CS 70	Discrete Mathematics and Probability Theory	
Spring 2024	Seshia, Sinclair	DIS 9A

1 Head Count

Note 15

Consider a coin with $\mathbb{P}[\text{Heads}] = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is $\mathbb{P}[X = k]$, for some $0 \leq k \leq 20$?

(b) Name the distribution of X and what its parameters are.

(c) What is $\mathbb{P}[X \geq 1]$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}[12 \leq X \leq 14]$?

Solution:

(a) There are a total of $\binom{20}{k}$ ways to select k coins to be heads. The probability that the selected k coins to be heads is $(\frac{2}{5})^k$, and the probability that the rest are tails is $(\frac{3}{5})^{20-k}$. Putting this together, we have

$$\mathbb{P}[X = k] = \binom{20}{k}\left(\frac{2}{5}\right)^k\left(\frac{3}{5}\right)^{20-k}.$$

(b) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(c)

$$\mathbb{P}[X \geq 1] = 1 - \mathbb{P}[X = 0] = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}[12 \leq X \leq 14] &= \mathbb{P}[X = 12] + \mathbb{P}[X = 13] + \mathbb{P}[X = 14] \\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$

2 Family Planning

Note 15

Mr. and Mrs. Johnson decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Johnsons have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C=1$	$C=2$	$C=3$
$G=0$			
$G=1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}[G=0]$	
$\mathbb{P}[G=1]$	

$\mathbb{P}[C=1]$	$\mathbb{P}[C=2]$	$\mathbb{P}[C=3]$

(d) Are G and C independent?

(e) What is the expected number of girls the Johnsons will have? What is the expected number of children that the Johnsons will have?

Solution:

(a) The sample space is the set of all possible sequences of children that the Johnsons can have: $\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:

$$\mathbb{P}[g] = \frac{1}{2}$$
$$\mathbb{P}[bg] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
$$\mathbb{P}[bbg] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$\mathbb{P}[bbb] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

(b)

	$C=1$	$C=2$	$C=3$
$G=0$	0	0	$\mathbb{P}[bbb]=1/8$
$G=1$	$\mathbb{P}[g]=1/2$	$\mathbb{P}[bg]=1/4$	$\mathbb{P}[bbg]=1/8$

(c) Marginal distribution for G:

$$\mathbb{P}[G=0] = 0+0+\frac{1}{8} = \frac{1}{8}$$
$$\mathbb{P}[G=1] = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$$

Marginal distribution for C:

$$\mathbb{P}[C=1] = 0+\frac{1}{2} = \frac{1}{2}$$
$$\mathbb{P}[C=2] = 0+\frac{1}{4} = \frac{1}{4}$$
$$\mathbb{P}[C=3] = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

(d) No, G and C are not independent. If two random variables are independent, then

$$\mathbb{P}[X=x, Y=y] = \mathbb{P}[X=x]\mathbb{P}[Y=y].$$

To show this dependence, consider an entry in the joint distribution table, such as $\mathbb{P}[G=0, C=3] = 1/8$. This is not equal to $\mathbb{P}[G=0]\mathbb{P}[C=3] = (1/8)\cdot(1/4) = 1/32$, so the random variables are not independent.

(e) We can apply the definition of expectation directly for this problem, since we've computed the marginal distribution for both random variables.

$$\mathbb{E}[G] = 0\cdot\mathbb{P}[G=0]+1\cdot\mathbb{P}[G=1] = 1\cdot\frac{7}{8} = \frac{7}{8}$$
$$\mathbb{E}[C] = 1\cdot\mathbb{P}[C=1]+2\cdot\mathbb{P}[C=2]+3\cdot\mathbb{P}[C=3] = 1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{4} = \frac{7}{4}$$

3 Pullout Balls

Note 15

Suppose you have a bag containing four balls numbered $1,2,3,4$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

Solution:

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathbb{E}[X] = \sum_x x\cdot\mathbb{P}[X=x] = 1\times\frac{1}{4}+2\times\frac{1}{4}+3\times\frac{1}{4}+4\times\frac{1}{4} = 2.5.$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 2.5$).

(b) Let Y be the product of two numbers that you pull out. Then

$$\mathbb{E}[Y] = \frac{1}{\binom{4}{2}}(1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4) = \frac{2+3+4+6+8+12}{6} = \frac{35}{6}.$$