HW #10 - Joseph Specht

1)

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$R = \text{const}$ $\qquad P_1 = (\theta_1 = 0) \quad \& \quad P_2 = (\theta_2, \varphi_2)$

a) since $r = R$, $\frac{dr}{dt} = 0$, so $dr^2 = 0$

$$\therefore \; dl^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2 = R^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

If we want the length of a path on a sphere, we take the path ~~integral~~ of $|dl|$, so for this sphere we can say the length is

$$L = \int_{\theta_1}^{\theta_2} |dl| = \int_{\theta_1}^{\theta_2} \sqrt{R^2(d\theta^2 + \sin^2\theta \, d\varphi^2)}$$

We can factor a $d\theta^2$ from each term in the $\sqrt{\ }$, so

$$L = \int_{\theta_1}^{\theta_2} \sqrt{R^2 d\theta^2 \left(1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2\right)} = \int_{\theta_1}^{\theta_2} R \, d\theta \sqrt{1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2}$$

This then gives ~~the~~ form

$$\boxed{L = R \int_0^{\theta_1} \sqrt{1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} \, d\theta}$$

b) $\frac{\partial f}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial f}{\partial \varphi'}\right) = 0$ is the E-L eq for ~~this~~, so...

where $f = \sqrt{1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2}$

$$\frac{\partial f}{\partial \varphi} = 0 \quad , \quad \frac{\partial f}{\partial \varphi'} = \frac{1}{2}\left(1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2\right)^{(-1/2)} \cdot 2\sin^2\theta \left(\frac{d\varphi}{d\theta}\right)$$

$$\frac{\partial f}{\partial \varphi'} = \frac{\sin^2\theta \left(\frac{d\varphi}{d\theta}\right)}{\sqrt{1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2}}$$

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now the last thing to find is $\frac{d}{dt}\left(\frac{\partial \ell}{\partial \varphi'}\right)$, so

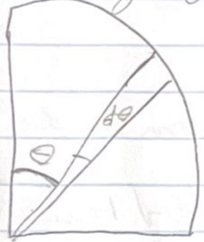
$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \varphi'}\right) = \frac{d}{dt}\left(\frac{\sin^2\theta\,\varphi'}{\sqrt{1+\sin^2\theta\,\varphi'^2}}\right) = 0\text{, no time dependence}$$

$\therefore \frac{\partial \ell}{\partial \varphi'} = C$ where C is a constant

To find C, apply initial conditions of $\theta_1 = 0$

$$\frac{\partial \ell}{\partial \varphi'} = \frac{\sin^2\theta\,\varphi'}{\sqrt{1+\sin^2\theta\,\varphi'^2}}\Bigg|_{\theta=0} = \frac{\sin^2 0\,\varphi'}{\sqrt{1+\sin^2 0\,\varphi'^2}} = \frac{0}{1} = C \quad \therefore C = 0$$

Looking for L, so need to solve L integral



φ = some ~~constant~~, but since we start @ the pole & move to any point on it so, $d\varphi = 0$ & $\frac{d\varphi}{d\theta} = 0$, so

$$\mathcal{L} = R\int_0^{\theta_2} \sqrt{1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)}\,d\theta = R\int_0^{\theta_2}\sqrt{1+\sin^2\theta\cdot 0}\,d\theta$$

$$\Rightarrow \mathcal{L} = R\int_0^{\theta_2} d\theta = R\theta_2$$

$$\boxed{L = R\theta_2}$$

* w/ φ constant @ φ_2 *

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4) Spherical coordinates only depend on the relative angle between two points, so we can rotate the coordinate system so that the north pole is aligned with the starting point so we can then see that the path is a great circle.

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2)

$$F[y(x)] = \int_0^1 f(x, y, y', y'')\,dx = \int_0^1 (y''(x))^2 dx$$

where $y' = dy/dx$

curve $y(x)$ goes from $(x=0, y=0)$ to $P = (x=1, y=1)$

a) $\delta S[q(t)] = S[q(t) + \delta q(t)] - S[q(t)]$

& we also know the multivariable taylor expansion is the following

$$\delta g(a, b) = g(a_0, b_0) + \frac{\partial g}{\partial a}\bigg|_{(a_0, b_0)} (a - a_0) + \frac{\partial g}{\partial b}\bigg|_{(a_0, b_0)} (b - b_0)$$

We can apply this line of reasoning to $\delta F[y(x)]$ & get the following

$$\delta F[y(x)] = F[y(x) + \delta y(x)] - F[y(x)]$$

$$F[y(x)] = \int_0^1 f(x, y, y', y'')\,dx$$

$$F[y(x) + \delta y(x)] = \int_0^1 f(x, y+\delta y, y'+\delta y', y''+\delta y'')\,dx$$

Taylor expanding $f(x, y+\delta y, y'+\delta y', y''+\delta y'')$

$$f(x, y+\delta y, y'+\delta y', y''+\delta y'') = \frac{\partial f}{\partial x}\bigg|_{(x,y,y',y'')} \overbrace{(\cancel{x} - x)}^{0}$$
$$+ \frac{\partial f}{\partial y}\bigg|_{(x,y,y',y'')} ((y+\delta y) - y)$$
$$+ \frac{\partial f}{\partial y'}\bigg|_{(x,y,y',y'')} ((y'+\delta y') - y')$$
$$+ \frac{\partial f}{\partial y''}\bigg|_{(x,y,y',y'')} ((y''+\delta y'') - y'')$$

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$$\delta F[y(x)] = \int_0^1 f(x,y,y',y'')\,dx + \int_0^1 \frac{\partial f}{\partial y}\Big|_{(x,y,y',y'')}(\delta y)\,dx + \int_0^1 \frac{\partial f}{\partial y'}\Big|_{(x,y,y',y'')}(\delta y')\,dx$$
$$+ \int_0^1 \frac{\partial f}{\partial y''}\Big|_{(x,y,y',y'')}(\delta y'')\,dx - \int_0^1 f(x,y,y',y'')\,dx$$

$P := (x, y, y', y'')$, so

$$\Rightarrow \delta F[y(x)] = \int_0^1 \frac{\partial f}{\partial y}\Big|_P (\delta y)\,dx + \int_0^1 \frac{\partial f}{\partial y'}\Big|_P (\delta y')\,dx + \int_0^1 \frac{\partial f}{\partial y''}\Big|_P (\delta y'')\,dx$$

To find the integral, solve each by parts

$$\int_0^1 \frac{\partial f}{\partial y''}\Big|_P \delta y''\,dx \qquad u = \frac{\partial f}{\partial y''} \quad v = \delta y' \qquad du = \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)dx \quad dv = \delta y''\,dx$$

$$\to \delta y' \frac{\partial f}{\partial y''}\Big|_0^1 - \int_0^1 \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\delta y'\,dx$$

$$\to \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\delta y\Big|_0^1 - \int_0^1 \delta y \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx \qquad u = \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) \quad v = \delta y \qquad du = \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx \quad dv = \delta y'\,dx$$

$$\to \delta y' \frac{\partial f}{\partial y''}\Big|_0^1 - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\delta y\Big|_0^1 + \int_0^1 \delta y \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx$$

$$\int_0^1 \frac{\partial f}{\partial y'}\Big|_P \delta y'\,dx \qquad u = \frac{\partial f}{\partial y'} \quad v = \delta y \qquad du = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx \quad dv = \delta y'\,dx$$

$$\to \delta y \frac{\partial f}{\partial y'}\Big|_0^1 - \int_0^1 \delta y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$\delta F[y(x)] = \int_0^1 \delta y \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx - \int_0^1 \delta y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx + \int_0^1 \frac{\partial f}{\partial y}\delta y\,dx$$
$$+ \delta y \frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\delta y + \delta y' \frac{\partial f}{\partial y''}$$

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We also know

$$f(x, y, y', y'') = (y''(x))^2$$

$\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial y'} = 0$ $\frac{\partial f}{\partial y''} = 2y''$

$\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = 2y'''$ $\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 2y''''$

Plugging everything in...

$$\delta F[y(x)] = \int_0^1 2y'''' \, \delta y \, dx + 0 + 0 + 0 - 2y''' \, \delta y \Big|_0^1 + 2y'' \, \delta y' \Big|_0^1$$

If we say $\eta = \delta y$ & $\eta' = \delta y'$, we can say

$$\boxed{\delta F[y(x)] = \int_0^1 2y'''' \, \eta \, dx + 2y'' \eta' \Big|_0^1 - 2y''' \eta \Big|_0^1}$$

This is what we want

b) ~~start~~ @ $(x=0, y=0) \to (x=1, y=1)$

$y'(0,0) = 0 = y''(0,0)$

$$0 = \int_0^1 2y'''' \, \eta \, dx + \left(2y''(1)\eta'(1) - \cancel{2y''(0)\eta'(0)}^{\,0}\right) + \left(\cancel{2y'''(0)\eta(0)}^{\,0} - \cancel{2y'''(1)\eta(1)}^{\,0}\right)$$

$$\Rightarrow \boxed{\int_0^1 2y'''' \, \eta \, dx + 2y''(1)\eta'(1) = 0}$$

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c) $y''''(x) = 0 \Rightarrow y'''(x) = A \Rightarrow y''(x) = Ax + B \Rightarrow y'(x) = \frac{1}{2}Ax^2 + Bx + C$

$$y = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D$$

$y'(0) = 0 \qquad y''(1) = 0 \qquad y(0) = 0 \qquad y(1) = 1$

$y(0) = D = 0 \qquad \therefore D = 0 \qquad y = Ax^3 + Bx^2 + Cx$

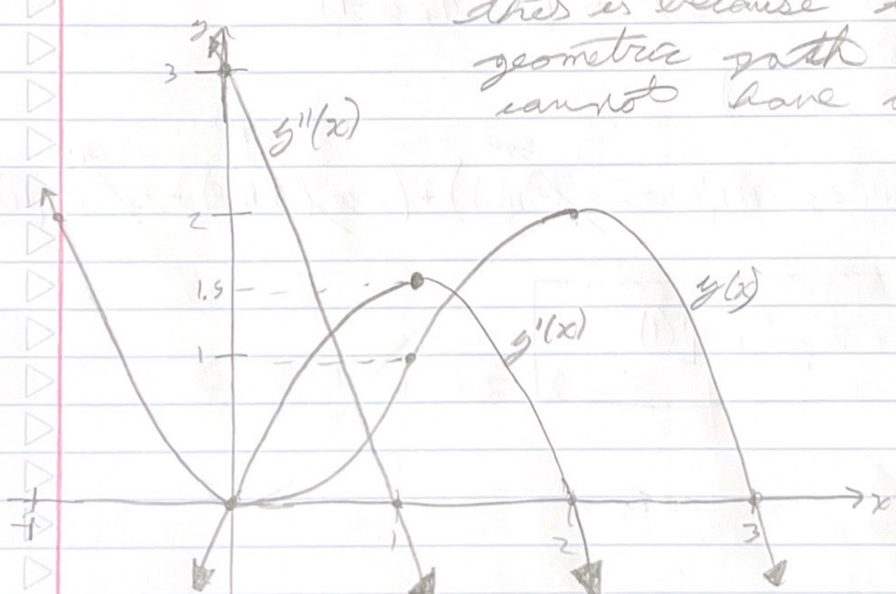
$y'(0) = C = 0 \qquad \therefore C = 0 \qquad y = Ax^3 + Bx^2$

$y(1) = \frac{1}{6}A + \frac{1}{2}B = 1 \Rightarrow A + 3B = 6 \Rightarrow A = 6 - 3B$

$y''(1) = A + B = 0 \Rightarrow A = -B \Rightarrow 6 - 3B = -B \Rightarrow 6 = 2B \Rightarrow B = 3$

$A = -3 \qquad \therefore y = -\frac{1}{2}x^3 + \frac{3}{2}Bx^2$

d) $y(x)$ is a path that minimizes $F[y(x)]$ this is because this is a geometric path & $\therefore$ you cannot have a maximum



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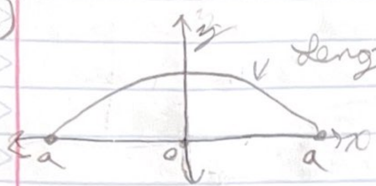
3)



a) Want to maximize the area, so have

$S_1 = \int_{-a}^{a} y\,dx$, but also have to constraint

$S_2 = \lambda \int_{-a}^{a} |ds|$ where λ is something & $|ds|$ is

$|ds| = \sqrt{dx^2 + dy^2} \Rightarrow$ w/ factoring out a dx, we get

$|ds| = \sqrt{dx^2\left(1+\left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\therefore S_2 = \lambda \int_{-a}^{a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = l$

Since we have two functionals, we subtract the arguments to get the total functional

$$S = \int_{-a}^{a} y\,dx - \lambda \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-a}^{a} \left(y - \lambda\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) dx$$

now that we have the functional, we know the function we apply the E-L eq to is

$f = y - \lambda\sqrt{1+(y')^2}$, so we have to

find the parts of the E-L eq,

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$$\frac{\partial f}{\partial y} = 1 \quad , \quad \frac{\partial g}{\partial y'} = -\lambda\left(\frac{1}{2}(1+y'^2)^{(-1/2)} \cdot 2y'\right) = \frac{-\lambda y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = \frac{d}{dx}\left(\frac{-\lambda y'}{\sqrt{1+y'^2}}\right)$$ plugging this into the eq gives

$$1 - \frac{d}{dx}\left(\frac{-\lambda y'}{\sqrt{1+y'^2}}\right) = 0 \quad \therefore 1 = \frac{d}{dx}\left(\frac{-\lambda y'}{\sqrt{1+y'^2}}\right)$$

If we integrate each side wrt x, we get

$$x = \frac{-\lambda y'}{\sqrt{1+y'^2}} - c$$, now solving for y' we get

$$x+c = \frac{-\lambda y'}{\sqrt{1+y'^2}} \Rightarrow (x+c)\left(\sqrt{1+y'^2}\right) = -\lambda y' \Rightarrow (x+c)^2(1+y'^2) = \lambda^2 y'^2$$

$$\Rightarrow (x+c)^2 + y'^2(x+c)^2 = \lambda^2 y'^2 \Rightarrow (x+c)^2 = \lambda^2 y'^2 - y'^2(x+c)^2$$

$$\Rightarrow (x+c)^2 = y'^2\left(\lambda^2 - (x+c)^2\right) \overset{\text{sqrt}}{\Rightarrow} (x+c) = y'\left(\lambda^2 - (x+c)^2\right)^{(1/2)}$$

$$\Rightarrow \frac{x+c}{\sqrt{\lambda^2 - (x+c)^2}}\,dx = dy \overset{\text{into calculator}}{\Rightarrow} y = \pm\sqrt{\lambda^2 - (x+c)^2} + D$$

$$\Rightarrow y - D = \pm\sqrt{\lambda^2 - (x+c)^2} \Rightarrow (y-D)^2 = \lambda^2 - (x+c)^2$$

$$\Rightarrow (y-D)^2 + (x+c)^2 = \lambda^2$$

This is the eq of a circle, but we know this circle is centered along the y axis $\therefore c = 0$, so the final equation is...

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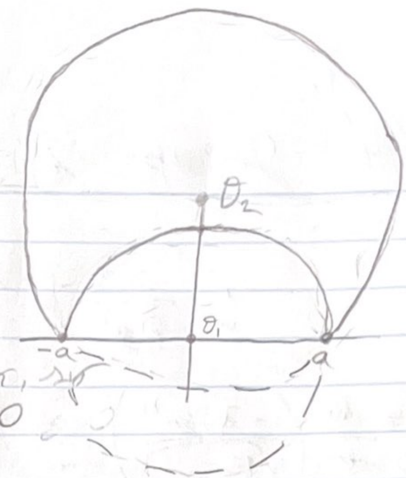
$$\boxed{(y-\theta)^2 + x^2 = \lambda^2}$$



Also area needs to be max, so take path above $y=0$

The shape of this should be a circle w/ radius of λ w/ a center @ $(0,\theta)$

e) λ is the radius of the circle

we know $l = \int_{-a}^{a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$, but we know

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\left(\sqrt{\lambda^2 - x^2} + \theta\right) = \frac{x}{\sqrt{\lambda^2 - x^2}}$$

$$\therefore\ l = \int_{-a}^{a} \sqrt{1 + \frac{x^2}{\lambda^2 - x^2}}\, dx = \int_{-a}^{a} \sqrt{\frac{\lambda^2 - x^2}{\lambda^2 - x^2} + \frac{x^2}{\lambda^2 - x^2}}\, dx$$

$$l = \int_{-a}^{a} \sqrt{\frac{\lambda^2}{\lambda^2 - x^2}}\, dx = \lambda \operatorname{asin}\left(\frac{x}{\lambda}\right)\Bigg|_{-a}^{a}$$ ← I used a robot

$$= \lambda\left(\operatorname{asin}\left(\frac{a}{x}\right) - \operatorname{asin}\left(\frac{-a}{x}\right)\right) = 2\lambda \operatorname{asin}\left(\frac{a}{\lambda}\right)$$

$$\boxed{l = 2\lambda \operatorname{asin}\left(\frac{a}{\lambda}\right)}$$

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4) $\Psi(x)$ passes through $(x=0, \Psi=0)$ & $(x=1, \Psi=0)$ on x-axis

$$F[\Psi(x)] = \int_0^1 \frac{\hbar^2}{2m}(\Psi'(x))^2 dx \; ; \; G[\Psi(x)] = \int_0^1 (\Psi(x))^2 dx = 1$$

a) $$\int_0^1 \left[\frac{\hbar^2}{2m}\left((\Psi'(x))^2\right) - E(\Psi(x))^2\right] dx$$

need to solve E-L eq in form of

$$\frac{\partial f}{\partial \Psi} - \frac{d}{dx}\left(\frac{\partial f}{\partial \Psi'}\right) = 0 \text{ where } f = \left[\frac{\hbar^2}{2m}\left((\Psi'(x))^2\right) - E(\Psi(x))^2\right]$$

$$\frac{\partial f}{\partial \Psi} = -2E\,\Psi(x) \quad ; \frac{d}{dx}\left(\frac{\partial f}{\partial \Psi'}\right) = \frac{d}{dx}\left(\frac{\hbar^2}{m}\Psi'(x)\right) = \frac{\hbar^2}{m}\Psi''(x)$$

putting it all together, we get...

$$-2E\,\Psi(x) - \frac{\hbar^2}{m}\Psi''(x) = 0 \Rightarrow \boxed{2E\,\Psi(x) + \frac{\hbar^2}{m}\Psi''(x) = 0}$$

b) $\frac{\hbar^2}{m}\Psi''(x) + 2E\,\Psi(x) = 0$, call $\Psi(x) = \exp(\lambda x)$

$$\Rightarrow \frac{\hbar^2}{m}\left(\lambda^2 \exp(\lambda x)\right) + 2E\exp(\lambda x) = 0 \Rightarrow \frac{\lambda^2\hbar^2}{m} + 2E = 0$$

$$\Rightarrow \frac{\lambda^2\hbar^2}{m} = -2E \Rightarrow \lambda^2 = \frac{-2Em}{\hbar^2} \Rightarrow \lambda = \pm i\sqrt{\frac{2Em}{\hbar^2}}$$

$$\therefore \Psi(x) = A\exp\left(i\sqrt{\frac{2Em}{\hbar^2}}x\right) + B\exp\left(-i\sqrt{\frac{2Em}{\hbar^2}}x\right)$$

$$\Psi(x) = A\cos\left(\sqrt{\frac{2Em}{\hbar^2}}x\right) + B\sin\left(\sqrt{\frac{2Em}{\hbar^2}}x\right)$$

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Plugging in $(x=0, \Psi=0)$ we have

$0 = A\cos(0) + B\sin(0) \Rightarrow 0 = A$

Plugging in $(x=1, \Psi=0)$ we have

$0 = B\sin\left(\sqrt{\frac{2Em}{\hbar^2}}\right)$ & we know $\sin(x)=0$ if $x = n\pi$ $n \in \mathbb{Z}$

$\therefore \sqrt{\frac{2Em}{\hbar^2}} = n\pi \Rightarrow \frac{2Em}{\hbar^2} = n^2\pi^2 \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2m}$

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2m} \text{ where } n \in \mathbb{Z}}$$

c) $\Psi = B\sin(n\pi x) \Rightarrow \int_0^1 \Psi^2(x)\,dx = 1$

$\Rightarrow \int_0^1 B^2\sin^2(n\pi x)\,dx = 1 \Rightarrow \int_0^1 \sin^2(n\pi x)\,dx = 1/B^2 = \frac{1}{2}$

$\Rightarrow B^2 = 2 \Rightarrow B = \sqrt{2}$

$$\boxed{\therefore \Psi(x) = \sqrt{2}\sin(n\pi x) \text{ where } n \in \mathbb{Z}}$$