Lax-Wendroff Scheme

- $u_j^n = e^{i(kj\Delta x - \omega n\Delta t)} \equiv \xi^n e^{i(kj\Delta x)}$, where $\xi \equiv e^{-i(\omega\Delta t)} = e^{\omega_I\Delta t}e^{-i\omega_R\Delta t}$

  $|\xi| = e^{\omega_I\Delta t}$

- $u_j^{n+1} = u_j^n - \dfrac{v\Delta t}{\Delta x}\left(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}\right)$ ...... ①

  $u_{j+1/2}^{n+1/2} = \dfrac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \dfrac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right)$ ...... ②

将②式代入①式，且 $r \equiv \dfrac{v\Delta t}{\Delta x}$

$$\Rightarrow u_j^{n+1} = u_j^n - r\left\{\left[\frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{r}{2}(u_{j+1}^n - u_j^n)\right] - \left[\frac{1}{2}(u_j^n + u_{j-1}^n) - \frac{r}{2}(u_j^n - u_{j-1}^n)\right]\right\}$$

$$\Rightarrow u_j^{n+1} = u_j^n - r\left[\frac{1}{2}(u_{j+1}^n - u_{j-1}^n) - \frac{r}{2}(u_{j+1}^n + u_{j-1}^n - 2u_j^n)\right]$$

$$\Rightarrow u_j^{n+1} = u_j^n - r\frac{(u_{j+1}^n - u_{j-1}^n)}{2} + r^2\frac{(u_{j+1}^n + u_{j-1}^n)}{2} - r^2u_j^n$$

$$\Rightarrow u_j^{n+1} = (1-r^2)\cdot u_j^n - r\cdot\frac{(u_{j+1}^n - u_{j-1}^n)}{2} + r^2\cdot\frac{(u_{j+1}^n + u_{j-1}^n)}{2}$$

将 $u_j^n = \xi^n \cdot e^{i(kj\Delta x)}$ 代入

$$\Rightarrow \xi^{n+1}\cancel{e^{i(kj\Delta x)}} = \xi^n\cdot\cancel{e^{ikj\Delta x}}\cdot\left[(1-r^2) - r\cdot\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2} + r^2\cdot\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}\right]$$

$$\Rightarrow \xi = (1-r^2) - ir\sin(k\Delta x) + r^2\cos(k\Delta x)$$

$$\begin{aligned}
\Rightarrow |\xi|^2 &= \left[(1-r^2) + r^2\cos(k\Delta x)\right]^2 + r^2\sin^2(k\Delta x) \\
&= (1-r^2)^2 + 2(1-r^2)r^2\cos(k\Delta x) + r^2\left[(r^2-1)\cos^2(k\Delta x) + 1\right] \\
&= 1 - 2r^2 + (r^2)^2 + r^2 + 2r^2(1-r^2)\cos(k\Delta x) + r^2(r^2-1)\cos^2(k\Delta x) \\
&= 1 - r^2(1-r^2) + r^2(1-r^2)\left[2\cos(k\Delta x) - \cos^2(k\Delta x)\right] \\
&= 1 - r^2(1-r^2)\left[1 - 2\cos(k\Delta x) + \cos^2(k\Delta x)\right] \\
&= 1 - r^2(1-r^2)\left(1 - \cos(k\Delta x)\right)^2
\end{aligned}$$

Crank Nicolson scheme

$\cdot\ u_j^n = \xi^n e^{i(kj\Delta x)}$ and let $\alpha \equiv \frac{D\Delta t}{2\Delta x^2}$ ...... ①

$\cdot\ u_j^{n+1} = u_j^n + \frac{D\Delta t}{2\Delta x^2}\left[(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right]$ ...... ②

将①代入②式:

$u_j^{n+1} = u_j^n + \alpha u_{j+1}^{n+1} - 2\alpha u_j^{n+1} + \alpha u_{j-1}^{n+1} + \alpha u_{j+1}^n - 2\alpha u_j^n + \alpha u_{j-1}^n$

$\Rightarrow (1+2\alpha)\, u_j^{n+1} = (1-2\alpha)\, u_j^n + \alpha\left[u_{j+1}^{n+1} + u_{j-1}^{n+1} + u_{j+1}^n + u_{j-1}^n\right]$

$\Rightarrow (1+2\alpha)\cdot \xi^{\cancel{n}+1} \cancel{e^{ikj\Delta x}} = (1-2\alpha)\cancel{\xi^n}\cdot \cancel{e^{ikj\Delta x}} + \alpha \cancel{\xi^n}\cdot \cancel{e^{ikj\Delta x}}\left[\xi(e^{ik\Delta x} + e^{-ik\Delta x}) + (e^{ik\Delta x} + e^{-ik\Delta x})\right]$

$\Rightarrow (1+2\alpha)\cdot \xi = (1-2\alpha) + 2\alpha\xi\cos(k\Delta x) + 2\alpha\cos(k\Delta x)$

$\Rightarrow \left[1 + 2\alpha - 2\alpha\cos(k\Delta x)\right]\cdot \xi = (1-2\alpha) + 2\alpha\cos(k\Delta x)$

$\Rightarrow \xi = \frac{1-2\alpha+2\alpha\cos(k\Delta x)}{1+2\alpha-2\alpha\cos(k\Delta x)} = \frac{1-2\alpha(1+\cos(k\Delta x))}{1+2\alpha(1-\cos(k\Delta x))} < 1$

$\therefore$ it is unconditionally stable