Unobserved Heterogeneity

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Unobserved heterogeneity

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At a population level this creates **heterogeneity**.

How can we model this?

Individual hazards

Two random variables:

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Survival function

$s(x|z) = e^{-\int_0^x \mu(t|z)dt}$,
$\mu(x|z) = -\dfrac{d \ln s(x|z)}{dx}$.

Population hazard

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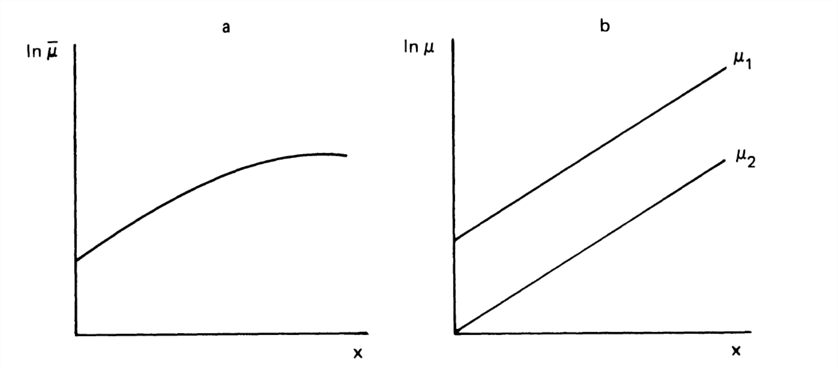
The population hazard at any age x is a function of the baseline $\mu(x)$ and the average frailty $\bar{z}(x)$ among survivors to this age.

individuals

≠

POPULATION

Individuals vs population hazards



Vaupel and Yashin (1985)

Remarks on $\bar{\mu}(x) = \bar{z}(x)\mu(x)$

- Because frailer individuals die out first, $\bar{z}(x)$ decreases with age x and, as a result, the **individual hazard** increases faster than the **population hazard**,

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- The population hazard **converges** to the hazard of the most robust individuals as age x goes to the maximum age (**mortality deceleration and levelling-off**).
- It is a model for **cohort mortality**,
- Individual hazards are **proportional** to the baseline hazard,
- Individual frailty **does not change with age**. It remains the same *for all ages*. It is often called **fixed-frailty model**.

Continuous frailty

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$Z > 0$: continous r.v. with density $\pi(z)$.

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Hazard and survival functions for populations

- $\bar{\mu}(x) = \int_0^\infty \mu(x|z)\pi(z)dz = \bar{z}(x)\mu(x)$
- $\bar{s}(x) = \int_0^\infty s(x|z)\pi(z)dz$

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Can we get a closed form for $\bar{s}(x)$?

A note on the Laplace transform

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In probability:

$\mathscr{L}_f(s) = \mathbb{E}(e^{-sX})$

and

$\mathbb{E}(x) = -\mathscr{L}_f'(0)$

$\mathbb{V}(x) = \mathscr{L}_f''(0) - [\mathscr{L}_f'(0)]^2$

Laplace transform in frailty models

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$$\bar{s}(x) = \int_0^\infty s(x|z)\pi(z)dz = \int_0^\infty e^{-H(x)z}\pi(z)dz = \mathbb{E}(e^{-H(x)z}) = \mathscr{L}_Z(H(x)).$$

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An extremely useful tool in modelling unobserved heterogeneity:

$$\bar{s}(x) = \mathscr{L}_Z(H(x))$$

What are the distributions of X and Z?

A parametric approach

$X \sim$ Gompertz or Gompertz-Makeham,

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$Z \sim$ Gamma, Inverse Gaussian, Log-Normal, etc.

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Frailty distributions with closed-form Laplace transforms are the most convenient to work with

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The gamma distribution has a flexible shape and converges to a normal distribution,

It has a simple Laplace transform.

Gamma frailty model

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$Z \sim \Gamma(k, \lambda)$ such that $k, \lambda > 0$

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Density function of Z

$$\pi(0, z) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}$$

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Laplace transform of Z and survival function of the population

$$\mathscr{L}_Z(H(x)) = \bar{s}(x) = \left(\frac{1}{1 + \frac{1}{k} H(x)}\right)^k$$

Average frailty of the population

If $\bar{z}(0) = 1$ and $k = \lambda = 1/\gamma$, where γ can be interpreted as the squared root of the coefficient of variation of Z at any age.

$$\bar{z}(x) = \frac{1}{1 + \gamma H(x)} = [\bar{s}(x)]^{\gamma}$$

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Thus γ tell us the amount of frailty in a population.

Gamma-Gompertz

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$X \sim Gompertz(a, b)$

Individual hazard

$\mu(x) = ae^{bx}$

$H(x) = \frac{a}{b}(e^{bx} - 1)$

If $Z \sim Gamma(\gamma)$

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Life expectancy: see Castellares, Patricio and Lemonte (2020)

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Individual hazard

$\mu(x) = be^{b(x-M)}$

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M: modal age at death,

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a: initial level of mortality,

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γ: amount of frailty in the population.

Fitting Gamma-Gompertz

Fitting Gamma-Gompertz - individual data

Maximize likelihood

$$L(\theta, \beta, \gamma) = \left(\Pi_{i=1}^{n} \frac{\mu(x_i; \theta) e^{y_i \beta}}{1 + \gamma H(x_i; \theta) e^{y_i \beta}} \right)^{\delta_i} \left(1 + \gamma H(x_i; \theta) e^{y_i \beta}\right)^{-\gamma}$$

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    - other frailty distributions: log-Normal, Beta, etc.

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Maximize Poisson log-likelihood

$$\ln L(\theta, \gamma) = \Sigma_x \left(D(x) \ln \bar{\mu}(x; \theta, \gamma) - E(x) \bar{\mu}(x; \theta, \gamma) \right)$$

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  - based on GLM models
  - other distributions: Kannisto, Gompertz, Weibull, etc.

Fitting Gamma-Gompertz - Example

My GitHub: `github.com/jssalvrz`

Directory: `EDSD/UnobsHeter`

- `GGFun.R`: Functions to fit the Gamma-Gompertz model (with and without the Mode) using aggregate data.
- `Dat.RData`: Data from the Human Mortality Database.
- `UnobserHeterFit.R`: Run the analysis.

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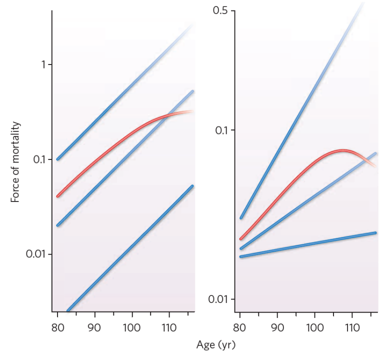

Vaupel (2010)

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  Extension of survival analysis

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  - stochastic vitality: changes in frailty over age (Manton and Yashin, 1997)

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    - Model "old-age" mortality, where does "old-age" start?

Further reading

- Vaupel, Manton and Stallard. *The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality.* Demography, 1979.
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Questions?