1: Probabilities

$$\boxed{Independent\ Events -\ \ P(A\ \cap\ B) = P(A)P(B)} \tag{1}$$

$$\boxed{Rule\ of\ Multiplication -\ \ P(A\ \cap\ B) = P(A)P(B|A)} \tag{2}$$

(1) Given $P(A_1) = P(A_2) = P(A_1|A_2) = \frac{1}{2}$, we want to prove that A_1 and A_2 are independent events.

Events A_1 and A_2 are independent if and only if Eq. (1) is satisfied.

$$P(A_2 \cap A_1) = P(A_2)P(A_1)$$

We can use Eq. (2) to restate the LHS of Eq. (1) in terms of probabilities we are given.

$$P(A_2)P(A_1|A_2) = P(A_2)P(A_1)$$

$$\frac{1}{2} * \frac{1}{2} = \frac{1}{2} * \frac{1}{2}$$

By showing Eq. (1) is satisfied, we have proven A_1 and A_2 are independent events.

(2) From lecture, we saw the Theorem of Total probability pertaining to mutually exclusive events $A_1, A_2, ..., A_n$ where $\sum_i P(A_i) = 1$. When this condition is met, we know the following is true:

$$P(B) = \sum_{i=1}^{n} P(B \cap A) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

We have mutually exclusive events A_1, A_2, and A_3 where the sum of their probabilities is 1. The condition is met, so we can use the Theorem of Total probability. All the needed probabilities to calculate $P(A_4)$ are provided in the problem.

$$P(A_4) = \sum_{i=1}^{3} P(A_4|A_i)P(A_i)$$

$$P(A_4) = \frac{1}{3} * (\frac{1}{6} + \frac{1}{3} + \frac{1}{2})$$

$\boxed{P(A_4) = \frac{1}{3}}$

$$\boxed{Binomial\ Distribution - \binom{n}{k} p^k (1-p)^{n-k}} \tag{3}$$

(3) Let X be a random variable representing the top of the six-sided die toss. The dice is a fair dice so we know $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = \frac{1}{6}$.

There are six possible events and the total probability of exactly two heads after n coin tosses is the sum of the probability of each of the six events happening.

$$\sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

Where $B(n, k, p)$ represents the binomial distribution from Eq. (3). This is the probability that given n trials, there are exactly k successes if the probability of success is p where $O \leq p \leq 1$ and $k \leq n$. Let Y be a random variable representing the exact number of heads after n coin tosses. Note that the probability of getting exactly two heads when only tossing one coin is 0.

$$P(Y = 2) = \sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6}\sum_{i=1}^{6} B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6}(0 + \frac{1}{4} + \frac{3}{8} + \frac{6}{16} + \frac{10}{32} + \frac{15}{64})$$

$$\boxed{P(Y = 2) = \frac{33}{128} = 0.2578}$$

Thus this is the probability of getting exactly 2 heads after n coin flips, where n is the result of a fair six-sided die toss.

$$\boxed{Rule\ of\ Addition -\ \ P(A\ \cup\ B) = P(A) + P(B) - P(A)P(B|A)} \tag{4}$$

(4) We want to prove that if $P(A_1) = a_1$ and $P(A_2) = a_2$ then $P(A_1|A_2) \geq \frac{a_1+a_2-1}{a_2}$.

Proof: we begin with Eq. (4) which is the rule for union of two events.

$$P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

$P(A_2 \cup A_1)$ is a probability so we know it has a upper bound of 1.

$$1 \geq P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

$$1 \geq P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

Rearranging terms and multiplying both sides by -1.

$$\frac{1 - P(A_2) - P(A_1)}{P(A_2)} \geq -P(A_1|A_2)$$

$$\frac{P(A_1) + P(A_2) - 1}{P(A_2)} \leq P(A_1|A_2)$$

Replacing $P(A_1) = a_1$ and $P(A_2) = a_2$ on the LHS of the inequality.

$$P(A_1|A_2) \geq \frac{a_1 + a_2 - 1}{a_2}$$

Thus arriving at the original inequality and proving it's correctness.

(5a) Given two independent random variables A_1 and A_2, we want to prove that $E[A_1 + A_2] = E[A_1] + E[A_2]$ is true. We will assume A_1 and A_2 are discrete for this proof. The equality also holds for continuous random variables with a slightly different proof.

Given a discrete random variable X that can take values $x_1,\ x_2,\ ...,\ x_k$, with respective probabilities $p_1,\ p_2,\ ...,\ p_k$, then the expected value of X is defined as:

$$\boxed{Expectation - \ E[X] = \sum_{i=1}^{k} x_i * P(X = x_i)} \tag{5}$$

Proof: Starting with the LHS of the equality we want to prove, we will use the definition of expectation to arrive at the RHS.

$$E[A_1 + A_2] = \sum_{i=1}^{k}\sum_{j=1}^{k}(a_{1i} + a_{2j}) * P(A_1 = a_{1i}, A_2 = a_{2j})$$

Multiply and split RHS into two sets of summations.

$$E[A_1 + A_2] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{1i} * P(A_1 = a_{1i}, A_2 = a_{2j}) + \sum_{i=1}^{k}\sum_{j=1}^{k} a_{2j} * P(A_1 = a_{1i}, A_2 = a_{2j})$$

$$E[A_1 + A_2] = \sum_{i=1}^{k} a_{1i} * P(A_1 = a_{1i}) + \sum_{j=1}^{k} a_{2j} * P(A_2 = a_{2j})$$

The RHS is the definition of expectation for A_1 summed with the expectation of A_2.

$$E[A_1 + A_2] = E[A_1] + E[A_2]$$

(5b) Given two independent random variables A_1 and A_2, we want to prove that $var[A_1 + A_2] = var[A_1] + var[A_2]$ is true.

Proof: Variance is defined as:

$$\boxed{Variance - \ var[X] = E[(X - E[X])^2} \tag{6}$$

Replacing X with $A_1 + A_2$ and using algebra, this can be rewritten as:

$$var[A_1 + A_2] = E[(A_1 + A_2)^2] - E[A_1 + A_2]^2$$

Expanding out the polynomials gives us:

$$var[A_1 + A_2] = E[A_1^2 + 2A_1A_2 + A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2]^2$$

Using the proof of linearity of expectation from part *a*, we can rewrite the first expectation:

$$var[A_1 + A_2] = E[A_1^2] + 2E[A_1A_2] + E[A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2]^2$$

When the covariance of two random variables is zero, the expected value operator is multiplicative. That is, $E[XY] = E[X]E[Y]$. We know the covariance of A_1 and A_2 is 0 as they are independent of each other.

$$var[A_1 + A_2] = E[A_1^2] + 2E[A_1]E[A_2] + E[A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2^2]$$
$$var[A_1 + A_2] = E[A_1^2] - E[A_1]^2 + E[A_2^2] - E[A_2^2]$$
$$var[A_1 + A_2] = var[A_1] + var[A_2]$$

2: Naïve Bayes

(1a) Given infinite data drawn from this distribution, the learned probabilities $\hat{P}$ will be identical to the true probabilities P. This is a result of the law of large numbers.

$$\hat{P}(y = -1) = 0.1 \; and \; \hat{P}(y = 1) = 0.9$$
$$\hat{P}(x_1 = -1|y = -1) = 0.8 \; and \; \hat{P}(x_1 = 1|y = -1) = 0.2$$
$$\hat{P}(x_1 = -1|y = 1) = 0.1 \; and \; \hat{P}(x_1 = 1|y = 1) = 0.9$$

(1b) Using the conditional independence assumption the naive Bayes model makes, we can calculate the generative distribution of the data using $\hat{P}(x_1, y) = \hat{P}(y)\hat{P}(x_1|y)$. A prediction y' is determined by taking the maximum of the $\hat{P}(x_1, y)$ for a given value of x_1.

x_1	$\hat{P}(x_1, y = -1)$	$\hat{P}(x_1, y = 1)$	**Prediction:** $y' = argmax_y \hat{P}(x_1, y)$
-1	$(0.1)(0.8) = \mathbf{0.08}$	$(0.9)(0.1) = \mathbf{0.09}$	1
1	$(0.1)(0.2) = \mathbf{0.02}$	$(0.9)(0.9) = \mathbf{0.81}$	1

(1c) To determine the error of the classifier, we calculate the probability of $P(y' \neq y)$ being true. To calculate this we can use the fact that $P(y' \neq y) = P(y' \neq y, x_1 = -1) + P(y' \neq y, x_1 = 1)$, using the values from the table in part b.

$$P(y' \neq y) = P(y = -1|x_1 = -1) + P(y = -1|x_1 = 1)$$
$$P(y' \neq y) = (0.1)(0.8) + (0.1)(0.2) = 0.1$$

If we trained a classifier on the given data, it would have an error rate of **0.1**.

(2a) The two features x_1 and x_2 are not conditionally independent given y.

Proof: From lecture we saw the definition of conditional independence of three random variables. The random variables are independent if and only if the equality is satisfied.

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$
$$P(x_1, x_2|y) = P(x_1|y)P(x_2|y)$$

As x_1 and x_2 are identical, $P(x_1, x_2|y)$ is the same as $P(x_1|y)$ as x_1 and x_2 always happen or don't happen together. Replacing $P(x_1, x_2|y)$ with $P(x_1|y)$ in the conditional independence formula.

$$P(x_1|y) = P(x_1|y)P(x_2|y)$$

We know our given probabilities are non-zero, thus the above equality does not hold. Proving x_1 and x_2 are not conditionally independent.

(2b) As in part 1, using the conditional independence assumption the naive Bayes model makes, we can use $\hat{P}(x_1|y)$, $\hat{P}(x_2|y)$, and $\hat{P}(y)$ to calculate the generative distribution of the data.

$$\hat{P}(x_1, x_2, y) = \hat{P}(y)\hat{P}(x_1|y)\hat{P}(x_2|y)$$

x_1	x_2	$\hat{P}(x_1, x_2, y = -1)$	$\hat{P}(x_1, x_2, y = 1)$	**Prediction:** $y' = argmax_y \hat{P}(x_1, x_2, y)$
-1	-1	$(0.1)(0.8)(0.8) = \mathbf{0.064}$	$(0.9)(0.1)(0.1) = \mathbf{0.009}$	-1
-1	1	$(0.1)(0.8)(0.2) = \mathbf{0.016}$	$(0.9)(0.1)(0.9) = \mathbf{0.081}$	1
1	-1	$(0.1)(0.2)(0.8) = \mathbf{0.016}$	$(0.9)(0.9)(0.1) = \mathbf{0.081}$	1
1	1	$(0.1)(0.2)(0.2) = \mathbf{0.004}$	$(0.9)(0.9)(0.9) = \mathbf{0.729}$	1

(2c) To determine the error of the classifier, we calculate the probability of $P(y' \neq y)$ being true. To calculate this we can use the fact that $P(y' \neq y) = P(y' \neq y, x_1 = -1, x_2 = -1) + P(y' \neq y, x_1 = -1, x_2 = 1) + P(y' \neq y, x_1 = 1, x_2 = -1) + P(y' \neq y, x_1 = 1, x_2 = 1)$, using the values from the table in part b.

$$P(y' \neq y) = 0.009 + 0.016 + 0.016 + 0.004 = 0.045$$

If we trained a classifier on the given data, it would have an error rate of **0.045**.

(2d) Both naïve Bayes (generative) and logistic regression (discriminative) compute the same posterior distribution over the outputs. But, discriminative models doesn't characterize the distribution of the inputs as a generative model does.

So no, I don't expected the duplication of a variable to have the same results in logistic regression as we saw in naïve Bayes.

3: Naïve Bayes and Linear Classifiers

Our classifier will predict a label of 1 if the following is true:

$$\frac{Pr(x|y=1)Pr(y=1)}{Pr(x|y=0)Pr(y=0)} \geq 1 \tag{7}$$

We saw in class that the naïve Bayes classifier makes the assumption that all features x_j are independent of each other. This allows for a simplified calculation for a conditional probability:

$$Pr(\mathbf{x}|y) = \prod_{j=0}^{d} Pr(x_j|y) \tag{8}$$

This allows us to rewrite the condition for predicting a label of 1 from Eq. (7):

$$\frac{Pr(y=1)}{Pr(y=0)} * \prod_{i=0}^{d} \frac{Pr(x_j|y=1)}{Pr(x_j|y=0)} \geq 1 \tag{9}$$

Suppose each $Pr(x_j|y)$ is defined using a Gaussian/Normal probability density function, one for each value of y and j. Each Gaussian distribution has mean $\mu_{j,y}$ and variance σ^2. We can rewrite the $Pr(x_j|y)$ in Eq. (9) in terms of their Gaussian functions. Additionally we will replace $Pr(y=1)$ with p and $Pr(y=0)$ with $p-1$.

$$\frac{p}{p-1} * \prod_{i=0}^{d} \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{\frac{-(x_j-\mu_{j,1})^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\sigma^2\pi}} e^{\frac{-(x_j-\mu_{j,0})^2}{2\sigma^2}}} \geq 1 \tag{10}$$

Our goal is to get Eq. (10) in the form of a familiar linear model. First, we group all the constant terms that don't rely on a x_i term.

$$\left(\frac{p}{p-1} \prod_{i=0}^{d} \frac{\frac{1}{\sqrt{2\sigma^2\pi}}}{\frac{1}{\sqrt{2\sigma^2\pi}}} \right) * \prod_{i=0}^{d} \frac{e^{\frac{-(x_j-\mu_{j,1})^2}{2\sigma^2}}}{e^{\frac{-(x_j-\mu_{j,0})^2}{2\sigma^2}}} \geq 1 \tag{11}$$

Apply the log rule $e^{-x}/e^{-y} = e^{x-y}$:

$$\left(\frac{p}{p-1} \prod_{i=0}^{d} \frac{\frac{1}{\sqrt{2\sigma^2\pi}}}{\frac{1}{\sqrt{2\sigma^2\pi}}} \right) * \prod_{i=0}^{d} e^{\frac{(x_j-\mu_{j,0})^2}{2\sigma^2} - \frac{(x_j-\mu_{j,1})^2}{2\sigma^2}} \geq 1 \tag{12}$$

Then we will take the natural log of both sides of the equation:

$$\left(\frac{p}{p-1} \prod_{i=0}^{d} \frac{\frac{1}{\sqrt{2\sigma^2\pi}}}{\frac{1}{\sqrt{2\sigma^2\pi}}} \right) * \sum_{i=0}^{d} \frac{(x_j-\mu_{j,0})^2}{2\sigma^2} - \frac{(x_j-\mu_{j,1})^2}{2\sigma^2} \geq 0 \tag{13}$$

Let denote the grouped constant terms as b and simplify the second term in the sum:

$$b + \sum_{i=0}^{d} \frac{x_j - \mu_{j,1}}{x_j - \mu_{j,0}} \geq 0 \tag{14}$$

4: Experiment

(1)

$$g(W) = log(1 + exp(-y_i \mathbf{w^T x_i}))$$

We want to calculate $\frac{dg}{dw}$, or the derivative of the function g in terms of $\mathbf{w}$. We can use the chain rule where $a = 1 + exp(-y_i \mathbf{w^T x_i})$ and $b = -y_i \mathbf{w^T x_i}$.

$$\frac{d}{da} \ log(a) = \frac{1}{a}$$

$$\frac{d}{db} \ 1 + exp(b) = exp(b)$$

$$\frac{d}{dw} \ -y_i \mathbf{w^T x_i} = -y_i \mathbf{x_i}$$

Computing the derivative of the composition of the three functions using the chain rule:

$$\frac{dg}{dw} = \frac{dg}{da} * \frac{da}{db} * \frac{db}{dw}$$

$$\frac{dg}{dw} = \frac{1}{a} * e^b * -y_i \mathbf{x_i}$$

$$\frac{dg}{dw} = \frac{1}{1 + exp(-y_i \mathbf{w^T x_i})} * exp(-y_i \mathbf{w^T x_i}) * -y_i \mathbf{x_i}$$

$$\boxed{\frac{dg}{dw} = \frac{-y_i \mathbf{x_i}}{1 + exp(y_i \mathbf{w^T x_i})}} \tag{15}$$

(2) When the entire dataset is composed of a single example, the objective is expressed as:

$$\boxed{objective: \ J(w) = log(1 + exp(-y_i \mathbf{w^t x_i})) + \frac{1}{\sigma^2} \mathbf{w^T w}} \tag{16}$$

This is equivalent to the original optimization problem as finding the min of a summation is redundant when taking only one example.

The gradient of this objective can be found using the derivative found in part 1 plus the derivative of $\frac{1}{\sigma^2} \mathbf{w^T w}$. We use the fact that $w^T w = w^2$.

$$\boxed{\nabla J(w) = \frac{-y_i \mathbf{x_i}}{1 + exp(y_i \mathbf{w^T x_i})} + \frac{2\mathbf{w}}{\sigma^2}} \tag{17}$$

(3) Using the objective $J(w)$ (Eq. 8) and the gradient $\nabla J(w)$ (Eq. 9) from the previous part, we can write pseudo code for a stochastic gradient descent algorithm. Recall in stochastic gradient descent we treat a random example as the full dataset.

Stochastic gradient descent for logistic regression classifier

Given a training set S = $\{(\mathbf{x_i}, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1, 1\}$:

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 ... T:
    1. For each training example $(\mathbf{x_i},\ y_i) \in S$:
    2. Take the derivative of $J(w)$, where the current example represents the entire dataset, at the current $\mathbf{w^{t-1}}$ to be $\nabla J^t(\mathbf{w^{t-1}})$
    3. Update: $\mathbf{w^t} = \mathbf{w^{t-1}} - \gamma \nabla J^t(\mathbf{w^{t-1}})$
3. Training is done, return the weight vector $\mathbf{w}$

(4) *Cross validation:* I performed cross validation to determine the best learning rate γ and σ^2 for the Adult data set. For each combination of $\gamma \in \{1,\ 0.5,\ 0.1,\ 0.05,\ 0.01,\ 0.001\}$ and $\sigma \in \{10,\ 20,\ 25,\ 50,\ 100,\ 1,000\}$ 6-cross validation was performed. Below is a table of interesting results including the best and worst performing hyper-parameters.

γ	σ	Epoch	Accuracy
1	25	1	0.594
0.01	10	1	0.835
0.01	50	1	0.845
0.01	1,000	1	0.844
0.0001	25	1	0.756

Table 1: Interesting results from 6-fold cross validation.

I found σ to have little to no weight on the accuracy of the classifier. This can be seen in the middle three results from the table, regardless of the σ chosen the accuracy remains the same. I found the choice of γ to be the determiner in the accuracy of the classifier. A large γ such as 1 or a smaller γ such as 0.0001 caused poor results. It should be noted that 0.76 is a baseline for this dataset as predicting all -1 gives this accuracy. **I found the best γ to be 0.01 when paired with a σ of 50**.

Experiment: With the found best γ of 0.01 and σ of 50, I trained stochastic gradient descent algorithm on the Adult training data. Once trained, I evaluated it's performance on the Adult training and Adult test data.

Data Set	γ	σ	Epoch	Accuracy
Adult training	0.01	50	200	0.85
Adult testing	0.01	50	200	0.847

Table 2: Classification accuracy on the Adult training and test set.

Below is a plot of the objective after each epoch of SGD. For this plot, SGD was ran on the Adult training set with a γ of 0.01, a σ of 50, and a epoch of 200.