HW #4

1) $d\vec{B} = \frac{\mu_0 I \, d\vec{l} \times \hat{r}}{4\pi r^2}$

a) find $\vec{B}$ @ center of loop, Biot-Savart @ $r=R$, find $\vec{B}$

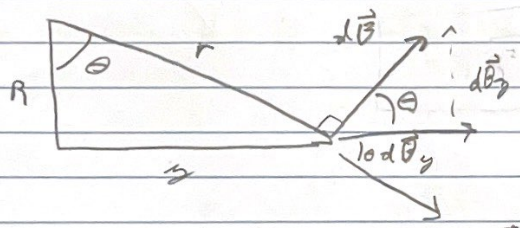
$d\vec{B} = \frac{\mu_0 I \, \vec{l} \times \hat{r}}{4\pi r^2}$ $\quad \vec{r} \to -\hat{r}$ $\quad$ & $\quad d\vec{l} = R\,d\phi\,\hat{\phi}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi R^2} \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & R\,d\phi & 0 \\ -1 & 0 & 0 \end{vmatrix} = \frac{\mu_0 I}{4\pi R^2}\left(R\,d\phi\right)\hat{z} = \frac{\mu_0 I}{4\pi R}\,d\phi\,\hat{z}$$

on axis

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0 I}{4\pi R}\,d\phi\,\hat{z} = \frac{\mu_0 I}{4\pi R}\,\hat{z}\int_0^{2\pi} d\phi = \frac{\mu_0 I}{4\pi R}(2\pi)\,\hat{z} = \boxed{\frac{\mu_0 I}{2R}\,\hat{z}}$$

b)



$\boxed{r = \sqrt{R^2 + z^2}}$

c) $\sin\theta = \frac{z}{r}$ $\qquad \sin\theta = \frac{d\vec{B}_z}{d\vec{B}} \Rightarrow \boxed{d\vec{B}\sin\theta = d\vec{B}_z}$

$\cos\theta = \frac{R}{r}$ $\qquad \cos\theta = \frac{d\vec{B}_y}{d\vec{B}} \Rightarrow \boxed{d\vec{B}\cos\theta = d\vec{B}_y}$

d) $\sin\theta = -\frac{d\vec{B}_z}{d\vec{B}'} \Rightarrow \boxed{-d\vec{B}'\sin\theta = d\vec{B}_z = -d\vec{B}'_z}$

$\cos\theta = \frac{d\vec{B}_y}{d\vec{B}'} \Rightarrow \boxed{d\vec{B}'\cos\theta = d\vec{B}_y = d\vec{B}'_y}$

The $\hat{z}$ components of $d\vec{B}$ & $d\vec{B}'$ will cancel out

$\therefore$ particle will only feel force in $\hat{z}$
(on-axis)

HW #4 - cont

e) $\cos\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}}$

f) $d\vec{B} = \frac{\mu_0 I}{4\pi r^2} d\vec{l} \times \hat{r} = \frac{\mu_0 I}{4\pi r^2} \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & R\,d\phi & 0 \\ -\cos\theta & 0 & 0 \end{vmatrix} = \frac{-\mu_0 I \, d\phi R \cos\theta \, \hat{z}}{4\pi r^2}$

$$\Rightarrow \frac{-\mu_0 I R}{4\pi r^2} \cdot \frac{R\hat{z}}{\sqrt{R^2+z^2}} \int_0^{2\pi} d\phi = -\frac{\mu_0 I R^2}{4\pi} \frac{1}{(R^2+z^2)^{3/2}} \cdot 2\pi \hat{z}$$

$$\Rightarrow \vec{B} = \frac{-\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \hat{z}$$

HW #4

2) uniform B : $\vec{B} = B_0 \hat{z}$ (I) & mirror B : $\vec{B} = B_0\left[1+\left(\frac{z}{a_0}\right)^2\right]\hat{z}$ (II)

a) mass m, charge q, find $v_{\parallel}(z)$ for I & II. assume particle mirrors @ $z = \pm z_t$

II) mirror ratio

$$B_{max} = B(z_t) = B_0\left[1+\left(z_t/a_0\right)^2\right]$$

$$B_{min} = B(0) = B_0\left[1+0\right] = B_0$$

$$\Rightarrow R_m = \frac{B_0\left[1+\left(z_t/a_0\right)^2\right]}{B_0} = 1+\left(z_t/a_0\right)^2$$

I) $m\dot{v} = q\vec{v}\times\vec{B} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix} = qB_0\left(v_y\hat{x} - v_x\hat{y}\right)$

$m\dot{v}_z = 0 \Rightarrow \boxed{v_z = v_{z0} = v_{\parallel,0}}$

II) $\frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2 = C_0 \Rightarrow v_\perp^2 + v_\parallel^2 = C$

$\Rightarrow v_\parallel = \sqrt{C - v_\perp^2}$, $\underset{\text{reflects}}{v_\parallel(\pm z_t) = 0} = \sqrt{C - v_\perp^2} \Rightarrow C = v_\perp^2(\pm z_t)$

$$v_\parallel = \sqrt{v_\perp^2(\pm z_t) - v_\perp^2(z)}$$

$\mu = \frac{1}{2}m\frac{v_\perp^2}{B} \Rightarrow v_\perp^2 = \frac{2\mu B}{m}$ $\quad v_\perp^2(z) = \frac{2\mu}{m}B(z)$

$\Rightarrow v_\perp^2(z) = \frac{2\mu}{m}\left[B_0 + B_0\left(\frac{z}{a_0}\right)^2\right]$, $v_\perp^2(\pm z_t) = \frac{2\mu}{m}\left[B_0 + B_0\left(\frac{\pm z_t}{a_0}\right)^2\right]$

$$\Rightarrow v_\parallel = \sqrt{\frac{2\mu B_0}{m}\left[1+\left(\frac{\pm z_t}{a_0}\right)^2\right] - \frac{2\mu B_0}{m}\left[1+\left(\frac{z}{a_0}\right)^2\right]}$$

HW#4

$$\Rightarrow \boxed{v_{\parallel} = \sqrt{\frac{2\mu B_0}{m a_0^2}\left(z_t - z\right)}}$$

b) Show $F_\parallel \propto \frac{\partial B}{\partial z}\hat{z}$

$F_\parallel = m\frac{dv_\parallel}{dt}$, $v_\parallel = \frac{dz}{dt}$ $\Rightarrow F_\parallel = m\frac{dv_\parallel}{dt} = m\frac{dv_\parallel}{dt}\left[v_\parallel \cdot \frac{dt}{dz}\right]$

$$F_\parallel = m v_\parallel \frac{dv_\parallel}{dz} = \frac{m}{2}\frac{d(v_\parallel^2)}{dz} = \frac{m}{2}\frac{d}{dz}\left[\frac{2\mu B_0}{m a_0^2}\left(z_t^2 - z^2\right)\right]$$

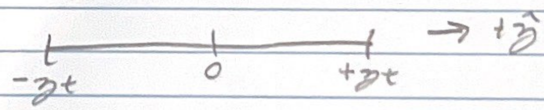
$$\Rightarrow F_\parallel = \frac{m}{2}\left(\frac{2\mu B_0}{m a_0^2}(-2z)\right) = \boxed{\frac{-2\mu B_0}{a_0^2} z = F_\parallel}$$

$\boxed{\frac{\partial B}{\partial z} = \frac{B_0 2}{a_0^2} z}$ $\boxed{\frac{F_\parallel(z)}{\frac{\partial B}{\partial z}} = \mu \quad \therefore F_\parallel(z) \propto \frac{\partial B}{\partial z}}$

c) I) $F_\parallel(z) = 0$ as $v_z \parallel B_z$

The force does nothing to the particle

II) $F_\parallel(z) = \frac{-2\mu B_0}{a_0^2} z$



as $z \to +z_t$, the force gets stronger in $-\hat{z}$ forcing particle back to center

as $z \to -z_t$, the force gets stronger in $+\hat{z}$ forcing particle back to center

HW#4

2d) $F_{\parallel} = m\frac{v_{\parallel}}{dt} = \frac{-2\mu B_0 z}{a_0^2} = m\ddot{z}$

$$\boxed{\Rightarrow \frac{d^2 z}{dt^2} + \underbrace{\left(\frac{2\mu B_0}{m a_0^2}\right)}_{\omega^2} z = 0 \quad w/ \quad \omega^2 = \frac{2\mu B_0}{m a_0^2}}$$

Period $\frac{2\pi}{T} = \omega = \sqrt{\frac{2\mu B_0}{m a_0^2}} \Rightarrow \boxed{\therefore T = 2\pi\sqrt{\frac{m a_0^2}{2\mu B_0}}}$

HW#4

3) find trajectory w/ $B=(0,0,1)$ & $E=(E_0 \sin(\omega t),0,0)$

$\omega_c = \frac{qB_0}{m} = 1$ in code

If $\omega \ll \omega_c$, the perturbations happen so quickly that it looks like the trajectory is constant like for constant E field.

If $\omega \sim \omega_c$, the local perturbations are much more noticeable & the x & y positions move about a guiding center w/ sinusoidal perturbations about the trajectory of constant E

For $\omega \ll \omega_c$ & $\omega \sim \omega_c$, the mean behavior is roughly the same. Although the trajectory for y is the same in all cases, the sinusoidally oscillating $\bar{E}$ fields have a lower final x position than a constant E field. This makes sense as the varying E will exert less force when averaged over a period of oscillation than a constant field over the same period.