A problem theory and its application to Model Theory and Logic

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August 20, 2024

Abstract

Problem Theory, as outlined in [1] and [2], is used to characterize and develop some basic theorems of Model Theory and Logic. We "represent" core concepts in Logic with problems and solutions. These include Godel's First Incompleteness Theorem, Godel's Completeness Theorem, and the Compactness Theorem. Further developments in this topic will be persued.

1 Introduction

Theophilus Agama wrote 2 papers, one called "On the theory of problems and their solution spaces" [1], and the other, "On the topology of problems and their solutions" [2]. The reader is advised to read [1] first. As both papers deal with the same concept(s), problem and solution spaces, I have decided to call those papers as belonging to "Problem Theory". For now we are not interested in the topology of Problem Theory, but are interested in characterizing and developing some small problems in Model Theory in terms of Problem Theory. Besides this and [1] and [2], there aren't any papers on this subject. Also, [1] does have some "holes", in which I may address in this paper.

Note that problems and/or solutions are not defined; they are treated as like sets or axioms; mathematical objects themselves.

2 Core Definitions

All definitions here will be given in Problem Theory unless explicitly stated otherwise.

- Definition. A **solution to a problem** is a relation $\mathbf{S}$ such that $a \mathbf{S} b$ iff a is a solution and b is a problem. Such a relation need not be an equivalence relation.
- Definition. M is a **model** of σ iff for each solution in M, there exists a corresponding problem in σ, such that each solution in M solves a corresponding problem in σ.

- Definition. ϕ is **valid** iff for every problem in ϕ, ϕ has a solution in any model of a language L.
- Definition. A language is a set of problems and solutions.
- Definition. A **proof** of a problem P in L is sequence of solutions $X := (x_1,\ x_2,\ x_3,\ ...\ \ x_n)$ in L such that (a) $\forall x \in X,\ x \in S_{x_n}(P)$, (b) x_n is a solution to L, and (c) under a rule(s) of inference r, (x_1 r x_2 r x_3, ... x_{n-1}) is equivalent to x_n. P is said to be a **theorem** of L.
- Definition. A negative problem $\neg P$ is a problem such that it cannot be solved. A negative solution is a statement that is not a solution to any problem.

Also, define the class **Pro** consisting of all problems, and **Sol** consisting of all solutions. Endow them with composition.

Note: the notion for the negation of problems and solutions cannot be formed.

3 Some elementary theorems in Model Theory and Logic

We start off by proving a practice problem in Chang and Keisler (p.16), to demonstrate our new system.

Theorem 1. A sentence ϕ is satisfiable iff $\neg\phi$ is not valid.

Proof. ($\leftarrow$) If ϕ is a problem, or set of problems, then if "ϕ cannot be solved" is invalid, then ϕ can be solved, proving this case. ($\rightarrow$) If ϕ has a solution, then our theorem is proved according to ($\leftarrow$) and the definitions. $\square$

Another, more important theorem is proved.

Theorem 2. Lindenbaum's Theorem.

Proof. Let Σ represent a set of problems, and that there is no 2 problems σ_1 and σ_2 with solutions τ_1 and τ_2 in Σ such that $S_{\sigma_1}(\tau_1)$ and $S_{\sigma_2}(\tau_2)$ do not have "contradictory" solutions. The rest continues in a way similar to Chang and Keisler (p. 10). $\square$

An alternate proof, and equivalence of Godel's First Incompleteness Theorem is given.

Theorem 3. A problem P may have a solution iff it has a proof.
Proof is given directly from the definition of Proof in this paper.

Theorem 4. "There exists a problem with no solution" is equivalent to Godel's First Incompleteness Theorem.

Proof. ($\rightarrow$) Let there exist a problem P in language L with no solution. As a problem P may have a solution iff it has a proof, P has no proof in L, implying Godel's First Incompleteness Theorem. ($\leftarrow$) Let there exist a statement S in a language L with no proof, represent S as a problem, per Theorem 3, S has no solution. $\square$

One may also prove Godel's Completeness Theorem using the definition we have developed, as well as its extended form.

Proof of the Compactness Theorem (countable case).

Proof. Let every problem of a subset of a set of problems $\mathcal{P}$ have a model. Then every problem of a subset of a set of problems in $\mathcal{P}$ has its corresponding set of solutions. Taking the union of the corresponding sets of solutions and subset of sets of problems in $\mathcal{P}$, we have arrived at the compactness theorem.

($\rightarrow$) Same way, but with partitions. $\square$

4 Motivation of Problem Theory

The subject in which I call "Problem Theory" was developed by T. Agama in 2 papers, in which in his first, he states the original motivation of Problem Theory was to "solve" P = NP. However, I add to this, saying that it could be used to develop, in addition to Model Theory, Proof Theory, among other branches of Logic. It also allowed us to prove existing problems in a much easier fashion.

References

[1] Agama, Theophilus. (2022) "On the theory of problems and their solution spaces"
[2] Agama, Theophilus. (2023) "On the topology of problems and their solutions"
[3] Chang, C.C, and Keisler, H. Jerome. *Model Theory.* 3rd ed, Dover, 2020.