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1) Binomial, $p \neq 1/2$. N_0 molecules in V_0. N molecules in sub-volume V. Equally distributed in space. Prob molecule in V is V/V_0.

a) $\langle N \rangle = N_0 \frac{V}{V_0} = N_0 p$

b) $\frac{\sigma_N^2}{\langle N \rangle^2} = \frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle^2} = \frac{\langle N^2 \rangle - 2\langle N \rangle^2 + \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle N^2 \rangle}{\langle N \rangle^2} - 1$

know $\langle N \rangle^2 = \left(N_0 \frac{V}{V_0}\right)^2$, so need $\underline{\langle N^2 \rangle}$

start w/ log derivative: $p \frac{\partial}{\partial p}(p^N) = p(N p^{N-1}) = Np$

$\Rightarrow$ know $(p+q)^{N_0} = \sum_{N=0}^{N_0} \binom{N_0}{N} p^N q^{N_0-N} \Rightarrow p\frac{\partial}{\partial p}(LHS) = p\frac{\partial}{\partial p}(RHS)$

$\Rightarrow$ apply log derivative: $pN_0 (p+q)^{N_0-1} = \sum_{N=0}^{N_0} \binom{N_0}{N} N p\, p^{N-1} q^{N_0-N} = \sum_{N=0}^{N_0} \binom{N_0}{N} N p^N q^{N_0-N}$

$\Rightarrow$ again: $p\frac{\partial}{\partial p}\left(pN_0 (p+q)^{N_0-1}\right) = pN_0 \left(\frac{\partial}{\partial p}\left(p(p+q)^{N_0-1}\right)\right)$

$= pN_0\left((p+q)^{N_0-1} + p(N_0-1)(p+q)^{N_0-2}\right) = \sum_{N=0}^{N_0} \binom{N_0}{N} N^2 p^N q^{N_0-N}$

$\Rightarrow RHS = \langle N^2 \rangle$ & set $p+q=1$: $\langle N^2 \rangle = pN_0(1 + p(N_0-1)) = pN_0(1 + pN_0 - p)$

$= pN_0 + p^2 N_0^2 - p^2 N_0 = \langle N^2 \rangle$

$\Rightarrow$ as $\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2$: $\sigma_N^2 = pN_0 + p^2N_0^2 - p^2N_0 - N_0^2 p^2 = pN_0 - p^2 N_0$

$\therefore \frac{\sigma_N^2}{\langle N \rangle^2} = \frac{pN_0 - p^2N_0}{N_0^2 p^2} = \frac{1-p}{N_0 p} = \frac{1}{N_0}\left(\frac{1-p}{p}\right) = \frac{1}{N_0}\left(\frac{1}{p} - 1\right)$ w/ $p = \frac{V}{V_0}$

$$\boxed{\frac{\sigma_N^2}{\langle N \rangle^2} = \frac{1}{N_0}\left(\frac{V_0}{V} - 1\right)}$$

c) if $V \ll V_0$, $\left(\frac{V_0}{V} - 1\right) = \left(\frac{V_0}{V} - \frac{V}{V}\right) = \left(\frac{V_0 - V}{V}\right) \approx \frac{V_0}{V}$ $\boxed{\therefore \frac{\sigma_N^2}{\langle N \rangle^2} = \frac{V_0}{N_0 V}}$

d) variance should be 0 if $V \to V_0$

$\Rightarrow \frac{1}{N_0}\left(\frac{V_0}{V_0} - 1\right) = \frac{1}{N_0}(1-1) = 0$ ✓ guess agrees w/ b)

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2) Poisson is binomial w/ $1 \ll N$ & $p \ll 1$ w/ $Np = C$, $x \ll N$

Show $P_p(n,N) \approx \frac{a^n}{n!} \exp(-a)$ w/ $a = Np = \langle n \rangle$

~~start~~ w/ $P_p(n,N) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{(N-n)!\,n!} p^n q^{N-n}$

$\Rightarrow$ w/ Stirling approx: $\ln\left(\frac{N!}{(N-n)!}\right) = \ln(N!) - \ln((N-n)!)$

$= N\ln(N) - N - (N-n)\ln(N-n) + (N-n) = N\ln(N) - (N-n)\ln(N-n) - n$

$=$ w/ $\ln(N-n) = \ln(N(1-\frac{n}{N})) = \ln(N) + \ln(1-\frac{n}{N}) \approx \ln(N) - \underbrace{\frac{n}{N}}_{\sim 0 \text{ as } n \ll N} \approx \ln(N)$

$\Rightarrow N\ln(N) - (N-n)\ln(N) - n = N\ln(N) - N\ln(N) + n\ln(N) - n = n\ln(N) - n = \ln(N^n) - n$

$\Rightarrow$ exp both sides: $\frac{N!}{(N-n)!} = \exp(\ln(N^n) - n) = N^n \exp(-n)$

~~as~~ $N \to \infty$, $Np = \langle n \rangle \to \infty$ w/ $1 \ll n \ll N$ & $\exp(-n) \approx 1$ $\therefore \frac{N!}{(N-n)!} = N^n$

$\Rightarrow P_p \approx \frac{N^n p^n}{n!} q^{N-n} = \frac{(Np)^n}{n!} q^{N-n} = \frac{a^n}{n!}\left(1 - \frac{a}{N}\right)^{N-n}$

Know: $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^{bn} = \exp(ab)$ & $\left(1 - \frac{a}{N}\right)^{N-n} \approx \left(1 - \frac{a}{N}\right)^{N} \approx \exp(-a)$

$$\boxed{\therefore \text{ as } N \to \infty \quad P(n,N) \approx \frac{a^n}{n!} \exp(-a)}$$

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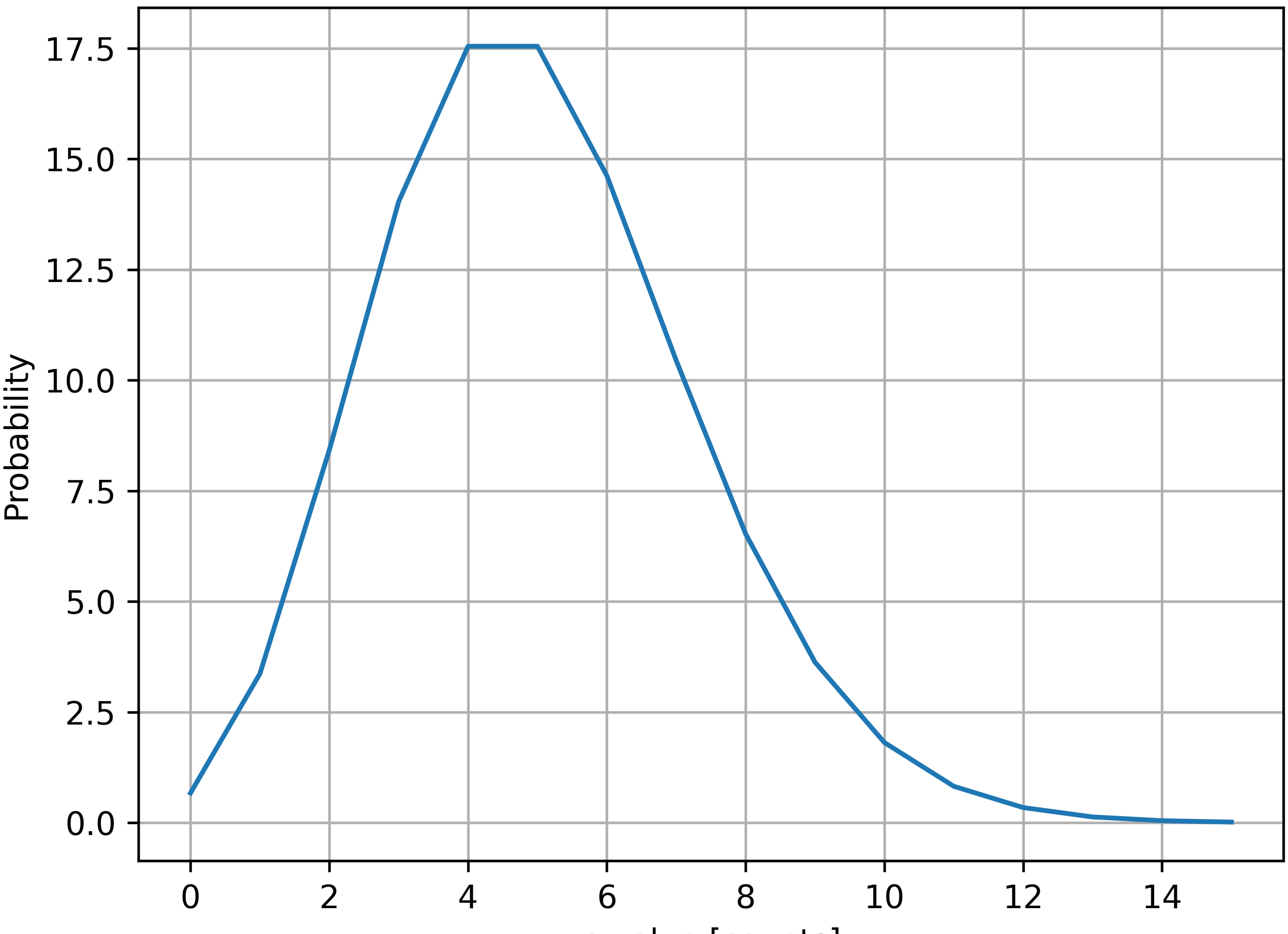
3) λ long, so rare event. 5 cpm = $\langle n \rangle = a$. Use Poisson distribution

a) $P_P(0,N) = \frac{(5)^0}{0!} \exp(-5) = \boxed{0.67379\%}$

b) $P_P(10,N) = \frac{(5)^{10}}{10!} \exp(-5) = \boxed{1.81328\%}$

c) See attached


Probability
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
2
4
6
8
10
12
14
n value [counts]

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4) cannot assume $n \gg N$, $n \ll N$. only $1 \ll n$ & $1 \ll N$. ignore zero-point energy.

a) $\Omega = \frac{(N+n-1)!}{n!(N-1)!} \Rightarrow \ln\Omega = \ln((N+n-1)!) - \ln(n!) - \ln((N-1)!)$

stirling approx: $\ln\Omega \approx (N+n-1)\ln(N+n-1) - (N+n-1) - n\ln(n) + n - (N-1)\ln(N-1) + (N-1)$

$\Rightarrow \ln\Omega \approx (N+n-1)\ln(N+n-1) - n\ln(n) - (N-1)\ln(N-1)$

$(x-a) \approx x$ if $x \gg a$

$\Rightarrow$ apply same reasoning as 2z): $\ln\Omega \approx (N+n)\ln(N+n) - n\ln(n) - N\ln(N)$

$\Rightarrow \ln\Omega = N\ln(N+n) + n\ln(N+n) - n\ln(n) - N\ln(N)$

$= N\ln\left(\frac{N+n}{N}\right) + n\ln\left(\frac{N+n}{n}\right) = N\ln\left(1+\frac{n}{N}\right) + n\ln\left(1+\frac{N}{n}\right)$

w/ $n = \frac{u}{\hbar\omega} \Rightarrow \boxed{\ln\Omega = \sigma = N\ln\left(1+\frac{u}{\hbar\omega N}\right) + \frac{u}{\hbar\omega}\ln\left(1+\frac{N\hbar\omega}{u}\right)}$

b) $\frac{1}{T} = k\frac{\partial\sigma}{\partial u} \Rightarrow \frac{\partial\sigma}{\partial u} = \overbrace{\frac{N}{\hbar\omega N^2}\cdot\frac{1}{1+u/\hbar\omega N}}^{① = \text{first term}} + \frac{1}{\hbar\omega}\ln\left(1+\frac{N\hbar\omega}{u}\right) \ldots$

$\ldots + \frac{u}{\hbar\omega}\left[\frac{1}{1+\frac{N\hbar\omega}{u}}\cdot\left(-\frac{N\hbar\omega}{u^2}\right)\right]$ } ③ = third term

w/ ③ $= -\frac{N}{u}\left(\frac{1}{1+\frac{N\hbar\omega}{u}}\right) = -\frac{N}{u+N\hbar\omega}$ & ① $= \frac{N}{N\hbar\omega+u}$ $\therefore$ ①+③ = 0

$\Rightarrow \frac{\partial\sigma}{\partial u} = \frac{1}{\hbar\omega}\ln\left(1+\frac{N\hbar\omega}{u}\right) \Rightarrow \frac{\hbar\omega}{kT} = \ln(1+N\hbar\omega/u)$

$\Rightarrow$ exp both sides: $\exp\left(\frac{\hbar\omega}{kT}\right) = 1 + N\hbar\omega/u$

$\Rightarrow \frac{N\hbar\omega}{u} = \exp\left(\frac{\hbar\omega}{kT}\right) - 1 \Rightarrow \boxed{u = \frac{N\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}}$

4c) say $\hbar\omega/kT = x$ $\quad \hbar\omega = kTx$

$\Rightarrow \lim_{x\to 0} = \frac{NkTx}{\exp(x)-1} \Rightarrow$ L'Hospital $\Rightarrow \lim_{x\to 0} \frac{NkT}{\exp(x)} = NkT$

w/ N as # of oscillators, U/N = energy per oscillator

$\Rightarrow U/N = \frac{NkT}{N} = \boxed{kT = U/N}$ for the high temp limit

d) w/ $\hbar\omega/kT = x$ $\quad \hbar\omega = kTx$

$\Rightarrow \lim_{x\to\infty} = \frac{NkTx}{\exp(x)-1} \Rightarrow$ L'Hospital $\Rightarrow \lim_{x\to\infty} \frac{NkT}{\exp(x)} = 0$

$\Rightarrow \boxed{\frac{U}{N} = 0}$ for the low temp limit, no oscillator has energy