

HW #7 - Joseph Specht

D) $m\ddot{x} + \alpha x = F_0 \exp(-\lambda t)$

Q) $w_n = \sqrt{k/m}$

E) $m\ddot{x} + \alpha x = F_0 \exp(-\lambda t) \Rightarrow m\lambda^2 D \exp(-\lambda t) - \alpha D \exp(-\lambda t) = F_0 \exp(-\lambda t)$

if $x(t) = D \exp(-\lambda t)$, canceling out $\exp(-\lambda t)$ gives

$m\lambda^2 D + \alpha D = F_0 \Rightarrow D(m\lambda^2 + \alpha) = F_0$

$$D = \frac{F_0}{m\lambda^2 + \alpha}$$

$$x_p = \frac{F_0 \exp(-\lambda t)}{m\lambda^2 + \alpha}$$

D) $m\ddot{x} + \alpha x = F_0 \exp(-\lambda t)$

$x_A = A \cos(w_n t) + B \sin(w_n t)$

$\dot{x}_A = -Aw_n \sin(w_n t) + Bw_n \cos(w_n t)$

$\ddot{x}_A = -Aw_n^2 \cos(w_n t) - Bw_n^2 \sin(w_n t)$

$x_p = D \exp(-\lambda t)$

$\dot{x}_p = -D\lambda \exp(-\lambda t)$

$\ddot{x}_p = D\lambda^2 \exp(-\lambda t)$

$x = A \cos(w_n t) + B \sin(w_n t) + D \exp(-\lambda t)$

$\dot{x} = -Aw_n \sin(w_n t) + Bw_n \cos(w_n t) + D\lambda \exp(-\lambda t)$

Plugging this in to original expression

$m(-Aw_n^2 \cos(w_n t) - Bw_n^2 \sin(w_n t) + D\lambda^2 \exp(-\lambda t)) +$

$\alpha(A \cos(w_n t) + B \sin(w_n t) + D \exp(-\lambda t)) = F_0 \exp(-\lambda t)$

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This can be written as

$$m(-A\omega_n^2 \cos(\omega_n t) - B\omega_n^2 \sin(\omega_n t)) + B(A \cos(\omega_n t) + B \sin(\omega_n t)) + m(Dx^2 \exp(-\omega_n t)) + B(D \exp(-\omega_n t)) = F_0 \exp(-\omega_n t)$$

This is the same as $L(x_p + x_h) = F_0 \exp(-\omega_n t)$ & this can be written as $L(x_p + x_h) = Q + F_0 \exp(-\omega_n t)$

$$\text{where } L(x) = m\ddot{x} + Bx$$

this can be seen as the superposition of solutions made & this sum is equal to the original force

$$d) x_0 = x(t=0) = 0 \quad \dot{x}_0 = \dot{x}(t=0) = 0$$

using homogeneous case, we get

$$x(t) = A \cos(\omega_n t) + B \sin(\omega_n t) + D \exp(-\omega_n t)$$

$$x(0) = A \cos(0) + B \sin(0) + D \exp(0)$$

$$x(0) = A + 0 = 0 \Rightarrow A = 0$$

$$\dot{x}(t) = -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t) + D\omega_n^2 \exp(-\omega_n t)$$

$$\dot{x}(0) = -A\omega_n \sin(0) + B\omega_n \cos(0) - D\omega_n^2 \exp(0)$$

$$\dot{x}(0) = B\omega_n + D\omega_n^2 = 0 \Rightarrow B = -D/\omega_n$$

This gives

$$x = x_p + x_h = -D \cos(\omega_n t) + \frac{D}{\omega_n} \sin(\omega_n t) + D \exp(-\omega_n t)$$

$$x = D \left(-\cos(\omega_n t) + \frac{1}{\omega_n} \sin(\omega_n t) + \exp(-\omega_n t) \right)$$

$$\text{w/ } D = \frac{F_0}{m\omega_n^2 + B}$$

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$$x = \frac{F_0}{m\gamma^2 - k} \left[-\cos\left(t\sqrt{\frac{k}{m}}\right) + \lambda\sqrt{\frac{m}{k}} \sin\left(t\sqrt{\frac{k}{m}}\right) + \exp(-\lambda t) \right]$$

e) $\lim_{t \rightarrow \infty} (x(t)) = D \left[-\cos(\infty) + \frac{\lambda}{\omega_n} \sin(\infty) + \exp(\infty) \right]$

$\lim_{t \rightarrow \infty} (\sin(\infty) \text{ or } \cos(\infty))$ is meaningless as they oscillate periodically,
so the just follow cos & sin rules

$\lim_{t \rightarrow \infty} (\exp(\infty)) = 0$, so adding these limits together we get

as time $\gg \frac{1}{\lambda}$, $x(t) = \frac{F_0}{m\gamma^2 - k} \left[-\cos\left(t\sqrt{\frac{k}{m}}\right) + \lambda\sqrt{\frac{m}{k}} \sin\left(t\sqrt{\frac{k}{m}}\right) \right]$

This simply acts as a steady state oscillator,
so this makes sense as there is no drag
so all the energy the force put it has to
stay there & cannot dissipate, so it is left
as a steady state oscillator

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2) $F = F_0 \sin(\omega t)$

$c = 6 \text{ kg/s}$; $\delta = 160 \text{ N/m}$; $m = 10 \text{ kg}$; $F_0 = 2 \text{ N}$; $\omega = 4 \text{ Hz}$

a) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160 \text{ N/m}}{10 \text{ kg}}} = 4 \text{ rad/s}$

$$\frac{\omega}{\omega_n} = \frac{4}{4} = 1$$

$$\boxed{\frac{\omega}{\omega_n} = 1}$$

b) Driving force $F = F_0 \sin(\omega t)$ is equal to $F_0 \cos(\omega t - \frac{\pi}{2})$

now we can use $x_p(t) = F_0 \delta(\omega) \cos(\omega t - \frac{\pi}{2} - \Phi(\omega))$

Solving for δ

$$\delta = \frac{1}{\sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \quad \text{but } \frac{\omega}{\omega_n} = 1, \text{ so}$$

$$\delta = \frac{1}{\sqrt{1 - 1^2 + (2\zeta)^2}} = \frac{1}{\sqrt{1 - 1 + 4\zeta^2}} = \frac{1}{\sqrt{4\zeta^2}} = \frac{1}{2\zeta}, \text{ but } \zeta = \frac{c}{2m\omega_n}$$

$$\delta = \frac{1}{2\zeta} = \frac{1}{2 \cdot \frac{6}{2m\omega_n}} = \frac{1}{2 \cdot \frac{6}{2 \cdot 10 \cdot 4}} = \frac{1}{24} = \boxed{\frac{1}{24} = \delta}$$

Solving for Φ

$$\Phi = \arctan \left(\frac{2\zeta \omega \omega_n}{\omega_n^2 - \omega^2} \right) \text{ but } \omega_n^2 - \omega^2 = 0, \text{ so } \Phi = \arctan(\infty)$$

so

$$\boxed{\Phi = \frac{\pi}{2}}$$

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$$x_p = \frac{F_0}{2\zeta} \cos(\omega t - \pi)$$

Q) $\zeta = \frac{3}{40} < 1$, \therefore system is underdamped

$$\zeta = \frac{6}{2 \cdot 10 \cdot 4} = \frac{3}{40}$$

d) $x_r(t) = \exp(-\omega_n \zeta t) [A \cos(\omega_n t) + B \sin(\omega_n t)]$

$$\omega_n = \sqrt{\frac{k}{m}} = 4 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{6}{2 \cdot 10 \cdot 4} = \frac{3}{40}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4 \sqrt{1 - \frac{9}{1600}} = 4 \sqrt{\frac{1591}{1600}} = \frac{4}{4} \sqrt{\frac{1591}{160}} = \sqrt{\frac{1591}{160}}$$

$$x_r = \exp\left(-\frac{3}{10}t\right) \left[A \cos\left(\sqrt{\frac{1591}{160}}t\right) + B \sin\left(\sqrt{\frac{1591}{160}}t\right) \right]$$

$$\therefore x = \exp\left(-\frac{3}{10}t\right) \left[A \cos\left(\sqrt{\frac{1591}{160}}t\right) + B \sin\left(\sqrt{\frac{1591}{160}}t\right) \right] + \frac{8}{24} \cos(4t - \pi)$$

Finding $x(0)$...

$$x(0) = \exp(0) [A \cos(0) + B \sin(0)] + \frac{8}{24} \cos(-\pi)$$

$$2 = A - \frac{1}{3} \Rightarrow A = \frac{7}{3} 2$$

\ddot{x} from a derivative calculator is

$$\ddot{x} = \frac{\exp\left(-\frac{3}{10}t\right)}{3 \cdot 10^{3/2}} \left[(9\sqrt{10}B + 30\sqrt{1591}A) \sin\left(\sqrt{\frac{1591}{160}}t\right) + (9\sqrt{10}A - 30\sqrt{1591}B) \cos\left(\sqrt{\frac{1591}{160}}t\right) \right]$$

$$\cos\left(\sqrt{\frac{1591}{160}}t\right) - 4 \cdot 10^{3/2} \exp\left(\frac{3}{10}t\right) \sin\left(4t\right)$$

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$$\ddot{x}(0) = 0 = \frac{\exp(0)}{3 \cdot 10^{3/2}} \left[(...) \sin(0) + (9\sqrt{10}A - 30\sqrt{159}B) \cos(0) - (...) \sin(0) \right]$$

$$\Rightarrow 0 = \frac{1}{3 \cdot 10^{3/2}} \left[9\sqrt{10}A - 30\sqrt{159}B \right] \Rightarrow 9\sqrt{10}A - 30\sqrt{159}B$$

$$\therefore 9\sqrt{10}A = 30\sqrt{159}B \Rightarrow \theta = \frac{9\sqrt{10}A}{30\sqrt{159}B} = \frac{9\sqrt{10} \cdot 7}{30\sqrt{159} \cdot 3} = \frac{21\sqrt{10}}{30\sqrt{159}}$$

$$= \frac{7\sqrt{10}}{10\sqrt{159}} = \boxed{\frac{7}{\sqrt{1590}}} = \theta$$

Plugging in we get

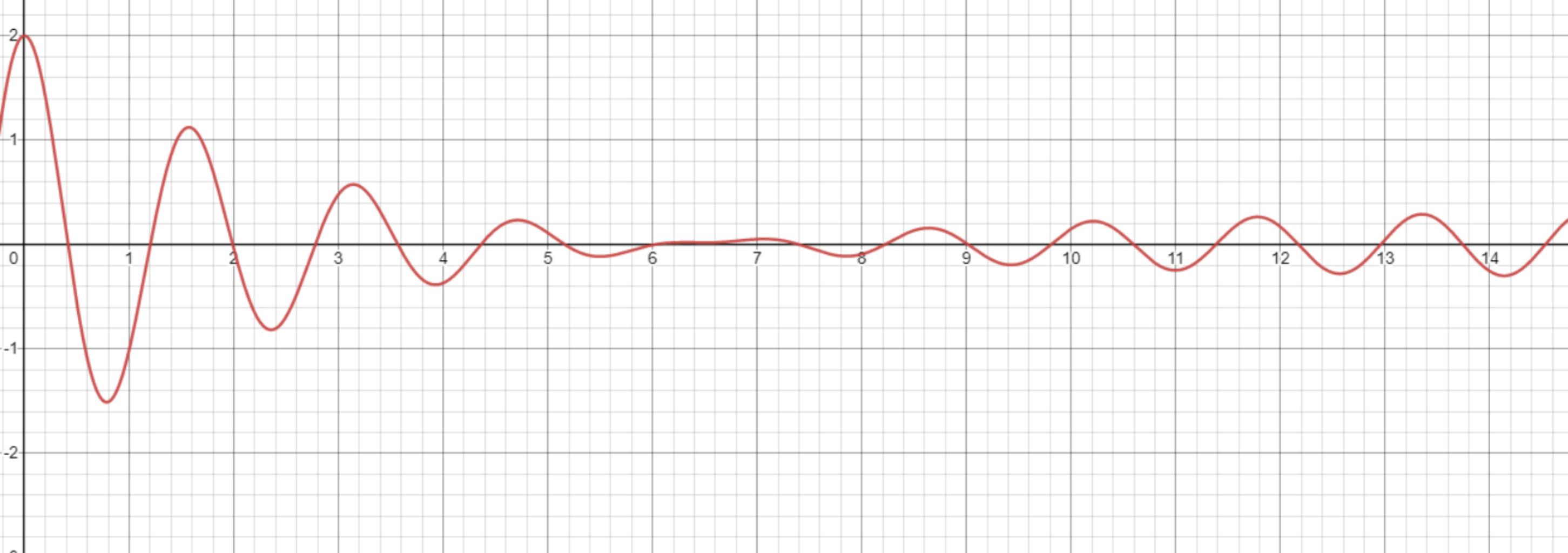
$$x = \exp\left(-\frac{3}{10}t\right) \left[\frac{7}{3} \cos\left(4t\right) + \frac{7}{\sqrt{159}} \sin\left(4t\right) \right] +$$

$$\frac{1}{3} \cos\left(4t - \pi\right)$$

a) condition satisfied ✓, $\theta + 0$, $x = 0$ & max, so $n=0$

b) the $\frac{1}{e}$ period for ω_0 is $t = \frac{10}{3} s = 3.3 s$

The homogeneous solution decays in about 11 seconds, so the decay time & these are the same order of magnitude



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3) $F(t) = F_0 \exp(-\alpha t) \cos(\omega t + \theta)$

Since we want to write F in terms of only a real part of a function

$$F(t) = \operatorname{Re} [F_0 \exp(-\alpha t) \exp(i(\omega t + \theta))] = \operatorname{Re} [F_0 \exp((-\alpha + i\omega)t + i\theta)]$$

but $-\alpha + i\omega = \tilde{\alpha}$, so

$$F = \operatorname{Re} [F_0 \exp(\tilde{\alpha}t + i\theta)]$$

using the given x guess, we get

$$x_p = \operatorname{Re} [A \exp(\tilde{\alpha}t + i\theta)]$$

$$\dot{x}_p = A \tilde{\alpha} \exp(\tilde{\alpha}t + i\theta)$$

$$\ddot{x}_p = A \tilde{\alpha}^2 \exp(\tilde{\alpha}t + i\theta)$$

Plugging these in we get

$$F_0 \exp(\tilde{\alpha}t + i\theta) = A \exp(\tilde{\alpha}t + i\theta) [\tilde{\alpha}^2 m + c \tilde{\alpha} + b]$$

$$F_0 = A [\tilde{\alpha}^2 m + c \tilde{\alpha} + b] \Rightarrow A = F_0 / (\tilde{\alpha}^2 m + c \tilde{\alpha} + b)$$

$$x_p = \operatorname{Re} \left[\frac{F_0}{\tilde{\alpha}^2 m + c \tilde{\alpha} + b} \exp(\tilde{\alpha}t + i\theta) \right]$$

Expanding $\tilde{\alpha}t$ to find real part

$$x_p = \operatorname{Re} \left[F_0 \frac{\exp((1-\alpha+i\omega)t + i\theta)}{(-\alpha + i\omega)^2 m + c(-\alpha + i\omega) + b} \right] = \operatorname{Re} \left[F_0 \frac{\exp(-\alpha t + i\omega t + i\theta)}{(\alpha^2 - 2\alpha i\omega - \omega^2)m + c(-\alpha + i\omega) + b} \right]$$

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$$\triangleright x_p = \operatorname{Re} \left[F_0 \exp(-\alpha t + i\omega t + i\theta) \right]$$

$$\frac{\alpha^2 m - 2\alpha i\omega m - \omega^2 m - c\alpha + ci\omega + b}{\alpha^2 - \omega^2 m - c\alpha + b - 2\alpha i\omega m + ci\omega}$$

Rearrange

$$\triangleright x_p = \operatorname{Re} \left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{\alpha^2 - \omega^2 m - c\alpha + b - 2\alpha i\omega m + ci\omega} \right] \text{ mult by } \bar{z} \text{ ob denominator}$$

$$\triangleright x_p = \operatorname{Re} \left[\frac{F_0 \exp(-\alpha t + i\omega t + i\theta)}{\alpha^2 - \omega^2 m - c\alpha + b - 2\alpha i\omega m + ci\omega} \times \frac{\alpha^2 - \omega^2 m - c\alpha + b + 2\alpha i\omega m - ci\omega}{\alpha^2 - \omega^2 m - c\alpha + b + 2\alpha i\omega m - ci\omega} \right]$$

$$\triangleright \theta = \alpha^2 - \omega^2 m - c\alpha + b$$

$$F = -\alpha t + i\omega t + i\theta$$

$$\triangleright H = -2\alpha i\omega m + ci\omega$$

$$\triangleright \therefore x_p = \operatorname{Re} \left[\frac{F_0 \exp(F)}{\theta + Hi} \times \frac{\theta - Hi}{\theta - Hi} \right] = \frac{F_0 \exp(F)(\theta - Hi)}{\theta^2 + H^2}$$

Expanding $\exp(F)$...

$$\exp(F) = \exp(-\alpha t + i\omega t + i\theta) = \exp(-\alpha t + (\omega t + \theta)i)$$

$$\Rightarrow \exp(-\alpha t) \exp((\omega t + \theta)i) = \exp(-\alpha t) (\cos(\omega t + \theta) + i \sin(\omega t + \theta))$$

Plugging that in we get

$$\triangleright x_p = \operatorname{Re} \left[\frac{F_0 \exp(-\alpha t) (\cos(\omega t + \theta) + i \sin(\omega t + \theta)) (\theta - Hi)}{\theta^2 + H^2} \right]$$

Taking the real

$$\triangleright x_p = F_0 \exp(-\alpha t) \operatorname{Re} \left[(\cos(\omega t + \theta) + i \sin(\omega t + \theta)) / (\theta - Hi) \right]$$

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$$x_p = \frac{F_0 \exp(-\alpha t)}{\omega^2 + \eta^2} \operatorname{Re} \left[\cos(\omega t + \theta) g - H_i \cos(\omega t + \theta) + S_i \sin(\omega t + \theta) + H_s \sin(\omega t + \theta) \right]$$

now we can take the real part of this

$$x_p = \frac{F_0 \exp(-\alpha t)}{\omega^2 + \eta^2} (\cos(\omega t + \theta) g + H_s \sin(\omega t + \theta))$$

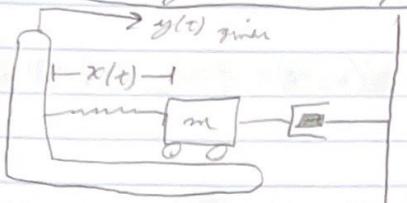
Plugging everything in we get

$$x_p = F_0 \exp(-\alpha t) \left[\begin{aligned} & (\alpha^2 - \omega^2 m - c\alpha + k) \cos(\omega t + \theta) \\ & (-2\alpha \omega m + cw) \sin(\omega t + \theta) \end{aligned} \right]$$

$$(\alpha^2 - \omega^2 m - c\alpha + k)^2 + (-2\alpha \omega m + cw)^2$$

HW#7-joseph Specht

A)



$$F_d = -cv$$

$$\ddot{x} = A_0 \cos(\omega t)$$

a) $F = ma = -cv - bx$

$$m\frac{d^2}{dt^2}(x + y) = -c\frac{dy}{dt}(x + y) - bx$$

We use $(x+y)$ for a & v because the relative position depends on both.

$$m\ddot{x} + my\ddot{y} = -c\dot{x} - cy - bx \Rightarrow m\ddot{x} + c\dot{x} + bx = -my\ddot{y} - cy$$

$$m\ddot{x} + c\dot{x} + bx = -my\ddot{y} - cy$$

This follows the form of the eq, so...

$$M_{eff} = m$$

$$C_{eff} = c$$

$$K_{eff} = b$$

$$D = -m$$

$$E = -c$$

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- $y(t) = 0$, if this happens, eq of motion is

$$m\ddot{x} + c\dot{x} + kx = 0$$

This makes sense as this is the general eq of damped harmonic oscillators

- positive C_{eff} & K_{eff} . These are both positive as $C_{eff} = C > 0$ & $K_{eff} = k > 0$

* Dimensional analysis

$$\left[\frac{kg \cdot m}{s^2} \right] + \left[\frac{kg \cdot m}{s \cdot s} \right] + \left[\frac{N}{m} = \frac{kg \cdot m \cdot m}{s^2 \cdot m} \right] = \left[\frac{kg \cdot m}{s} \right] - \left[\frac{kg \cdot m}{s \cdot s} \right]$$

all of these units are equal to N as they should be

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c) obtain $X_p(t)$ for $y(t) = A_0 \cos(\omega t)$
 find $\mathcal{E}(w)$ & $\Phi(w)$

$$\mathcal{E}(w) = \frac{1}{\sqrt{\left(1 - \left(\frac{w}{\omega_n}\right)^2\right)^2 + \left(\frac{25}{\omega_n} w\right)^2}}$$

$$\Phi(w) = \tan \left[\frac{2 \zeta \omega_n w}{(\omega_n^2 - w^2)} \right]$$

Now we need to find ω_n, ω, ζ To get these solutions

$$\omega_n = \sqrt{\frac{k}{m}} \quad \omega = \omega \quad \zeta = \frac{c}{2m\omega_n}$$

because this spans the (wt) trig space
 & we need this bc the force is in the form of $(\cos(\omega t) + \sin(\omega t))$

We know \mathcal{E}, Φ , & w , so

$$\mathcal{E}(w) = \frac{1}{\sqrt{\left[\left(1 - \frac{w^2}{\omega_n^2}\right)^2 + \frac{4\zeta^2 w^2}{\omega_n^2} \right]^{1/2}}} \Rightarrow 4\zeta^2 w^2 = \frac{4c^2 w^2}{\omega_n^2} = \frac{c^2 w^2}{k^2}$$

$$\therefore \mathcal{E}(w) = \frac{1}{\sqrt{k}} \left[\left(1 - \left(\frac{w}{\omega_n}\right)^2\right)^2 + \left(\frac{cw}{k}\right)^2 \right]^{-1/2}$$

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Simplifying $\varphi(w)$ gives...

$$\begin{aligned}\varphi(w) &= \text{atan} \left[\frac{2\zeta w w_n}{(w_n^2 - w^2)} \right] = \text{atan} \left[\frac{c w w_n}{m w_n (w_n^2 - w^2)} \right] = \text{atan} \left[\frac{c w}{m (w_n^2 - w^2)} \right] \\ &= \text{atan} \left[\frac{c w}{m \left(\frac{\zeta^2}{m} - w^2 \right)} \right] = \text{atan} \left[\frac{c w}{\zeta^2 - m w^2} \right]\end{aligned}$$

a guess for x_p , so we can assume a superposition of forces $-m\ddot{x} - c\dot{x}$, so we get

$$\begin{aligned}F_1 &= -c(-A_0 w \sin(wt)) = A_0 w c \sin(wt) \\ F_2 &= -m(-A_0 w^2 \cos(wt)) = A_0 w^2 c \cos(wt)\end{aligned}$$

now we can say the guesses are

$$\begin{aligned}x_{p1} &= g F_1 = g A_0 w c \sin(wt + \varphi) \\ x_{p2} &= g F_2 = g A_0 w^2 c \cos(wt + \varphi)\end{aligned}$$

knowing $x_p = x_{p1} + x_{p2}$, we can write the final solution...

$$x_p = g A_0 w \left[w_m \cos(wt + \varphi) + c \sin(wt + \varphi) \right]$$

$$\text{where } g = \frac{1}{\zeta} \left[\left(1 - \left(\frac{w}{w_n} \right)^2 \right) + \left(\frac{c w}{\zeta} \right)^2 \right]^{-\frac{1}{2}}$$

$$\& \quad \varphi = \text{atan} \left[\frac{c w}{\zeta^2 - m w^2} \right]$$