } dq = \rho\, d\forall = \rho \cdot 4\pi r^2 \cdot dr$$

$$\therefore \vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho \cdot 4\pi r^2\, dr\,\hat{r}}{r^2} = \frac{4\pi}{4\pi\epsilon_0}\int kr\, dr\,\hat{r} = \frac{k}{\epsilon_0}\int r\, dr\,\hat{r}$$

$$\Rightarrow \vec{E} = \frac{k\hat{r}}{\epsilon_0}\left[\frac{1}{2}r^2\right] = \frac{kr^2}{2\epsilon_0}\hat{r}$$

$$\boxed{\therefore \vec{E} = \frac{kr^2}{2\epsilon_0}\hat{r}}$$