1: Probabilities

$$\boxed{Independent\ Events -\ \ P(A\ \cap\ B) = P(A)P(B)} \tag{1}$$

$$\boxed{Rule\ of\ Multiplication -\ \ P(A\ \cap\ B) = P(A)P(B|A)} \tag{2}$$

$$\boxed{Rule\ of\ Addition -\ \ P(A\ \cup\ B) = P(A) + P(B) - P(A)P(B|A)} \tag{3}$$

$$\boxed{Binomial\ Distribution - \binom{n}{k} p^k (1-p)^{n-k}} \tag{4}$$

(1) Given $P(A_1) = P(A_2) = P(A_1|A_2) = \frac{1}{2}$, we want to prove that A_1 and A_2 are independent events.

Events A_1 and A_2 are independent if and only if Eq. (1) is satisfied.

$$P(A_2 \cap A_1) = P(A_2)P(A_1)$$

We can use Eq. (2) to restate the LHS of Eq. (1) in terms of probabilities we are given.

$$P(A_2)P(A_1|A_2) = P(A_2)P(A_1)$$

$$\frac{1}{2} * \frac{1}{2} = \frac{1}{2} * \frac{1}{2}$$

By showing Eq. (1) is satisfied, we have proven A_1 and A_2 are independent events.

(2)

(3) Let X be a random variable representing the top of the six-sided die toss. The dice is a fair dice so we know $P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6) = \frac{1}{6}$. There are six possible events and the total probability of exactly two heads after n coin tosses is the sum of the probability of each of the six events happening.

$$\sum_{i=1}^{6} P(X=i) * B(n=i, k=2, p=0.5)$$

Where $B(n,k,p)$ represents the binomial distribution from Eq. (4). This is the probability that given n trials, there are exactly k successes if the probability of success is p where $O \leq p \leq 1$ and $k \leq n$. Let Y be a random variable representing the exact number of heads after n coin tosses. Note that the probability of getting exactly two heads when only tossing one coin is 0.

$$P(Y=2) = \sum_{i=1}^{6} P(X=i) * B(n=i, k=2, p=0.5)$$

$$P(Y=2) = \frac{1}{6}\sum_{i=1}^{6} B(n=i, k=2, p=0.5)$$
$$P(Y=2) = \frac{1}{6}(0 + \frac{1}{4} + \frac{3}{8} + \frac{6}{16} + \frac{10}{32} + \frac{15}{64})$$
$$P(Y=2) = \frac{33}{128} = 0.2578$$

Thus this is the probability of getting exactly 2 heads after n coin flips, where n is the result of a fair six-sided die toss.

$\boxed{P(heads=2) = \frac{33}{128} = 0.2578}$

(4) We want to prove that if $P(A_1) = a_1$ and $P(A_2) = a_2$ then $P(A_1|A_2) \geq \frac{a_1+a_2-1}{a_2}$.
Proof: we begin with Eq. (3) which is the rule for union of two events.

$$P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

$P(A_2 \cup A_1)$ is a probability so we know it has a upper bound of 1.

$$1 \geq P(A_2 \cup A_1) = P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$
$$1 \geq P(A_2) + P(A_1) - P(A_2)P(A_1|A_2)$$

Rearranging terms and multiplying both sides by -1.

$$\frac{1 - P(A_2) - P(A_1)}{P(A_2)} \geq -P(A_1|A_2)$$
$$\frac{P(A_1) + P(A_2) - 1}{P(A_2)} \leq P(A_1|A_2)$$

Replacing $P(A_1) = a_1$ and $P(A_2) = a_2$ on the LHS of the inequality.

$$\boxed{P(A_1|A_2) \geq \frac{a_1 + a_2 - 1}{a_2}} \tag{5}$$

Thus arriving at the original inequality and proving it's correctness.

(5a)