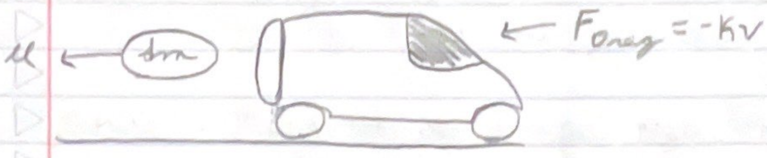
HW#3 - Joseph Specht

1)

$K > 0$, constant
m_i = initial mass
m_f = final mass (no fuel)
u = ejection speed



a) $F_{net} = F_{drag}$

- Deriving & applying the impulse-momentum Theorem to get a useful expression for F_{net}.

$F = \frac{dp}{dt} \Rightarrow F\,dt = dp$

dp represents the instantaneous change in momentum the momentum at some time (dt) after t.

$dp = p_f - p_i = p(t+dt) - p(t)$

The momentum @ t is simply $m_i v_0$ because it is the initial mass of the system times the initial v.

$p(t) = m_i v_0$

The momentum @ $(t+dt)$ is the momentum of the car plus the momentum of the ejected dm.

$$p(t+dt) = \underbrace{(m_i - dm)}_{a}\underbrace{(v_0 + dv)}_{b} + dm\underbrace{(v_0 - u)}_{c}$$

a) The mass of the car after dm of fuel is ejected
b) The velocity of the car after some change of velocity
c) Relative velocity of the mass unit dm.

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Since dP is the change in momentum, $dp = p(t+dt) - p(t)$

$\therefore dp = m_i v_0 + m_i\, dv - dm\, v_0 - dm\, dv + dm\, v_0 - u\, dm - m_i v_0$

Since $dm\, dv$ is two small things multiplied $\therefore dm\,dv \approx 0$

$\therefore dp = m_i\, dv + v_0\, dm$ & w/ $F = \frac{dp}{dt}$, $F = m_i \dot{v} + v_0 \dot{m}$

w/ this expression for force, we set it equal to F_{drag}

$-kv = m_i \dot{v} + v_0 \dot{m}$ or $m_i \dot{v} = u\dot{m} - kv$

This gives the final expression or multiplying through by dt changing m_i to M to make the expression valid for $0 < t < t_f$

a) $\boxed{M\dot{v} = u\dot{m} - kv \quad \text{or} \quad M\,dv = u\,dm - kv\,dt}$

b) We are given $\dot{M} = -\alpha$, which is the rate the car is losing mass. This is also equal and opposite to the rate the exhaust is gaining mass ($\dot{m}$)

$$\therefore -\dot{M} = \dot{m} = \alpha$$

Substituting this into the previous expression gives...

$$M\dot{v} = u\alpha - kv$$

And multiplying by dt on each side gives

$$M\,dv = (u\alpha - kv)\,dt$$

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We know M changes w/ time at a rate $-\alpha \left[\frac{kg}{s}\right]$

$$\therefore M = m_i - \alpha t$$

Plugging this into the previous expression & seperating variables...

$$\frac{dv}{u\alpha - kv} = \frac{dt}{m_i - \alpha t}$$

With bounds of v from v_0 to v_f & t from t_0 to t_f, integrating, & combining the terms in from of each expression

$$\ln\left(\frac{u\alpha - kv_f}{u\alpha - kv_0}\right) = \frac{K}{\alpha} \ln\left(\frac{m_i - \alpha t_f}{m_i - \alpha t_0}\right)$$

We know the final time is when all fuel is used & this usage happens @ a constant rate $\alpha \left[{kg}/{s}\right]$
This logic gives rise to the expression for t_f of...

$$t_f = \frac{m_i - m_f}{\alpha}$$

Plugging in all the conditions we know, $v_0 = 0$, $t_0 = 0$, t_f

$$\ln\left(\frac{u\alpha - kv_f}{u\alpha}\right) = \frac{K}{\alpha} \ln\left(\frac{m_f}{m_i}\right)$$

Then using the rule $A \ln(B) = \ln(B^A)$

$$\ln\left(\frac{u\alpha - kv_f}{u\alpha}\right) = \ln\left(\frac{m_f}{m_i}\right)^{K/\alpha}$$

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Cancelling the ln gives us the expression...

$$\frac{u\alpha - kv_f}{u\alpha} = \left(\frac{m_f}{m_i}\right)^{K/\alpha}$$

Solving for v_f gives us the following expression

(b) $$\boxed{v_f = \frac{u\alpha}{K}\left(1 - \left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)}\right)}$$

c) Analyzing the units, if $F = -kv$, $\left[\frac{kg\, m}{s^2}\right] = [k]\left[\frac{m}{s}\right] \therefore [k] = \left[\frac{kg}{s}\right]$
This is good because this means the exponential K/α is unitless. This also means $\left[\frac{u\alpha}{K}\right] = [m/s]$, which lines up with the expected velocity.

We can also see this makes sense by realizing when we increase α, we get a higher final speed. This makes sense because if we discharge more mass quicker, we have drag acting for a shorter time, which would give a higher v.

Increasing K also makes sense b/c it results in a lower speed. This makes sense because if the coefficient of drag is higher, you would move slower.

Finally, the more mass that is fuel in the beginning, means we would have a lower m_f. This would give a higher final speed, which makes sense if we have more fuel.

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d) If we want to expand $\left(\frac{m_f}{m_i}\right)^{K/\alpha}$, we can realize it takes the form A^x, which has the following taylor expansion.

$$A^x \approx 1 + A^x \ln(A)\, x + \ldots$$

Plugging in for $A^x = \left(\frac{m_f}{m_i}\right)^{K/\alpha}$, we get

$$1 + \left(\frac{m_f}{m_i}\right)^{K/\alpha} \ln\left(\frac{m_f}{m_i}\right)\left(\frac{K}{\alpha}\right) + \ldots$$

Taking $\frac{K}{\alpha} \approx 0$, $\left(\frac{m_f}{m_i}\right)^{\frac{K}{\alpha}} \approx 1$ & plugging this in gives

$$v_f = -u \ln\left(\frac{m_f}{m_i}\right)$$

We want to equate this to eq 6 $v = v_0 - u\ln\left(\frac{m}{m_0}\right)$, but $v_0 = 0$, so

$$\boxed{v_f = -u\ln\left(\frac{m_f}{m_i}\right) \equiv (6)\ v = -u\ln\left(\frac{m}{m_0}\right)}$$

e) If k is very small, you should get eq. 6 because the drag force is inconsequential compared to the thrust force, so the impact of drag is insignificant.

This also works for $\alpha \gg k$ because if the rate of discharge of mass acts so quickly that the drag force doesn't have enough time to act.

Also if $\alpha \gg k$ $A^{\frac{k}{\alpha}} \approx 1$, so the expression reduces.

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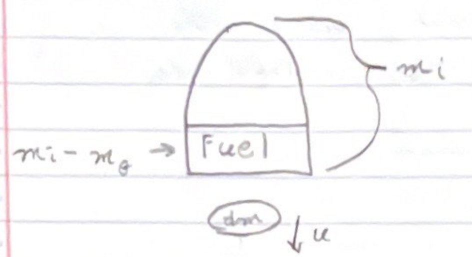
2)



$g/6 = g_e$ where $g = -9.81 \ \frac{m}{s^2}$

a) We know force is equal to the thrust force & mg_e, so

$$F = u\dot{m} - mg_e$$

But $F = 0$ because we are hovering, so

$$u\dot{m} = mg_e$$

$$u\,dm = mg_e\,dt$$

$$dt = \frac{u\,dm}{mg_e}$$

Integrating from t, 0 to t_f & m, m_0 to m_f

$$t_f = \frac{u}{g_e}\ln\left(\frac{m_f}{m_i}\right)$$

b) The units make sense being $\left[\frac{m}{s}\right] / \left[\frac{m}{s^2}\right] = [s]$.
This also makes sense when increasing fuel used would cause $mass_f$ to decrease & result in a higher final time.

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c) The rover using 1/3 of its weight means the $m_\theta / m_0 = 2/3$ because $1 - \frac{1}{3} = \frac{2}{3} = m_\theta$. $u = 1500$ m/s

$$t_\theta = \frac{u}{g_0} \ln\left(\frac{2}{3}\right)$$

$$\boxed{t_\theta = 371.99\ s}$$

3) I will be using the fact that the curl need to be 0 for the vector field to be conservative. Also $\vec{F} = -\vec{\nabla} U$

Also, since potential change in energy is relative to another position, I will be omitting the integration constant C.

If curl = 0, then all of these have to be true.

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad ; \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \quad ; \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$$

a) $$\frac{\partial F_x}{\partial y} = 0 = \frac{\partial F_y}{\partial x} \quad ; \quad \frac{\partial F_x}{\partial z} = 0 = \frac{\partial F_z}{\partial x} \quad ; \quad \frac{\partial F_y}{\partial z} = 0 = \frac{\partial F_z}{\partial y}$$

If $F = -\nabla U$, we integrate F_x wrt x, F_y wrt y, & F_z wrt z.

$F_x = a\frac{\partial U}{\partial x} \quad \therefore U_x = a\int x\,dx = -\frac{x^2}{2}$

$F_y = a\frac{\partial U}{\partial y} \quad \therefore U_y = a\int 2y\,dy = -y^2$

$F_z = a\frac{\partial U}{\partial z} \quad \therefore U_z = a\int 3z\,dz = -\frac{3}{2}z^2$

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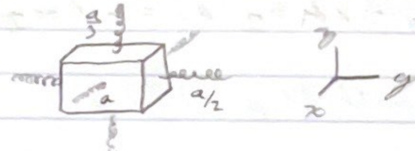
$\mathcal{U} = \mathcal{U}_x \cup \mathcal{U}_y \cup \mathcal{U}_z = a\left(\frac{x^2}{2} + y^2 + \frac{3}{2} z^2\right)$

$$\boxed{\mathcal{U} = \frac{a}{2}\left(x^2 + 2y^2 + 3z^2\right)}$$

a system that experiences this type of force is a box suspended w/ different strength springs in each axis.



b) $\frac{\partial F_x}{\partial y} = -1 \neq \frac{\partial F_y}{\partial x} = 1$; fails test, so not conservative

c) $\frac{\partial F_x}{\partial y} = 6axy^2 = \frac{\partial F_y}{\partial x}$; $\frac{\partial F_x}{\partial z} = 6axz^2 = \frac{\partial F_z}{\partial x}$; $\frac{\partial F_y}{\partial z} = 6ayz^2 = \frac{\partial F_z}{\partial y}$

$\therefore$ conservative vector field

$F_x = -\frac{\partial \mathcal{U}}{\partial x}$ $\therefore \mathcal{U}_x = -a\int(2xz^3 + 2xy^3)\,dx = -a(x^2z^3 + x^2y^3)$

$F_y = \frac{-\partial \mathcal{U}}{\partial y}$ $\therefore \mathcal{U}_y = -a\int(2yz^3 + 3x^2y^2 + 5y^4)\,dy = -a(y^2z^3 + x^2y^3 + y^5)$

$F_z = -\frac{\partial \mathcal{U}}{\partial z}$ $\therefore \mathcal{U}_z = -a\int(3x^2z^2 + 3y^2z^2)\,dz = -a(x^2z^3 + y^2z^3)$

$\mathcal{U} = \mathcal{U}_x \cup \mathcal{U}_y \cup \mathcal{U}_z$

$$\boxed{\mathcal{U} = -a\left(x^2y^3 + x^2z^3 + y^5 + y^2z^3\right)}$$

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4a) $\frac{\partial F_x}{\partial y} = 0 = \frac{\partial F_y}{\partial x}$; $\frac{\partial F_x}{\partial z} = 2Az = \frac{\partial F_z}{\partial x}$; $\frac{\partial F_y}{\partial z} = 0 = \frac{\partial F_z}{\partial y}$

$\boxed{\text{Passes the tests} \therefore \text{it is conservative}}$

$F_x = \frac{-\partial U}{\partial x} \quad \therefore U_x = -\int A z^2 dx = -Axz^2$

$F_y = \frac{-\partial U}{\partial y} \quad \therefore U_y = -\int B y^3 dy = -B y^4/4$

$F_z = \frac{-\partial U}{\partial z} \quad \therefore U_z = -\int 2A xz \, dz = -Axz^2$

$$U = U_x \cup U_y \cup U_z$$

$$U = -(Axz^2 + By^4/4)$$

b) We know in a conservative field, $\Delta T = -\Delta U$, knowing this we can use $T = \frac{1}{2}mv^2$ to find v.

$T = (Axz^2 + By^4/4)$

$\Delta T = T(-1, 2, 1) - T(0, 0, 0)$

$\Delta T = (-A + 4B) - 0 \qquad \sqrt{\frac{2T}{m}} = v$

Solving $\frac{1}{2}mv^2 = T$ for v & plugging T in gives

$$v = \sqrt{\frac{2}{m}(4B - A)} \quad w/ \; v = v_0 + v_0, \text{ so}$$

$$\boxed{v_f = \sqrt{v_0^2 + \frac{2}{m}(4B - A)}}$$

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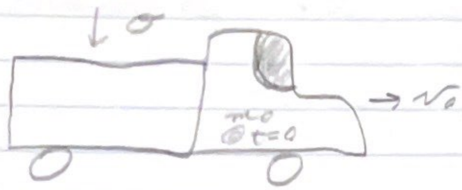
5)



a) Using the expression $F_{ext}dt = dp$, we can realize that $F_{ext} = 0$ because there is no external force acting on the truck because ice has no friction & $\Sigma F_y = 0$.

$$\therefore dp = 0 \quad \text{\& Finding the change in momentum}$$

$p(t+dt) = (M + dm)(v + dv)$
$p(t) = Mv$

w/ $dp = M\,dv + v\,dm + dm\,dv$

But $dm\,dv \approx 0$, so

$$dp = M\,dv + v\,dm$$

With this we plug into the original expression to find

$$M\,dv = -v\,dm$$

Seperating variables & integrating gives

$$\ln\left(\frac{v_f}{v_0}\right) = -\ln\left(\frac{m_f}{m_0}\right)$$

But the final mass after t seconds will be $m_0 + \sigma t$, so

$$\ln\left(\frac{v_f}{v_0}\right) = \ln\left(\frac{m_0}{m_0 + \sigma t}\right)$$

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Removing the ln gives...

$$\frac{v_\theta}{v_0} = \frac{m_0}{m_0 + \sigma t}$$

$$\boxed{v_\theta = \frac{v_0 m_0}{m_0 + \sigma t}}$$

e) $m\dot{v} = dp = p(t+dt) - p(t)$

$p_\theta = m(v + dv) + dm(v - u)$
$p_0 = mv$

$$dp = m dv + v dm - u dm$$

This is equal to 0 because there is no external forces so

$$m dv = (u - v) dm$$

Seperating variables gives the expression

$$\frac{m dv}{u - v} = dm \qquad \sigma = \frac{dm}{dt} \therefore dm = \sigma dt$$

$$\frac{m dv}{u - v} = \sigma dt$$

We can then integrate this expression to get

$$-m \ln\left(\frac{u - v_\theta}{u - v_0}\right) = \sigma(t_\theta - t_0)$$

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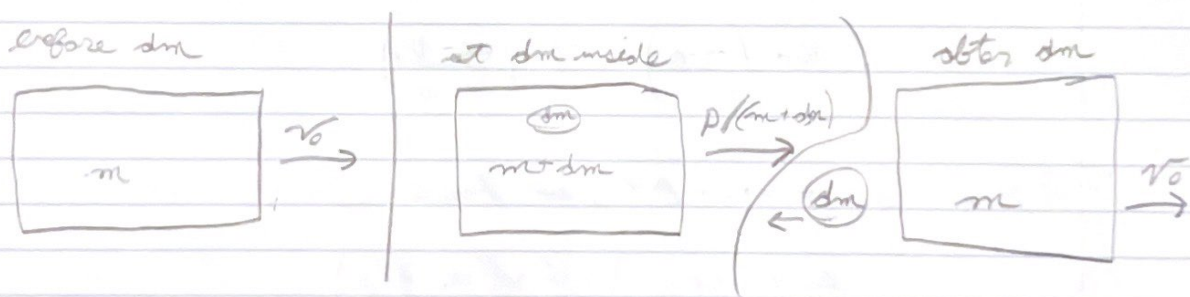
Exponentiating both sides & setting $t_0 = 0$,

$$\frac{u - v_f}{u - v_0} = \exp\left(\frac{-\sigma t_f}{m}\right)$$

Solving for v_f gives the expression

$$\boxed{v_f = u - (u - v_0)\exp\left(\frac{-\sigma t}{m}\right)}$$

c) The reason the speed of the truck approaches u is because the truck has some total momentum before the dm of water is added (call it p). Then the speed is p/m. Once the mass dm is added, the total momentum doesn't change, but the speed drops to $p/(m+dm)$. Once the mass dm is ejected the final momentum is $m(v+dv) - u\,dm$. For this to be equal to the initial momentum, dv needs to be equal to u.



p is constant, so $mv + m\,dv - u\,dm = mv$ $\therefore$ $u\,dm = m\,dv$

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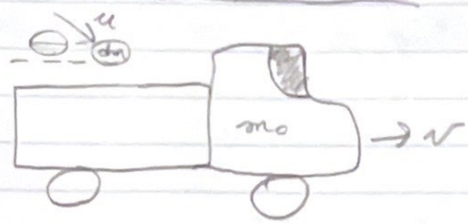
d)



Force is once again equal to zero because there is no net force.

$\therefore dp = 0$

Finding $dp = p(t+dt) - p(t)$

$p(t+dt) = (m + dm)(v + dv)$

$p(t) = mv + u_x dm$

$\therefore dp = m\,dv + v\,dm - u_x dm$

Rearranging this equation gives

$$\frac{dm}{m} = \frac{dv}{u_x - v}$$

integrating gives

$$\ln\left(\frac{m_f}{m_i}\right) = -\ln\left(\frac{u_x - v_f}{u_x - v_0}\right)$$

Since $m_f = m_i + \sigma t$, we have

$$\ln\left(\frac{m_i}{m_i + \sigma t}\right) = \ln\left(\frac{u_x - v_f}{u_x - v_0}\right)$$

Simplifying this gives

$$\boxed{v_f = u_x - \frac{m_i(u_x - v_0)}{m_i + \sigma t}}$$ ← ! where $u_x = u\cos\theta$!

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The rain will speed the truck up when u_x / $u\cos\theta$ is greater than the speed of the truck.

The rain will have no effect on speed when $u_x/u\cos\theta$ is the same speed as the truck.

The rain will slow down the truck when $u_x/u\cos\theta$ is lower than the speed of the truck.

Much like part b, the speed will approach that of the x component of u. It is only different here because u has some y component that doesn't increase p.