## 1: Probabilities

$$\boxed{Independent\ Events -\ \ P(A\ \cap\ B) = P(A)P(B)} \tag{1}$$

$$\boxed{Rule\ of\ Multiplication -\ \ P(A\ \cap\ B) = P(A)P(B|A)} \tag{2}$$

$$\boxed{Binomial\ Distribution - \binom{n}{k} p^k (1-p)^{n-k}} \tag{3}$$

**(1)** Given $P(A_1) = P(A_2) = P(A_1|A_2) = \frac{1}{2}$, we want to prove that $A_1$ and $A_2$ are independent events.

Events $A_1$ and $A_2$ are independent if and only if Eq. (1) is satisfied.

$$P(A_2 \cap A_1) = P(A_2)P(A_1)$$

We can use Eq. (2) to restate the LHS of Eq. (1) in terms of probabilities we are given.

$$P(A_2)P(A_1|A_2) = P(A_2)P(A_1)$$

$$\frac{1}{2} * \frac{1}{2} = \frac{1}{2} * \frac{1}{2}$$

By showing Eq. (1) is satisfied, we have proven $A_1$ and $A_2$ are independent events.

**(2)**

**(3)** Let $X$ be a random variable representing the top of the six-sided die toss. The dice is a fair dice so we know $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = \frac{1}{6}$. There are six possible events and the total probability of exactly two heads after $n$ coin tosses is the sum of the probability of each of the six events happening.

$$\sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

Where $B(n, k, p)$ represents the binomial distribution from Eq. (3). This is the probability that given $n$ trials, there are exactly $k$ successes if the probability of success is $p$ where $O \leq p \leq 1$ and $k \leq n$. Let $Y$ be a random variable representing the exact number of heads after $n$ coin tosses. Note that the probability of getting exactly two heads when only tossing one coin is 0.

$$P(Y = 2) = \sum_{i=1}^{6} P(X = i) * B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6} \sum_{i=1}^{6} B(n = i, k = 2, p = 0.5)$$

$$P(Y = 2) = \frac{1}{6}(0 + \frac{1}{4} + \frac{3}{8} + \frac{6}{16} + \frac{10}{32} + \frac{15}{64})$$

$$P(Y = 2) = \frac{33}{128} = 0.2578$$

Thus this is the probability of getting exactly 2 heads after $n$ coin flips, where $n$ is the result of a fair six-sided die toss.

$\boxed{P(heads = 2) = \frac{33}{128} = 0.2578}$