HW#3 - Joseph Specht

1)

$\leftarrow dm$

$F_{drag} = -Kv$

$K > 0$

(no fuel)

m_i = initial mass, m_f = final mass

u = speed of fuel

$(m_i - m_f) \to kg$ $\quad \alpha \to \frac{kg}{s}$

a) $F = ma$ = thrust - drag

$$ma = u\frac{dm}{dt} - kv$$

$$F = \frac{dP}{dt} \Rightarrow F\,dt = dP \Rightarrow dP = (P(t+dt) - P(t))$$

$$P(t+dt) = \underbrace{(m_i - dm)}_{\text{mass car}}\underbrace{(v_0 + dv)}_{v \text{ car}} + \underbrace{dm}_{m \text{ fuel}}\underbrace{(v_0 - u)}_{v \text{ fuel}}$$

$$P(t) = m_i v_0$$

$dm\,dv \approx 0$ 2 small things time each other

$$dP = m_i v_0 + m_i\,dv - \cancel{dm\,v_0} - \underbrace{dm\,dv}_{} + \cancel{dm\,v_0} - dm\,u - m_i v_0$$

$$F\,dt = m\,dv - dm\,u$$

$$-kv\,dt = m\,dv - dm\,u \Rightarrow dm\,u = m\,dv + kv\,dt$$

$$\Rightarrow \dot{m}u = m\dot{v} + kv \Rightarrow \boxed{m\dot{v} = u\dot{m} - kv}$$

$dM = -dm$

b) $\dot{M} = -\alpha \Rightarrow \dot{m} = \alpha$

$$M\dot{v} = u\alpha - kv \Rightarrow M\frac{dv}{dt} = u\alpha - kv \Rightarrow \frac{M\,dv}{u\alpha - kv} = dt$$

$m_i - m_f = \alpha t$; t need to go to when fuel is all used up $\quad t_f = \frac{m_i - m_f}{\alpha}$

$M = m_i - \alpha t$ ← M is a function of time

$$\frac{dv}{u\alpha - kv} = \frac{dt}{m_i - \alpha t} \Rightarrow \int_0^{v_f}\frac{dv}{u\alpha - kv} = \int_0^{t_f}\frac{dt}{m_i - \alpha t} \Rightarrow \frac{-1}{k}\left[\ln(u\alpha - kv_f) - \ln(0)\right]$$

$$= \frac{-1}{\alpha}\left[\ln(m_i - \alpha t)\right]_0^{t_f} \Rightarrow \frac{-1}{\alpha}\left(\ln\left(m_i - \frac{\alpha(m_i - m_f)}{\alpha}\right) - \ln(m_i - 0)\right)$$

Crete-Monee High School

(2) Dimensional analysis for $\left(\frac{K}{\alpha}\right)$

(1) need to use v_f, v_0 & t_f, t_0 for unitless dimension in $\ln(\ldots)$

HW 3 - cont

1 e) $\cancel{\frac{-1}{k}\left(\ln(u\alpha - kv_f) - 1\right) = \frac{-1}{\alpha}\left(\ln(m_i - m_i + m_f) - \ln(m_i)\right)}$

$\cancel{= \frac{-1}{k}(\ln(u\alpha - kv_f) - 1) = \frac{-1}{\alpha}\ln\left(\frac{m_f}{m_i}\right) \Rightarrow \frac{\alpha}{k}\ln(u\alpha - kv_f) - \frac{\alpha}{k} = \ln\left(\frac{m_f}{m_i}\right)}$

$\cancel{\Rightarrow (u\alpha - kv_f)^{\frac{\alpha}{k}} - e^{\frac{\alpha}{k}} = \frac{m_f}{m_i} \Rightarrow (u\alpha - kv_f)^{\frac{\alpha}{k}} = \frac{m_f}{m_i} + e^{\frac{\alpha}{k}}}$

$M\dot{v} = u\alpha - kv$, $M = m_i - \alpha t$, $\alpha t_f = m_i - m_f \Rightarrow t_f = \frac{m_i - m_f}{\alpha}$

$(m_i - \alpha t)\frac{dv}{dt} = u\alpha - kv \Rightarrow (m_i - \alpha t)\,dv = (u\alpha - kv)\,dt$

$\Rightarrow \frac{dv}{(u\alpha - kv)} = \frac{dt}{(m_i - \alpha t)}$ (1) w/ these bounds, easier to solve $\Rightarrow \int_{v_0}^{v_f} \frac{dv}{(u\alpha - kv)} = \int_{t_0}^{t_f} \frac{dt}{m_i - \alpha t}$

$\Rightarrow \frac{-1}{k}\left(\ln(u\alpha - kv_f) - \ln(u\alpha - kv_0)\right) = \frac{-1}{\alpha}\left(\ln(m_i - \alpha t_f) - \ln(m_i - \alpha t_0)\right)$

$\Rightarrow \ln\left(\frac{u\alpha - kv_f}{u\alpha - kv_0}\right) = \frac{k}{\alpha}\ln\left(\frac{m_i - \alpha t_f}{m_i - \alpha t_0}\right) \Rightarrow \ln\left(\frac{u\alpha - kv_f}{u\alpha - kv_0}\right) = \ln\left(\frac{m_i - \alpha t_f}{m_i - \alpha t_0}\right)^{\left(\frac{K}{\alpha}\right)}$ (2)

$\Rightarrow \frac{u\alpha - kv_f}{u\alpha - kv_0} = \left(\frac{m_i - \alpha t_f}{m_i - \alpha t_0}\right)^{\left(\frac{K}{\alpha}\right)}$ plugging in initial conditions $v_0 = 0$, $t_f = \frac{m_i - m_f}{\alpha}$, $t_0 = 0$

$\Rightarrow \frac{u\alpha - kv_f}{u\alpha} = \left(\frac{m_i - \alpha\left(\frac{m_i - m_f}{\alpha}\right)}{m_i}\right)^{\left(\frac{K}{\alpha}\right)} \Rightarrow 1 - \frac{kv_f}{u\alpha} = \left(\frac{m_i - m_i + m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)}$

$\Rightarrow 1 - \frac{kv_f}{u\alpha} = \left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)} \Rightarrow u\alpha - kv_f = u\alpha\left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)}$

$\Rightarrow u\alpha - u\alpha\left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)} = kv_f \Rightarrow v_f = \frac{u\alpha}{k} - \frac{u\alpha}{k}\left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)}$

$\Rightarrow \boxed{v_f = \frac{u\alpha}{K}\left(1 - \left(\frac{m_f}{m_i}\right)^{\left(\frac{K}{\alpha}\right)}\right)}$ ← dimensions work

HW#3

Phys 325 - Joseph Specht

1(c) $v_f = \frac{u\alpha}{k}\left(1-\left(\frac{m_f}{m_i}\right)^{\left(\frac{k}{\alpha}\right)}\right)$ $\quad \frac{m}{s}\cdot\frac{kg}{s}/\frac{kg}{s}\left(1-\left(\frac{kg}{kg}\right)^{\left(\frac{kg}{s}/\frac{kg}{s}\right)}\right)$

units work out & makes physical sense,
$\uparrow\alpha$, faster : $\uparrow k$, slower

$\frac{m_f}{m_i}$ tells ratio of how much fuel we can use & this to the exponent $\left(\frac{k}{\alpha}\right)$ & subtracted from 1 means the higher k is, the slower we go $\left(1-\left(\frac{m_f}{m_i}\right)^{\frac{k}{1}}\right)$ means we'll have a lower total which makes sense
increasing α means higher end total $\left(1-\frac{m_f}{m_i}^{\left(\frac{1}{\alpha}\right)}\right)$ & this makes sense b/c the faster we burn fuel, the less time drag has to act

d) match eq 6 . $v = v_0 - u\ln\left(\frac{m}{m_0}\right)$, use Taylor approx of $\ln(x)$

Taylor expansion of A^x about $x=0$

Derivative of $A^x = A^x \ln A$

$f(x) \approx 1 + f'(x)x + \ldots \qquad A^x \approx 1 + A^x\ln(A)\,x + \ldots$

$\therefore \left(\frac{m_f}{m_i}\right)^{\left(\frac{k}{\alpha}\right)} \approx 1 + \left(\frac{m_f}{m_i}\right)^{\left(\frac{k}{\alpha}\right)}\ln\left(\frac{m_f}{m_i}\right)\left(\frac{k}{\alpha}\right)+\ldots$

Plugging in
$v_f = \frac{u\alpha}{k}\left(1-\left(1+\left(\frac{m_f}{m_i}\right)^{\left(\frac{k}{\alpha}\right)}\ln\left(\frac{m_f}{m_i}\right)\left(\frac{k}{\alpha}\right)\right)\right)$ since $\frac{k}{\alpha}\approx 0$, $\left(\frac{m_f}{m_i}\right)^{\left(\frac{k}{\alpha}\right)}\approx 1$

$\Rightarrow v_f = \frac{u\alpha}{k}\left(1-\left(1+1\cdot\ln\left(\frac{m_f}{m_i}\right)\left(\frac{k}{\alpha}\right)\right)\right) \Rightarrow v_f = \frac{u\alpha}{k}\left(\frac{k}{\alpha}\ln\left(\frac{m_f}{m_0}\right)\right)$

$\Rightarrow v_f = u\ln\left(\frac{m_f}{m_0}\right)$ $\qquad t_f\alpha = m_i - m_f \qquad \alpha = \frac{m_i - m_f}{t_f}$

$\Rightarrow v_f = -u\ln\left(\frac{m_0}{m_f}\right)$

HW#3

1a) $v_f = -u \ln\left(\frac{m_f}{m_0}\right)$ make equal to $v = v_0 - u \ln\left(\frac{m}{m_0}\right)$

for us, $v_0 = 0$ & $m = m_f$, so $v = -u \ln\left(\frac{m_f}{m_0}\right)$

in this case $v_f = v = -u \ln\left(\frac{m_f}{m_0}\right)$

these equations are the same in this situation

e) generally $v = v_0 - u \ln\left(\frac{m_0}{m_f}\right)$

v_f reduces to this because the force equation is

eq w/o $-kv$ force $\rightarrow m\dot{v} = u\dot{m} \Rightarrow m\frac{dv}{dt} = u\frac{dm}{dt} \Rightarrow m\,dv = u\,dm$

$\Rightarrow dv = \frac{u\,dm}{m} \Rightarrow \int_{v_0}^{v_f} dv = u\int_{m_0}^{m_f} \frac{dm}{m} \Rightarrow v_f - v_0 = u \ln\left(\frac{m_f}{m_0}\right)$

$\Rightarrow v_f = v_0 + u \ln\left(\frac{m_f}{m_0}\right) \Rightarrow v_f = v_0 - u \ln\left(\frac{m_0}{m_f}\right)$

if k is small, then the $(-kv)$ term drops out in the original equation & then it simplifies down to eq 6

k/a also ≈ 0 if $k \ll a$, so the old expression is still valid

HW #3

$$3)\ F(x,y,z) = -a(\overset{M}{x\,i} + \overset{N}{2y\,j} + \overset{P}{3z\,\hat{k}}) \qquad \vec{F} = -a\vec{\nabla} f \qquad U = -af$$

a) $F(x,y,z)$ conservative if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \qquad \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y} \qquad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$$

$$F_x = -ax \Rightarrow f_x = \int x\,dx = \frac{x^2}{2} + g(y,z) + C$$

$$F_y = -2ay \Rightarrow f_y = \int 2y\,dy = y^2 + g(x,z) + C$$

$$F_z = -3az \Rightarrow f_z = \int 3z\,dz = \frac{3}{2}z^2 + g(x,y) + C$$

$$f = f_x + f_y + f_z = \frac{x^2}{2} + y^2 + \frac{3}{2}z^2 + C$$

Potential function $U = -a\left(\frac{x^2}{2} + y^2 + \frac{3}{2}z^2\right) + C$

A 3-D spring w/ different spring constants in each direction would give this potential function

HW#3

3a) $F(x,y,z) = (-y, x, 0)$

$\frac{\partial F_x}{\partial y} = -1 \neq \frac{\partial F_y}{\partial x} = 1$ Fails ~~test~~, ~~so~~ not conservative

c) $F(x,y,z) = (2axz^3 + 2axy^3,\ 2ayz^3 + 2ay^4 + 3ax^2y^2 + 3ay^4,$
$3ax^2z^2 + 3ay^2z^2)$

$\frac{\partial F_x}{\partial y} = 6axy^2 = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = 6axz^2 = \frac{\partial F_z}{\partial x}$, $\frac{\partial F_y}{\partial z} = 6ayz^2 = \frac{\partial F_z}{\partial y}$

$\vec{F}(x,y,z) = -a \vec{\nabla} f(x,y,z)$ $U = a f(x,y,z)$

$F_x = 2axz^3 + 2axy^3 \Rightarrow f_x = \int(2xz^3 + 2xy^3)\,dx = x^2z^3 + x^2y^3 + g(y,z) + C$

$F_y = 2ayz^3 + 2ay^4 + 3ax^2y^2 + 3ay^4 \Rightarrow f_y = \int(2yz^3 + 3x^2y^2 + 5y^4)\,dy$

$= y^2z^3 + x^2y^3 + y^5 + g(x,z) + C$

$F_z = 3ax^2z^2 + 3ay^2z^2 \Rightarrow f_z = \int(3x^2z^2 + 3y^2z^2)\,dz = x^2z^3 + y^2z^3 + g(x,y) + C$

$f = (f_x \cup f_y \cup f_z) + C = x^2z^3 + x^2y^3 + y^2z^3 + y^5 + C$

$$\boxed{U = a(x^2y^3 + x^2z^3 + y^5 + y^2z^3 + y^5) + C}$$

? do we need a constant

HW#3 - Joseph Spedto

4) $\vec{F}(\vec{r}) = (Az^2, By^3, 2Axz)$ $\qquad \vec{F} = \vec{\nabla} U$

a) $\frac{\partial F_x}{\partial y} = 0 = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = 2Az = \frac{\partial F_z}{\partial x}$, $\frac{\partial F_y}{\partial z} = 0 = \frac{\partial F_z}{\partial y}$ ✓ conservative

$F_x = Az^2$, $f_x = A\int z^2\,dx \Rightarrow Axz^2 + g(y,z) + C$

$F_y = By^3$, $f_y = B\int y^3\,dy \Rightarrow By^4/4 + g(x,z) + C$

$F_z = 2Axz$, $f_z = 2A\int xz\,dz \Rightarrow Axz^2 + g(x,y) + C$

$\boxed{U = f_x \cup f_y \cup f_z = Axz^2 + B\frac{y^4}{4}}$ $\quad C = 0$

b) $\vec{F} = m\vec{a} = m\dot{\vec{v}} = m\frac{d\vec{v}}{dt}$

$m\,d\vec{v} = \vec{F}\,dt \Rightarrow m\int_{\vec{v}_0}^{\vec{v}_f} d\vec{v} = \int_0^t (Az^2, By^3, 2Axz)\,dt$

$\Rightarrow m(\vec{v}_f - \vec{v}_0) = (Az^2t, By^3t, 2Axzt)$

$\Rightarrow \vec{v}_f = \vec{v}_0 + (Az^2t, By^3t, 2Axzt)/m$

@ $\vec{r} = -\hat{i} + 2\hat{j} + \hat{k}$

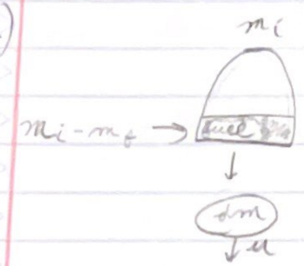
$\Rightarrow \vec{v}_f = \vec{v}_0 + (At, 8Bt, -2At)/m$

$\Rightarrow |\vec{v}_f| = \sqrt{v_0^2 + \frac{A^2t^2 + 64B^2t^2 + 4A^2t^2}{m^2}} = \sqrt{v_0^2 + \frac{5A^2t^2 + 64B^2t^2}{m^2}}$

$|\vec{v}_f| = \sqrt{v_0^2 + \frac{t^2}{m^2}(5A^2 + 64B^2)}$

HW #3

2)



$\downarrow g/6 = g_e \quad g = -9.81$

a) ~~$m\dot{v} = u\dot{m} - mg_e \Rightarrow m\frac{dv}{dt} = u\frac{dm}{dt} - mg_e \Rightarrow m\,dv = u\,dm - mg_e\,dt$~~

~~$\Rightarrow m\,dv = 0$ b/c v doesn't change $dv = 0$~~

~~$\Rightarrow u\,dm = mg_e\,dt \Rightarrow \int_{m_i}^{m_f} u\,dm = g_e \int_0^{t_f} m\,dt$~~

$F = ma = u\dot{m} - mg_e \Rightarrow a = \frac{u\dot{m}}{m} - g_e$, $a = 0$ want to stay still

$m = m_0 - dm$

$\Rightarrow g_e = \frac{u}{m}\frac{dm}{dt} \Rightarrow g_e\,dt = \frac{u}{m}\,dm \Rightarrow g_e \int_0^{t_f} dt = u \int_{m_i}^{m_f} \frac{1}{m}\,dm$

$\Rightarrow g_e t_f = u(\ln(m_f) - \ln(m_i)) \Rightarrow g_e t_f = u \ln\left(\frac{m_f}{m_i}\right) \Rightarrow t_f = \frac{u}{g_e}\ln\left(\frac{m_f}{m_i}\right)$

$$\boxed{t_f = \frac{u}{g_e}\ln\left(\frac{m_f}{m_i}\right)}$$

b) units make sense b $\left[\frac{m}{s} / \frac{m}{s^2} = s\right]$ & the less mass @ the end, the more fuel is used the greater the mag of ln, but it is negative. However g_e is negative, so they cancel out to give a $t_f > 0$

c) $t_f = \frac{1500}{g_e}\ln\left(\frac{2}{3}\right) = \frac{1500\ m/s}{-1.635\ m/s^2} \cdot -.40546 = 371.99\,s$

$$\boxed{t_f = 371.99\,s}$$

HW#3 - Joseph Spallo

5)

d) $F = \frac{dp}{dt} \Rightarrow F\,dt = dp \Rightarrow dp = (p_f - p_i)$

$\Rightarrow p_f = (m_0 + \sigma dt)(v_0 + dv) = m_0 v_0 + v_0 \sigma dt + m dv + \cancel{\sigma dt\, dv}^{\,0}$

$p_i = m_0 v_0$

$dv < 0$, dv needs to be negative otherwise $\int$ doesn't work

$dp = v_0 \sigma dt + m dv$

$\Rightarrow F = 0$, on ice so no friction & $\Sigma F_y = 0$

$\Rightarrow 0 = v_0 \sigma dt + m dv \Rightarrow m dv = -v_0 \sigma dt \Rightarrow m_0 \int_{v_0}^{v_f} dv = -v_0 \sigma \int_0^{t_f} dt$

$\Rightarrow m_0 (v_f - v_0) = -v_0 \sigma t_f \Rightarrow \boxed{v_f = v_0 - \frac{v_0 \sigma t_f}{m_0}}$

e) $m \dot{v} dt = dp = (p_f - p_i)$

$p_f = m(v + dv) + dm(v - u) = mv + m dv + v dm - u dm$

$p_i = mv$

$dp = m dv + v dm - u dm$

$\Rightarrow 0 = m dv + v dm - u dm \Rightarrow m dv = u dm - v dm \Rightarrow m dv = (u - v) dm$

$\Rightarrow \frac{m}{u - v} dv = dm \Rightarrow \int_{v_0}^{v_f} \frac{m}{u - v} dv \int_{m_0}^{m_f} \overset{\sigma dt}{dm} \Rightarrow \int_{v_0}^{v_f} \frac{m_0}{u - v} dv = \sigma \int_0^{t_f} dt$

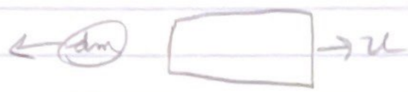
$\Rightarrow -m_0 (\ln(u - v_f) - \ln(u - v_0)) = \sigma t_f \Rightarrow \ln\left(\frac{u - v_f}{u - v_0}\right) = \frac{-\sigma t_f}{m_0}$

$\Rightarrow \frac{u - v_f}{u - v_0} = e^{\left(-\frac{\sigma t_f}{m_0}\right)} \Rightarrow u - v_f = (u - v_0) e^{-\frac{\sigma t_f}{m_0}} \Rightarrow v_f = u - (u - v_0) \exp\left(-\frac{\sigma t_f}{m_0}\right)$

$\boxed{v_f = u - (u - v_0) \exp\left(-\frac{\sigma t_f}{m_0}\right)}$ $\quad v_f = u - \frac{(u - v_0)}{\exp\left(\frac{\sigma t_f}{m_0}\right)}$

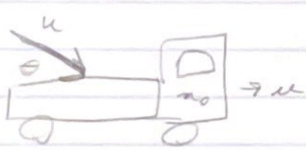
HW#3

5c) you always have at least dm in the truck, so



If the car is traveling at speed u already, once it gets another dm inside the pump, the speed slows to $P/(m_0+dm)$ because momentum is conserved, Then once it ejects the dm, the speed goes to u again (P/m_0).

d)



$F\,dt = dp = (P_f - P_i)$ $\qquad$ $P_f = (m_0 + \sigma dt)(v_0 + dv)$

$P_i = (m_0 v_0) + (u\sigma\,dt)$

$$P_f = m_0 v_0 + m_0\,dv + \sigma v_0\,dt + \overset{0}{\cancel{\sigma dt\,dv}}$$
$$P_i = m_0 v_0 + u\sigma\,dt$$

$$dP = m_0\,dv + \sigma v_0\,dt - u\sigma\,dt$$

$$F = m\dot{v} \Rightarrow \overset{0}{F}\,dt = dp \Rightarrow 0 = m_0\,dv + \sigma v_0\,dt - u\sigma\,dt$$

$$\Rightarrow -m_0\,dv = \sigma(v_0 - u)\,dt \Rightarrow -m_0\int_{v_0}^{v_f} dv = \sigma\int_0^{t_f}(v_0 - u)\,dt$$

$$\Rightarrow v_f - v_0 = \frac{-\sigma}{m_0}\int_0^{t_f}(v_0 - u)\,dt \Rightarrow v_f = v_0 - \frac{\sigma(v_0 - u)t_f}{m_0}$$

$$v_f = v_0 - \frac{\sigma t_f(v_0 - u)}{m_0} \qquad v_f = v_0 - \frac{\sigma t_f}{m_0}(v_0 - u)$$

If the component of u in the direction the truck is moving, is greater than the speed of the truck, it will help