HW #4 - Joseph Specht

1) $V = Kr^4,\ K>0$ & $U = mV$

a) $F = -\frac{mv^2}{r}$ in a circular orbit

$rF = -mv^2 \Rightarrow mr^2F = -m^2v^2r^2$ & w/ $\mathcal{L} = mvr$ this gives

$mr^3F = -\mathcal{L}^2$ w/ force, we know $F = -\nabla U$, so

$-\mathcal{L}^2 = mr^3\left(-\frac{\partial U}{\partial r}\right) = mr^3\left(-4mKr^3\right)$

$\mathcal{L}^2 = 4m^2r^6K \Rightarrow \boxed{\mathcal{L} = 2mr^3\sqrt{K}}$

We also know $E = \frac{1}{2}\dot{r}^2 + \frac{\mathcal{L}^2}{2r^2} + U$, but $\dot{r} = 0$ b/c circular orbit

$E = \frac{\mathcal{L}^2}{2r^2m} + Kr^4m = \frac{4m^2r^6K}{2r^2m} + Kr^4m = 3mr^4K$

$@ r = a \Rightarrow \boxed{E = 3ma^4K \quad \& \quad \mathcal{L} = 2ma^3\sqrt{K}}$

b) $P = \frac{2\pi}{\omega}$ & $\omega = \frac{v}{r}$ since we know $\mathcal{L} = mvr$, we can factor out a factor of mr to get v

$\therefore v = 2a^2\sqrt{K}$ & $\omega = 2a\sqrt{K}$

$$\boxed{P = \frac{\pi}{a\sqrt{K}}}$$

c) Taylor expanding $U_{eff}(a+\delta r)$ gives...

$U_{eff}(a+\delta r) = U_{eff}(a) + U'_{eff}(a+\delta r) + \frac{1}{2}U''_{eff}(a+\delta r)^2 + \ldots$

We know $U'_{eff} = 0$ because this is a circular orbit, so it is at the bottom of a potential well

HW#4 - Joseph Specht

$\therefore U_{eff}(a+\delta r) = U_{eff}(a+\delta r) + \frac{1}{2} U_{eff}''(a+\delta r)^2$

This new potential takes the form of a gravitational potential plus a spring potential ($\frac{1}{2} k x^2$). Since $\frac{1}{2} U_{eff}''(a+\delta r)^2$ is in this form, we get

$$k = U_{eff}'' = \frac{\partial^2 U_{eff}}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial U_{eff}}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(\frac{l^2}{2mr^2} + mKr^4\right)\right)$$

$$= \frac{\partial^2}{\partial r^2}\left(\frac{(2mr^3\sqrt{K})^2}{2mr^2} + mKr^4\right) = \frac{\partial^2}{\partial r^2}\left(2mr^4K + mKr^4\right) = \frac{\partial^2}{\partial r^2}\left(3mr^4K\right)$$

$$\Rightarrow k = 36mr^2K$$

We disregard δr b/c $\delta r \ll a$, we also know $\omega = \sqrt{k/m}$, so

$$\omega = \sqrt{\frac{36mr^2K}{m}} = 6r\sqrt{K}$$

$$\omega @ r = a+\delta r) = \boxed{6a\sqrt{K}}$$

HW #4

2) $V = -\frac{K \exp(-\alpha r)}{r}$ $\quad K>0 \quad \alpha \geq 0$

a) $\vec{F}(r) = \frac{\partial(mV)}{\partial r} = -\left(-m\left(\frac{-r\alpha K \exp(-\alpha r) - mK\exp(-\alpha r)}{r^2}\right)\right)$

$= -\frac{mK\exp(-\alpha r)(r\alpha+1)}{r^2}$

$$\boxed{F = -\frac{mK\exp(-\alpha r)(r\alpha+1)}{r^2}}$$

if $\alpha = 0$, then $\exp(-\alpha r) = 1$ & $(r\alpha+1) = 1$

$$\boxed{\therefore F(\alpha=0) = -\frac{mK}{r^2}}$$

c) For stable circular states, $U'_{eff} = 0$ & $U''_{eff} > 0$.

$U_{eff} = U(r) + \frac{\mathcal{L}^2}{2mr^2} = -\frac{Km\exp(-\alpha r)}{r} + \frac{\mathcal{L}^2}{2mr^2}$

$U'_{eff} = -F(r) - \frac{\mathcal{L}^2}{mr^3} = \frac{mK\exp(-\alpha r)(r\alpha+1)}{r^2} - \frac{\mathcal{L}^2}{mr^3}$

U'_{eff} needs to equal 0 for this to be a stable point, so

$\frac{mK\exp(-\alpha r)(r\alpha+1)}{r^2} = \frac{\mathcal{L}^2}{mr^3} \Rightarrow mK\exp(-\alpha r)(r\alpha+1) = \frac{\mathcal{L}^2}{mr}$

HW#4

$$\frac{\ell^2}{m^2 k} = r\exp(-\alpha r)(1+\alpha r) \Rightarrow \frac{\ell^2}{m^2 k} = \exp(-\alpha r)(r+\alpha r^2)$$

This is the expression we wanted, but we need to find the right side has a max @ r_c desired

$$\frac{d}{dr}\left(\exp(-\alpha r)(r+\alpha r^2)\right) = 0 = -\exp(-\alpha r)(\alpha^2 r^2 - \alpha r - 1)$$

$-\exp(-\alpha r)$ is only 0 @ $r=\infty$, which is what we don't want, so we need to find when $(\alpha^2 r^2 - \alpha r - 1) = 0$

$$0 = \alpha^2 r^2 - \alpha r - 1 \quad \therefore \quad r = \frac{\alpha \pm \sqrt{\alpha^2 - 4\alpha^2(-1)}}{2\alpha^2} = \frac{\alpha \pm \sqrt{5\alpha^2}}{2\alpha^2}$$

$$r = \frac{\alpha \pm \alpha\sqrt{5}}{2\alpha^2} = \frac{1 \pm \sqrt{5}}{2\alpha}$$ we only care about positive b/c $r<0$ doesn't make sense

$$\boxed{\therefore r_c = \frac{1+\sqrt{5}}{2\alpha}}$$ found r_c, now find concavity

$$U''_{eff}(r) = -F'(r) - \frac{d}{dr}\left(-\ell^2/mr^3\right) = -F'(r) + 3\ell^2/mr^4$$

$$= -\left(\frac{mk\exp(-\alpha r)(\alpha^2 r^2 + 2\alpha r + 2)}{r^3}\right) + \frac{3\ell^2}{mr^3}$$

$$0 < \frac{mk\exp(-\alpha r)(\alpha^2 r^2 + 2\alpha r + 2)}{r^3} + \frac{3\ell^2}{mr^3}$$ plugging in r_c will always give a positive because this is an exp which can't be negative, times a positive number, divided by a positive, and plus another positive

$$\boxed{\therefore U''_{eff} > 0 \text{ @ } r_c \text{ \& } U'_{eff} = 0, \text{ so it is a stable orbit, } r < r_c}$$

HW#4

1) assuming conditions are satisfied,

$U(r) = -\frac{m\exp(-\alpha r)}{r}$ $\quad F(r) = -\frac{mK\exp(-\alpha r)(r\alpha+1)}{r^2}$

$\frac{F}{m} = a = \frac{-|\vec{v}|^2}{r}$ $\quad \therefore -|\vec{v}|^2 = -\frac{K\exp(-\alpha r)(r\alpha+1)}{r}$

$|\vec{v}|^2 = K\exp(-\alpha r)(r\alpha+1) \Rightarrow |\vec{v}| = \sqrt{\frac{K\exp(-\alpha r)(\alpha r+1)}{r}}$

90° out of phase

$\mathcal{L} = mrv\sin(\pi/2) = mrv = mr\sqrt{\frac{K\exp(-\alpha r)(\alpha r+1)}{r}}$

$\mathcal{L} = m\sqrt{rK\exp(-\alpha r)(\alpha r+1)}$

$$\boxed{|\mathcal{L}| = m\sqrt{rK\exp(-\alpha r)(\alpha r+1)}}$$

$E = \frac{1}{2}mv^2 + U = \frac{mK\exp(-r\alpha)(r\alpha+1)}{2r} - \frac{mK\exp(-\alpha r)}{r}$

$E = \frac{mK\exp(-\alpha r)}{r}\left(\frac{1}{2}(r\alpha+1) - 1\right) = \frac{mK\exp(-\alpha r)}{r}\left(\frac{r\alpha}{2} - \frac{1}{2}\right)$

$E = \frac{mK\exp(-\alpha r)}{2r}(r\alpha - 1)$

$$\boxed{E(@\, r=b) = \frac{mK\exp(-\alpha b)}{2b}(\alpha b - 1)}$$

HW#4

1) $U_{eff}(\ell+\delta r) = U_{eff}(\ell+\delta r) + U'_{eff}(\ell+\delta r)\cdot(\ell+\delta r) + \frac{1}{2}U''_{eff}(\ell+\delta r)\cdot(\ell+\delta r)^2 + \ldots$

$U'_{eff} = 0$ because it is a stable orbit

$\therefore U_{eff} = U_{eff}(\underset{\delta r \approx 0}{\ell}) + \frac{1}{2}U''_{eff}(\underset{\delta r \approx 0}{\ell})(\ell+\delta r)^2 + \ldots$

$\frac{1}{2}U''_{eff}(\ell+\delta r)^2$ takes form of spring potential $\frac{1}{2}kx^2$, so $U''_{eff} = k$. Also $\omega = \sqrt{k/m}$

$$U_{eff} = -\frac{mK\exp(-\alpha r)}{r} + \frac{\mathcal{L}}{2mr^2} = -\frac{mK\exp(-\alpha r)}{r} + \frac{m^2 r K\exp(-\alpha r)(\alpha r+1)}{2mr^2}$$

$$= -\frac{mK\exp(-\alpha r)}{r} + \frac{mK\exp(-\alpha r)(\alpha r+1)}{2r} = \frac{-2mK\exp(-\alpha r) + mK\exp(-\alpha r+1)}{2r}$$

$$= \frac{mK\exp(-\alpha r)}{2r}(\alpha r - 1)$$

from a calculator online

$$U''_{eff} = \frac{mK\exp(-\alpha r)(\alpha^3 r^3 - \alpha^2 r^2 - 2\alpha r - 2)}{2r^3} = K$$

$$\omega = \sqrt{\frac{K(\alpha^3 r^3 - \alpha^2 r^2 - 2\alpha r - 2)}{2r^3 \exp(\alpha r)}}$$

$$\boxed{\omega(@\, r = \ell) = \sqrt{\frac{K(\alpha^3 b^3 - \alpha b^2 - 2\alpha b - 2)}{2b^3 \exp(-\alpha b)}}}$$

HW #4

3) 

$$\rho = \frac{M}{L} = \frac{dM}{dx} \qquad s = \sqrt{x^2+y^2}$$

$$\Phi = -\int \frac{G\,dM}{s}$$

$$\Phi = -G\int_{-L/2}^{L/2} \frac{\rho\,dx}{\sqrt{x^2+y^2}} = -G\rho\int_{-L/2}^{L/2}\frac{dx}{\sqrt{x^2+y^2}}$$ given the integral...

$$\Phi = -G\rho\left(\ln\left(x+\sqrt{x^2+y^2}\right)\right)\Big|_{-L/2}^{L/2} = -G\rho\ln\left(\frac{\frac{L}{2}+\sqrt{L^2/4+y^2}}{-L/2+\sqrt{L^2/4+y^2}}\right)$$

multiplying through by 2...

$$\Phi = -G\rho\ln\left(\frac{L+\sqrt{L^2+4y^2}}{-L+\sqrt{L^2+4y^2}}\right)$$ since $y \gg L$, $\sqrt{L^2+y^2} \approx y$

$$\therefore \Phi = -G\rho\ln\left(\frac{L+2y}{-L+2y}\right)$$

both ≈ 0 b/c $y \gg L$

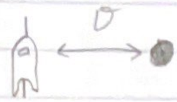
The taylor expansion of $\ln\left(\frac{L+2y}{-L+2y}\right)$ is $\left(\frac{L}{y} + \frac{L^3}{12y^3} + \frac{L^5}{80y^5} + \dots\right)$

$\therefore \Phi = -\frac{G\rho L}{y}$ but $\rho = \frac{M}{L}$, so $\boxed{\Phi = -\frac{GM}{y}}$

This is the expression we wanted

HW#4

A)



a) $F = ma \Rightarrow a = \frac{F}{m}$ w/ $F = \frac{GMm}{r^2}$

$\therefore a = -\frac{GM}{r^2}$, but @ radius D

$$\boxed{a = \frac{GM}{D^2}}$$

b) $a = \frac{v^2}{r}$, $v^2 = ar = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$

$\omega r = v$, $\omega = \frac{v}{r}$ w/ $P = \frac{2\pi}{\omega}$ we get $P = \frac{2\pi r}{v}$

$P = 2\pi D \sqrt{\frac{D}{GM}}$ $\boxed{P = 2\pi \sqrt{\frac{D^3}{GM}}}$

c) $M = 5$ solar $= 5 \cdot 2e30$ kg $= 10e30$ kg

$D = 5e3$ km $= 5e6$ m $G = 6.67e\text{-}11$ N m²/kg² } Black Hole

$\boxed{\begin{array}{l} P_{\text{Black Hole}} = 2.72\text{ s} \quad \text{much less than Period } \cancel{\text{ISS}} \\ \text{Period}_{ISS} = 5325\text{ s} = 88.76\text{ minutes} \checkmark \quad \text{about the period of ISS} \end{array}}$

HW#4

d) $\vec{a} = \frac{GM}{D^2}\hat{r}$ $\quad \Delta a = \vec{a}(D-R) - \vec{a}(D)$

$$\Delta a = \frac{GM}{(D-R)^2} - \frac{GM}{D^2} = GM\left(\frac{1}{(D-R)^2} - \frac{1}{D^2}\right) = \frac{GM}{D^2}\left(\frac{1}{\left(1-\frac{R}{D}\right)^2} - 1\right)$$

The taylor expansion of $(1-x)^{-2}$ is — used taylor series calculator

$$(1-x)^{-2} \approx 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$\therefore \left(1-\frac{R}{D}\right)^2 \approx 1 + 2\frac{R}{D} + \ldots$ anything more is very small

$$\therefore \Delta a = \frac{GM}{D^2}\left(\left(1+2\frac{R}{D}\right)-1\right) = \frac{GM}{D^2}\left(\frac{2R}{D}\right) = \frac{2GMR}{D^3}$$

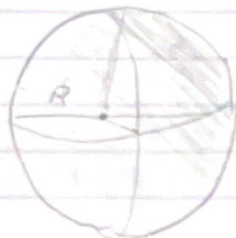
$$\boxed{\Delta a = \frac{2GMR}{D^3}}$$

e) $$\Delta a_{\text{black hole}} = \frac{\overset{G}{(6.67e{-11})}\,\overset{M}{(1e31)}\,\overset{R}{(1)} \cdot 2}{\underset{D}{(5e6)^3}}$$

$\boxed{\Delta a = 10.672 \text{ m/s}^2}$ This would feel like every ~3 feet or so you would experience about 2× earth gravity, so if you're about 6 feet tall, your legs would feel 3 times heavier & your midsection would feel 2 times heavier.

HW#4

5)



Radius R — assuming low thickness

Mass M

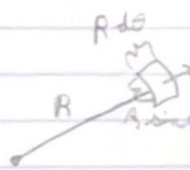
we know $\rho = \frac{M}{\text{surface area}} = \frac{dM}{dA}$ $\therefore \rho\, dA = dM$

w/ spheres $sa = 4\pi R^2$ $\therefore \rho = \frac{M}{4\pi R^2}$

Starting w/ the outside potential (guided by prelectures)

$\Phi = -G \int_A \frac{\rho}{s}\, dA$ but $\rho\, dA = dM$, so

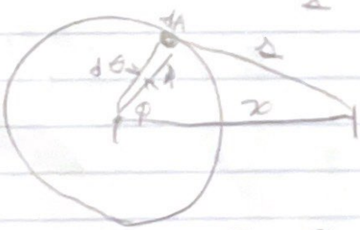
$\Phi = -G \int \frac{dM}{s}$



area density

$dM = \rho\, R\, d\theta\, R \sin\theta\, d\phi = \rho R^2 \sin\theta\, d\theta\, d\phi$

$\Phi = -G \iint \frac{\rho R^2 \sin\theta}{s}\, d\theta\, d\phi$ according to the law of cosines...



$s^2 = R^2 + x^2 - 2Rx\cos\theta$

$2s\frac{ds}{d\theta} = 2Rx\sin\theta \Rightarrow \frac{\sin\theta\, d\theta}{s} = \frac{ds}{Rx}$

$\Phi = -G\rho R^2 \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\sin\theta}{s}\, d\theta\, d\phi = -2\pi G\rho R^2 \int_{\theta=0}^{\pi} \frac{\sin\theta}{s}\, d\theta$

plugging in expression for $\frac{\sin\theta\, d\theta}{s}$ we get

$\Phi = -2\pi G\rho R^2 \int_{s_{min}}^{s_{max}} \frac{ds}{Rx} = \frac{-2\pi G\rho R}{x}(s_{max} - s_{min})$

HW#4

we know $^{\text{we are}}$ outside the shell, so

$s_{max} = x + R$ & $s_{min} = x - R$

$$\therefore \Phi = \frac{-2\pi G \rho R}{x}\left(x + R - (x - R)\right) = \frac{-2\pi G\rho R}{x}(2R) = -\frac{4\pi G \rho R^2}{x}$$

we know $M = 4\pi R^2 \cdot \rho$ $\therefore \Phi = -\frac{GM}{x}$ for outside

$$\Phi^{shell}_{outside} = -\frac{GM}{r} \quad \& \quad a = \frac{\partial \Phi}{\partial r} \quad \left(\frac{\vec{F}}{m} = a = -\frac{\vec{\nabla} U}{m} = -\vec{\nabla}\Phi\right)$$

$a^{shell}_{outside} = -\frac{GM_s}{r^2}\hat{r}$ $\quad a^{point}_{outside} = -\frac{GM_p}{r^2}\hat{r}$ ✓ just known

$$\boxed{\therefore a_{outside} = -\frac{G}{r^2}(M_s + M_p)\hat{r}}$$

acceleration inside the sphere

we know from lecture $\Phi_{shell} = -\frac{GM_s}{R}$ which is constant, so

$a_{shell} = -\nabla\Phi_{shell} = 0$, but point mass still has $-\frac{GM_p}{r^2}\hat{r}$

$a_{inside} = 0 + -\frac{GM_p}{r^2}\hat{r}$

$$\boxed{a_{inside} = -\frac{GM_p}{r^2}\hat{r}}$$

$\Phi_{outside} = -\frac{GM_s}{r} - \frac{GM_p}{r}$ & $\Phi_{inside} = -\frac{GM_s}{R} - \frac{GM_p}{r}$

but @ the shell boundary, $r = R$, so

$\Phi_{outside} = \Phi_{inside} = -\frac{2G}{R}(M_s - M_p)$ this shows Φ is continuous!