HW#2

1) paramagnet, $N \gg 1$ and spin $\pm 1/2$ particles, $\vec{B} = B\hat{z}$

$$\mathcal{U} = -\sum_{i=1}^{N} \vec{\mu}(i) \cdot \vec{B} = -\sum_{i=1}^{N} \mu_z(i) B$$

2 energy levels for each spin $\pm\mu B$. aligned w/ $B(-)$, anti aligned w/ $B(+)$

$\therefore \mathcal{U} = -\mu B (N_\uparrow - N_\downarrow)$

a) sketch $\sigma(\mathcal{U})$ as fn$(\mathcal{U})$. find max value

know $N = N_\uparrow + N_\downarrow \Rightarrow N_\downarrow = N - N_\uparrow$ & $N_\uparrow = N - N_\downarrow$

know $\Omega = \binom{N}{N_\uparrow} = \dfrac{N!}{N_\uparrow!(N-N_\uparrow)!} = \dfrac{N!}{N_\uparrow! N_\downarrow!}$

$\underline{N_\uparrow(N,\mathcal{U})} \Rightarrow \mathcal{U} = -\mu B(N_\uparrow - (N - N_\uparrow)) = -\mu B(2N_\uparrow - N)$

$$\Rightarrow \frac{-\mathcal{U}}{\mu B} = 2N_\uparrow - N \Rightarrow \underline{\frac{N}{2} - \frac{\mathcal{U}}{2\mu B} = N_\uparrow}$$

Similarly, $N_\downarrow(N,\mathcal{U}) \Rightarrow \mathcal{U} = -\mu B(N - 2N_\downarrow) \Rightarrow \underline{\dfrac{N}{2} + \dfrac{\mathcal{U}}{2\mu B} = N_\downarrow}$

w/ $\sigma = \ln\Omega = \ln(N!) - \ln(N_\uparrow)! - \ln(N_\downarrow)!$

Stirling as $N, N_\uparrow, N_\downarrow \gg 1 \Rightarrow \sigma \approx N\ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - N_\downarrow \ln N_\downarrow + N_\downarrow$

$\rightarrow N_\uparrow + N_\downarrow - N = 0$

$\sigma = N\ln N - N_\uparrow \ln N_\uparrow - N_\downarrow \ln N_\downarrow$; plug in $N_\uparrow(\mathcal{U},N), N_\downarrow(\mathcal{U},N)$

$$\sigma = N\ln N - \left(\frac{N}{2} - \frac{\mathcal{U}}{2\mu B}\right)\ln\left(\frac{N}{2} - \frac{\mathcal{U}}{2\mu B}\right) - \left(\frac{N}{2} + \frac{\mathcal{U}}{2\mu B}\right)\ln\left(\frac{N}{2} + \frac{\mathcal{U}}{2\mu B}\right)$$

$$\frac{\partial\sigma}{\partial\mathcal{U}} = \frac{\partial}{\partial\mathcal{U}}\left(\frac{\mathcal{U}}{2\mu B}\ln\left(\frac{N}{2} - \frac{\mathcal{U}}{2\mu B}\right) - \frac{\mathcal{U}}{2\mu B}\ln\left(\frac{N}{2} + \frac{\mathcal{U}}{2\mu B}\right) + \text{not fn}(\mathcal{U})\right)$$

$$= \frac{\partial}{\partial\mathcal{U}}\left(\frac{\mathcal{U}}{2\mu B}\left[\ln\left(\frac{N}{2} - \frac{\mathcal{U}}{2\mu B}\right) - \ln\left(\frac{N}{2} + \frac{\mathcal{U}}{2\mu B}\right)\right]\right) = 0 \text{ for max}$$

HW#2-cont

1a) $$\frac{\partial\sigma}{\partial U} = \frac{1}{2\mu B}\left[\ln\left(\frac{N}{2} - \frac{U}{2\mu B}\right) - \ln\left(\frac{N}{2} + \frac{U}{2\mu B}\right)\right] + \frac{U}{2\mu B}\left[\frac{1}{\frac{N}{2} - \frac{U}{2\mu B}} \cdot \frac{-1}{2\mu B} - \frac{1}{\frac{N}{2} + \frac{U}{2\mu B}} \cdot \frac{1}{2\mu B}\right] = 0$$

$$\frac{1}{2\mu B}\left[\ln\left(\frac{N}{2} - \frac{U}{2\mu B}\right) - \ln\left(\frac{N}{2} + \frac{U}{2\mu B}\right)\right] = \frac{U}{4\mu^2 B^2}\left[\frac{1}{\frac{N}{2} - \frac{U}{2\mu B}} + \frac{1}{\frac{N}{2} + \frac{U}{2\mu B}}\right]$$

$$\ln\left(\frac{N}{2} - \frac{U}{2\mu B}\right) - \ln\left(\frac{N}{2} + \frac{U}{2\mu B}\right) = \frac{U}{2\mu B}\left[\frac{1}{N_\uparrow} + \frac{1}{N_\downarrow}\right]$$

Know $\frac{U}{2\mu B} = -\frac{\mu B}{2\mu B}(N_\uparrow - N_\downarrow) = \frac{N_\downarrow - N_\uparrow}{2} = \frac{N - 2N_\uparrow}{2} = \frac{N}{2} - N_\uparrow$

ln always >0 & $N_\uparrow \propto U$, so only 1 intercept b/w LHS & RHS only occurs when both sides $= 0$.

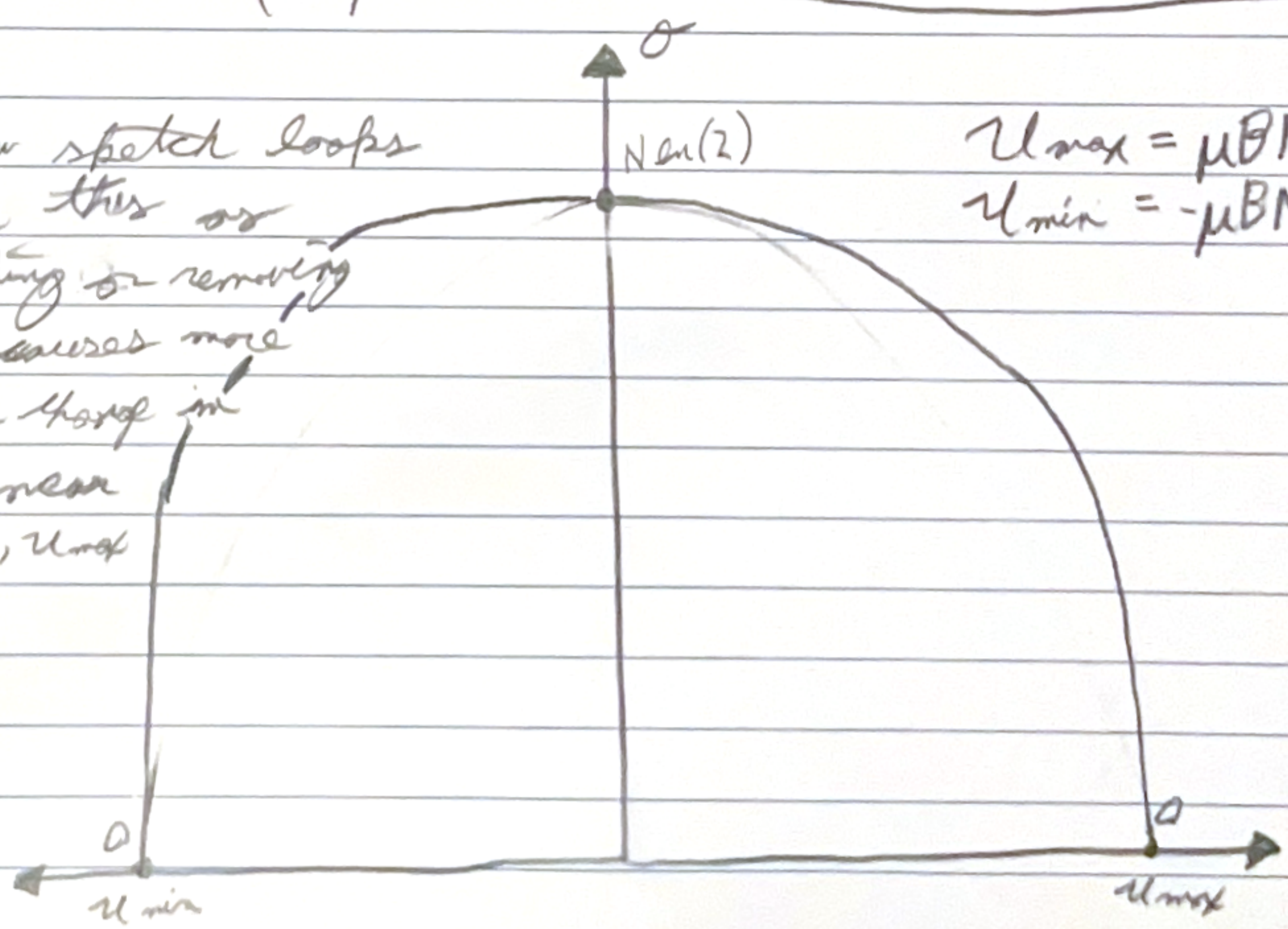
$\Rightarrow \frac{N}{2} - N_\uparrow = 0$, $N_\uparrow = N/2 = N_\downarrow \Rightarrow U = -\mu B(N_\uparrow - N_\downarrow) = -\mu B\left(\frac{N}{2} - \frac{N}{2}\right) = 0$

$\Rightarrow \ln\left(\frac{N}{2}\right) - \ln\left(\frac{N}{2}\right) = 0 \Rightarrow$ max σ @ $U = 0$

$$\sigma_{max} = N\ln N - \left(\frac{N}{2}\ln\left(\frac{N}{2}\right) - \frac{N}{2}\ln\left(\frac{N}{2}\right)\right) = N\ln N - N\ln(N/2)$$

$$\sigma_{max} = N\ln\left(\frac{2N}{N}\right) = N\ln(2) \qquad \boxed{\sigma_{max} = N\ln(2) \text{ @ } U = 0}$$

Know sketch looks like this as adding or removing U causes more of a change in σ near U_{min}, U_{max}



$U_{max} = \mu B N$
$U_{min} = -\mu B N$

HW#2

1b) Derive & sketch Temp as fn(U). $S \equiv k_B \sigma$, $1/T \equiv \left(\frac{\partial S}{\partial U}\right)_N = k\left(\frac{\partial \sigma}{\partial U}\right)_N$

from a) $$\frac{\partial \sigma}{\partial U} = \frac{1}{2\mu B}\left[\ln(N_\uparrow) - \ln(N_\downarrow)\right] - \frac{U}{4\mu^2B^2}\left(\frac{1}{N_\uparrow} + \frac{1}{N_\downarrow}\right)$$

$$= \frac{1}{2\mu B}\left[\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\right] - \frac{U}{4\mu^2B^2}\left(\frac{1}{N_\uparrow} + \frac{1}{N_\downarrow}\right)$$

$N_\uparrow$ can be written as $\frac{U_{max} - U}{2\mu B}$ & $N_\downarrow$ as $\frac{U_{max} + U}{2\mu B}$

$$\Rightarrow \frac{N_\uparrow}{N_\downarrow} = \frac{U_{max} - U}{U - U_{min}} \quad \& \quad \frac{1}{N_\uparrow} + \frac{1}{N_\downarrow} = \frac{2\mu B}{U - U_{min}} + \frac{2\mu B}{U_{max} - U}$$

$$\Rightarrow \frac{U}{4\mu^2B^2}\left(\frac{1}{N_\uparrow} + \frac{1}{N_\downarrow}\right) = \frac{U}{2\mu B}\left(\frac{1}{U_{max} - U} + \frac{1}{U_{max} + U}\right) = \frac{U}{2\mu B}\left(\frac{(U_{max} - U) + (U_{max} + U)}{(U_{max} - U)(U_{max} + U)}\right)$$

$$= \frac{U}{2\mu B}\left(\frac{U_{max} - U_{min}}{UU_{max} - U^2 + U_{min}U_{max} + UU_{min}}\right) \quad w/\ U_{min} = -U_{max}$$

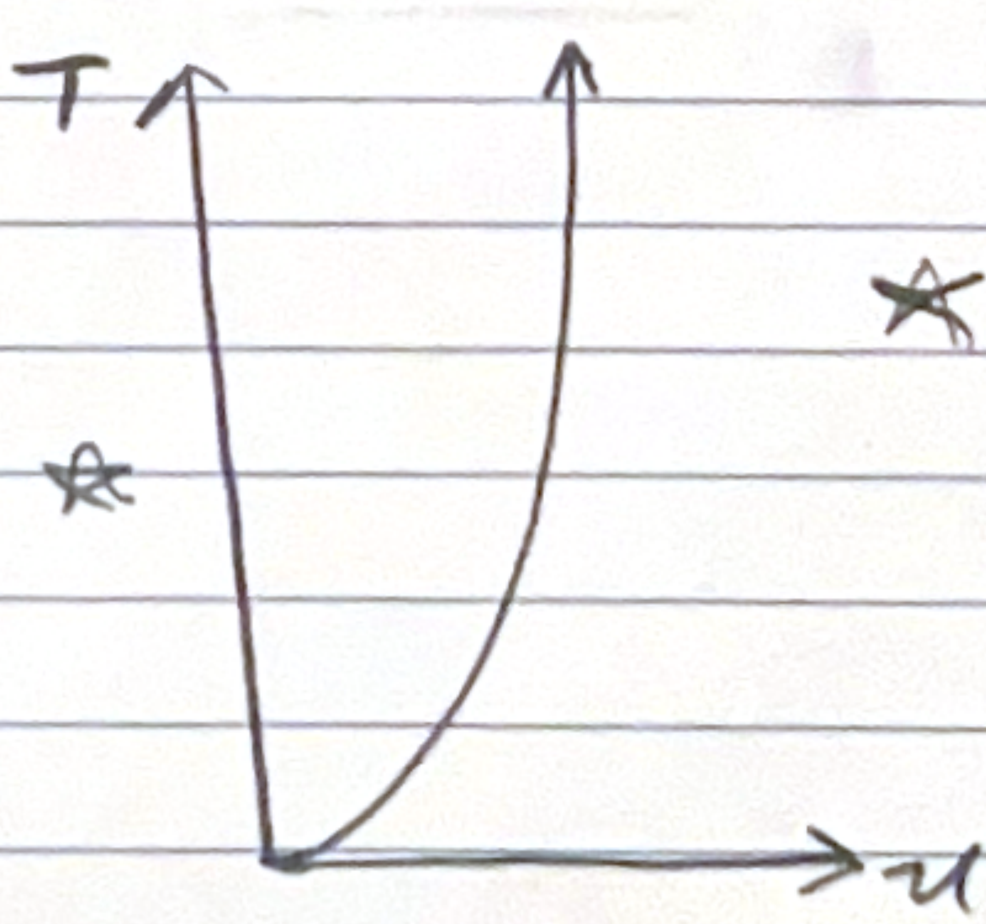
$$= \frac{U}{2\mu B}\left(\frac{2U_{max}}{UU_{max} - U^2 + U_{max}^2 - UU_{max}}\right) = \frac{-U}{\mu B}\left(\frac{\mu B N}{U^2 - U_{max}^2}\right) = \frac{-UN}{U^2 - U_{max}^2}$$

N fixed & $U \gg 0$, so $\frac{U}{U^2 - U_{max}^2} \approx 0$ as $U^2 \gg U$, so

$$\Rightarrow \frac{\partial S}{\partial U} = \frac{k}{2\mu B}\ln\left(\frac{U_{max} - U}{U_{max} + U} \times \frac{\frac{1}{\mu B}}{\frac{1}{\mu B}}\right) = \frac{k}{2\mu B}\ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right)$$

$$\Rightarrow \frac{1}{T} = \frac{\partial S}{\partial U} \Rightarrow T = \frac{2\mu B}{k}\ln\left(\frac{N + U/\mu B}{N - U/\mu B}\right) = \frac{2\mu B}{k}\ln\left(\frac{1 + U/N\mu B}{1 - U/N\mu B}\right)$$

$$\boxed{T = \frac{\mu B}{k}\,\mathrm{atanh}\left(\frac{U}{N\mu B}\right)}$$



HW #2

1c) energy flows in a way that maximizes entropy, so

if PM w/ $T=0$ put next to another system of T_B, the energy will flow into the other system $\boxed{T \to T_B}$ because there is a single microstate w/ T & the PM can increase the # of microstates by reducing its energy

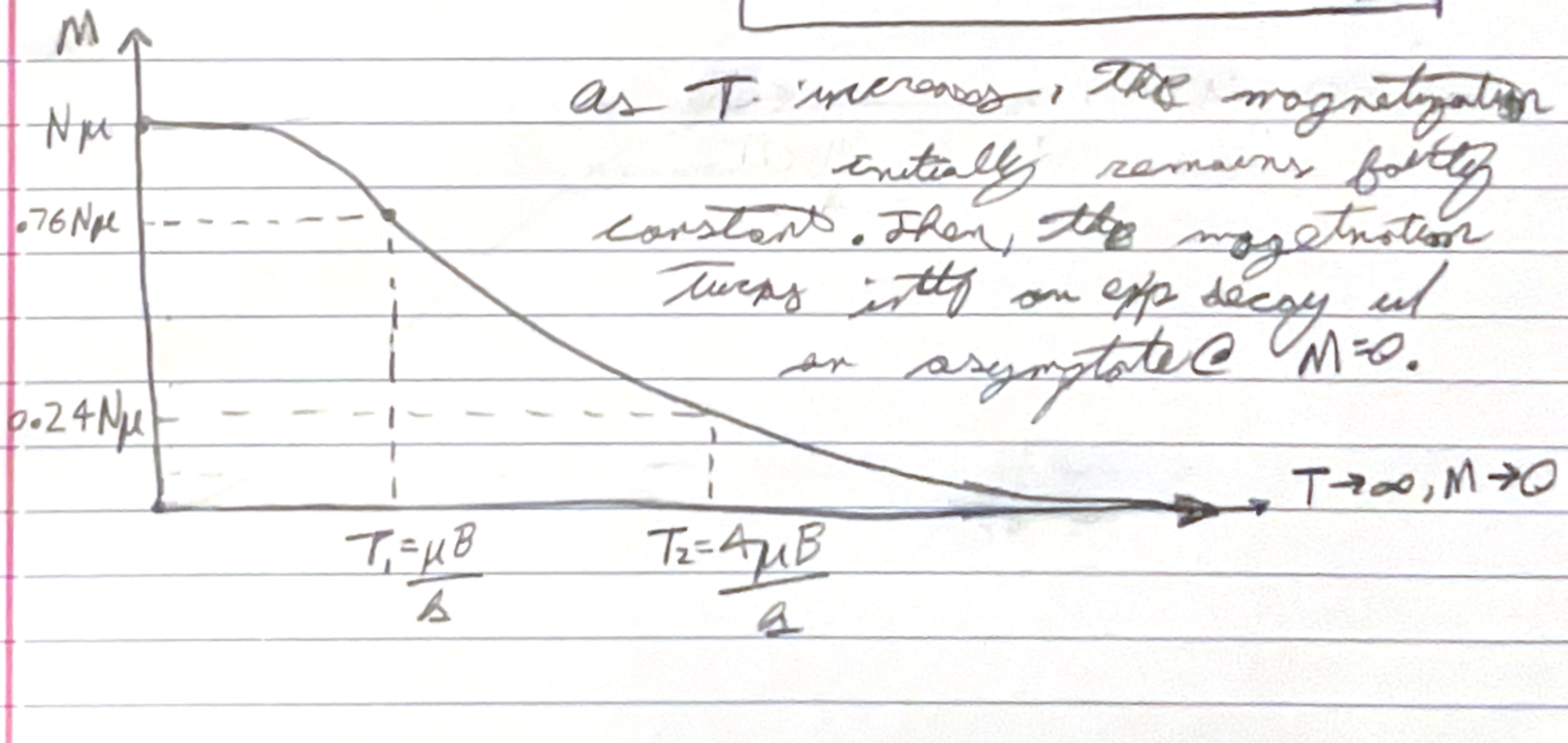
if PM w/ $T<0$, assuming $T_B>0$, the energy will flow from $\boxed{T_B \to T}$ as the paramagnet has max entropy @ $T=0$, so the system will move towards $T=0$. T will increase towards 0 until the increase in T causes fewer new microstates than T_B will lose by giving up that energy.

d) $M \equiv -U/B$. Find $M(T)$.

Know from $\frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_N = \left(\frac{\partial (k\sigma)}{\partial U}\right)_N = \left(\frac{\partial (k \ln \Omega)}{\partial U}\right)_N$

that $T = \frac{\mu B}{k} \operatorname{atanh}\left(U/N\mu B\right) \Rightarrow \frac{U}{N\mu B} = \tanh\left(\frac{kT}{\mu B}\right)$

$\Rightarrow U = N\mu B \tanh\left(\frac{kT}{\mu B}\right) \Rightarrow \boxed{M \equiv -U/B = N\mu \tanh\left(\frac{\mu B}{kT}\right)}$



as T increases, the magnetization initially remains fairly constant. Then, the magnetization turns into an exp decay w/ an asymptote @ $M=0$.

HW#2

1e) from a) $\frac{S}{k} = \sigma = N\ln N - \left(\frac{N}{2} - \frac{U}{2\mu B}\right)\ln\left(\frac{N}{2} - \frac{U}{2\mu B}\right) - \left(\frac{N}{2} + \frac{U}{2\mu B}\right)\ln\left(\frac{N}{2} + \frac{U}{2\mu B}\right)$

$$= N\ln N + \frac{U}{2\mu B}\ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right) - \frac{N}{2}\left[\ln(N - U/\mu B) + \ln(N + U/\mu B)\right]$$

w/ $\ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right) = -\ln\left(\frac{1 + U/N\mu B}{1 - U/N\mu B}\right) = -2\,\text{atanh}(U/N\mu B)$

part d) $U = -N\mu B \tanh\left(\frac{\mu B}{kT}\right)$

$$\sigma = N\ln N - \frac{U}{\mu B}\,\text{atanh}(U/N\mu B) - \frac{N}{2}\ln\left[(N - U/\mu B)(N + U/\mu B)\right]$$

w/ $\frac{U}{\mu B}\,\text{atanh}(U/N\mu B) = -N\tanh\left(\frac{\mu B}{kT}\right)\text{atanh}\left(-\tanh\left(\frac{\mu B}{kT}\right)\right) = \underline{+\frac{N\mu B}{kT}\tanh\left(\frac{\mu B}{kT}\right)}$

& $\ln\left[N^2 + NU/\mu B + NU/\mu B - U^2/\mu^2B^2\right] = \ln(N^2 - U^2/\mu^2B^2)$

$$= \ln\left(N^2 - \frac{\left(N\mu B\tanh\left(\frac{\mu B}{kT}\right)\right)^2}{\mu^2B^2}\right) = \ln\left(N^2 - N^2\tanh\left(\overset{a}{\overset{\shortparallel}{\frac{\mu B}{kT}}}\right)^2\right) = \ln\left(N^2\left(1 - \frac{\sinh^2 a}{\cosh^2 a}\right)\right)$$

$$= \ln\left(\frac{N^2}{\cosh^2 a}\left(\cosh^2 a - \sinh^2 a\right)\right) = \ln\left(\frac{N^2}{\cosh^2 a}\right) = \underline{2\ln(N) - 2\ln(\cosh a)}$$

$$\Rightarrow \sigma = N\ln N - \frac{N\mu B}{kT}\tanh\left(\frac{\mu B}{kT}\right) - N\ln N + N\ln\left(2\cosh\left(\frac{\mu B}{kT}\right)\right)$$

$$\Rightarrow \boxed{S = Nk\left[\ln\left(2\cosh\left(\frac{\mu B}{kT}\right)\right) - \frac{\mu B}{kT}\tanh\left(\frac{\mu B}{kT}\right)\right]} \checkmark$$

$$\lim_{T\to\infty} S = Nk\left[\ln\left(2\cosh(0)\right) - \frac{1}{\infty}\tanh(\infty)\right] = Nk\left[\ln(2) - \frac{1}{\infty}\right] = \underline{Nk\ln 2}$$

L'Hopital for $\lim_{T\to 0} S = Nk\left[\frac{\sinh(\mu B/kT)\mu B}{2\cosh(\mu B/kT)\,kT} + \frac{\mu B}{k\cosh\left(\frac{\mu B}{kT}\right)}\right] = Nk\left[\frac{\tanh(\mu B/kT)}{2kT} + 0\right]$

$$= Nk\left[\frac{-\mu B}{2k^2\cosh(\mu B/kT)T^2}\right] = \frac{-N\mu B}{2k\cosh(\infty)T^2} = \underline{0}$$ as cosh goes way faster than T^2

HW #2

1e) $\lim_{T\to\infty} S = Nk\ln(2)$, which makes sense if the number of particles is $\gg$ energy put into the system. aka, the system is a heat bath.

as $N\ln(2)$ was the max value for σ, $kN\ln(2)$ should be the max value for S. If we are heating an ∞ reservoir, we cannot excite all states. Therefore, never reach $N_\uparrow = N/2$. So, adding more energy gets closer to the max S of $k\,\sigma_{max} = kN\ln 2$.

$\lim_{T\to 0} S = 0$, which makes sense because w/ no energy, no magnet can be anti-aligned w/ $\vec{B}$. As no magnet has the energy to be anti-aligned, all magnets are aligned w/ $\vec{B}$. If all aligned, then there is only one possible state. Therefore, $k\ln(\Omega) = k\ln(1) = 0 = S$ as $T\to 0$.

HW#2

2) $TdS = dU - Fd\ell$ & $F = -T\left(\frac{\partial S}{\partial \ell}\right)_U$ • N links a length • $x_R \rightarrow$, $x_L \leftarrow$ • $P_L = P_R = .5$

a) $\ell = a|x_R - x_L|$

$$\Omega = \binom{N}{x_L} = \frac{N!}{x_L!\,(N-x_L)!} = \frac{N!}{x_L!\,x_R!} \qquad x_L + x_R = N$$

$\ell = a|x_R - x_L| = a|N - x_L - x_L| = a|N - 2x_L|$, say $N - 2x_L > 0$

$$\Rightarrow \frac{\ell}{a} - N = -2x_L \Rightarrow x_L = \frac{N}{2} - \frac{\ell}{2a}$$

similarly for N_R, $\ell = a|x_R - N + x_R| \Rightarrow \frac{\ell}{a} = |-N + 2x_R|$, say $-N + 2x_R > 0$

$$\Rightarrow \frac{\ell}{a} + N = 2x_R \Rightarrow x_R = \frac{N}{2} + \frac{\ell}{2a}$$, plug into Ω

$$\Omega = \frac{N!}{\left(\frac{N}{2} - \frac{\ell}{2a}\right)!\left(\frac{N}{2} + \frac{\ell}{2a}\right)!}$$, but said $N - 2x_L > 0$ & $-N + 2x_R > 0$, which is only 1/2 of the solutions, so x2

$$\Rightarrow \boxed{\Omega(\ell, N) = \frac{2N!}{\left(\frac{N}{2} + \frac{\ell}{2a}\right)!\left(\frac{N}{2} - \frac{\ell}{2a}\right)!}}$$

b) $Na \gg \ell$, $S = k \ln(\Omega)$

$$\frac{S}{k} = \ln\left(\frac{2N!}{x_L!\,x_R!}\right) = \ln(2N!) - \ln(x_L!) - \ln(x_R!)$$

as $Na \gg \ell$, $\Rightarrow \frac{S}{k} \approx 2N\ln(2N) - 2N - x_L\ln(x_L) + x_L - x_R\ln(x_R) + x_R$
Sterling

$$\Rightarrow \frac{S}{k} = 2N\ln(2N) - N - x_L\ln(x_L) - x_R\ln(x_R)$$

$$= 2N\ln(2N) - N - \left(\frac{N}{2} - \frac{\ell}{2a}\right)\ln\left(\frac{N}{2} - \frac{\ell}{2a}\right) - \left(\frac{N}{2} + \frac{\ell}{2a}\right)\ln\left(\frac{N}{2} + \frac{\ell}{2a}\right)$$

$$= 2N\ln(2N) - N - \frac{N}{2}\ln\left(\frac{N - \ell/a}{N + \ell/a}\right) + \frac{\ell}{2a}\ln\left(\frac{N}{2} - \frac{\ell}{2a}\right) - \frac{\ell}{2a}\ln\left(\frac{N}{2} + \frac{\ell}{2a}\right)$$

HW#2

2b) $\frac{S}{k} = 2N\ln(2N) - N - \frac{N}{2}\ln\left(\frac{N-l/a}{N+l/a}\right) + \frac{l}{2a}\left(\ln\left(\frac{N}{2}-\frac{l}{2a}\right) - \ln\left(\frac{N}{2}-\frac{l}{2a}\right)\right)$

$$= 2N\ln(2N) - N - \frac{N}{2}\ln\left(\frac{N-l/a}{N+l/a}\right) + \frac{l}{2a}\ln\left(\frac{N-l/a}{N+l/a}\right)$$

$$= 2N\ln(2N) - N - \frac{N}{2}\ln\left(\frac{1-l/Na}{1+l/Na}\right) + \frac{l}{2a}\ln\left(\frac{1-l/Na}{1+l/Na}\right)$$

$$= 2N\ln(2N) - N + N\,\text{atanh}(l/Na) - \frac{l}{2a}\,\text{atanh}(l/Na)$$

since $l \ll Na$, $\Rightarrow \text{atanh}(l/Na) \approx l/Na + \text{H.O.T.}$, $\text{HOT} \approx 0$

Taylor Expand

$$\Rightarrow 2N\ln(2N) - N + \frac{Nl}{Na} - \frac{l^2}{2Na^2} = 2N\ln(2N) - N + \frac{l}{a} - \frac{l^2}{2Na^2}$$

$l/a \approx 0$ as all other terms are $\gg$

$$\boxed{\Rightarrow S(l) = 2kN\ln(2N) - Nk - \frac{kl^2}{2Na^2} \quad w/ \quad S(0) = 2k\ln(2N) - Nk}$$

c) $F = -T\left(\frac{\partial S}{\partial l}\right)_u \Rightarrow \frac{\partial S}{\partial l} = \frac{\partial}{\partial l}\left(S(0) - \frac{kl^2}{2Na^2}\right) = 0 - \frac{2kl}{2Na^2} = -\frac{kl}{Na^2}$

$$\boxed{\Rightarrow F = \frac{Tkl}{Na^2} = \frac{Tkl}{Na^2}}$$