1. Post Correspondence Problem: (PCP)

* An instance of post correspondence Problem (PCP) consists of two lists.

$$A = w_1, \ldots, w_k \text{ and}$$

$$B = x_1, \ldots, x_k$$

of strings over some alphabet Σ.

* This instance of PCP has a solution if there any sequence of integers

$i_1, i_2, \ldots, i_m$ with $m \geq 1$ such that

$$w_{i_1}, w_{i_2}, \ldots w_{i_m} = x_{i_1} x_{i_2} \ldots x_{i_m}$$

* The sequence $i_1, \ldots i_m$ is a solution to this PCP.

Example:

Let $\Sigma = \{0, 1\}$. Let A and B be lists of three strings.

	List A	List B
i	w_i	x_i
1	1	111
2	10111	10
3	10	0

* This PCP has a solution

$m = 4$

$i_1 = 2, \quad i_2 = 1 \quad i_3 = 1$ and $i_4 = 3$

$$w_{i_1} w_{i_2} w_{i_3} w_{i_4} = x_{i_1} x_{i_2} x_{i_3} x_{i_4}$$

$$101111110 = 101111110.$$

$\therefore$ The solution

$$\boxed{\begin{array}{l} i_1 = 2 \\ i_2 = 1 \\ i_3 = 1 \\ i_4 = 3 \end{array}}$$

A modified version of PCP [MPCP]

* The modified version of PCP (MPCP) is the following.

  Given lists A and B, of k strings each from Σ^* say

  $$A = w_1, w_2, \ldots w_k$$

  $$B = x_1, x_2, \ldots x_k$$

* This problem has a solution if there exists sequence of integers

  $i_1, i_2, \ldots i_r$ such that

$w_1 w_{i_1} w_{i_2} \ldots w_{i_r} = x_1 x_{i_1} x_{i_2} \ldots x_{i_r}$

* The difference between the MPCP and PCP is that in the MPCP, a solution required to start with the first string on each list.

<u>Lemma</u>

If PCP were decidable, then MPCP would be decidable. That is,

MPCP reduces to PCP

<u>Proof</u>

Let

$A = w_1, w_2, \ldots, w_k$ and

$B = x_1, x_2, \ldots, x_k$

be an instance of the MPCP

* We convert this instance of MPCP to an instance of PCP that has a solution, iff our MPCP instance has a solution.
* If PCP were decidable then MPCP would be decidable
* Let

$\Sigma \rightarrow$ is the alphabet containing all the symbols in List A and B.

* Let ¢ and \$ not be in Σ

* Let $y_i \rightarrow$ be obtained from w_i by inserting the symbol ¢ after each character of w_i

* Let $z_i \rightarrow$ be obtained from x_i by inserting the symbol ¢ ahead of each character of x_i.

* Create new words

$$y_0 = ¢\, y_1 \qquad z_0 = z_1$$

$$y_{k+1} = \$ \qquad z_{k+1} = ¢\$$$

Example:

Reduce the following MPCP into PCP.

	List A	List B
	w_i	x_i
1	1	111
2	10111	10
3	10	0

Solution

* The Lists C and D are constructed from the Lists A and B.

* Let

$$C = y_0, y_1, \ldots y_{k+1}$$

$$D = z_0, z_1, \ldots z_{k+1}$$

	List C	List D
	y_i	z_i
0	¢1¢	¢1¢1¢1
1	1¢	¢1¢1¢1
2	1¢0¢1¢1¢¢	¢1¢0
3	1¢0¢	¢0
4	$	¢$

* Lists C and D represents an instance of PCP
* This instance of PCP has a solution iff the instance of MPCP represented by Lists A and List B has a solution

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Undecidability of PCP

Theorem:

PCP is undecidable

Proof:

* It is sufficient to show that if MPCP were decidable, then PCP would be decidable whether a TM accepts a given word.
* We reduce L_u to MPCP, which again can be reduced to PCP.

of MPCP that has a solution iff M accepts w.

* We do this by constructing an instance of MPCP that if it has a solution has one starts with

$\# q_0 w \# \alpha_1 q_1 \beta_1 \# \dots \# \alpha_k q_k \beta_k$

where

* strings between successive #'s are successive ID's in a computation of M with input w.
* q_k is the final state.
* Formally the pairs of strings forming lists A and B of the instance of MPCP are given below.
* The first pair is

List A	List B
#	$\# q_0 w \#$

* The remaining pairs are grouped as.

Group 1:

List A	List B	
X	X	for each X in Γ
#	#	

For each q in $Q-F$, p in Q, and X, Y and Z in Γ

List A	List B	
qX	Yp	if $\delta(q, X) = (p, Y, R)$
ZqX	pZY	if $\delta(q, X) = (p, Y, L)$
$q\#$	$Yp\#$	if $\delta(q, B) = (p, Y, R)$
$Zq\#$	$pZY\#$	if $\delta(q, B) = (p, Y, L)$

Group - III

For each q in F, and X and Y in Γ

List A	List B
XqY	q
Xq	q
qY	q

Group - IV

List A	List B	
$q\#\#$	$\#$	for each q in F

* (x, y) is a partial solution to MPCP with lists A and B

→ if x is a prefix of y.

→ x and y are the concatenation of corresponding strings of lists A and B respectively.

* If $xz = y$ then

z is the remainder of (x, y).

* $(x, y) =$

$$(\#q_0 w \# \alpha_1 q_1 \beta_1 \# \alpha_2 q_2 \beta_2 \# \ldots \# \alpha_{k-1} q_{k-1} \beta_{k-1},$$

$$\#q_0 w \# \alpha_1 q_1 \beta_1 \# \ldots \# \alpha_k q_k \beta_k \#)$$

Example:

$M = (\{q_1, q_2, q_3\}, \{0, 1, B\}, \{0, 1\}, \delta, q_1, B, \{q_3\})$.

and δ is given as.

	0	1	B
q_1	$(q_2, 1, R)$	$(q_2, 0, L)$	$(q_2, 1, L)$
q_2	$(q_3, 0, L)$	$(q_1, 0, R)$	$(q_2, 0, R)$
q_3	–	–	–

Construct an instance of MPCP with lists A and B.

$w = 01$.

Solution.

* The first pair is

List A	List B
#	$\#q_1 01 \#$

* The remaining pairs are

Group I

List A	List B
0	0
1	1
#	#

Group II

List A	List B	
$q_1 0$	$1 q_2$	from $\delta(q_1, 0) = (q_2, 1, R$
$0 q_1 1$ $1 q_1 1$	$q_2 0 0$ $q_2 1 0$	from $\delta(q_1, 1) = (q_2, 0, L$
$0 q_1 \#$ $1 q_1 \#$	$q_2 0 1 \#$ $q_2 1 1 \#$	from $\delta(q_1, B) = (q_2, 1, L$
$0 q_2 0$ $1 q_2 0$	$q_3 0 0$ $q_3 1 0$	from $\delta(q_2, 0) = (q_3, 0,$
$q_2 1$ $q_2 \#$	$0 q_1$ $0 q_2 \#$	from $\delta(q_2, 1) = (q_1, 0, R$ $\delta(q_2, B) = (q_2, 0, R$

Group III

List A	List B
$0 q_3 0$	q_3
$0 q_3 1$	q_3
$1 q_3 0$	q_3
$1 q_3 1$	q_3

$0q_3$	q_3
$1q_3$	q_3
$q_3 0$	q_3
$q_3 1$	q_3

Group - IV.

List A	List B
q_3 # #	#

* M accepts input $w = 01$ by the sequence of ID's

$q_1 01$, $1q_2 1$, $10q_1$, $1q_2 01$, $q_3 101$, $q_3 01$, $q_3 1$, q_3

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