HW #2 - Joseph Specht

1) $U(x) = \frac{a}{x^2} + bx^2 \qquad a, b \in \mathbb{R} > 0$

a) $\frac{dU}{dx} = -2ax^{-3} + 2bx$ want to find crit points, so $\frac{dU}{dx} = 0$

$0 = -2ax^{-3} + 2bx \Rightarrow 2bx = \frac{2a}{x^3} \Rightarrow bx^4 = a \Rightarrow x^4 = \frac{a}{b} \Rightarrow x = \pm\sqrt[4]{a/b}$

since we care about $x > 0$, disregard $-\sqrt[4]{a/b}$

$\frac{d^2U}{dx^2} = 6ax^{-4} + 2b \Rightarrow \frac{d^2U}{dx^2}(\sqrt{\frac{a}{b}}) = 6a\frac{b}{a} + 2b = 6b + 2b = 8b > 0$

$\boxed{\text{There is a min at } x = \sqrt[4]{a/b}\text{, found concavity w/ double derivative test}}$

b) Taylor expansion of $U(x) \approx U(x_0) + U'(x_0)(x - x_0) + \frac{U''(x_0)(x-x_0)^2}{2!} + \dots$

$U(x) \approx U(x_0) + U'(x_0)y + \frac{U''(x_0)}{2}y^2 + \dots$

$U(x) \approx U(\sqrt[4]{a/b}) + U'(\sqrt[4]{a/b})y + \frac{U''(\sqrt[4]{a/b})}{2}y^2 + \dots$

$\approx a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} + \underset{\text{know its a cp}}{0} + \frac{8b}{2}y^2 = a\sqrt{b/a} + b\sqrt{a/b} + 4by^2$

$\boxed{k = 8b} \qquad c = a\sqrt{b/a} + b\sqrt{a/b}$

c) $\omega = \sqrt{k/m} = \sqrt{8b/m} \qquad T = 2\pi/\sqrt{8b/m}$

$T = 2\pi \cdot \sqrt{\frac{m}{8b}} = 2\pi \cdot \frac{1}{2}\sqrt{\frac{m}{2b}} = \pi\sqrt{\frac{m}{2b}}$

$\boxed{T = \pi\sqrt{m/2b}}$

k is the spring constant, so $\omega = \sqrt{k/m}$

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2)

$cv \uparrow \downarrow mg$ $\quad F_{drag} = cv \quad mg > 0 \quad K = c/m$

a) $F = -cv - mg = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -Kv - g \Rightarrow \frac{dv}{-Kv-g} = dt$

$\Rightarrow \int_{v_0}^{v} \frac{dv}{-Kv-g} = \int_0^t dt \Rightarrow -\frac{1}{K} \ln\left(\frac{Kv+g}{Kv_0+g}\right) = t + 0 \Rightarrow \ln\left(\frac{Kv+g}{Kv_0+g}\right) = -Kt$

$\Rightarrow \frac{Kv+g}{Kv_0+g} = e^{-Kt} \Rightarrow Kv+g = e^{-Kt}(Kv_0+g) \Rightarrow \boxed{v = \frac{e^{-Kt}(Kv_0+g) - g}{K}}$

b) max height when $v = 0$, so

$0 = \frac{e^{-Kt}(Kv_0+g) - g}{K} \Rightarrow g = e^{-Kt}(Kv_0+g) \Rightarrow \frac{g}{Kv_0+g} = e^{-Kt}$

$\Rightarrow \ln\left(\frac{g}{Kv_0+g}\right) = -Kt \Rightarrow t = -\frac{1}{K} \ln\left(\frac{g}{Kv_0+g}\right)$

now integrate velocity to find the position function

$\frac{dy}{dt} = \frac{e^{-Kt}(Kv_0+g) - g}{K} = \left(v_0 + \frac{g}{K}\right)e^{-Kt} - \frac{g}{K} \Rightarrow \int_0^y dy = \int_0^t \left(v_0 + \frac{g}{K}\right)e^{-Kt}\,dt - \int \frac{g}{K}\,dt$

$\Rightarrow y = \left(v_0 + \frac{g}{K}\right)\int_0^t e^{-Kt}\,dt - \frac{gt}{K} \Rightarrow y = \left(v_0 + \frac{g}{K}\right)\left(-\frac{1}{K} e^{-Kt} - 1\right) - \frac{gt}{K}$

$-\frac{1}{K} e^{\ln(\dots)}$

$\Rightarrow y(t=0) = \left(v_0 + \frac{g}{K}\right)\left(-\frac{1}{K}\left(\frac{g}{Kv_0+g}\right) - 1\right) + \frac{g \ln(g/(Kv_0+g))}{K^2}$

$= \left(v_0 + \frac{g}{K}\right)\left(\frac{-g}{K^2v_0+Kg} - 1\right) + \frac{g \ln\left(\frac{g}{Kv_0+g}\right)}{K^2} = \frac{-gv_0}{K^2v_0+Kg} - \frac{g^2}{K^3v_0+K^2g} - v_0 - \frac{g}{K} + \square$

$\square = \frac{g \ln(g/(Kv_0+g))}{K^2}$

HW#2 cont - Joseph Specht

2b) $y(t=0) = \frac{-gv_0}{K^2v_0+Kg} - \frac{g^2}{K^3v_0+K^2g} - v_0 - \frac{g}{K} + \frac{g\ln(g/(Kv_0+g))}{K^2}$

$= \frac{-gKv_0}{K^3v_0+K^2g} - \frac{g^2}{K^3v_0+K^2g} - v_0 - \frac{g}{K} + \frac{g\ln(g/(Kv_0+g))}{K^2}$

$= \frac{-g^2-gKv_0}{K^3v_0+K^2g} - \frac{v_0K^2}{K^2} - \frac{gK}{K^2} + \frac{g\ln(g/(Kv_0+g))}{K^2}$

$\boxed{y = \frac{-g^2-gKv_0}{K^3v_0+K^2g} + \frac{g\ln\left(\frac{g}{Kv_0+g}\right) - gK - v_0K^2}{K^2}}$ where $K = c/m$

c) $v = \frac{e^{-Kt}(Kv_0+g)-g}{K}$ exponential part e^{-Kt}

$e^{-x} = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ around $x=0$

$e^{-Kt} = 1 - Kt + \frac{K^2t^2}{2} + \frac{K^3t^3}{6} + \ldots = 1 - \frac{ct}{m} + \frac{c^2t^2}{2m^2} + \frac{c^3t^3}{6m^3} + \ldots$

if $c \ll 1$, the expression for e^{-Kt} simplifies to

$1 - \frac{ct}{m} + \ldots \approx 1 - \frac{ct}{m}$ & plugging back into the expression for v,

$v = \frac{(1-Kt)(Kv_0+g)-g}{K} = \frac{Kv_0+g-K^2v_0t-Ktg-g}{K}$

$\Rightarrow \frac{Kv_0-K^2v_0t-Ktg}{K} = v_0 - \overset{\approx 0}{Kv_0t} - tg \approx v_0 - gt$

$\boxed{v \approx v_0 - gt}$ This is what is expected because this is the velocity formula for a projectile thrown in the air

HW #2 - Joseph Spadaro

3) $F(v,t) = -btv^2$ $b>0$ @ $t=0$ $x=0$ & $v=v_0$

a) $F = -btv^2 = m\frac{dv}{dt} \Rightarrow -\frac{bt}{m}dt = v^{-2}dv \Rightarrow \int_0^t -\frac{bt}{m}dt = \int_{v_0}^{v} v^{-2}dv$

$\Rightarrow -\frac{b}{2m}t^2 = -v^{-1} + v_0^{-1} \Rightarrow v^{-1} = v_0^{-1} + \frac{bt^2}{2m} \Rightarrow \frac{1}{v} = \frac{1}{v_0} + \frac{bt^2}{2m} \times \frac{vv_0 2m}{vv_0 2m}$

$\Rightarrow \frac{vv_0 2m}{v} = \frac{vv_0 2m}{v_0} + \frac{bt^2 vv_0 2m}{2m} \Rightarrow 2mv_0 = 2mv + bt^2 v v_0$

$\Rightarrow 2mv_0 = v(2m + bt^2 v_0) \Rightarrow \boxed{v = \frac{2mv_0}{2m + bt^2 v_0}}$

b) This shows the particle doesn't come to a stop because this is a continuous function with no discontinuities, so it goes on forever in the form of $\frac{1}{t^2}$, which asymptotes eventually

c) integrate velocity to find x & then take limit

$\frac{dx}{dt} = \frac{2mv_0}{2m + bt^2 v_0} \Rightarrow dx = \frac{2mv_0}{2m + bt^2 v_0}dt \Rightarrow \int_0^{x_f} dx = 2mv_0 \int_0^{t} \frac{dt}{2m + bt^2 v_0}$

$\Rightarrow x_f = 2mv_0 \int_0^t \frac{1}{2m}\frac{dt}{1 + \frac{bt^2 v_0}{2m}} = v_0 \int_0^t \frac{dt}{1 + \frac{bt^2 v_0}{2m}}$ let $u = \sqrt{\frac{bv_0}{2m}}\,t$ $du = \sqrt{\frac{bv_0}{2m}}\,dt$

$\Rightarrow dt = du\sqrt{\frac{2m}{bv_0}}$ plugging back in $x_f = v_0 \int_0^{\sqrt{\frac{2m}{bv_0}}u} \sqrt{\frac{2m}{bv_0}} \frac{1}{1+u^2}du$

$\Rightarrow x_f = v_0\sqrt{\frac{2m}{bv_0}} \int_0^{\sqrt{\frac{2m}{bv_0}}u} \frac{1}{1+u^2}du$ ← derivative of atan(u) $\Rightarrow x_f = v_0\sqrt{\frac{2m}{bv_0}}\left(\operatorname{atan}(u)\right)\Big|_0^{\sqrt{\frac{2m}{bv_0}}u}$

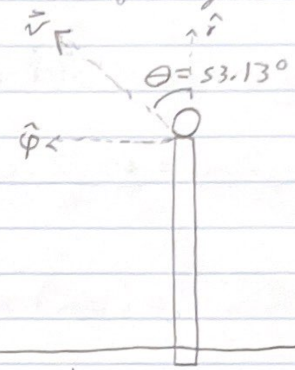
$x_f = v_0\sqrt{\frac{2m}{bv_0}}\left(\operatorname{atan}\left(\sqrt{\frac{bv_0}{2m}}\,t\right)\Big|_0^{\infty}\right)$ $\boxed{x_f = v_0\sqrt{\frac{2m}{bv_0}} \cdot \frac{\pi}{2}}$

$\operatorname{atan}(\infty) = \frac{\pi}{2}$

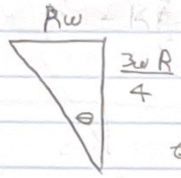
HW#2 - Joseph Spotts

4) ω = ang vel @ $t=t_0$, $\varphi = 90°$, $r = R$, $\dot{r}(t=t_0) = \frac{3}{4}\omega R$, $\vec{F}_f = -K\vec{v}$

a) @ $t=t_0$ once the bead leaves, its position & $\vec{F}_f$ is only acting, so v is constantly colinear w/ initial



$$\vec{v}_0 = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} = \frac{3}{4}\omega R\hat{r} + R\omega\hat{\varphi}$$



$$\tan\theta = \left(\frac{R\omega}{\frac{3}{4}\omega R}\right) = \left(\frac{4}{3}\right)$$

$$\theta = \operatorname{atan}(4/3) = 53.13°$$

$\boxed{\text{The bead travels } 53.13° \text{ counterclockwise off the } \hat{r} \text{ vector @ } t=t_0\text{, trajectory stays on this line}}$

c) $F = -K\vec{v} \Rightarrow m\frac{d\vec{v}}{dt} = -K\vec{v} \Rightarrow -\frac{m}{K}\frac{d\vec{v}}{\vec{v}} = dt \Rightarrow -\frac{m}{K}\int_{v_0}^{v}\frac{d\vec{v}}{\vec{v}} = \int_{t_0}^{t} dt$

$\Rightarrow -\frac{m}{K}\left(\ln(\vec{v}) - \ln(\vec{v}_0)\right) = t - t_0 \Rightarrow \ln\left(\frac{\vec{v}}{\vec{v}_0}\right) = -\frac{K}{m}(t-t_0)$

$\Rightarrow \frac{\vec{v}}{\vec{v}_0} = \exp\left(\frac{-K}{m}(t-t_0)\right) \Rightarrow \vec{v}(t) = \vec{v}_0\left(\exp\left(\frac{-K}{m}(t-t_0)\right)\right)$

$\Rightarrow \vec{v}(t) = \exp\left(\frac{-K}{m}(t-t_0)\right)\left(\frac{3}{4}\omega R\hat{r} + R\omega\hat{\varphi}\right)$

$$\boxed{\vec{v}(t) = \left(\frac{3}{4}\omega R\hat{r} + R\omega\hat{\varphi}\right)\exp\left(\frac{-K}{m}(t-t_0)\right)}$$

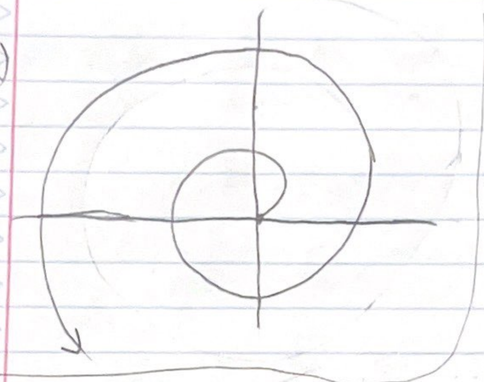
*assuming $\hat{r}$ & $\hat{\varphi}$ are constant from point of bead leaving

This is never 0 because exp(A) can never be 0, it has an asymptote @ $t=\infty$ of 0, but it never reaches 0. A constant times an exponential can "not be 0 ever.

HW#2 - cont - Joseph Spetto

5) $r = be^{kt}$

a)



b) $r = be^{kt}$
$\dot{r} = bke^{kt}$
$\ddot{r} = bk^2e^{kt}$

$\theta = ct$
$\dot{\theta} = c$
$\ddot{\theta} = 0$

$r(t) = r\hat{r} = be^{kt}\hat{r}$
$\dot{r}(t) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = bke^{kt}\hat{r} + be^{kt}c\hat{\theta}$
$\ddot{r}(t) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = (bk^2e^{kt} - be^{kt}c^2)\hat{r} + (0 + 2bke^{kt}c)\hat{\theta}$

$$\boxed{\begin{aligned} r(t) &= be^{kt}\hat{r} \\ \dot{r}(t) &= bke^{kt}\hat{r} + bce^{kt}\hat{\theta} \\ \ddot{r}(t) &= (bk^2e^{kt} - be^{kt}c^2)\hat{r} + 2bkce^{kt} \end{aligned}}$$

c) $u \cdot v = |u||v|\cos\theta$

$\dot{r} \cdot \ddot{r} = b^2k^3e^{2kt} - b^2kc^2e^{2kt} + 2b^2c^2ke^{2kt}$
$= b^2k^3e^{2kt} + b^2c^2ke^{2kt}$

$|\dot{r}| = \sqrt{b^2k^2e^{2kt} + b^2c^2e^{2kt}} = be^{kt}\sqrt{c^2+k^2}$

$|\ddot{r}| = \sqrt{b^2k^4e^{2kt} - 2b^2c^2k^2e^{2kt} + b^2c^4e^{2kt} + 4b^2c^2k^2e^{2kt}}$

$= \sqrt{b^2k^4e^{2kt} + b^2c^4e^{2kt} + 2b^2c^2k^2e^{2kt}}$

$= be^{kt}\sqrt{k^4 + 2k^2c^2 + c^4} = be^{kt}\sqrt{(c^2+k^2)^2}$

$= be^{kt}(c^2+k^2)$

HW #2

5c) $\cos\theta = \frac{u \cdot v}{|u||v|}$

$$\cos\theta = \frac{b^2 k^3 e^{2kt} + b^2 c^2 k e^{2kt}}{b^2 e^{2kt} (k^2+c^2)\sqrt{k^2+c^2}}$$

$$= \frac{k^3 + c^2 k}{(k^2+c^2)^{3/2}}$$

θ b/w $\dot{r}$ & $\ddot{r}$ $\boxed{\theta = a\cos\left(\frac{k^3 + c^2 k}{(k^2+c^2)^{3/2}}\right)}$

This expression is constant because it has no dependence on time & is only constants which never change

HW#2 cont - Joseph Spett

6) $\vec{R}(t) = b\sin\omega t\,\hat{\imath} + b\cos\omega t\,\hat{\jmath} + ct^2\hat{k}$ $\qquad b, c, \omega \in \mathbb{R}$

a) $r = \sqrt{x^2+y^2} = \sqrt{b^2\sin^2(\omega t) + b^2\cos^2(\omega t)} = \sqrt{b^2(\sin^2(\omega t) + \cos^2(\omega t))} = \sqrt{b^2} = b$

$\vec{r} = b\hat{r}$

$\theta = \operatorname{atan}\left(\frac{y}{x}\right) = \operatorname{atan}\left(\frac{b\cos(\omega t)}{b\sin(\omega t)}\right) = \operatorname{atan}(\cot(\omega t)) = \operatorname{atan}\left(\tan\left(\frac{\pi}{2} - \omega t\right)\right)$

$\Rightarrow \theta = \frac{\pi}{2} - \omega t$

$R(t) = b\hat{r} + ct^2\hat{k}$

$$\boxed{\begin{aligned} r(t) &= b\hat{r} \\ \theta(t) &= \frac{\pi}{2} - \omega t \\ z(t) &= ct^2 \end{aligned}}$$

b) $r = b$ $\qquad \theta = \frac{\pi}{2} - \omega t$ $\qquad z = ct^2$ $\qquad R = r\hat{r} + z\hat{k}$

$\dot{r} = 0$ $\qquad \dot{\theta} = -\omega$ $\qquad \dot{z} = 2ct$ $\qquad \dot{R} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{k} = r\dot{\theta}\hat{\theta} + \dot{z}\hat{k}$

$\ddot{r} = 0$ $\qquad \ddot{\theta} = 0$ $\qquad \ddot{z} = 2c$ $\qquad \ddot{R} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{k}$

$= -r\dot{\theta}^2\hat{r} + \ddot{z}\hat{k}$

$$\boxed{\begin{aligned} R(t) &= b\hat{r} + ct^2\hat{k} \\ \dot{R}(t) &= -b\omega\hat{\theta} + 2ct\hat{k} \\ \ddot{R}(t) &= -b\omega^2\hat{r} + 2c\hat{k} \end{aligned}}$$

$\boxed{\text{c) } |\ddot{R}| = \sqrt{b^2\omega^4 + 4c^2} \text{ this is the same as } |\vec{a}(t)| = \sqrt{b^2\omega^4 + 4c^2} \text{ in cartesian coords}}$