1: Balls and Bins

Markov's Inequality: $Pr(X \geq a) \leq \frac{E[x]}{a}$

(a) Given n bins and $4n \log n$ balls, we want to prove that the probability that there exists an empty bin is $< 1/n$.

Proof We will prove this using Markov's Inequality. Let X be a random variable representing the number of empty bins. We will show that the probability that at least one bin is empty is $< 1/n$.

$$Pr(X \geq 1) \leq E[X]$$

From class, we saw the expectation of X is equivalent to the summation of each of the bins. Let y_i represent each bin where a 1 represent an empty bin and a 0 otherwise.

$$Pr(X \geq 1) \leq E[X] = E[\sum_{i=1}^{n} y_i]$$

The probability a bin is empty is given by $(1 - 1/n)^m$ where m is the number of balls. That is, there are $1 - 1/n$ other bins each ball can be placed in.

$$Pr(X \geq 1) \leq n * (1 - \frac{1}{n})^{4nlogn}$$

From Bernoulli's inequality we know that $1 + y \leq e^y$ for all for all y. Letting $y = \frac{-1}{n}$ will allow us to substitute $1 + y$ for e^y. This is valid as $1 + y \leq e^y$ so Markov's Inequality still holds.

$$Pr(X \geq 1) \leq n * (1 + y)^{4nlogn}$$
$$Pr(X \geq 1) \leq n * (e^y)^{4nlogn}$$
$$Pr(X \geq 1) \leq n * (e^{-1/n})^{4nlogn}$$

Applying exponent and natural log identities.

$$Pr(X \geq 1) \leq n * e^{-4logn}$$
$$Pr(X \geq 1) \leq \frac{1}{n^3} < \frac{1}{n}$$

Thus proving the probability that at least one bin is empty is $< 1/n$.

(b.a) When $m = \frac{1}{2}nlogn$, the logic follows as in part a.

$$Pr(X \geq 1) \leq E[X] = E[\sum_{i=1}^{n} y_i] = n * (1 - \frac{1}{n})^{\frac{1}{2}nlogn}$$

Where y_i is a bin and 1 represents an empty bin and 0 otherwise. Making the similar substitution and applying similar identities as part a.

$$Pr(X \geq 1) \leq n * (e^{-1/n})^{\frac{1}{2}nlogn}$$
$$Pr(X \geq 1) \leq n * \frac{1}{\sqrt{n}}$$
$$Pr(X \geq 1) \leq \sqrt{n}$$

Given $\frac{1}{2}nlogn$ balls, we can say that the probability that at least one bin is empty is $\leq \sqrt{n}$.

$\boxed{Pr(X \geq 1) \leq \sqrt{n}}$

(b.b) When $m = 100nlogn$, a similar logic follows.

$$Pr(X \geq 1) \leq E[X] = E[\sum_{i=1}^{n} y_i] = n * (1 - \frac{1}{n})^{100nlogn}$$
$$Pr(X \geq 1) \leq n * (e^{-1/n})^{100nlogn}$$
$$Pr(X \geq 1) \leq n * \frac{1}{n^{100}}$$
$$Pr(X \geq 1) \leq \frac{1}{n^{99}}$$

Given $100nlogn$ balls, we can say that the probability that at least one bin is empty is $\leq \frac{1}{n^{99}}$.

$\boxed{Pr(X \geq 1) \leq \frac{1}{n^{99}}}$

(c)

(d)

2: Estimating the Mean and Median

(a)

(b)

(c)

(d)

3: Quick-sort with Optimal Comparisons

(a)

(b)

(c)

4: Randomized Min-Cut

(a)

(b)

(c)

(d)

5: Valiant-Vazirani Lemma