Unravelling the contribution of financial and longevity risks to changes over time in life annuities

Jesús-Adrián Álvarez
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stochastic change life annuities

interest or mortality?

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Duration

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How annuities respond to **changes in interest rates**? (Milevsky, 2013; Charupat, Kamstra, Milevsky, 2015)

What about changes in mortality?

Entropy

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Haberman et al (2011) applied the concept of **entropy** to **life annuities**:

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They showed that

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Changes over time in $\bar{a}_x(t)$

Derivative of $\bar{a}_x(t)$ with respect to time t:

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putting all the pieces together…

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No assumptions about the functional form of δ and μ (entirely data-driven).

Changes in life annuities in the UK

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HISTORICAL CONTRIBUTIONS OF MORTALITY AND INTEREST RATES

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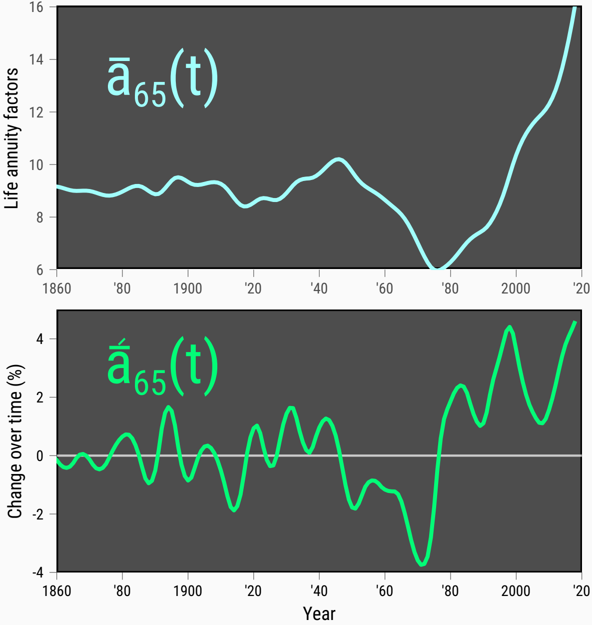
- **1841-2018.**

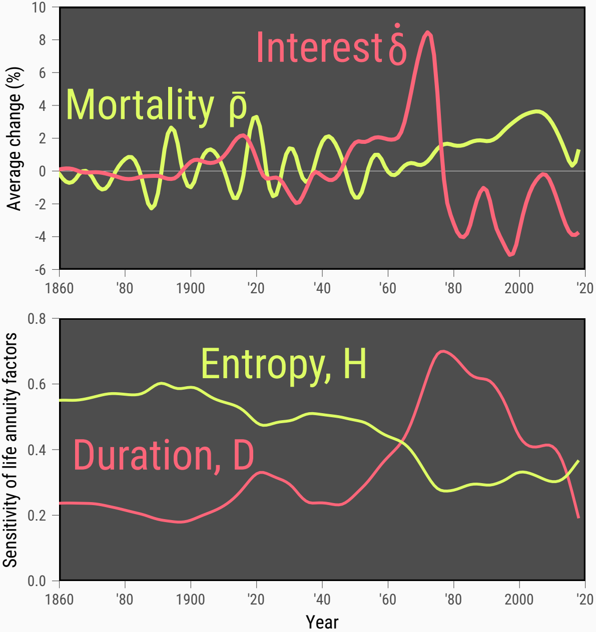
Life annuity factors

LIFE ANNUITY FACTORS AT AGE 65 FOR MALES. UK, 1841-2018



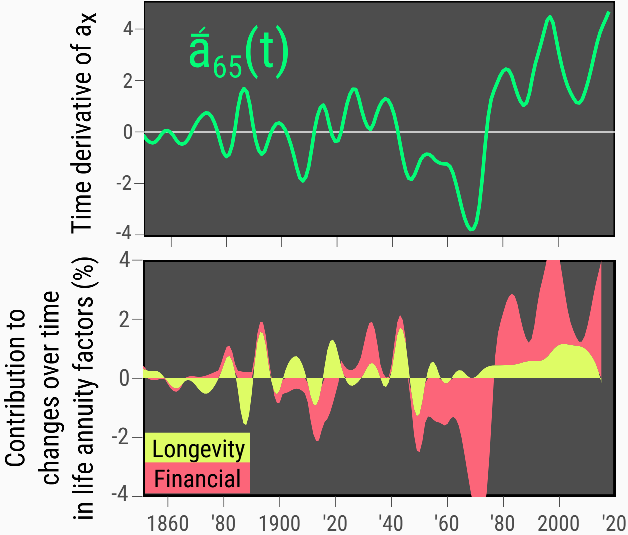
Stochastic changes and sensitivities

STOCHASTIC CHANGES AND SENSITIVITIES. UK, 1841-2018



Decomposition

DECOMPOSITION OF $\acute{\bar{a}}_x(t)$ AT AGE 65. MALES IN THE UK, 1841-2018



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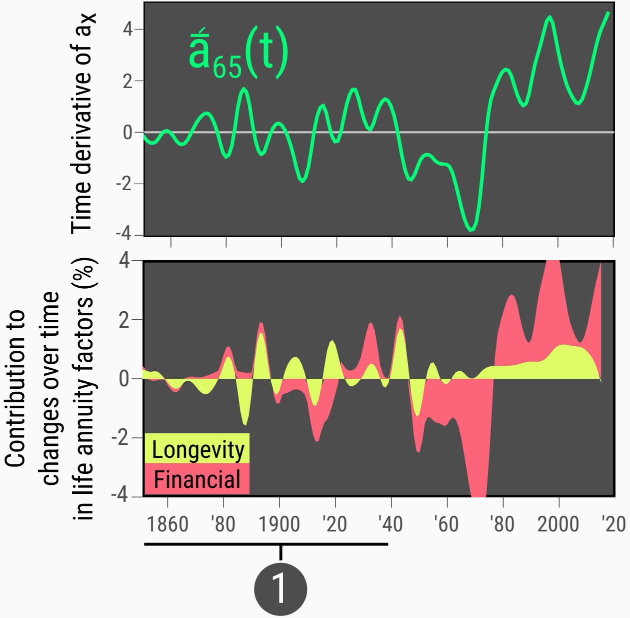

1 Stable a_x

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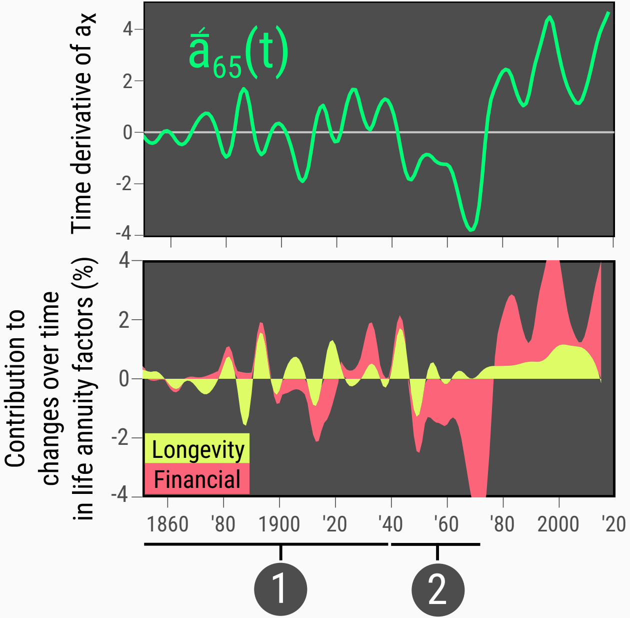

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- financial $\downarrow$

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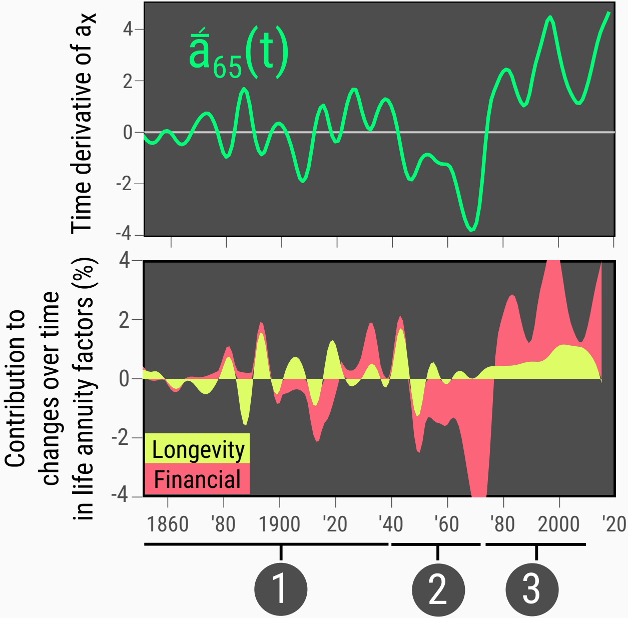

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3 Increasing a_x
- δ↓, μ↓
- financial ↑
- longevity ↑

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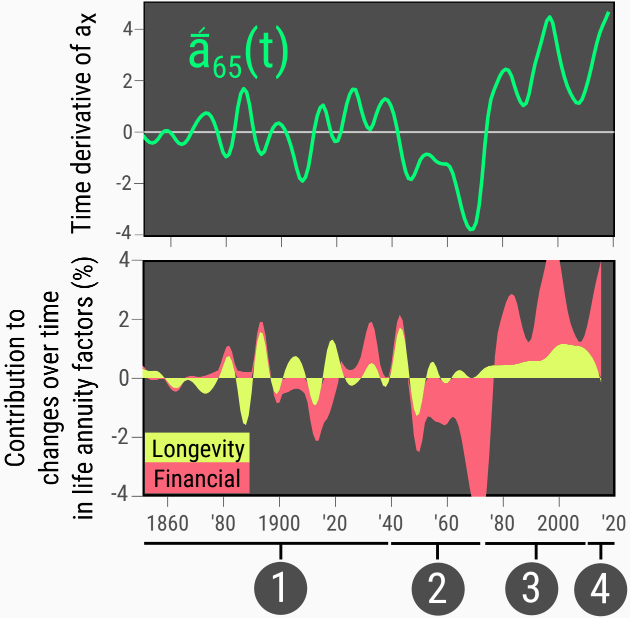

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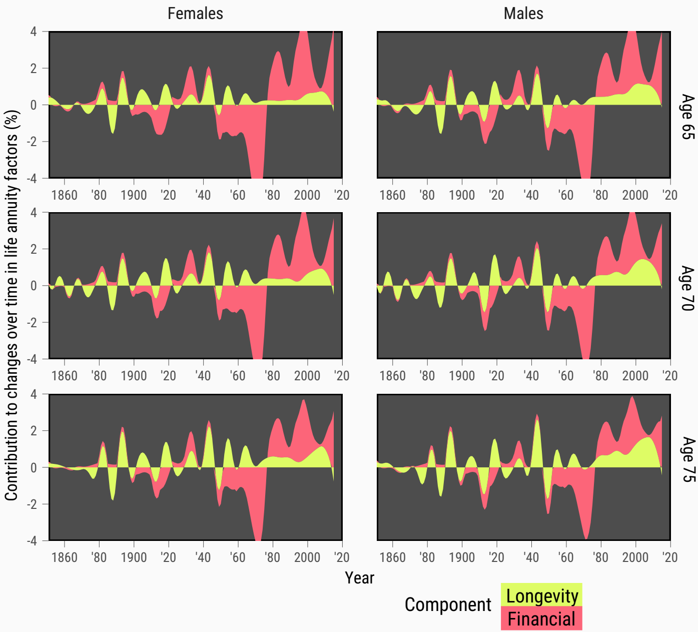
- $\delta \downarrow$, $\mu \downarrow$
- financial $\uparrow$
- longevity $\uparrow$

4 Increasing a_x
- financial $\uparrow$
- longevity $\uparrow$
- μ decelerates

DECOMPOSITION OF $\acute{\bar{a}}_x(t)$ AT AGES 65, 70 AND 75. UK, 1841-2018



To sum up

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  - Policies aiming at **increasing retirement ages entail higher longevity risk** (e.g. Denmark, Alvarez et al (2020)).

Next steps

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- **Causes of death** Vaupel and Canudas-Romo (2003).

Unravelling the contribution of financial and longevity risks to changes over time in life annuities

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Rate of mortality improvement

$$\rho(x,t) = -\frac{\frac{\mu(x,t)}{\partial t}}{\mu(x,t)} = -\frac{\dot{\mu}(x,t)}{\mu(x,t)}. \tag{1}$$

Change in interest rates over time

$$\varphi(s,t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)} = -\frac{\dot{\delta}(s,t)}{\delta(s,t)}. \tag{2}$$

Entropy

$$H_x^p(t) = \frac{\int_0^\infty \mu(x+s,t)_s | \bar{a}_x(t) ds}{\bar{a}_x(t)} \tag{3}$$

Duration

$$D_x^p(t) = \frac{\int_0^\infty \delta(s,t)_s | \bar{a}_x(t) ds}{\bar{a}_x(t)} \tag{4}$$

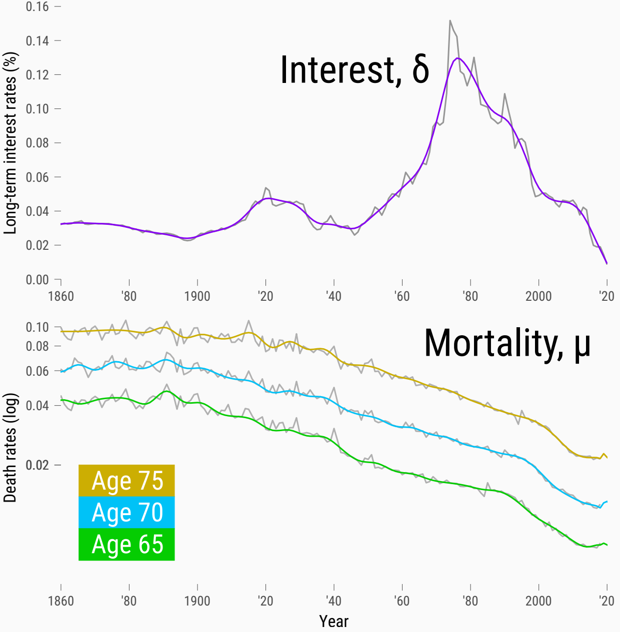
Time derivative of $\bar{a}_x(t)$

$$\begin{aligned}\dot{\bar{a}}_x(t) &= \int_0^\infty \rho(s,t)\mu(s,t)_s|\bar{a}_x(t)ds + \int_0^\infty \varphi(s,t)\delta(s,t)_s|\bar{a}_x(t)ds \\ &= \int_0^\infty \rho(s,t)_s M_x(t)ds + \int_0^\infty \varphi(s,t)_s W_x(t)ds,\end{aligned} \tag{5}$$

where ${}_sM_x(t) = \mu(s,t)_s|\bar{a}_x(t)$ and ${}_sW_x(t) = \delta(s,t)_s|\bar{a}_x(t)$.

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)} = \underbrace{\bar{\rho}(t)H_x^p(t)}_{\text{longevity component}} + \underbrace{\bar{\varphi}(t)D_x^p(t),}_{\text{financial component}} \tag{6}$$

INTEREST AND MORTALITY RATES FOR FEMALES IN THE UK, 1841-2018



INTEREST AND MORTALITY RATES FOR MALES IN THE UK, 1841-2018