

## 1 Stationary Distributions of Markov Chains

1. Suppose that in semester  $t = 0$  Alice passes the class with probability 0.5. Compute the probability that she passes in semester  $t = 1$  and semester  $t = 2$ .

We are given an initial distribution and transition matrix as follows:

$$\pi_0 = [0.5 \quad 0.5]$$

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

To compute the probabilities at  $t = 1$  we apply matrix multiplication:

$$\pi_0 P = \pi_1 = [0.6 \quad 0.4]$$

To compute the probabilities at  $t = 2$  we apply matrix multiplication again this time to  $\pi_1$  and  $P$ :

$$\pi_1 P = \pi_2 = [0.64 \quad 0.36]$$

$P(t_1 = \textit{pass}) = 0.6, \quad P(t_2 = \textit{pass}) = 0.64$

2. Compute the stationary distribution of this chain. (Hint: the easiest way to do this is to start with some “guess” – say 50/50 – and then keep simulating the chain for a while until it seems to settle down. Once you think you’ve got a guess at what the stationary distribution might be, try transitioning from there to see if you get back to the stationary distribution.) (Not as helpful hint: if you like linear algebra, you can directly compute the stationary distribution from the transition matrix using an eigenvalue decomposition: take a look at Wikipedia for how!)

I’ll use the technique of starting the distribution at  $\pi_0 = [0.5 \ 0.5]$ , simulating the chain for a while and waiting for it to settle down. I simulated the chain for  $t = 100$  using a program and arrived at  $\pi = [0.6667 \ 0.3333]$ . We can verify this by transitioning and checking if we arrive at the stationary distribution:

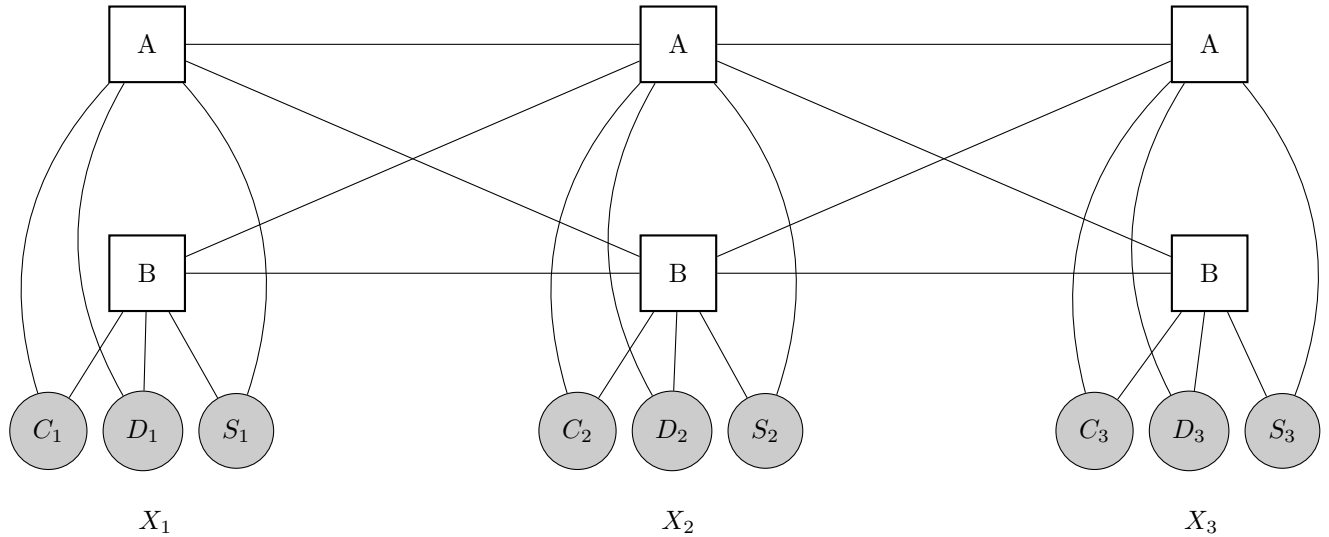
$$\pi P = [0.6667 \ 0.3333] \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = [0.6667 \ 0.3333] = \pi$$

Thus the stationary distribution is:

$\pi = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

## 2 Alice and the Crazy Coke Machine

1. Draw a state space lattice for this soda machine (as on the final slide of day 20) for three time steps.



Where square nodes are states and circle nodes are observations.

2. Suppose that Alice doesn't know what state the machine is in currently (specifically, she believes it's equally likely to be in either state), but puts money in and gets a Sprite out. What is the probability distribution over states that it was in when Alice put her money in? What is the probability distribution over states that it is in now?

Before Alice puts any money into the machine she believes there is a uniform distribution over possible states:

$$P(A_0) = 0.5, \quad P(B_0) = 0.5$$

Now Alice puts in money and gets a Sprite. We will compute the probability of being in each state given we observed a Sprite using Bayes' rule:

$$P(A_1|S_1) = \frac{P(S_1|A_1)P(A_1)}{P(S_1)}$$

$$P(B_1|S_1) = \frac{P(S_1|B_1)P(B_1)}{P(S_1)}$$

We know  $P(S_1|A_1) = 1/4$  and  $P(S_1|B_1) = 1/3$  from the problem. We must compute  $P(A_1)$  and  $P(B_1)$  first:

$$P(A_1) = P(A_1|A_0)P(A_0) + P(A_1|B_0)P(B_0) = 0.8 * 0.5 + 0.4 * 0.5 = 0.6$$

$$P(B_1) = P(B_1|B_0)P(B_0) + P(B_1|A_0)P(A_0) = 0.6 * 0.5 + 0.2 * 0.5 = 0.4$$

Now that we have these, we can compute  $P(S_1)$ :

$$P(S_1) = P(S_1|A_1)P(A_1) + P(S_1|B_1)P(B_1) = 0.25 * 0.6 + 0.333333 * 0.4 = 0.28333$$

Putting it all together:

$$P(A_1|S_1) = \frac{P(S_1|A_1)P(A_1)}{P(S_1)} = \frac{0.25 * 0.6}{0.28333} = 0.5294$$

$$P(B_1|S_1) = \frac{P(S_1|B_1)P(B_1)}{P(S_1)} = \frac{0.3333 * 0.4}{0.28333} = 0.4706$$

$$\boxed{P(A_1|S_1) = 0.5294, \quad P(B_1|S_1) = 0.4706}$$

3. **Suppose Alice comes back the next day (so again she doesn't know what state the machine is in) and really wants a diet coke. Unfortunately, the machine isn't being particularly nice to her and it produces the following series of emissions upon taking money from Alice: C S S D. What is the most likely sequence of states the machine went through in this process?**

We start with a uniform prior over the two states A and B:

$$P(A_1) = 0.5, \quad P(B_1) = 0.5$$

First we compute  $m_1[A]$  and  $m_1[B]$  using the prior and observing a Coke:

$$m_1[A] = P(C_1|A_1)P(A_1) = 0.5 * 0.5 = 0.25$$

$$m_1[B] = P(C_1|B_1)P(B_1) = 0.16667 * 0.5 = 0.083334$$

The second observation was a Sprite. Using  $m_1$  we can compute  $m_2$  for both states:

$$m_2[A] = P(S_2|A_2) * \max\{P(A_2|A_1) * m_1[A], P(A_2|B_1) * m_1[B]\}$$

$$m_2[A] = 0.25 * \max\{\mathbf{0.2}, 0.033336\}$$

$$m_2[A] = 0.05$$

$$m_2[B] = P(S_2|B_2) * \max\{P(B_2|B_1) * m_1[B], P(B_2|A_1) * m_1[A]\}$$

$$m_2[B] = 0.3333 * \max\{0.05, 0.05\}$$

$$m_2[B] = 0.01666$$

The third observation was another Sprite:

$$m_3[A] = P(S_3|A_3) * \max\{P(A_3|A_2) * m_2[A], P(A_3|B_2) * m_2[B]\}$$

$$m_3[A] = 0.25 * \max\{\mathbf{0.04}, 0.006664\}$$

$$m_3[A] = 0.01$$

$$m_3[B] = P(S_3|B_3) * \max\{P(B_3|B_2) * m_2[B], P(B_3|A_2) * m_2[A]\}$$

$$m_3[B] = 0.3333 * \max\{0.01, 0.01\}$$

$$m_3[B] = 0.003333$$

The final observation was a Diet Coke:

$$m_4[A] = P(D_4|A_4) * \max\{P(A_4|A_3) * m_3[A], P(A_4|B_3) * m_3[B]\}$$

$$m_4[A] = 0.25 * \max\{\mathbf{0.008}, 0.0013332\}$$

$$m_4[A] = 0.002$$

$$m_4[B] = P(D_4|B_4) * \max\{P(B_4|B_3) * m_3[B], P(B_4|A_3) * m_3[A]\}$$

$$m_4[B] = 0.5 * \max\{0.002, 0.002\}$$

$$m_4[B] = 0.001$$

Based on the  $m_4$  values, A is the most likely state on purchase 4. We can trace backwards and see which max was taken to recover the most likely sequence of states the machine went through. I've bolded the path above. It's most probable that the machine was in state A for all four purchases.

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