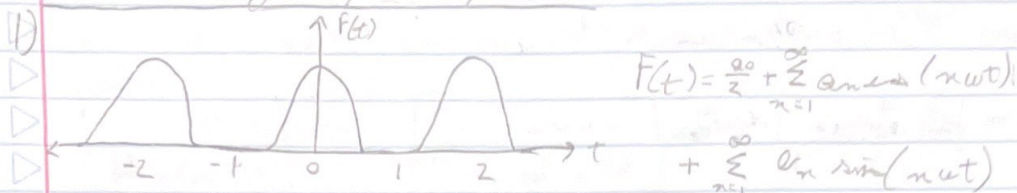


HW#8 - Joseph Specter



a) $T = 2$ $\omega = \frac{2\pi}{T} \therefore \omega = \frac{2\pi}{2} = \pi$

$T = 2$ $\omega = \pi \text{ rad/s}$

b) a_0 will not vanish b/c the average value of $F(t)$ is not 0

a_n will not be 0 as this is an even function

b_n will be 0 as this is an even function & b_n is the coefficient on the odd periodic basis vectors

c) $m\ddot{x} + c\dot{x} + kx = F(t)$ from § 16 slide 19

$$x_p(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n \mathcal{B}(n\omega) \cos(n\omega t - \phi(n\omega))$$

$$\mathcal{B}(n\omega) = \frac{1}{k} \left[\left(1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right)^2 + \left(2\zeta \left(\frac{n\omega}{\omega_n} \right) \right)^2 \right]^{-1/2}$$

$$\phi(n\omega) = \tan^{-1} \left(\frac{2\zeta \left(\frac{n\omega}{\omega_n} \right)}{1 - \left(\frac{n\omega}{\omega_n} \right)^2} \right)$$

$$a_0 = \frac{2}{T} \int_T F(t) dt \quad a_n = \frac{2}{T} \int_T F(t) \cos(n\omega t) dt$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2m\omega_n}$$

HW #8 - Joseph Spectt

- 1) We know average position is $\bar{x} = \frac{a_0}{2L}$ as this is the value of x after a long time

2) $F_{\text{rec}} = |F_0 \sin(\omega t)|$

3) graphed & $T = \frac{\pi}{\omega}$ $\omega = \sqrt{k/m}$

4) $F_{\text{rec}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$

This function is even, so $b_n = 0$, so

$F_{\text{rec}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$, so need to find a_0 & a_n , so

$$a_0 = \frac{2}{T} \int_0^{\frac{T}{2}} F(t) dt = \frac{4}{T} \int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} F(t) dt = \frac{4}{T} \int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} |F_0 \sin(\omega t)| dt$$

but $\sin(\omega t) \geq 0$ here so we can remove abs, so

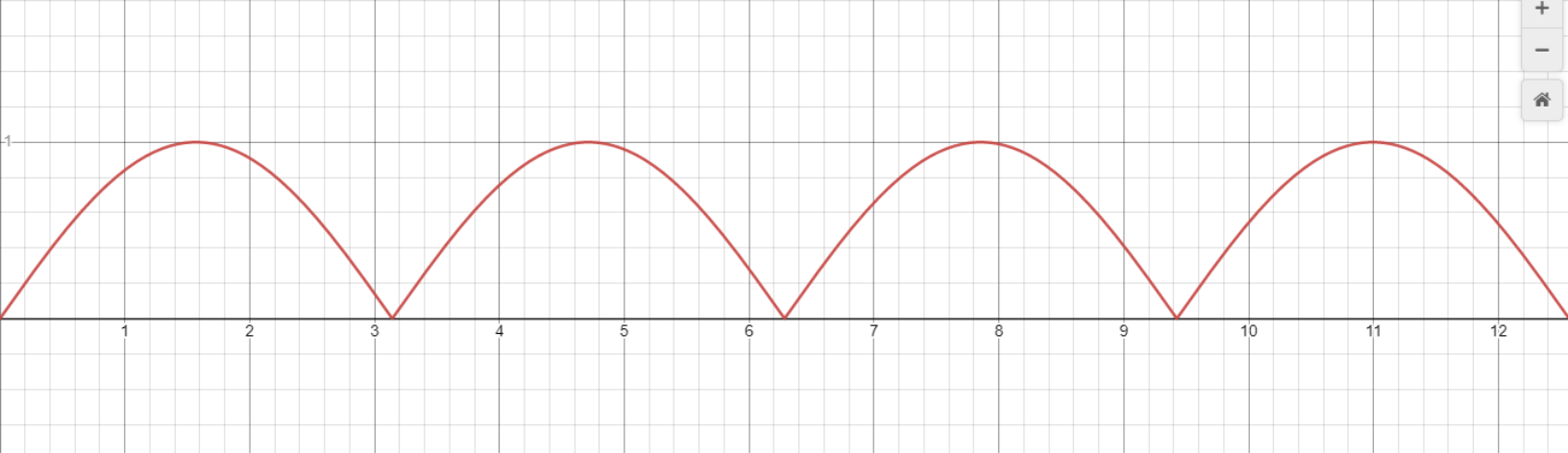
$$a_0 = \frac{4F_0}{T} \int_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}} \sin(\omega t) dt = \frac{4F_0}{T\omega} \left[-\cos(\omega t) \right]_{\frac{\pi}{2\omega}}^{\frac{\pi}{\omega}}$$

$$= \frac{4F_0}{T\omega} \left(-\cos(\pi) + \cos\left(\frac{\pi}{2}\right) \right) = \frac{4F_0}{T\omega} \left(-(-1) + 0 \right) = \frac{4F_0}{T\omega} = \frac{4F_0}{\pi} = a_0$$

now solving for a_n , we know

$$a_n = \frac{2}{T} \int_0^{\frac{T}{2}} F(t) \cos(n\omega t) dt$$

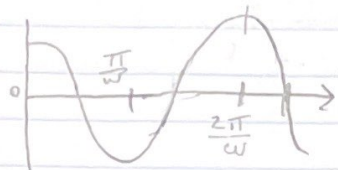
$$a_n = \frac{2}{T} \int_0^{\frac{T}{2}} |F_0 \sin(\omega t)| \cos(n\omega t) dt$$



HW 48 - Joseph Spectro

To find ω_2 , we graph $\cos(n\omega t)$

$$\boxed{\Omega = 2\omega}$$



$T_2 = 2T_1$, so to get an entire period in a single period, Ω needs to be 2ω & since we know the period is 2π as long, the frequency is $1/2$ as much & this means there are 2 periods of the force in 1 of the cos, so we need it to go 2π as fast, so

$$a_n = \frac{2}{T} \int_0^{\frac{T}{2}} (F_0 \sin(\omega t) \cos(2n\omega t)) dt, \text{ but at this interval, } \sin > 0, \text{ so}$$

$$a_n = \frac{2}{T} \int_0^{\frac{T}{2}} F_0 \sin(\omega t) \cos(2n\omega t) dt \quad \text{applying an identity series}$$

$$a_n = \frac{2F_0}{T} \int_0^{\frac{T}{2}} \frac{1}{2} (\sin[(2n+1)\omega t] - \sin[(2n-1)\omega t]) dt$$

$$a_n = \frac{F_0}{T} \left(\int_0^{\frac{T}{2}} \sin[(2n+1)\omega t] dt - \int_0^{\frac{T}{2}} \sin[(2n-1)\omega t] dt \right)$$

$$= \frac{F_0}{T} \left[\left. \frac{-\cos[(2n+1)\omega t]}{(2n+1)\omega} \right|_0^{\frac{T}{2}} + \left. \frac{\cos[(2n-1)\omega t]}{(2n-1)\omega} \right|_0^{\frac{T}{2}} \right]$$

$$= \frac{F_0}{T} \left(\frac{-\cos(\text{odd } 2 \cdot \pi)}{(2n+1)\omega} + \frac{\cos(0)}{(2n+1)\omega} + \frac{\cos(\text{odd } 2 \cdot \pi)}{(2n-1)\omega} - \frac{\cos(0)}{(2n-1)\omega} \right)$$

$$= \frac{F_0 \omega}{\pi \omega} \left(\frac{-(-1) + 1}{2n+1} + \frac{(-1) - 1}{2n-1} \right) = \frac{F_0}{\pi} \left(\frac{2}{2n+1} - \frac{2}{2n-1} \right) \quad \text{getting common denominator}$$

$$= \frac{2F_0}{\pi} \left(\frac{(2n-1)}{(2n+1)(2n-1)} - \frac{(2n+1)}{(2n-1)(2n+1)} \right) = \frac{2F_0}{\pi} \left(\frac{-2}{4n^2-1} \right) = \frac{-4F_0}{\pi(4n^2-1)} = a_n$$

HWF8 - Joseph Spect

Now we know every coefficient is in

$$F_{\text{res}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t)$$

$$a_0 = \frac{4F_0}{\pi} \quad \Omega = 2\omega \quad b_n = 0 \quad a_n = \frac{-4F_0}{\pi(4n^2 - 1)}$$

To find the particular solution, use form

$$x_p = \frac{a_0}{2K} + \sum_{n=1}^{\infty} a_n g(2n\omega) \cos(2n\omega t - \phi(2n\omega))$$

now we need to find $g(2n\omega)$ & $\phi(2n\omega)$, so

$$g(2n\omega) = \frac{1}{b} \left[\left(1 - \left(\frac{2n\omega}{\omega_n} \right)^2 \right)^2 + \left(2\zeta \left(\frac{2n\omega}{\omega_n} \right) \right)^2 \right]^{-1/2}$$

but we know $\omega = \omega_n$ & $\zeta = 0$ b/c it is undamped, so

$$g(2n\omega) = \frac{1}{b} \left[\left(1 - \left(\frac{2n\omega_n}{\omega_n} \right)^2 \right)^2 + 0 \right]^{-1/2} = \frac{1}{b} \left[1 - (2n)^2 \right]^{-1/2}$$

$$g(2n\omega) = \frac{1}{b} \frac{1}{\sqrt{1 - 4n^2}} = \frac{1}{b} \frac{1}{|1 - 4n^2|} = g(2n\omega) \leftarrow \text{all assemble @ the end}$$

Finding $\phi(2n\omega)$

$$\phi(2n\omega) = \tan^{-1} \left(\frac{2\zeta \left(\frac{\omega}{\omega_n} \right)}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right) \text{ but } \zeta = 0, \text{ so } \phi(2n\omega) = \tan^{-1}(0) = 0$$

HW1#2 - Joseph Specht

now plugging into the expression for $x_p(t)$

$$x_p = \frac{a_0}{2B} + \sum_{n=1}^{\infty} a_n B(2n\omega) \cos(2n\omega t - \phi(2n\omega)) \quad \text{we get}$$

$$x_p = \frac{2F_0}{8\pi} + \sum_{n=1}^{\infty} \frac{-4F_0}{\pi(4n^2-1)} \frac{1}{|1-4n^2|} \cos(2n\omega t - 0)$$

$$x_p = \frac{2F_0}{8\pi} \left(1 + \sum_{n=1}^{\infty} \frac{-2 \cos(2n\omega t)}{(4n^2-1)|1-4n^2|} \right)$$

but we know $n \geq 1$, so we can rewrite

$$|1-4n^2| \text{ as } |(-1)(4n^2-1)| = (1-1)(4n^2-1) = |4n^2-1| \text{ but } n \geq 1, \text{ so}$$

$(4n^2-1) > 0$, so we can disregard the absolute value, so in this case

$$|1-4n^2| = (4n^2-1) \text{ which we can combine as}$$

$$x_p = \frac{2F_0}{8\pi} \left(1 + \sum_{n=1}^{\infty} \frac{-2 \cos(2n\omega t)}{(4n^2-1)^2} \right)$$

d) now finding F_{res} these n 's we have

$$F_{res} = \frac{2F_0}{\pi} + \sum_{n=1}^{\infty} \frac{-4F_0}{(4n^2-1)} \cos(2n\omega t) = \frac{2F_0}{\pi} \left(1 + \sum_{n=1}^{\infty} \frac{-2 \cos(2n\omega t)}{(4n^2-1)} \right)$$

now we need to plug in the desired n values

HW #2 - Joseph Specter

$$\text{amplitude } n=1 = \max_{\text{term}} \left(\frac{-4F_0 \cos(2\omega t)}{\pi(4-1)} \right) = \frac{4F_0}{3\pi} = A_1$$

$$\text{amplitude } n=2 = \max_{\text{term}} \left(\frac{-4F_0 \cos(4\omega t)}{\pi(16-1)} \right) = \frac{4F_0}{15\pi} = A_2$$

$$\text{amplitude } n=3 = \max_{\text{term}} \left(\frac{-4F_0 \cos(6\omega t)}{\pi(36-1)} \right) = \frac{4F_0}{35\pi} = A_3$$

now we can find the ratios $\left| \frac{A_2}{A_1} \right|$ & $\left| \frac{A_3}{A_1} \right|$

$$\left| \frac{A_2}{A_1} \right| = \frac{4F_0}{15\pi} \cdot \frac{3\pi}{4F_0} = \frac{1}{5}$$

$$\left| \frac{A_3}{A_1} \right| = \frac{4F_0}{35\pi} \cdot \frac{3\pi}{4F_0} = \frac{3}{35}$$

We see $A_1 > A_2 > A_3$ & this is because $n=1$ is the lowest integer multiple of the resonance frequency available. We also know the largest response happens when $n\omega$ is as small as possible, which is what happened here.

HW# 8 - Joseph Specter

3) $T = 1 \text{ s}$ $F(t) = 2F_0 t$ for $-0.5 \leq t \leq 0.5$ & then periodic

a) $\Omega = \frac{2\pi}{T} = 2\pi$, just use the formula relating T & ω

$a_0 = 0$, because the average value is 0 of the function & this is centered @ $F=0$

$a_n = 0$, because this is an odd function, so the even part cancels out

$$e_n = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \sin(n\omega t) dt = \frac{2}{T} \int_{-0.5}^{0.5} 2F_0 t \sin(n\omega t) dt$$

$$= \frac{4F_0}{T} \int_{-0.5}^{0.5} t \sin(n\omega t) dt = \frac{4F_0}{1} \left(\frac{\sin(n\omega t) - n\omega t \cos(n\omega t)}{n^2 \omega^2} \right) \Big|_{-0.5}^{0.5}$$

but we know $\omega = 2\pi$, so

$$e_n = 4F_0 \left(\frac{\sin(2\pi n t) - 2\pi n t \cos(2\pi n t)}{4\pi^2 n^2} \right) \Big|_{-0.5}^{0.5} = \cos(\pi n)$$

$$= 4F_0 \left(\frac{\sin(n\pi) + \sin(-n\pi) - n\pi \cos(\pi n) - \pi n \cos(-\pi n)}{4\pi^2 n^2} \right)$$

but we know $\sin(n\pi) = 0$, so either -1 or 1

$$e_n = \frac{F_0}{\pi^2 n^2} (0 + 0 - 2n\pi \cos(\pi n)) = \frac{-2F_0 \cos(\pi n)}{\pi \pi} = \frac{-2F_0 (-1)^n}{\pi n} = e_n$$

$$\Omega = 2\pi \text{ rad/s}$$

$$a_0 = 0$$

$$a_n = 0$$

$$e_n = \frac{-2F_0 (-1)^n}{\pi n}$$

HW #8 - Joseph Spectro

a) find $x(t)$. know it follows form below

$$x_p = \sum_{n=1}^{\infty} \frac{1}{\omega_n} g(n\omega) \sin(n\omega t - \phi(n\omega)) \quad \text{w/ } a_0 = a_n = 0$$

now we need to find g & ϕ , so we just plug in $\omega = 2\pi$

$$x_p = \sum_{n=1}^{\infty} \frac{-2F_0(-1)^n}{\pi n} g(2\pi n) \sin(2\pi n t - \phi(2\pi n))$$

$$g(2\pi n) = \frac{1}{2} \left[\left(1 - \left(\frac{2\pi}{\omega_n} \right)^2 \right)^2 + \left(2\zeta \left(\frac{2\pi}{\omega_n} \right) \right)^2 \right]^{-1/2}$$

$$\phi(2\pi n) = \arctan \left(\frac{2\zeta \left(\frac{2\pi}{\omega_n} \right)}{1 - \left(\frac{2\pi}{\omega_n} \right)^2} \right) \quad \zeta = \frac{c}{2m\omega_n}$$

c) resonance is $\omega_d = \omega_n \sqrt{1 - 2\zeta^2}$ where ω_n is

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{200}{0.6}} = \sqrt{\frac{1000}{3}} \quad \text{to find } \zeta \text{ we do}$$

$$\zeta = \frac{c}{2m\omega_n} \quad \& \quad 2\zeta^2 = \frac{c^2}{2m^2} \cdot \frac{3}{1000} = \frac{3c^2}{2000 m^2} \quad \text{plugging in values we get}$$

$$2\zeta^2 = \frac{3 \cdot 5^2}{2000 \cdot 0.6^2} = \frac{1}{960}$$

$$\omega_d = \sqrt{\frac{1000}{3}} \sqrt{1 - \frac{1}{960}} = \sqrt{\frac{1000}{3}} \sqrt{\frac{959}{960}} = \sqrt{\frac{23,915}{72}}$$

$$\omega_d = \sqrt{\frac{23,915}{72}} \text{ rad/s} \approx 18.2479 \text{ rad/s}$$

HW#8 - Joseph Spectro

I used a computer program to complete these calculations. I will list it below.

n	$\max F [N]$	Ω in $\sin(\Omega t - \phi/2\pi n)$ [rad/s]
1	.00361 F_0	6.28
2	.00302 F_0	12.57
→ 3	.01309 F_0	18.85
4	.00089 F_0	25.13
5	.00032 F_0	31.42

The amplitude of $n=3$ was the largest & this makes sense as the frequency of oscillations Ω is closest to the resonance frequency $\omega \approx 18.2479 \text{ rad/s}$

$$18.85 \text{ rad/s} = \Omega_3 \approx \omega_d = 18.2479 \text{ rad/s}$$

These values are closer for $n=3$ than any other n

```
1 from sympy import *
```

```
1 w, z, m, c, k, F, n = S('w, z, m, c, k, F, n')
2
3 m = 0.6 #kg
4 c = 0.5 #kg/s
5 k = 200 #N/m
6
7 w = (k/m)**(1/2)
8 z = c / (2*m*w)
9
10 bn = 2*F*(-1)**(n+1) / (pi*n)
11 G = 1/k * (((1-(2*pi*n/w)**(2))**2) + (2*z*(2*pi*n/w))**(2))**(-1/2)
12
13 x = bn * G
14
15 print('n=1', x.subs(n,1).evalf())
16 print('n=2', x.subs(n,2).evalf())
17 print('n=3', x.subs(n,3).evalf())
18 print('n=4', x.subs(n,4).evalf())
19 print('n=5', x.subs(n,5).evalf())
```

```
n=1 0.00361016441477838*F
n=2 -0.00301889587202642*F
n=3 0.0130944349267146*F
n=4 -0.000886986242011769*F
n=5 0.000324399913827974*F
```

```
1 print('n=1', (2*pi*1).evalf())
2 print('n=2', (2*pi*2).evalf())
3 print('n=3', (2*pi*3).evalf())
4 print('n=4', (2*pi*4).evalf())
5 print('n=5', (2*pi*5).evalf())
```

```
n=1 6.28318530717959
n=2 12.5663706143592
n=3 18.8495559215388
n=4 25.1327412287183
n=5 31.4159265358979
```


HW#8 - Joseph Speck

4a)

$$\int_0^{\infty} \delta(2t^{1/3} - 54) \exp(2t) dt$$

$$\text{let } u = 2t^{1/3} \Rightarrow du = \frac{2}{3} t^{-2/3} dt \Rightarrow dt = \frac{3}{2} t^{2/3} du$$

$$= \frac{3}{2} \int_0^{\infty} \delta(u - 54) \exp(2t) t^{2/3} du, \text{ since } u = 2t^{1/3} \quad t = \left(\frac{u}{2}\right)^3$$

$$= \frac{3}{2} \int_0^{\infty} \delta(u - 54) \exp\left(2\left(\frac{u}{2}\right)^3\right) \left(\left(\frac{u}{2}\right)^3\right)^{2/3} du = \frac{3}{2} \int_0^{\infty} \delta(u - 54) \exp\left(2\left(\frac{u}{2}\right)^3\right) \left(\frac{u}{2}\right)^2 du$$

we know $\int_0^{\infty} \delta(u - 54) du = 1$ if $u = 54$ & 0 if else, so we can say $f(x) = \exp\left(2\left(\frac{u}{2}\right)^3\right) \left(\frac{u}{2}\right)^2$, so we know the integral is simply $f(54)$

if $f(x) = \exp\left(2\left(\frac{u}{2}\right)^3\right) \left(\frac{u}{2}\right)^2$, then the integral evaluates to

$$f(54) = \exp\left(2\left(\frac{54}{2}\right)^3\right) \left(\frac{54}{2}\right)^2 = \exp(2(27)^3) (27)^2$$

$$\int_0^{\infty} \delta(2t^{1/3} - 54) \exp(2t) dt = \exp(2 \cdot 27^3) (27)^2 \approx 1.992 \text{ e } 17,099$$

HW# 8 - Joseph Spectro

2) $\int_{-\infty}^{\infty} \left(\frac{d^2}{dt^2} (\delta(t-4)) \right) \sin\left(\frac{t^2}{4}\right) dt$; $\int u dv = uv - \int v du$

$$u = \sin\left(\frac{t^2}{4}\right) \quad v = \frac{d}{dt} \delta(t-4)$$

$$du = \frac{t}{2} \cos\left(\frac{t^2}{4}\right) dt \quad dv = \frac{d^2}{dt^2} \delta(t-4)$$

$$= \left. \frac{d}{dt} (\delta(t-4)) \sin\left(\frac{t^2}{4}\right) \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dt} (\delta(t-4)) \frac{t}{2} \cos\left(\frac{t^2}{4}\right) dt$$

0 b/c $\delta(t-4) = 0 @ -\infty$ & the derivative of $\delta(t-4) = 0$ too

$$= - \int_{-\infty}^{\infty} \frac{d}{dt} (\delta(t-4)) \frac{t}{2} \cos\left(\frac{t^2}{4}\right) dt$$

$$u = \frac{t}{2} \cos\left(\frac{t^2}{4}\right) \quad v = \delta(t-4)$$

$$du = \frac{\cos\left(\frac{t^2}{4}\right)}{2} - \frac{t^2 \sin\left(\frac{t^2}{4}\right)}{4} dt \quad dv = \frac{d}{dt} (\delta(t-4))$$

$$= - \delta(t-4) \frac{t}{2} \cos\left(\frac{t^2}{4}\right) + \int_{-\infty}^{\infty} \delta(t-4) du(t) dt$$

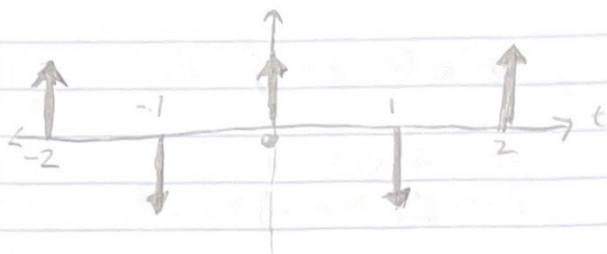
"same reason as before"

This expression is only non-zero when $t=4$, so the answer is $du(4)$

$$\int_{-\infty}^{\infty} \frac{d^2}{dt^2} (\delta(t-4)) \sin\left(\frac{t^2}{4}\right) dt = \frac{\cos(4)}{2} - 4 \sin(4)$$

HW #8 - Joseph Specht

5)



$$F(t) = \sum_{d \in \mathbb{Z}} (-1)^d \delta(t-d), \quad d \in \mathbb{Z}$$

a) even function w/ average value of $F(t) = 0$,
we know $a_0 = 0 = b_n$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \cos(n\omega t) dt, \quad T = 2 \quad \therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \boxed{\pi = \omega}$$

We can rewrite $F(t)$ as another function that follows the same values of $(-1)^d$, so we have

$$F(t) = \sum_{d \in \mathbb{Z}} \cos(n\pi t) \delta(t-d) \quad d \in \mathbb{Z}$$

$$\therefore a_n = \frac{2}{2} \int_{-1.5}^{1.5} \cos(d\pi t) \delta(t-d) \cos(n\pi t) dt$$

We use these bounds to not cut a δ in half

$$a_n = \int_{-1.5}^{1.5} \cos(d\pi t) \cos(n\pi t) \delta(t-d) dt, \text{ but } \delta(t-d) \neq 0 \text{ @ } d = 0 \text{ \& } 1, \text{ so}$$

$$a_n = \cos(0) \cos(0) + \cos(d\pi) \cos(n\pi) = 1 + \cos(d\pi) \cos(n\pi)$$

but d is either 0 or 1 w/ values 1 & -1 @ these times, so

$$a_n = 1 + (-1)^n$$

HW#2 - Joseph Spect

$$\omega = \pi$$

$$a_0 = 0$$

$$a_n = 1 + (-1)^n \quad a_2 = 0$$

$$F(t) = \sum_{n=1}^{\infty} (1 + (-1)^n) \cos(n\pi t)$$

guess for x_p is

$$x_p = \sum_{n=1}^{\infty} (1 + (-1)^n) g(n\pi) \cos(n\pi t - \varphi(n\pi))$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\varphi(n\pi) = \arctan\left(\frac{2\zeta\left(\frac{n\pi}{\omega_n}\right)}{1 - \left(\frac{n\pi}{\omega_n}\right)^2}\right)$$

$$g(n\pi) = \frac{1}{k} \left[\left(1 - \left(\frac{n\pi}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{n\pi}{\omega_n}\right)\right)^2 \right]^{-1/2}$$

$$n\omega = \omega_n \sqrt{1 - 2\zeta^2}$$

d) we know it resonates when we have

$$n\omega = \omega_n \sqrt{1 - 2\zeta^2}$$

$$n\omega = \omega_n \sqrt{1 - \frac{2c^2}{4m^2\omega_n^2}} = \omega_n \sqrt{1 - \frac{c^2}{2mk^2}}$$

$$\therefore \omega = \sqrt{\frac{k}{m}} \sqrt{1 - \frac{c^2}{2mk^2}} \left(\frac{1}{n}\right) \text{ where } \frac{n}{2} \in \mathbb{Z}$$

we need an even n because when n is odd, $a_n = 0$, so we have no amp, so not resonant