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1a) $\mathcal{U} = \int_0^\infty d\epsilon\, D(\epsilon)\, \epsilon\, f \qquad w/ \quad d\epsilon\, D(\epsilon) = \dfrac{\frac{1}{8} 4\pi k^2 dk}{\pi^3/V} = \dfrac{V}{2\pi^2} k^2 dk$

$w/ \quad k = \sqrt{2m/\hbar^2}\sqrt{\epsilon} \qquad d\epsilon\, D(\epsilon) = \dfrac{V}{4\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon$

$dk = \sqrt{2m/\hbar^2}\, \frac{1}{2}\frac{d\epsilon}{\sqrt{\epsilon}}$

$$\mathcal{U} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{d\epsilon\, \epsilon^{3/2}}{\exp(\beta\epsilon)-1} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{\beta^{5/2}} \int_0^\infty \frac{dx\, x^{3/2}}{\exp(x)-1}$$

$$w/ \int_0^\infty \frac{dx\, x^{3/2}}{\exp(x)-1} = \Gamma(5/2)\,\zeta(5/2) \qquad \boxed{\mathcal{U} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \Gamma(5/2)\,\zeta(5/2)\,(kT)^{5/2}}$$

b) $$\mathcal{U} = \underbrace{\left[\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} k^{5/2}\, \Gamma(5/2)\,\zeta(5/2)\right]}_{\alpha} T^{5/2} \equiv \alpha T^{5/2}$$

$$C_V = \left.\frac{\partial \mathcal{U}}{\partial T}\right|_V = \alpha\, \partial_T\left(T^{5/2}\right) = \alpha \frac{5}{2} T^{3/2}$$

$$\boxed{\therefore C_V = \frac{5V}{8\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} k^{5/2}\, \Gamma(5/2)\,\zeta(5/2)\, T^{3/2}}$$

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2a)

$N = \int d\epsilon\, D(\epsilon) f$ w/ $d\epsilon\, D(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon$ from 1a

$$N = \overbrace{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2}}^{\alpha} \int_0^\infty \sqrt{\epsilon}\, d\epsilon \left[\exp(-\beta(\epsilon-\mu)) + \exp(-2\beta(\epsilon-\mu))\right]$$

$$= \alpha \int_0^\infty \sqrt{\epsilon}\, d\epsilon \left[\exp(\beta\mu)\exp(-\beta\epsilon) + \exp(2\beta\mu)\exp(-2\beta\epsilon)\right]$$

$$= \alpha \left[\exp(\beta\mu) \int_0^\infty \frac{\sqrt{\epsilon}\, d\epsilon}{\exp(\beta\epsilon)} + \exp(2\beta\mu) \int_0^\infty \frac{\sqrt{\epsilon}\, d\epsilon}{\exp(2\beta\epsilon)}\right]$$

$$= \alpha\left[\exp(\beta\mu)\beta^{-3/2} + \exp(2\beta\mu)(2\beta)^{3/2}\right] \int_0^\infty \frac{\sqrt{x}\, dx}{\exp(x)} = \frac{\alpha\sqrt{\pi}}{2\beta^{3/2}}\left[\exp(\beta\mu) + \frac{\exp(2\beta\mu)}{2^{3/2}}\right]$$

$$\text{w/ } \frac{\alpha\sqrt{\pi}}{2\beta^{3/2}} = \frac{V\sqrt{\pi}}{4\pi^2} \frac{2^{3/2}}{2}\left(\frac{m}{\beta\hbar^2}\right)^{3/2} = V\left(\frac{m}{\beta\pi\hbar^2}\right)^{3/2} \frac{2^{3/2}}{2^3} = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} = n_Q V$$

$$N = n_Q V\left[\exp(\beta\mu) + \frac{\exp(2\beta\mu)}{2^{3/2}}\right] \Rightarrow \text{w/ } n = \frac{N}{V}, \quad \frac{n}{n_Q} = \exp(\beta\mu) + \frac{\exp(2\beta\mu)}{2^{3/2}}$$

$$\Rightarrow \exp(\beta\mu) = \frac{n}{n_Q} - \frac{\exp(\beta\mu)^2}{2^{3/2}}$$

we know from the classical limit, $\exp(\beta\mu) \approx n/n_Q$, so make same approx here

$$\exp(\beta\mu) = \frac{n}{n_Q} - \frac{1}{2^{3/2}}\left(\frac{n}{n_Q}\right)^2 \Rightarrow \boxed{\exp(\beta\mu) = \frac{n}{n_Q}\left[1 - \frac{1}{\sqrt{8}}\frac{n}{n_Q}\right]}$$

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2b) $$\mathcal{U} = \underbrace{\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{\alpha} \int_0^\infty \epsilon^{3/2}\, d\epsilon \left[\exp(-\beta(\epsilon-\mu)) + \exp(-2\beta(\epsilon-\mu))\right]$$

$$= \alpha\left[\exp(\beta\mu)\int_0^\infty \frac{\epsilon^{3/2}\, d\epsilon}{\exp(\beta\epsilon)} + \exp(2\beta\mu)\int_0^\infty \frac{\epsilon^{3/2}\, d\epsilon}{\exp(2\beta\epsilon)}\right]$$

$$= \alpha\left[\frac{\exp(\beta\mu)}{\beta^{5/2}} + \frac{\exp(2\beta\mu)}{(2\beta)^{5/2}}\right]\int_0^\infty \frac{x^{3/2}\, dx}{\exp(x)} = \frac{3\alpha\sqrt{\pi}}{4\beta^{5/2}}\left[\exp(\beta\mu) + \frac{\exp(2\beta\mu)}{2^{5/2}}\right]$$

w/ $$\frac{3\alpha\sqrt{\pi}}{4\beta^{5/2}} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{3\sqrt{\pi}}{4\beta^{5/2}} = \frac{3V}{\beta}\frac{2^{3/2}}{2^4}\left(\frac{m}{\beta\pi\hbar^2}\right)^{3/2} = \frac{3V}{2\beta}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} = \frac{3V n_Q}{2\beta}$$

$$\boxed{\Rightarrow \mathcal{U} = \frac{3V}{2\beta} n_Q\left[\exp(\beta\mu) + \frac{\exp(2\beta\mu)}{2^{5/2}}\right]}$$

$$N = V n_Q\left(1 + \frac{\exp(\beta\mu)}{2^{3/2}}\right)\exp(\beta\mu)$$

$$\Rightarrow \mathcal{U} = \frac{3}{2}NkT\, V n_Q\left[\left(1 + 2^{-3/2}\exp(\beta\mu)\right) - 2^{-5/2}\exp(\beta\mu)\right]\exp(\beta\mu)$$

$$= \frac{3}{2}kT\left[N - V n_Q 2^{-5/2}\exp(2\beta\mu)\right] \quad \boxed{\Rightarrow \mathcal{U} = \frac{3}{2}NkT\left[1 - \frac{n}{n_Q}2^{-5/2}\right]}$$

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3a) **Spacing**

$\epsilon_0 = (0+0+0)\hbar\omega = 0$ $\qquad \epsilon_1 - \epsilon_0 = \epsilon_2 - \epsilon_1 = \frac{1}{2}(\epsilon_2 - \epsilon_0) = \hbar\omega$

$\epsilon_1 = (1+0+0)\hbar\omega = \hbar\omega$

$\epsilon_2 = (2+0+0)\hbar\omega = 2\hbar\omega$ $\qquad \therefore$ spacing b/w energy levels is $\hbar\omega$

degeneracy

know there are $\Rightarrow \binom{n+2}{n} = \frac{(n+2)!}{(n+2-n)!\,n!} = \underline{\frac{(n+1)(n+2)}{2}}$ @ high $n \approx \underline{\frac{n^2}{2}}$
states for each n_i

density

know # of states $= \sum_i \frac{n_i^2}{2} \approx \int \frac{n^2\,dn}{2} = \frac{n^3}{3}$ w/ $n = \frac{E}{\hbar\omega}$, $= \frac{1}{6}\frac{E^3}{(\hbar\omega)^3}$

~~take~~ ~~derivative~~ wrt E $\Rightarrow dE\,D(E) = \partial_E\left(\frac{1}{6}\frac{E^3}{(\hbar\omega)^3}\right) = \boxed{\frac{1}{2(\hbar\omega)^3}E^2\,dE = dE\,D(E)}$

b) $N = \frac{1}{2(\hbar\omega)^3}\int \frac{E^2\,dE}{\exp(\beta E)-1} = \frac{1}{2}(\hbar\omega\beta)^{-3}\,\Gamma(3)\zeta(3) = \zeta(3)(\beta\hbar\omega)^{-3}$

$\Rightarrow \frac{N(\hbar\omega)^3}{\zeta(3)} = (kT)^3 \Rightarrow kT_E = \left(\frac{N}{\zeta(3)}\right)^{1/3}\hbar\omega \Rightarrow \boxed{T_E = \frac{\hbar\omega}{k}\left(\frac{N}{\zeta(3)}\right)^{1/3}}$