1: Margins

(1) Consider the following input to the XOR function in two dimensions (x_1, x_2) and the input mapped into the space (x_1, x_1x_2).

x_1	x_2	label	x_1x_2
-1	-1	-1	1
-1	1	1	-1
1	-1	1	-1
1	1	-1	1

Table 1: Inputs to the XOR function in two dimensions and mapping.
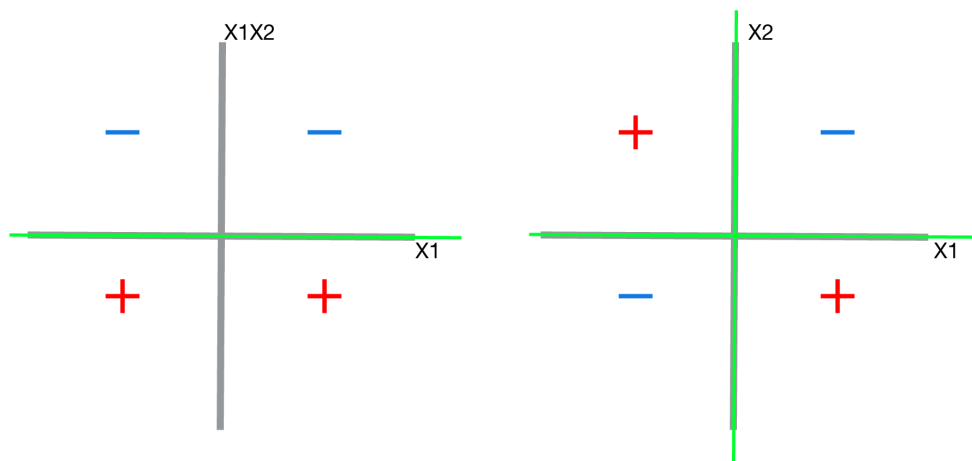

Figure 1: The separating line producing the maximal margin in both spaces.

The XOR function in x_1, x_1x_2 space has a maximum margin of 1. This is achieved with the separating line $x_1x_2 = 0$.

Translating the line $x_1x_2 = 0$ back to the original x_1, x_2 Euclidean space will be the lines $x_1 = 0$ and $x_2 = 0$. This is because $x_1x_2 = 0$ is true when at least one of x_1 or x_2 is 0. Which is satisfied by the lines running along the x1-axis and the x2-axis.
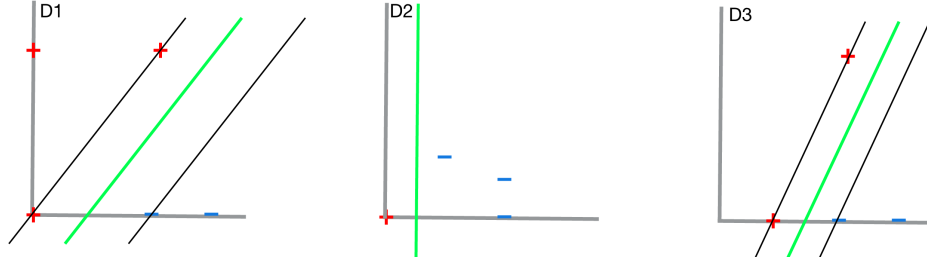
Maximum margin of 1.

Figure 2: Geometry of the three training sets.

(2a) D_1 **-** Points x_1 and x_3 form the line $y - x = 0$ which has a slope of 1. A line with slope 1 that passes through point x_5 is given by $y - x + 1 = 0$. The parallel separating line between these two lines will give the maximum margin. It intersects the x-axis at $(0.5, 0)$, has a slope of 1 and is given by the formula $y - x + 0.5 = 0$. We can use the distance formula for a line and a point to determine the margin between this separating line and the nearest points.

$$distance(ax + by + c = 0, (x_0, y_0)) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{1}$$

The separating line is defined by $a = -1$, $b = 1$, and $c = 0.5$.

$$distance(-x + y + 0.5 = 0, (0,0)) = \frac{0.5}{\sqrt{2}}$$

$$distance(-x + y + 0.5 = 0, (1,1)) = \frac{0.5}{\sqrt{2}}$$

$$distance(-x + y + 0.5 = 0, (1,0)) = \frac{0.5}{\sqrt{2}}$$

$$\boxed{maximum \ \gamma = \frac{1}{2\sqrt{2}}}$$

D_2 **-** The separating line that produces the maximum margin is evident from the geometry of the training points. As seen in figure 2, the line $x = 0.25$ separates the points with the maximum margin.

The separating line is defined by $a = 1$, $b = 0$, and $c = -0.25$.

$$distance(x - 0.5 = 0, (0,0)) = 0.25$$

$$distance(x - 0.5 = 0, (\frac{1}{2}, \frac{\sqrt{3}}{2})) = 0.25$$

$$\boxed{maximum \ \gamma = 0.25}$$

D_3 **-** Points x_3 and x_4 form the line $y - 2x + 1 = 0$ which has a slope of 2. A line with slope 2 that passes through point x_5 is given by $y - 2x + 2 = 0$. The parallel separating line between these two lines will give the maximum margin. It intersects the x-axis at $(\frac{3}{4}, 0)$, has a slope of 2 and is given by the formula $y - 2x + 1.5 = 0$.

The separating line is defined by $a = -2$, $b = 1$, and $c = 1.5$.

$$distance(-2x + y + 1.5 = 0, (\tfrac{1}{2}, 0)) = \frac{0.5}{\sqrt{5}}$$

$$distance(-2x + y + 1.5 = 0, (1, 1)) = \frac{0.5}{\sqrt{5}}$$

$$distance(-2x + y + 1.5 = 0, (1, 0)) = \frac{0.5}{\sqrt{5}}$$

$$\boxed{maximum \; \gamma = \tfrac{1}{2\sqrt{5}}}$$

(2b)

$$\boxed{Perceptron \; Mistake \; Bound \; = (\frac{R}{\gamma})^2} \tag{2}$$

D_1 - From part a we found $\gamma = \frac{1}{2\sqrt{2}}$. To find R, we find the training point furthest away from the origin as the line from the origin to this point defines the radius of the training examples. Using the Pythagorean Theorem, we determine x_7 is the furthest point from the origin with a distance of 1.5.

$$PMB(\gamma, R) = PMB(\frac{1}{2\sqrt{2}}, 1.5) = (\frac{R}{\gamma})^2 = 18$$

$$\boxed{Mistake \; bound \; of \; 18}$$

D_2 - We found $\gamma = 0.25$. The furthest point from the origin is point x_8 with a distance of $\frac{\sqrt{5}}{2}$.

$$PMB(\gamma, R) = PMB(0.25,$$

$\sqrt{5}\tfrac{}{2}) = (\text{R}\underset{\gamma)^2=20}{\overline{\qquad}}$

$$\boxed{Mistake \; bound \; of \; 20}$$

D_3 - We found $\gamma = \frac{1}{2\sqrt{5}}$. The furthest point from the origin is the point x_7 with a distance of 1.5.

$$PMB(\gamma, R) = PMB(0.25,$$

$\sqrt{5}\tfrac{}{2}) = (\text{R}\underset{\gamma)^2=20}{\overline{\qquad}}$

(2c)

2: Kernels

(1a)

(1b)

(2)

(3)

3: Support Vector Machines

(1)

(2)

(3)

4: Ensemble of Decision Trees

(1)

(2a)

(2b)