Problem 6 - Convexity Basics

For this problem, let f be a convex function defined over a convex set K, and suppose the diameter of K is 1.

(a) Let $x \in K$, and suppose $f(x) = 2$ and $||\nabla f(x)|| = 1$. Give a lower bound on $\min_{z \in K} f(z)$.

□

(b) Let x^* be the minimizer of f over K (suppose it is unique), and let x be any other point. The intuition behind gradient descent is that the vector: $-\nabla f(x)$ points *towards* x^*. Prove that this is indeed true, in the sense that $\langle \nabla f(x), x - x^* \rangle \geq 0$ (i.e., the negative gradient makes an acute angle with the line to the optimum).

From the lecture 7 scribe notes we have the following property of convex functions

$$f(y) - f(x) \geq \langle y - x, \nabla f(x) \rangle \quad \forall x, y$$

We will apply this property to a unique $x^* \in K$ and any other point $x \in K$

$$f(x^*) - f(x) \geq \langle x^* - x, \nabla f(x) \rangle$$

Our goal is to show $\langle \nabla f(x), x - x^* \rangle \geq 0$, so we will multiply both sides by -1 to change the direction of the inequality

$$-\langle x^* - x, \nabla f(x) \rangle \geq f(x) - f(x^*)$$

Applying the commutative property of dot products and distributing the negative gives us

$$\langle \nabla f(x), x - x^* \rangle \geq f(x) - f(x^*)$$

We know that x^* is a minimizer of f over K, it follows that

$$f(x) - f(x^*) \geq 0 \quad \forall x \in K$$

Thus we have shown that $\langle \nabla f(x), x - x^* \rangle \geq 0$, proving the vector $-\nabla f(x)$ points *towards* x^* for all $x \in K$.

(c) Suppose now that the function f is *strictly convex* i.e.,
$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda) f(y)$ (strictly), for all $x \neq y$, and $0 < \lambda < 1$. Prove that all the *maximizers* of f over K lie on the boundary of K. [*Hint:* You many want to use the definition that a point x is not on the boundary iff there exists points $y, z \in K$ such that $x = (y + z)/2$.]

proof by contradiction: Let's $x \in K$ be a *maximizer* of f and assume that x is **not** on the boundary of K.

Take $y, z \in K$ such that $y \neq z$ and $x = (y+z)/2$. We know $y, z \in K$ exist as x is not on the boundary and using the provided hint. Since x is a *maximizer* of f, then it follows that $f(x) \geq f(y)$ and $f(x) \geq f(z)$. Applying the definition of *strictly convex* to y and z

$$f(\lambda y + (1-\lambda)z) < \lambda f(y) + (1-\lambda) f(z)$$

Using $f(x) \geq f(y)$ and $f(x) \geq f(z)$ we can give an upper-bound

$$f(\lambda y + (1-\lambda)z) < \lambda f(y) + (1-\lambda) f(z) \leq f(x)$$

Ignoring the middle term gives us

$$f(\lambda y + (1-\lambda)z) < f(x)$$

We know that $x = (y+z)/2$. By making this substitution for x and taking $\lambda = 1/2$

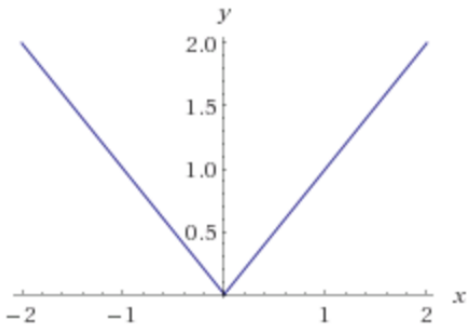
$$f(\frac{y}{2} + \frac{z}{2}) < f(\frac{y+z}{2})$$
$$f(\frac{y+z}{2}) < f(\frac{y+z}{2})$$

We arrive at a contradiction as the inequality is strict. Thus all the *maximizers* of f over K lie on the boundary of K.

Problem 7 - Gradient Descent Basics

(a) Give an example of a function defined over $\mathbb{R}$, for which for *any* step-size $\eta > 0$ (no matter how small), gradient descent with step-size η oscillates around the optimum point (i.e., never gets to distance $< \eta/4$ to it), for some starting point $x \in \mathbb{R}$.

Consider the absolute value function $g(x) = |x|$ defined over $\mathbb{R}$.



The absolute value function is convex but it is not smooth. It is not differentiable as the derivative at $x = 0$ is not defined. For any arbitrarily small step-size η, we can find a starting point $x \in \mathbb{R}$ such that gradient descent will oscillate around the optimum point (never getting to distance $< \eta/4$ to it).

Each iteration gradient descent takes a step in the direction of the negative gradient at the current point, with a step-size of η.

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \eta \nabla g(\mathbf{w}^{(t)})$$

The $\nabla g(x)$ is defined as

$$\nabla g(x) = \begin{cases} -1 & x < 0 \\ undefined & x = 0 \\ 1 & x > 0 \end{cases}$$

As such, we can see at each iteration of gradient descent we will take a η sized step from the current point. This step will be in the positive direction if x is negative and vice-versa. For example take the starting point $x = \eta/2$. At each iteration of gradient descent, we will oscillate between $x = -\eta/2$ and $x = \eta/2$. Thus we will never get within $\eta/4$ of the optimum.

(b) Consider the function $f(x, y) = x^2 + \frac{y^2}{4}$, and suppose we run gradient descent with starting point $(1, 1)$, and $\eta = 1/4$. Do we get arbitrarily close to the minimum? Experimentally, find the *threshold* for η, beyond which the gradient descent starts to oscillate.

Given the function $f(x, y) = x^2 + \frac{y^2}{4}$ with a starting point of $\mathbf{w}^{(1)} = (1, 1)$ and $\eta = 1/4$ we get arbitrarily close to the minimum. The minimum of $f(x, y)$ is 0 when $\mathbf{w} = (0, 0)$. To explore this gradient descent scenario, I wrote a program to calculate $\mathbf{w}^{(t)}$ and defined arbitrarily close to being within 0.0001 of the minimum. I found after 39 steps that $\mathbf{w}^{(39)} = (3.6379\text{e}{-12}, 0.0063)$ and $f(x, y) = 9.7848\text{e}{-06}$. Given enough iterations, the gradient descent found the optimum point (barring floating-point precision issues).

Using a program, I found the threshold for η. I found that gradient descent does not find the optimum point when $\eta \geq 1$. When $\eta = 1$, the behavior is interesting. The x component of $\mathbf{w}^{(t)}$ alternates between the values 1 and -1 and the y component converges to 0. As such, $f(x, y) = 1$ which is not arbitrarily close to the minimum.

(c) Why is the behavior similar to that in part (a) (oscillation for every η) not happening in part (b)?

In part (a) we observed that the absolute value function $g(x) = |x|$ oscillates around the optimum point despite choosing an arbitrarily small step-size η. In part (b) we concluded that $f(x, y) = x^2 + \frac{y^2}{4}$ with reach the optimum point given a step-size $\eta < 1$.

If we consider the gradient of $f(x, y)$ which is defined as

$$\nabla f(x, y) = (2x, \frac{y}{2})$$

and the update for each iteration of gradient descent

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \eta \nabla f(\mathbf{w}^{(t)})$$

We can observe that $\mathbf{w}^{(t+1)}$ is dependent on the value of x_t, y_t, and a fixed η. As x_t and y_t get closer to 0, the update "size" at each iteration decreases (since $||\nabla f(x, y)||$ decreases). This was not the case with $g(x)$, as the direction of the update is dependent on x but the "size" of the update is only dependent on η. Thus, with a reasonably small η, $f(x, y)$ will reach the optimum point.

Problem 8 - Stochastic Gradient Descent

Suppose we have points $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$ in the plane and suppose that $|a_i| \leq 1$, and $|b_i| \leq 1$ for all i. Let $f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y)$, where $f_i(x, y) = (x - a_i)^2 + (y - b_i)^2$.

(a) What is the point (x, y) that minimizes $f(x, y)$?

We can observe that $f_i(x, y) = (x - a_i)^2 + (y - b_i)^2$ is minimized when $x = a_i$ and $y = b_i$. This is true as $f_i(x = a_i, y = b_i) = 0$ for all $|a_i| \leq 1$ and $|b_i| \leq 1$ and $f_i(x, y)$ is a non-negative function.

The intuition is we want to minimize the average distance from a point (x, y) to each point (a_i, b_i) from the sampled $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$ points in the plane. This is from an observation that $f_i(x, y)$ is minimized the closer $x - a_i$ and $x - b_i$ are to zero. To find the point (x, y) that minimizes $f(x, y)$, we take

$$x = \frac{1}{n} \sum_{i=1}^{n} a_i \qquad y = \frac{1}{n} \sum_{i=1}^{n} b_i$$

or the average value of the n sampled a_i and b_i.

(b) Suppose we perform gradient descent (on f) with step size $0 < \eta < 1$. Give the geometric interpretation for one iteration.

(c) Now suppose we perform stochastic gradient descent with fixed step-size $0 < \eta < 1$, and by picking i at random in $\{1, ..., n\}$, and moving along the gradient of f_i (as in SGD seen in class). After T steps, for T large enough, can we say that we get arbitrarily close to the optimum? (Provide a clear explanation). [*Hint:* Remember η is fixed.]

□

(d) Pick $n = 100$ random points in $[-1, 1]^2$ (uniformly), and run SGD for fixed $\eta = 1/2$, as above. Write down what the distance to optimum is, after $T = 10$, $T = 100$, and $T = 1,000$ iterations (if you want to be careful, you should average over 5 random choices for the initialization). Now consider dropping step size $\eta_t = 1/t$, and write down the result for T as above.

Experiment 1: I ran SGD with 100 random points in $[-1, 1]^2$ and a fixed $\eta = 1/2$. I randomly initialized (x, y) in $[-100000, 100000]^2$ and took an average of the distance from the optimum point (keeping the same 100 random points for each instance).

t	distance
10	0.005107
100	0.005202
1,000	0.00495

Table 1: Results at $t = 10, 100$, and $1,000$ for a fixed η.

Experiment 2: Same as in experiment 1 but now at each iteration η is updated such that $\eta_t = 1/t$.

t	distance
10	0.00166
100	0.0003
1,000	0.000009

Table 2: Results at $t = 10, 100$, and $1,000$ for $\eta_t = 1/t$.

We can observe in experiment 1 we experience oscillation around the optimum point when η is fixed at $1/2$. Alternatively, if we decrease the step-size at each iteration it allows us to get arbitrarily close to the optimum point.

Problem 9 - Numeric Accuracy in MW Updates

Consider the randomized experts setting we saw in class (we maintain a distribution over experts at each time, and the loss of the algorithm at that time is the expected loss over the distribution). Consider a setting where the experts predict 0/1, and the loss is either 0 or 1 for each expert. We saw how to update the probabilities (multiply by $e^{-\eta}$ if an expert makes a mistake, keep unchanged otherwise, and renormalize).

One of the common issues here is that numeric errors in such computations tend to compound if not done carefully. Suppose we have N experts, and we start with a uniform distribution over them. Let $p_t^{(i)}$ denote the probability of expert i at time t, for the "true" (infinite precision) multiplicative weight algorithm, and let $q_t^{(i)}$ denote the probabilities that the 'real life' algorithm uses (due to precision limitations).

(a) One simple way to deal with limited precision is to zero out weights that are "too small". Specifically, suppose we set $q_t^{(i)} = 0$ if $q_t^{(i)} / \max_j q_t^{(j)} < \epsilon$, for some precision parameter ϵ (such

changes frequently occur due to roundoff). Other than this, suppose that the $q_t^{(i)}$ are updated accurately. Prove that in this case, we cannot hope to achieve any non-trivial regret bound. Specifically, for large enough T, the algorithm could have error $T(1 - o(1))$, while the best expert may have error $o(T)$. [Hint: in this case, we are "losing" all information about an expert.]

proof: Take two experts A and B. Suppose that expert A is the best expert with an error of $\log_2(T)$ which is $< o(T)$ and expert B who is poor and has an error of $T/2$. We will show that the algorithm could have error $T(1 - o(1))$ which for convenience is of the order $O(T)$.

Consider the adversarial scenario where expert A makes all of their mistakes before expert B and we will choose $\eta = 0.693147$ such at $e^{-\eta} = 1/2$ (to keep the calculations convenient).

Initially at $t = 0$ we have $q_0^{(A)} = 1/2$ and $q_0^{(B)} = 1/2$. At $t = 1$ assume A makes a mistake and B doesn't, giving $q_1^{(A)} = 1/2 * e^{-\eta} = 1/4$. After normalization, we have $q_0^{(A)} = 1/3$ and $q_0^{(B)} = 2/3$. At $t = 2$ expert A makes another mistake where expert B does not. Again giving $q_2^{(A)} = 1/4 * e^{-\eta} = 1/8$ or after normalization $q_2^{(A)} = 1/5$ and $q_2^{(B)} = 4/5$.

This scenario where expert A always makes a mistake and expert B does not can be generalized. At step t and after normalization

$$q_t^{(A)} = \frac{1}{t^2 + 1}$$

$$q_t^{(B)} = max_j q_t^{(j)} = \frac{t^2}{t^2 + 1}$$

We will set $q_t^{(A)} = 0$ and "lose" all information from expert A after a small number of steps for a reasonable ϵ

$$\frac{\frac{1}{t^2+1}}{\frac{t^2}{t^2+1}} = \frac{1}{t^2} < \epsilon$$

For example, let $T = 64$ and $\epsilon = 1/16$. At $t = 5$ we will lose expert A by setting $q_t^{(A)} = 0$ as $1/t^2 < \epsilon$. We know that expert A makes $\log_2(T)$ mistakes and 5 is of the order $\log_2(T)$. For $t = 6$ to $t = 32$ classification will only depend on expert B who makes $T/2$ mistakes. Thus our algorithm will make roughly $(T - \log_2(T))/2 + \log_2(T)$ mistakes which is the order of $O(T)$ despite having a best expert who makes $o(T)$ mistakes.

We have proven that in a worst-cast scenario, having a single good expert can still lead to a poor mistake bound.

(b)

☐

(c)

☐

(d)

□

Collaboration and Code

I collaborated with Sierra Allred, Maks Cegielski-Johnson, and Dietrich Geisler on problem 6 (all parts) and problem 9 (parts a and b).

Experiments were conducted with programs written in standard Python3. I can e-mail you my codebase if need be.