Problem 6 - Convexity Basics

For this problem, let f be a convex function defined over a convex set K, and suppose the diameter of K is 1.

(a) Let $x \in K$, and suppose $f(x) = 2$ and $||\nabla f(x)|| = 1$. Give a lower bound on $\min_{z \in K} f(z)$.

□

(b) Let x^* be the minimizer of f over K (suppose it is unique), and let x be any other point. The intuition behind gradient descent is that the vector: $-\nabla f(x)$ points *towards* x^*. Prove that this is indeed true, in the sense that $\langle \nabla f(x), x - x^* \rangle \geq 0$ (i.e., the negative gradient makes an acute angle with the line to the optimum).

□

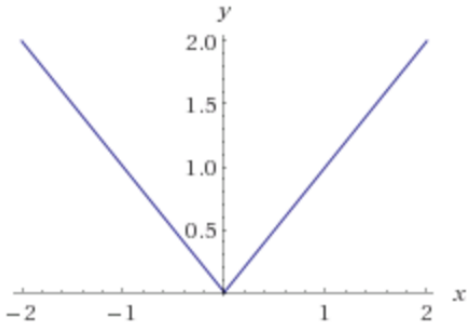
(c) Suppose now that the function f is *strictly convex* i.e.,
$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$ (strictly), for all $x \neq y$, and $0 < \lambda < 1$. Prove that all the *maximizers* of f over K lie on the boundary of K. [*Hint:* You many want to use the definition that a point x is not on the boundary iff there exists points $y, z \in K$ such that $x = (y+2)/2$.]

□

Problem 7 - Gradient Descent Basics

(a) Give an example of a function defined over $\mathbb{R}$, for which for *any* step-size $\eta > 0$ (no matter how small), gradient descent with step-size η oscillates around the optimum point (i.e., never gets to distance $< \eta/4$ to it), for some starting point $x \in \mathbb{R}$.

Consider the absolute value function $g(x) = |x|$ defined over $\mathbb{R}$.



The absolute value function is convex but it is not smooth. It is not differentiable as the derivative at $x = 0$ is not defined. For any arbitrarily small step-size η, we can find a starting point $x \in \mathbb{R}$ such that gradient descent will oscillate around the optimum point (never getting to distance $< \eta/4$ to it).

Each iteration gradient descent takes a step in the direction of the negative gradient at the current point, with a step-size of η.

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \eta \nabla g(\mathbf{w}^{(t)})$$

The $\nabla g(x)$ is defined as

$$\nabla g(x) = \begin{cases} -1 & x < 0 \\ undefined & x = 0 \\ 1 & x > 0 \end{cases}$$

As such, we can see at each iteration of gradient descent we will take a η sized step from the current point. This step will be in the positive direction if x is negative and vice-versa. Thus, regardless of how small η is, we can easily arrive at a situation where gradient descent will oscillate around the optimum point found at the origin.

(b) Consider the function $f(x, y) = x^2 + \frac{y^2}{4}$, and suppose we run gradient descent with starting point $(1, 1)$, and $\eta = 1/4$. Do we get arbitrarily close to the minimum? Experimentally, find the *threshold* for η, beyond which the gradient descent starts to oscillate.

Given the function $f(x, y) = x^2 + \frac{y^2}{4}$ with a starting point of $\mathbf{w}^{(1)} = (1, 1)$ and $\eta = 1/4$ we get arbitrarily close to the minimum. The minimum of $f(x, y)$ is 0 when $\mathbf{w} = (0, 0)$. To explore this gradient descent scenario, I wrote a program to calculate $\mathbf{w}^{(t)}$ and defined arbitrarily close to being within 0.0001 of the minimum. I found after 39 steps that $\mathbf{w}^{(39)} = (3.6379\text{e}{-12}, 0.0063)$ and $f(x, y) = 9.7848\text{e}{-06}$. Given enough iterations, the gradient descent found the optimum point.

Using a program, I found the threshold for η. I found the gradient descent starts to oscillate when $\eta \geq 1$. When $\eta = 1$, the behavior is interesting. The x component of $\mathbf{w}^{(t)}$ alternates between the values 1 and -1 and the y component converges to 0. As such, $f(x, y)$ oscillates between -1 and 1.

(c) Why is the behavior similar to that in part (a) (oscillation for every η) not happening in part

(b)?

In part (a) we observed that the absolute value function $g(x) = |x|$ oscillates around the optimum point despite choosing an arbitrarily small step-size η. In part (b) we concluded that $f(x, y) = x^2 + \frac{y^2}{4}$ with reach the optimum point given a step-size $\eta < 1$.

If we consider the gradient of $f(x, y)$ which is defined as

$$\nabla f(x, y) = (2x, \frac{y}{2})$$

and the update for each iteration of gradient descent

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \eta \nabla f(\mathbf{w}^{(t)})$$

We can observe that $\mathbf{w}^{(t+1)}$ is dependent on the value of x_t, y_t, and a fixed η. As x_t and y_t get closer to 0, the update "size" at each iteration decreases (since $||\nabla f(x, y)||$ decreases). This was not the case with $g(x)$, as the direction of the update is dependent on x but the "size" of the update is only dependent on η. Thus, with a reasonably small η, $f(x, y)$ will reach the optimum point.

Problem 8 - Stochastic Gradient Descent

Suppose we have points $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$ in the plane and suppose that $|a_i| \leq 1$, and $|b_i| \leq 1$ for all i. Let $f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y)$, where $f_i(x, y) = (x - a_i)^2 + (y - b_i)^2$.

(a) What is the point (x, y) that minimizes $f(x, y)$?

☐

(b) Suppose we perform gradient descent (on f) with step size $0 < \eta < 1$. Give the geometric interpretation for one iteration.

☐

(c) Now suppose we perform stochastic gradient descent with fixed step-size $0 < \eta < 1$, and by picking i at random in $\{1, ..., n\}$, and moving along the gradient of f_i (as in SGD seen in class). After T steps, for T large enough, can we say that we get arbitrarily close to the optimum? (Provide a clear explanation). [*Hint:* Remember η is fixed.]

☐

(d) Pick $n = 100$ random points in $[-1, 1]^2$ (uniformly), and run SGD for fixed $\eta = 1/2$, as above. Write down what the distance to optimum is, after $T = 10$, $T = 100$, and $T = 1,000$ iterations (if you want to be careful, you should average over 5 random choices for the initialization). Now consider dropping step size $\eta_t = 1/t$, and write down the result for T as above.

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Problem 9 - Numeric Accuracy in MW Updates

Consider the randomized experts setting we saw in class (we maintain a distribution over experts

at each time, and the loss of the algorithm at that time is the expected loss over the distribution). Consider the simple setting where the experts predict 0/1, and the loss is either 0 or 1 for each expert. We saw how to update the probabilities (multiply by $e^{-\eta}$ if an expert makes a mistake, keep unchanged otherwise, and renormalize). One of the common problems here is that numeric errors in such computations tend to compound if not done carefully.

(a) Consider one simple suggestion: say we zero out weights that are "too small", specifically, suppose we set $q_i^{(i)} = 0$ if $q_i^{(i)} / \max_j q_t^{(i)} < \epsilon$, for some precision parameter (such changes frequently occur due to roundoff). Other than this, suppose that the $q_t^{(i)}$ are updated accurately. Prove that in this case, we cannot hope to achieve any non-trivial regret bound. Specifically, for large enough T, the algorithm can have error $T(1 - o(1))$, while the best expert may have error $o(T)$. [Hint: in this case, we are "losing" all information about an expert.]

(b) A simple way to avoid this (in this setting) is to avoid storing probabilities, but instead maintaining only the number of mistakes $m_t^{(i)}$. Prove how this suffices to recover the probabilities $p_t^{(i)}$ (assuming infinite precision arithmetic).

□

(c) Suppose we use the idea in part (b) to construct a distribution q_t that differs from p_t by $< \epsilon$ in the ℓ_1 norm, i.e., $\sum_i |p_t^{(i)} - q_t^{(i)}| < \epsilon$. Then, assuming we construct such a q_t at time t to sample, show that the expected number of mistakes of the algorithm is bounded by $(1 + \eta)$ $\min_i m_T^{(i)} + O(\log N/\eta) + \epsilon T$.

□

(d) The bound above is not great if there is an expert who makes very small number of mistakes (compared to T). Noting that we are dealing with binary predictions, can you come up with a way to run the algorithm, so as to obtain a mistake bound of $(1 + \eta + 2\epsilon)$ $\min_i m_T^{(i)} + O(\log N/\eta)$?

□