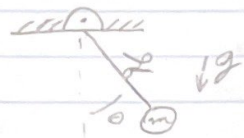
HW#6 - Joseph Spadaro

1)



$U(\theta=0)=0$, Energy E

a) E_{max} when 0 potential energy & all kinetic energy.
We know

$$T=\frac{1}{2}m\vec{v}^2 \quad \& \text{ in polar } T=\frac{1}{2}m(\dot{r}+r\dot{\theta})^2$$

but $\dot{r}=0$ b/c fixed pendulum, so

$$T=\frac{1}{2}mr^2\dot{\theta}^2=E, \text{ plugging in to this problem}$$

$$E=\frac{1}{2}m\mathcal{L}^2\dot{\theta}^2 \Rightarrow \dot{\theta}^2=\frac{2E}{\mathcal{L}^2 m} \Rightarrow \boxed{\dot{\theta}=\sqrt{\frac{2E}{\mathcal{L}^2 m}}}$$

b) θ_{max} is when $T=0$ or all energy is potential, so

$$E=mgh(1-\cos\theta), \text{ but for this problem}$$

$$E=mg\mathcal{L}(1-\cos\theta) \Rightarrow \frac{E-mg\mathcal{L}}{mg\mathcal{L}}=-\cos\theta \Rightarrow \frac{E}{mg\mathcal{L}}-1=-\cos\theta$$

$$\Rightarrow \cos\theta=1-E/mg\mathcal{L} \Rightarrow \boxed{\theta_{max}=\operatorname{acos}\left(1-\frac{E}{mg\mathcal{L}}\right)}$$

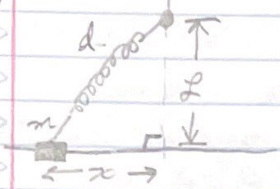
c) To swing in a circle, $\theta_{max}>\pi$, so

$$\operatorname{acos}\left(1-\frac{E}{mg\mathcal{L}}\right)>\pi \Rightarrow 1-\frac{E}{mg\mathcal{L}}>-1 \Rightarrow \frac{E}{mg\mathcal{L}}>2$$

$$\boxed{E>2mg\mathcal{L}}$$

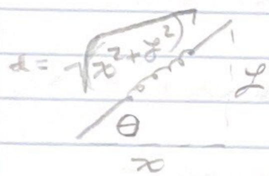
HW #6 – Joseph Specht

2)



$L_0 = L/2$

a) Force of spring = $-kx$ & eq length is $L_0 = L/2$, so this $x \rightarrow$ is the stretched length



$$U = \frac{1}{2}k\left(d - \frac{L}{2}\right)^2$$

$$U = \frac{1}{2}k\left(\sqrt{x^2+L^2} - \frac{L}{2}\right)^2$$

$$\boxed{U = \frac{1}{2}k\left(\sqrt{x^2+L^2} - \frac{L}{2}\right)^2}$$

b) Using a derivative calculator, I found U' to be

$$U' = kx\frac{\left(\sqrt{x^2+L^2} - \frac{L}{2}\right)}{\sqrt{x^2+L^2}} = kx\left(\frac{\sqrt{x^2+L^2}}{\sqrt{x^2+L^2}} - \frac{L}{2\sqrt{x^2+L^2}}\right)$$

$$= kx\left(1 - \frac{L}{2\sqrt{x^2+L^2}}\right) = kx - \frac{Lkx}{2\sqrt{x^2+L^2}}$$, this is only 0 when $x=0$

$$\boxed{\therefore x_{min} = 0}$$

HW#6 - Joseph Szott

c) using an online calculator we get

$$u''(x) = \frac{k\left(2(x^2+L^2)^{3/2} - L^3\right)}{2(x^2+L^2)^{3/2}}$$, but since we are expanding around $x_{min} = 0$, we plug in 0 for $u''(x)$

$$u''(x) = \frac{k\left(2(L^2)^{3/2} - L^3\right)}{2(L^2)^{3/2}}$$ & this simplified gives

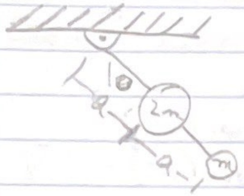
$u''(0) = k/2$, $\boxed{\therefore\ k_{eff} = K/2}$

d) $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{2m}} = \sqrt{\frac{k}{8\pi^2 m}}}$$

HW #6 - Joseph Spadaro

3)



$\tau = I\alpha$

$I = I_1 + I_2 = 2m(a)^2 + m(2a)^2 = 2ma^2 + 4ma^2$
$= 6ma^2$

$\tau = -g\sin\theta\,(m_1 r_1 + m_2 r_2)$

$= -g\sin\theta\,(2ma + m2a) = -g\sin\theta\,(4ma)$

$\tau = -4mag\sin\theta = I\alpha = 6ma^2\ddot{\theta}$

$-4mag\sin\theta = 6ma^2\ddot{\theta} \Rightarrow -4g\sin\theta = 6a\ddot{\theta}$

Since θ is really small, $\sin\theta \approx \theta$, so

$-4g\theta = 6a\ddot{\theta} \Rightarrow 6a\ddot{\theta} + 4g\theta = 0$

$\Rightarrow \ddot{\theta} + \frac{4g}{6a}\theta = 0 \Rightarrow \ddot{\theta} + \frac{2g}{3a}\theta = 0$ guess $\theta = \exp(\lambda t)$

so, $\ddot{\theta} = \lambda^2 \exp(\lambda t)$ & plugging this in gives

$\lambda^2 \exp(\lambda t) + \frac{2g}{3a}\exp(\lambda t) = 0$, so $\lambda^2 + \frac{2g}{3a} = 0$

$\therefore \lambda = \pm i\sqrt{\frac{2g}{3a}}$ & $\omega = \sqrt{\frac{2g}{3a}}$, but $T = \frac{2\pi}{\omega}$, so

$$\boxed{T = 2\pi\sqrt{\frac{3a}{2g}} = \sqrt{\frac{6\pi^2 a}{g}}}$$

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4a) $x(t) = C_1 \cos(\omega t - \varphi_1)$

$C_1 = 5$, amplitude is $= 5$ & this is C

$\varphi_1 = \pi/3$, @ $x=0$, $C_1\cos(-\varphi_1) = \frac{1}{2}C_1$ $\therefore$ $\varphi_1 = -\mathrm{acos}(1/2) = \pm\pi/3$, but we know the graph is almost @ a max, so it has to be positive (1st quadrant)

$\omega = 2$, @ $x = \pi/6$, we have a max, so $C_1\cos(\omega t - \pi/3) = C_1$
$\therefore \cos(\omega t - \pi/3) = 1$, so $\omega t - \pi/3 = 0$, so $\omega t = \pi/3$
$\therefore \omega = \pi/3t = \pi/3 \cdot \frac{6}{\pi} = 2$

$$\boxed{x(t) = 5\cos(2t - \pi/3)}$$

b) sin lags $\pi/2$ behind cos, so $\varphi_2 = \varphi_1 - \frac{\pi}{2} = \frac{\pi}{3} - \frac{\pi}{2} = \frac{-\pi}{6}$

$$\boxed{\therefore x(t) = 5\sin(2t + \pi/6)}$$

c) we know $x(t) = A\cos(2t) + B\sin(\omega t)$ & $C_1 = \sqrt{A^2+B^2}$

also $\tan(\varphi) = \frac{B}{A} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, so $B = \sqrt{3}A$

Plugging this in we get $C_1 = \sqrt{A^2 + 3A^2} = \sqrt{4A^2} = 2A$, so $A = C/2$

& we know $B = \sqrt{3}\frac{C}{2}$, so $A = \frac{5}{2}$ & $B = \frac{5}{2}\sqrt{3}$

$$\boxed{\therefore x(t) = \frac{5}{2}\cos(2t) + \frac{5}{2}\sqrt{3}\sin(2t)}$$

HW#6 - Joseph Specht

d) If x takes the form $\theta_1 \exp(i\omega t) + \theta_2 \exp(-i\omega t)$, then using euler's identity, we get

$$x = \theta_1 \cos(\omega t) + \theta_1 i \sin(\omega t) + \theta_2 \cos(\omega t) - \theta_2 i \cos(\omega t)$$

& grouping these terms gives,

$$x = (\theta_1 + \theta_2)\cos(\omega t) + (\theta_1 i - \theta_2 i)\sin(\omega t)$$

but we know from part c the constants in front of each, so we know,

$$\tfrac{5}{2} = \theta_1 + \theta_2 \quad \& \quad \tfrac{5}{2}\sqrt{3} = \theta_1 i - \theta_2 i$$

$$\theta_1 = \tfrac{5}{2} - \theta_2 \Rightarrow \tfrac{5}{2}\sqrt{3} = \tfrac{5}{2}i - \theta_2 i - \theta_2 i \Rightarrow \tfrac{5}{2}\sqrt{3} - \tfrac{5}{2}i = -2\theta_2 i$$

$\Rightarrow \theta_2 = \frac{5}{4} + \frac{5}{4}\sqrt{3}\,i$ since $i = \frac{1}{i} = \sqrt{-1} = \frac{1}{\sqrt{-1}}$, so θ_1 is

$$\theta_1 = \tfrac{5}{2} - \left(\tfrac{5}{4} + \tfrac{5}{4}\sqrt{3}\,i\right) = \tfrac{5}{4} - \tfrac{5}{4}\sqrt{3}\,i$$

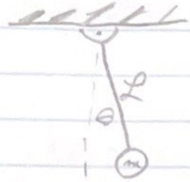
$$\boxed{\therefore x(t) = \left(\frac{5}{4} - \frac{5i\sqrt{3}}{4}\right)\exp(2it) + \left(\frac{5}{4} + \frac{5i\sqrt{3}}{4}\right)\exp(-2it)}$$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$\zeta = \frac{c}{2m\omega_n}$

HW#6 - Joseph Specht

5)



$F_{drag} = -cv$

a) $\tau = I\alpha$ $\qquad I = L^2 m$

$I\alpha = \tau_{drag} + \tau_g = L^2 m\alpha = L(F_{drag}) + L(mg\sin\theta)$

$L^2 m\alpha = -Lc(\dot r + r\dot\theta) - Lmg\sin\theta$, but $\dot r = 0$, so

$L^2 m\ddot\theta = -L(cL\dot\theta) - Lmg\sin\theta \Rightarrow mL\ddot\theta = -cL\dot\theta - mg\sin\theta$

$$\Rightarrow \boxed{mL\ddot\theta + cL\dot\theta + mg\sin\theta = 0}$$

b) if $\theta \ll 0$, $\sin\theta \approx \theta$, so $(\sin\theta = \theta + \dots)$

$$mL\ddot\theta + cL\dot\theta + mg\theta = 0 \Rightarrow \ddot\theta + \frac{c}{m}\dot\theta + \frac{g}{L}\theta = 0$$

guessing $\theta = \exp(\lambda t)$, we get

$$\lambda^2\exp(\lambda t) + \frac{c}{m}\lambda\exp(\lambda t) + \frac{g}{L}\exp(\lambda t) = 0, \text{ so } \lambda^2 + \frac{c}{m}\lambda + \frac{g}{L} = 0$$

use quadratic formula to get λ, so

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{g}{L}}}{2}$$ & using $\zeta = \frac{c}{2m\omega_n}$, so

$$\lambda = \frac{2\omega_n\left(-\zeta \pm \sqrt{4\omega_n^2\zeta^2 - 4\omega_n^2\left(\frac{g}{L}\right)\frac{1}{\omega_n^2}}\right)}{2}$$

This is equal to... (next page)

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$$\lambda = \frac{2\omega_n\left(-\zeta \pm 2\omega_n\sqrt{\zeta^2 - \left(\frac{g}{L}\right)\frac{1}{\omega_n^2}}\right)}{2}$$

however, for pendulums $\omega_n = \sqrt{\frac{g}{L}}$, so

$\frac{g}{L} \cdot \frac{1}{\omega_n^2} = \frac{g}{L} \cdot \frac{L}{g} = 1$, so $\lambda = \omega_n\left(-\zeta \pm \omega_n\sqrt{\zeta^2 - 1}\right)$

$\Rightarrow \lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = -\zeta\omega_n \pm \omega_n i\sqrt{1-\zeta^2} = -\zeta\omega_n \pm i\omega_d$, so

$\omega_d = \omega_n\sqrt{1-\zeta^2}$, and $\zeta = \frac{c}{2m\omega_n}$, so

$$\omega_d = \omega_n\sqrt{1 - \frac{c^2}{4m^2\omega_n^2}} = \sqrt{\omega_n^2 - \frac{c^2\omega_n^2}{4m^2\omega_n^2}} = \sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}$$

$$\boxed{\omega_d = \sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}}$$

d) $\theta(t=0) = \theta_0$ & $\dot{\theta}(t=0) = \Omega$ find $\theta(t)$

we know $\theta(t) = \exp(-\zeta\omega_n t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$

but $\theta(0) = A\cos(0) = A = \theta_0$ $\therefore A = \theta_0$

$$\dot{\theta}(t) = \exp(-\zeta\omega_n t)\left[-A\omega_d\sin(\omega_d) + B\omega_d\cos(\omega_d t)\right] - \zeta\omega_n\exp(-\zeta\omega_n t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]$$

$\dot{\theta}(0) = B\omega_d\cos(0) - \zeta A\cos(0) = B\omega_d - \zeta A = \Omega$

$\Rightarrow B = \frac{\Omega + \zeta A}{\omega_d} = \frac{\Omega + \zeta\theta_0}{\omega_d}$

$$\omega_d = \sqrt{\frac{g}{\ell} - \frac{c^2}{4m^2}} \qquad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

$$\zeta = \frac{c}{2m\omega_n} \qquad \omega_n = \sqrt{g/\ell}$$

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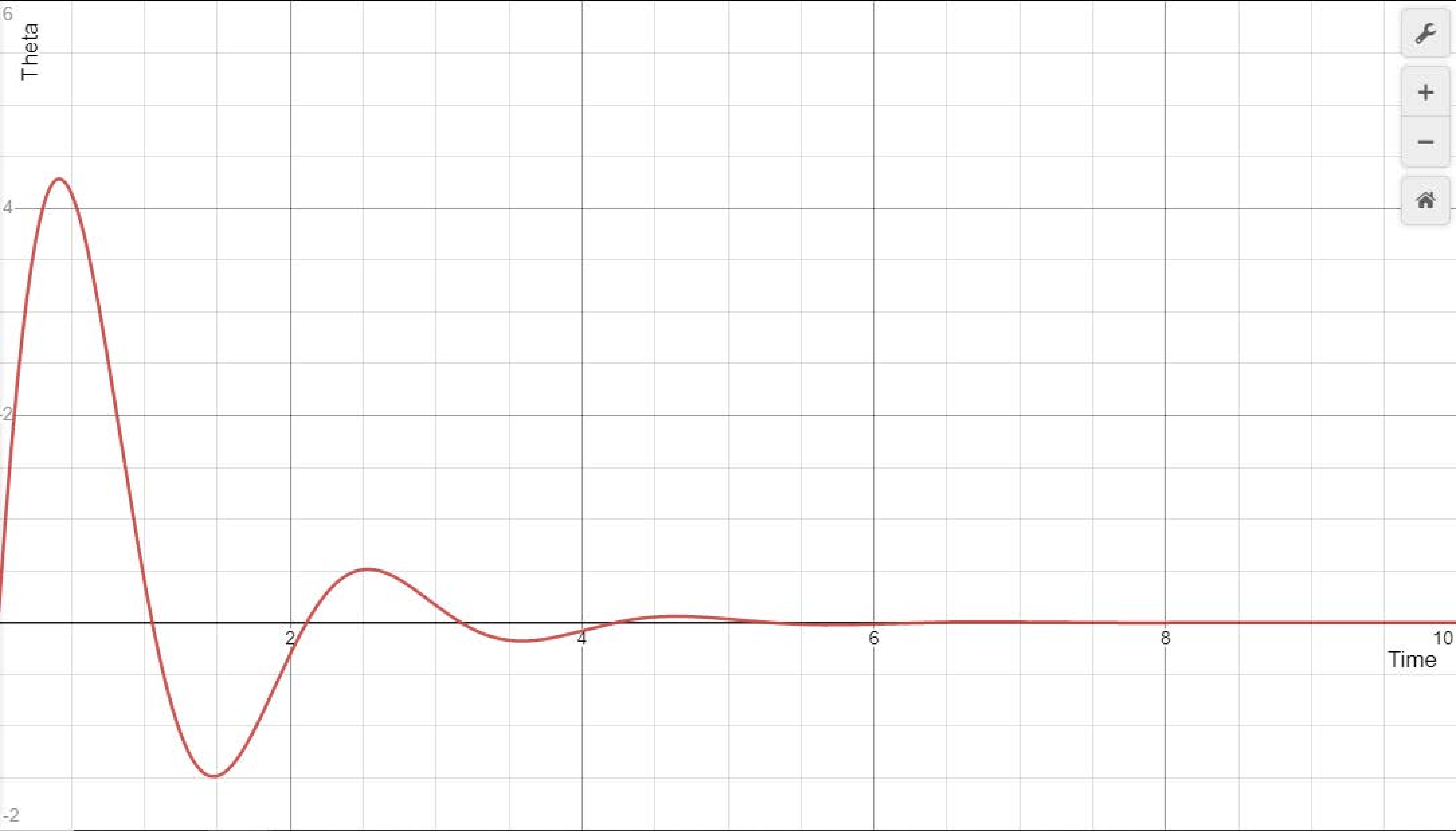
Plugging everything in, we get,

$$\theta(t) = \exp(-\zeta\omega_n t)\left[\theta_0 \cos(\omega_d t) + \left(\frac{\Omega + \zeta\theta_0}{\omega_d}\right)\sin(\omega_d t)\right]$$

now plugging in $\omega_n = \sqrt{g/\ell}$, $\zeta = \frac{c\sqrt{\ell}}{2m\sqrt{g}}$, &

$\omega_d = \sqrt{\frac{g}{\ell} - \frac{c^2}{4m^2}}$, we get

$$\boxed{\theta(t) = \exp\left(\frac{-ct}{2m}\right)\left[\theta_0 \cos\left(\sqrt{\frac{g}{\ell} - \frac{c^2}{4m^2}}\,t\right) + \left(\frac{\Omega + \frac{c\theta_0}{2m}\sqrt{\frac{g}{\ell}}}{\sqrt{\frac{g}{\ell} - \frac{c^2}{4m^2}}}\right)\sin\left(t\sqrt{\frac{g}{\ell} - \frac{c^2}{4m^2}}\right)\right]}$$


Theta
6
4
2
-2
2
4
6
8
10
Time