1: Probabilities

$$\boxed{Independent\ Events -\ \ P(A\ \cap\ B) = P(A)P(B)} \tag{1}$$

$$\boxed{Rule\ of\ Multiplication -\ \ P(A\ \cap\ B) = P(A)P(B|A)} \tag{2}$$

(1) Given $P(A_1) = P(A_2) = P(A_1|A_2) = \frac{1}{2}$, we want to prove that A_1 and A_2 are independent events.

Events A_1 and A_2 are independent if and only if Eq. (1) is satisfied.

$$P(A_2 \cap A_1) = P(A_2)P(A_1)$$

We can use Eq. (2) to restate the LHS of Eq. (1) in terms of probabilities we are given.

$$P(A_2)P(A_1|A_2) = P(A_2)P(A_1)$$

$$\frac{1}{2} * \frac{1}{2} = \frac{1}{2} * \frac{1}{2}$$

By showing Eq. (1) is satisfied, we have proven A_1 and A_2 are independent events.

(2) From lecture, we saw the Theorem of Total probability pertaining to mutually exclusive events $A_1, A_2, ..., A_n$ where $\sum_i P(A_i) = 1$. When this condition is met, we know the following is true:

$$P(B) = \sum_{i=1}^{n} P(B \cap A) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

We have mutually exclusive events A_1, A_2, and A_3 where the sum of their probabilities is 1. The condition is met, so we can use the Theorem of Total probability. All the needed probabilities to calculate $P(A_4)$ are provided in the problem.

$$P(A_4) = \sum_{i=1}^{3} P(A_4|A_i)P(A_i)$$

$$P(A_4) = \frac{1}{3} * (\frac{1}{6} + \frac{1}{3} + \frac{1}{2})$$

$\boxed{P(A_4) = \frac{1}{3}}$

$$\boxed{Binomial\ Distribution - \binom{n}{k} p^k (1-p)^{n-k}} \tag{3}$$

(3) Let X be a random variable representing the top of the six-sided die toss. The dice is a fair dice so we know $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = \frac{1}{6}$.

There are six possible events and the total probability of exactly two heads after n coin tosses is the sum of the probability of each of the six events happening.

$$\sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

Where $B(n, k, p)$ represents the binomial distribution from Eq. (3). This is the probability that given n trials, there are exactly k successes if the probability of success is p where $O \leq p \leq 1$ and $k \leq n$. Let Y be a random variable representing the exact number of heads after n coin tosses. Note that the probability of getting exactly two heads when only tossing one coin is 0.

$$P(Y = 2) = \sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6} \sum_{i=1}^{6} B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6}(0 + \frac{1}{4} + \frac{3}{8} + \frac{6}{16} + \frac{10}{32} + \frac{15}{64})$$

$$\boxed{P(Y = 2) = \tfrac{33}{128} = 0.2578}$$

Thus this is the probability of getting exactly 2 heads after n coin flips, where n is the result of a fair six-sided die toss.

$$\boxed{Rule\ of\ Addition - \ P(A\ \cup\ B) = P(A) + P(B) - P(A)P(B|A)} \tag{4}$$

(4) We want to prove that if $P(A_1) = a_1$ and $P(A_2) = a_2$ then $P(A_1|A_2) \geq \frac{a_1+a_2-1}{a_2}$.

Proof: we begin with Eq. (4) which is the rule for union of two events.

$$P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

$P(A_2 \cup A_1)$ is a probability so we know it has a upper bound of 1.

$$1 \geq P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$
$$1 \geq P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

Rearranging terms and multiplying both sides by -1.

$$\frac{1 - P(A_2) - P(A_1)}{P(A_2)} \geq -P(A_1|A_2)$$
$$\frac{P(A_1) + P(A_2) - 1}{P(A_2)} \leq P(A_1|A_2)$$

Replacing $P(A_1) = a_1$ and $P(A_2) = a_2$ on the LHS of the inequality.

$$P(A_1|A_2) \geq \frac{a_1 + a_2 - 1}{a_2}$$

Thus arriving at the original inequality and proving it's correctness.

(5a) Given two independent random variables A_1 and A_2, we want to prove that $E[A_1 + A_2] = E[A_1] + E[A_2]$ is true. We will assume A_1 and A_2 are discrete for this proof. The equality also holds for continuous random variables with a slightly different proof.

Given a discrete random variable X that can take values $x_1,\ x_2,\ ...,\ x_k$, with respective probabilities $p_1,\ p_2,\ ...,\ p_k$, then the expected value of X is defined as:

$$\boxed{Expectation - \ E[X] = \sum_{i=1}^{k} x_i * P(X = x_i)} \tag{5}$$

Proof: Starting with the LHS of the equality we want to prove, we will use the definition of expectation to arrive at the RHS.

$$E[A_1 + A_2] = \sum_{i=1}^{k}\sum_{j=1}^{k}(a_{1i} + a_{2k}) * P(A_1 = a_{1i}, A_2 = a_{2k})$$

Multiply and split RHS into two sets of summations.

$$E[A_1 + A_2] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{1i} * P(A_1 = a_{1i}, A_2 = a_{2k}) + \sum_{i=1}^{k}\sum_{j=1}^{k} a_{2k} * P(A_1 = a_{1i}, A_2 = a_{2k})$$

$$E[A_1 + A_2] = \sum_{i=1}^{k} a_{1i} * P(A_1 = a_{1i}) + \sum_{j=1}^{k} a_{2k} * P(A_2 = a_{2k})$$

The RHS is the definition of expectation for A_1 summed with the expectation of A_2.

$$E[A_1 + A_2] = E[A_1] + E[A_2]$$

(5b) Given two independent random variables A_1 and A_2, we want to prove that $var[A_1 + A_2] = var[A_1] + var[A_2]$ is true.

Proof: Variance is defined as:

$$\boxed{Variance - \ var[X] = E[(X - E[X])^2} \tag{6}$$

Replacing X with $A_1 + A_2$ and using algebra, this can be rewritten as:

$$var[A_1 + A_2] = E[(A_1 + A_2)^2] - E[A_1 + A_2]^2$$

Expanding out the polynomials gives us:

$$var[A_1 + A_2] = E[A_1^2 + 2A_1A_2 + A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2]^2$$

Using the proof of linearity of expectation from part a, we can rewrite the first expectation:

$$var[A_1 + A_2] = E[A_1^2] + 2E[A_1A_2] + E[A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2]^2$$

When the covariance of two random variables is zero, the expected value operator is multiplicative. That is, $E[XY] = E[X]E[Y]$. We know the covariance of A_1 and A_2 is 0 as they are independent of each other. Using this:

$$var[A_1 + A_2] = E[A_1^2] + 2E[A_1]E[A_2] + E[A_2^2] - E[A_1]^2 - 2E[A_1]E[A_2] - E[A_2^2]$$
$$var[A_1 + A_2] = E[A_1^2] - E[A_1]^2 + E[A_2^2] - E[A_2^2]$$
$$var[A_1 + A_2] = var[A_1] + var[A_2]$$

2: Naïve Bayes