Unobserved Heterogeneity

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At a population level this creates **heterogeneity**.

How can we model this?

Two random variables:

Individual hazards

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X: Age at death,

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Survival function

$s(x|z) = e^{-\int_0^x \mu(t|z)dt}$,
$\mu(x|z) = -\frac{d \ln s(x|z)}{dx}$.

Population hazard

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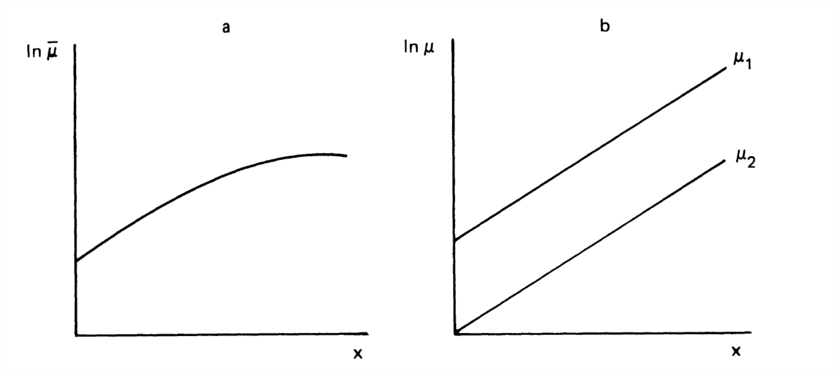
The population hazard at any age x is a function of the baseline $\mu(x)$ and the average frailty $\bar{z}(x)$ among survivors to this age.

individuals

≠

POPULATION

Individuals vs population hazards



Vaupel and Yashin (1985)

REMARKS ON $\bar{\mu}(x) = \bar{z}(x)\mu(x)$

- Because frailer individuals die out first, $\bar{z}(x)$ decreases with age x and, as a result, the force of mortality for **individuals increases faster** than the force of mortality for **the population** as a whole,

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- It is a model for **cohort mortality**,
- Frailty acts **multiplicatively** on the baseline hazard,
- Frailty **does not change with age**. It remains the same *for all ages*. It is often called **fixed-frailty model**.

Continuous frailty

Continous frailty

X: continous r.v.
$Z > 0$: continous r.v. with density $\pi(z)$.
Hazard and survival functions for populations

- $\bar{\mu}(x) = \int_0^\infty \mu(x|z)\pi(z)dz = \bar{z}(x)\mu(x)$
- $\bar{s}(x) = \int_0^\infty s(x|z)\pi(z)dz$

A note about the Laplace transform

A function of functions

It helps us to transform a function $f(t)$ to another function $f(s)$.

The Laplace transform of a function $f(t), t \geq 0$ is

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and

$\mathbb{E}(x) = -\mathscr{L}_f'(0)$

$\mathbb{V}(x) = \mathscr{L}_f''(0) - [\mathscr{L}_f'(0)]^2$

Laplace transform in frailty models

The survival function for the total population is the Laplace transform of the frailty distribution at $x = 0$, calculated for the cumulative baseline hazard $H(x)$

$$\bar{s}(x) = \int_0^\infty s(x|z)\pi(z)dz = \int_0^\infty e^{-H(x)z}\pi(z)dz = \mathbb{E}(e^{-H(x)z}) = \mathscr{L}_Z(H(x)).$$

An extremely useful tool in modelling unobserved heterogeneity:

$$\bar{s}(x) = \mathscr{L}_Z(H(x))$$

What are the distributions of X and Z?

A parametric approach

$X \sim$ Gompertz or Gompertz-Makeham,

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$Z \sim$ Gamma, Inverse Gaussian, Log-Normal, etc.

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The gamma distribution has a flexible shape and converges to a normal distribution,

It has a simple Laplace transform.

Gamma frailty model

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Density function of Z

$$\pi(0, z) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}$$

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Laplace transform of Z and survival function of the population

$$\mathscr{L}_Z(H(x)) = \bar{s}(x) = \left(\frac{1}{1 + \frac{1}{k} H(x)}\right)^k$$

Average frailty of the population

If $\bar{z}(0) = 1$ and $k = \lambda = 1/\gamma$, where γ can be interpreted as the squared root of the coefficient of variation of Z at any age.

$$\bar{z}(x) = \frac{1}{1 + \gamma H(x)} = [\bar{s}(x)]^{\gamma}$$

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Mortality hazard for populations

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Using the Laplace transform:

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Thus γ tell us the amount of frailty in a population.

Gamma-Gompertz

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$X \sim Gompertz(a, b)$

Individual hazard

$\mu(x) = ae^{bx}$

$H(x) = \frac{a}{b}(e^{bx} - 1)$

If $Z \sim Gamma(\gamma)$

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a: initial level of mortality,

b: individual rate of ageing,

γ: amount of frailty in the population.

Gamma-Gompertz via the Mode

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$X \sim Gompertz(b, M)$

Individual hazard

$\mu(x) = be^{b(x-M)}$

$H(x) = e^{-bM}(e^{bx} - 1)$

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Gamma-Gompertz-Makeham

$X \sim Gompertz - Makeham(a, b, c)$

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γ: amount of frailty in the population.

Fitting Gamma-Gompertz

Fitting Gamma-Gompertz - individual data

Maximize likelihood

$$L(\theta, \beta, \gamma) = \left(\Pi_{i=1}^{n} \frac{\mu(x_i; \theta) e^{y_i \beta}}{1 + \gamma H(x_i; \theta) e^{y_i \beta}} \right)^{\delta_i} \left(1 + \gamma H(x_i; \theta) e^{y_i \beta}\right)^{-\gamma}$$

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    - other frailty distributions: log-Normal, Beta, etc.

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Maximize Poisson log-likelihood

$$\ln L(\theta, \gamma) = \Sigma_x \left(D(x) \ln \bar{\mu}(x; \theta, \gamma) - E(x)\bar{\mu}(x; \theta, \gamma)\right)$$

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- R-Package: `MortalityLaws`
  - based on GLM models
  - other distributions: Kannisto, Gompertz, Weibull, etc.

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  - stochastic vitality: changes in frailty over age (Manton and Yashin, 1997)

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- Many options for choosing the frailty distribution.
- Variance of frailty determines the degree of unobserved heterogeneity.

Further reading

- Vaupel, Manton and Stallard. *The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality.* Demography, 1979.
- Vaupel and Yashin. *Heterogeneity's ruses: Some surprising effects of selection on population dynamics*. American Statistician, 1985.
- Vaupel and Missov. *Unobserved population heterogeneity: A review of formal relationships.* Demographic Research, 2014.
- Steinsaltz and Wachter. *Understanding mortality rate deceleration and heterogeneity*. Mathematical Population Studies, 2006.
- Horiuchi and Wilmoth. *Deceleration in the age pattern of mortality at olderages*. Demography, 1998.
- Wienke. *Frailty models in survival analysis*, 2016.

Questions?