Unravelling the contribution of financial and longevity risks to changes over time in life annuities

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Abstract

Actuaries and risk managers are interested in developing strategies to ensure that changes in interest rates do not affect the value of a portfolio (commonly known as immunization). Similarly, there is a long-standing tradition among demographers to measure how changes over time in mortality affect summary measures such as life expectancy. In this paper, we bring these two perspectives together. We develop a new decomposition method to quantify the contribution of changes in mortality and interest rates to the change in life annuity prices. We introduce neat and intuitive formulations that allow actuaries and risk managers to easily asses stochastic changes in financial and longevity risks embedded in their life annuities' portfolios.

To illustrate our method, we look at the long-term development of life annuity prices using financial and mortality data from the United Kingdom since 1841. We found that there is clear interplay between longevity and financial risk, where the former one is at times masked by high financial risk.

1 Introduction

I commented all the text in the introduction, we should work on it at the end, when we have the empirical results to see what is the route to take.

2 Preliminaries

All of the quantities expressed here vary over time t. According to standard actuarial notation, we define the following quantities:

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- $\mu(x,t)$ is the force of mortality at age x.
- ${}_sp_x(t) = e^{-\int_0^s \mu(x+y,t)dy}$ is the probability of surviving from age x to age $x+s$.
- $\delta(s,t)$ is the force of interest at time s. This measure is a general form to express the term-structure of interest rates.
- $v(s,t) = e^{-\int_0^s \delta(s,t)dy}$ is the discount factor, where s is the length of the interval from issue of the life annuity to death.

The derivative with respect to time t is denoted by adding a point on top of the function of interest. For example, time derivatives for the forces of mortality and interest are expressed as:

$$\dot{\mu}(x,t) \equiv \frac{\partial \mu(x,t)}{\partial t}, \tag{1}$$

and

$$\dot{\delta}(s,t) \equiv \frac{\partial \delta(s,t)}{\partial t}. \tag{2}$$

The rate of mortality improvement (or progress in reducing mortality) is defined as

$$\rho(x,t) = -\frac{\frac{\mu(x,t)}{\partial t}}{\mu(x,t)} = -\frac{\dot{\mu}(x,t)}{\mu(x,t)}. \tag{3}$$

Similarly, the relative change in interest rates over time is captured by

$$\varphi(s,t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)} = -\frac{\dot{\delta}(s,t)}{\delta(s,t)}. \tag{4}$$

The actuarial present value of a life annuity at age x evaluated at time t is given by

$$\bar{a}_x(t) = \int_0^\infty {}_sp_x(t)v(s,t)ds = \int_0^\infty {}_sE_x(t)ds, \tag{5}$$

where ${}_sE_x(t) = {}_sp_x(t)v(s,t)$.

A life annuity deferred s years starting to be paid at age $x+s$ is expressed as

$${}_{s|}\bar{a}_x(t) = {}_sE_x(t)\bar{a}_{x+s}(t) \tag{6}$$

3 Dynamics of a life annuity

We are interested on measuring changes in $\bar{a}_x(t)$ with respect to the time variable t. To achieve this aim, we first need to describe how $\bar{a}_x(t)$ reacts to changes in the forces of mortality and interest. We denote the *entropy of a life annuity*[1] as the measure that captures changes in $\bar{a}_x(t)$ with respect to $\mu(x,t)$. Formally,

[1] Lin and Tsai (2020); Tsai and Chung (2013); Tsai et al. (2011) denoted this measure as *mortality duration*. We reserve the term duration to denote changes in $\bar{a}_x(t)$ with respect to interest rates.

it is defined as

$$H_x(t) = \frac{\frac{\partial \bar{a}_x(t)}{\partial \mu(x,t)}}{\bar{a}_x(t)}. \tag{7}$$

The measure that captures the sensitivity of $\bar{a}_x(t)$ to changes in interest rates is commonly known as *duration* and it is the foundation of interest rates' immunization. For a life annuity, it is defined as the relative derivative of the annuity factor with respect to changes in the force of interest (Milevsky, 2012):

$$D_x(t) = \frac{\frac{\partial \bar{a}_x(t)}{\partial \delta(s,t)}}{\bar{a}_x(t)}. \tag{8}$$

Greater values for $H_x(t)$ and $D_x(t)$ indicate that $\bar{a}_x(t)$ is highly sensitive to changes in $\mu(x,t)$ and $\delta(s,t)$ respectively. The entropy and the duration of a life annuity can be measured either by assuming constant or proportional changes in $\mu(x,t)$ and $\delta(s,t)$. In the following section we develop formulations for both cases.

3.1 Changes in $\bar{a}_x(t)$ with respect to $\mu(x,t)$

The entropy of a life annuity is denoted by $H_x^c(t)$ when changes in $\mu(x,t)$ are held constant at all ages and by $H_x^p(t)$ when changes are performed proportional. Based on the results developed by Tsai and Chung (2013) and Lin and Tsai (2020), when $\mu(x,t)$ is changed constantly to $\mu(x,t)+\gamma$ such that γ is a small number (see proof in the Appendix), the entropy of $\bar{a}_x(t)$ becomes

$$H_x^c(t) = -\frac{\int_0^\infty s\, {}_sp_x(t) v(s,t) ds}{\bar{a}_x(t)} = \frac{h_x^c(t)}{\bar{a}_x(t)}, \tag{9}$$

where $h_x^c(t) = -\int_0^\infty s\, {}_sp_x(t) v(s,t) ds$. The term $h_x^c(t)$ is expressed in absolute (monetary) terms, whereas the entropy $H_x^c(t)$ is dimensionless because it does not depend on the absolute value of $\bar{a}_x(t)$.

In two separate articles, Haberman et al. (2011) and Tsai and Chung (2013) show that when changes in $\mu(x,t)$ are assumed to be proportional to a small number γ such that $\mu(x,t)(1+\gamma)$, the entropy of $\bar{a}_x(t)$ becomes

$$H_x^p(t) = -\frac{\int_0^\infty {}_sp_x(t) \ln[{}_sp_x(t)] v(s,t) ds}{\int_0^\infty {}_sp_x(t) v(s,t) ds}. \tag{10}$$

Alternatively, we show (see proof in Section A.2 of the Appendix) that Equation 10 can be expressed as

$$H_x^p(t) = \frac{\int_0^\infty \mu(x+s,t)\, {}_sE_x(t) \bar{a}_{x+s}(t) ds}{\bar{a}_x(t)} = \frac{h_x^p(t)}{\bar{a}_x(t)}, \tag{11}$$

where $h_x^p(t) = \int_0^\infty \mu(x+s,t)\, {}_sE_x(t) \bar{a}_{x+s}(t) ds$. Analogous to the case where changes are assumed to be constant, quantities $h_x^p(t)$ and $H_x^p(t)$ are expressed in absolute and relative terms respectively. The formulations shown in this section are closely related to the ones developed in the mortality immunization literature (Lin and Tsai, 2020; Tsai and Chung, 2013). We extended them to the continuous case.

3.2 Changes in $\bar{a}_x(t)$ with respect to $\delta(s,t)$

Similar to the entropy, changes in $\bar{a}_x(t)$ with respect to $\delta(s,t)$ can be assumed to be either constant or proportional. Duration assuming constant changes in $\delta(s,t)$ is denoted by $D_x^c(t)$, whereas $D_x^p(t)$ refers to the duration assuming proportional changes. For the former case we have that:

$$\begin{aligned} D_x^c(t) &= -\frac{\int_0^\infty s\,_sp_x(t)v(s,t)ds}{\bar{a}_x(t)} \\ &= \frac{d_x^c(t)}{\bar{a}_x(t)}, \end{aligned} \tag{12}$$

where $d_x^c(t) = -\int_0^\infty s\,_sp_x(t)v(s,t)ds$. Thus, assuming constant changes in $\delta(s,t)$ results into common types of duration known in finance as *dollar duration*, $d_x^c(t)$, and *modified duration*, $D_x^c(t)$ (see Milevsky (2012) and Tsai and Chung (2013) for further details). It is worth noting that Equations 9 and 12 are identical such that $d_x^c(t) = h_x^c(t)$. This means that constant (parallel) changes in the force of mortality have essentially the same effect as parallel changes in the force of interest.

Assuming proportional changes in $\delta(s,t)$ (see proof in Section A.4 of the Appendix) results in:

$$D_x^p(t) = -\frac{\int_0^\infty \,_sp_x(t)v(s,t)\ln(v(s,t))ds}{\bar{a}_x(t)}. \tag{13}$$

Note that when assuming constant force of mortality such that $v(s,t) = e^{-\delta(t)s}$, we have that $D_x^p(t) = \delta(t)D_x^c(t)$,

$$\begin{aligned} D_x^p(t) &= -\frac{\int_0^\infty \,_sp_x(t)v(s,t)\ln(v(s,t))ds}{\bar{a}_x(t)} \\ &= -\delta(t)\frac{\int_0^\infty s\,_sp_x(t)e^{-\delta(t)s}ds}{\bar{a}_x(t)} \\ &= \delta(t)D_x^c(t). \end{aligned} \tag{14}$$

Equation 13 can also be re-expressed as:

$$\begin{aligned} D_x^p(t) &= \frac{\int_0^\infty \delta(s,t)\,_sE_x(t)\bar{a}_{x+s}(t)ds}{\bar{a}_x(t)} \\ &= \frac{d_x^p(t)}{\bar{a}_x(t)}. \end{aligned} \tag{15}$$

where $d_x^p(t) = \int_0^\infty \delta(s,t)\,_sE_x(t)\bar{a}_{x+s}(t)ds$.

4 Time derivative of $\bar{a}_x(t)$

We compute the derivative of $\bar{a}_x(t)$ with respect to the time variable t, $\dot{\bar{a}}_x(t) = \frac{\partial \bar{a}_x(t)}{\partial t}$, such that

$$\dot{\bar{a}}_x(t) = \int_0^\infty {}_s\dot{p}_x(t) v(s,t) ds + \int_0^\infty {}_s p_x(t) \dot{v}(s,t) ds. \tag{16}$$

To develop a closed-form solution for Equation 16 we consider the general case where ${}_s p_x(t) = e^{-\int_0^s \mu(x+y,t)dy}$ and $v(s,t) = e^{-\int_0^s \delta(s,t)dy}$. We analyse separately each of the two terms in the right side of Equation 16. Let us first focus on the first term:

$$\begin{aligned}
\int_0^\infty {}_s\dot{p}_x(t) v(s,t) &= \int_0^\infty v(s,t) e^{-\int_0^s \dot{\mu}(x+y,t)dy} ds \\
&= -\int_0^\infty v(s,t)\, {}_s p_x(t) \int_0^s \dot{\mu}(x+y,t) dy ds \\
&= -\int_0^\infty \dot{\mu}(x+s,t) \int_s^\infty v(y,t)\, {}_y p_x(t) dy ds \\
&= -\int_0^\infty \dot{\mu}(x+s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t) ds \\
&= \int_0^\infty \rho(x+s,t) \mu(x+s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t) ds
\end{aligned} \tag{17}$$

The second part equals:

$$\begin{aligned}
\int_0^\infty {}_s p_x(t) \dot{v}(s,t) ds &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \dot{\delta}(y,t)dy} ds \\
&= -\int_0^\infty {}_s p_x(t) v(s,t) \int_0^s \dot{\delta}(y,t) dy ds \\
&= -\int_0^\infty \dot{\delta}(s,t) \int_s^\infty {}_y p_x(t) v(y,t) dy ds \\
&= \int_0^\infty \varphi(s,t) \delta(s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t) ds
\end{aligned} \tag{18}$$

Thus, $\dot{\bar{a}}_x(t)$ can be expressed as

$$\begin{aligned}
\dot{\bar{a}}_x(t) &= \int_0^\infty \rho(s,t) \mu(s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t) ds + \int_0^\infty \varphi(s,t) \delta(s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t) ds \\
&= \int_0^\infty \rho(s,t)\, {}_s M_x(t) ds + \int_0^\infty \varphi(s,t)\, {}_s W_x(t) ds,
\end{aligned} \tag{19}$$

where ${}_s M_x(t) = \mu(s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t)$ and ${}_s W_x(t) = \delta(s,t)\, {}_s E_x(t) \bar{a}_{x+s}(t)$. We can express Equation 19 in terms of the duration and entropy

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)} = \underbrace{\bar{\rho}(t) H_x^p(t)}_{\text{longevity component}} + \underbrace{\bar{\varphi}(t) D_x^p(t),}_{\text{financial component}} \tag{20}$$

where $\bar{\rho}(t) = \frac{\int_0^\infty \rho(s,t)_s M_x(t) ds}{\int_0^\infty {}_s M_x(t) ds}$ and $\bar{\varphi}(t) = \frac{\int_0^\infty \varphi(s,t)_s W_x(t) ds}{\int_0^\infty {}_s W_x(t) ds}$ are the average paces of change in mortality and interest rates respectively. Functions $\bar{\rho}(t)$ and $\bar{\varphi}(t)$ capture the stochastic change in the forces of mortality and interest whereas $H_x^p(t)$ and $D_x^p(t)$ capture the sensitivity due to changes in μ and δ. In other words, Equation 20 entails that changes over time in $\bar{a}_x(t)$ are driven by $\bar{\rho}(t)$ and $\bar{\varphi}(t)$, which are modulated by $H_x^p(t)$ and $D_x^p(t)$ respectively. Equation 20 is an extension of the decomposition formula to changes over time in life expectancy developed by Vaupel and Canudas-Romo (2003). Here we extended it to life annuities.

4.1 Assuming constant force of interest

It is common to use a single interest rate (without using the term-structure of interest rates, $\delta(s,t)$) for the calculation of life annuity factors. Thus, we re-express the decomposition formula for $\acute{\bar{a}}_x(t)$ by assuming $v(s,t) = e^{-\delta(t)s}$. This assumption affects only the second part of Equation 16:

$$\begin{aligned}
\int_0^\infty {}_s p_x(t) \dot{v}(s,t) ds &= \int_0^\infty {}_s p_x(t) \frac{\partial \left[e^{-\delta(t)s} \right]}{\partial t} ds \\
&= -\dot{\delta}(t) \int_0^\infty s_s p_x(t) e^{-\delta(t)s} ds \\
&= \dot{\delta}(t) d_x^c(t).
\end{aligned} \tag{21}$$

Substituting Equations 17 and 21 in Equation 16 results into:

$$\acute{\bar{a}}_x(t) = \bar{\rho}(t) H_x^p(t) + \dot{\delta}(t) D_x^c(t). \tag{22}$$

Given that $D_x^p(t) = \delta(t) D_x^c(t)$ (see Equation 14), it is straight-froward to show that in the case of a single interest rate, Equations 20 and 22 are equivalent. Thus, it suffices to use the entropy ($H_x^p(t)$) and the modified duration ($D_x^c(t)$) together with $\bar{\rho}(t)$ and $\dot{\delta}(t)$ to determine the contribution of financial and longevity risks to changes over time in life annuities.

4.2 Recap of formulations

[Here I am planning to add a table with all the formulas]

5 Historical contribution of mortality and interest to changes in life annuities

In this section we illustrate the decomposition method developed in Section 4 by examining the long-term development of life annuities in the UK. We use two centuries of data from 1841 to 2018. Age-specific death rates come from the (Human Mortality Database, 2020). Long-term interest rates are represented by the yield on 2.5% Consols up to 1977, then by the yield on FTSE Actuaries Government Securities

Irredeemable stocks up to 2014 and thereafter by the yield on FTSE Actuaries Government Securities 45 years stock (INCLUDE REFERENCE). We calculated life annuity factors at ages 65, 70 and 75 to understand how the longevity and financial components react to different onset ages.

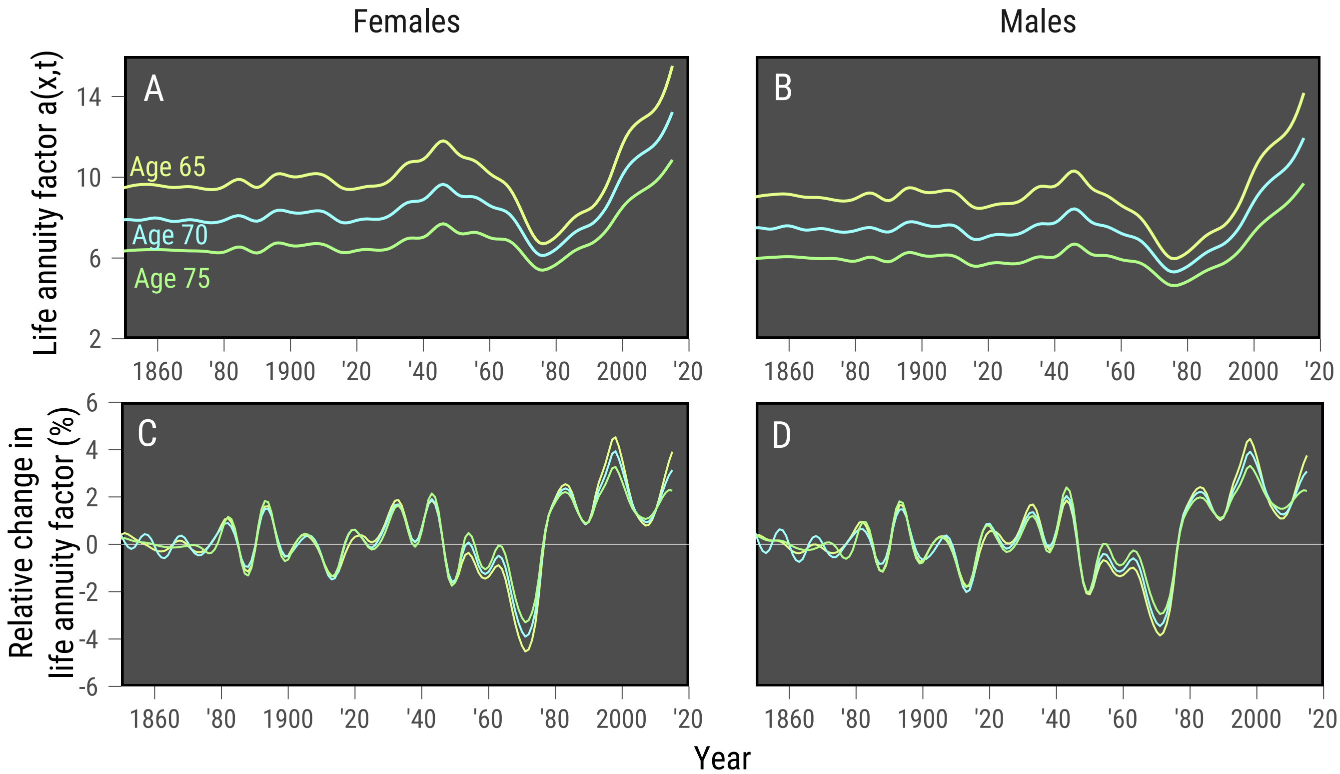

Figure 1: *Trends over time in life annuity factors and relative change in $\bar{a}_x(t)$ calculated at ages 65, 70 and 75. Both sexes, 1841-2018.*

How has the value of life annuities changed over time? Panels A and B of Figure 1 depict values for $\bar{a}_x(t)$ from 1841 to 2018 calculated at ages 65, 70 and 75 for females and males respectively. We observe that during the second half of the 19th century and up to the decade of 1940s, $\bar{a}_x(t)$ remained at similar levels with small fluctuations. Thereafter, $\bar{a}_x(t)$ exhibited a sharp decline up to the decade of 1980s followed by an increasing pattern that has remained until recent years.

Sex-specific trends appear to be very similar between sexes with the absolute level of $\bar{a}_x(t)$ for females being slightly higher than the one depicted for males. Similar trends over time also replicate at any onset age of calculation analysed here (e.g. age 65, 70 and 75). However, a closer look reveal that $\bar{a}_x(t)$ changes more abruptly at younger ages (age 65) than at older ages (age 75). This pattern is clear when analysing the time derivative of $\bar{a}_x(t)$ portrayed by $\acute{\bar{a}}_x(t)$ (see Panels C and D of Figure 1). In the following sections we analyse the sources of the time trend in $\acute{\bar{a}}_x(t)$ by making use of the decomposition method developed in Section 4.

5.1 Contribution of financial and longevity risks to changes in $\bar{a}_x(t)$

According to the decomposition formula developed (Equation 22), changes over time in $\bar{a}_x(t)$ can be explained in terms of longevity and financial components. Each component depends on the stochastic fluctuations of $\mu(x,t)$ and $\delta(s,t)$ (captured by $\bar{\rho}(t)$ and $\dot{\delta}(t)$ respectively) which are modulated by the sensitivity of $\bar{a}_x(t)$ to changes in the forces of mortality and interest (entropy, $H_x^p(t)$ and modified duration, $D_x^c(t)$). We first describe sex-specific trends in $\dot{\delta}(t)$ and $\bar{\rho}(t)$ (see Figure 2).

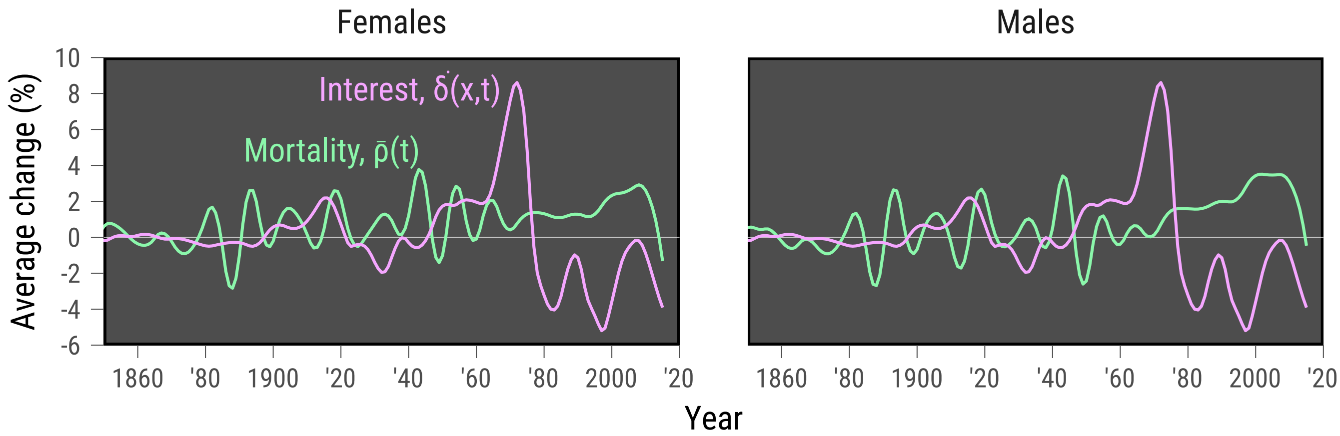

Figure 2: *Average change in mortality ($\bar{\rho}(t)$) calculated for all ages above age 65 and rate of change in interest rates ($\dot{\delta}(t)$). Note that $\dot{\delta}(t)$ is expressed in basis points to ease readability. Both sexes, 1841-2018.*

Time trends for $\bar{\rho}(t)$ are very much alike at ages 65, 70 and 75. Hence, we only focus on $\bar{\rho}(t)$ at age 65 to illustrate the general change in mortality. Values for rho at ages 70 and 75 can be found in the Appendix (ADD THIS). During most of the observation period, $\bar{\rho}(t)$ fluctuated between 0% and 2%. As of 1980s, $\bar{\rho}(t)$ remained steady indicating an average mortality improvement above age 65 of 2% and 3% for the decade of 2000s. In recent years, $\bar{\rho}(t)$ trended downwards, which coincides with the mortality deterioration currently prevailing in the United Kingdom (REF).

Regarding changes over time in the force of interest, we observe that $\dot{\delta}(t)$ has also remained at similar levels for most of the observation period with some fluctuations between 1930 and 1950. Since the 1960s, $\dot{\delta}(t)$ went up rapidly reaching the highest increase (of around 8% in the decade of 1970s). Thereafter, $\dot{\delta}(t)$ declined faster and moved towards negative values entailing a decline in interest rates, which is still ongoing until recent time. As mentioned above, $\bar{\rho}(t)$ and $\dot{\delta}(t)$ are responsible for the time trend we observe in $\bar{a}_x(t)$, which is modulated by the entropy, $H_x^p(t)$, and modifed duration, $D_x^c(t)$.

From Figure 3, we observe that $H_x^p(t)$ and $D_x^c(t)$ have fluctuated over time. The entropy went down for most of the observation period and up to the 1980s where it started to rise. Modified duration, on the other hand, remained somewhat constant over time with a small peak in the 1920s. However, as of the 1950s $D_x^c(t)$ trended upwards reaching its maximum in the 1980s (of about 0.8) and then begin to drop. It is worth noting that there are some specific points in time where $H_x^p(t)$ and $D_x^c(t)$ are equal. This means that $\bar{a}_x(t)$ is equally sensitive to changes in mortality and interest rates.

From Figure 3, the interplay between $H_x^p(t)$ and $D_x^c(t)$ is clear. We observe two regularities. First, $H_x^p(t)$ is lower for females than for males whereas $D_x^c(t)$ is higher for females than for males. This means that, in general, $\bar{a}_x(t)$ for females is less sensitive to μ and more sensitive to δ than for males. Second, $\bar{a}_x(t)$ becomes more sensitive to μ (higher $H_x^p(t)$) and less sensitive to δ (lower $D_x^c(t)$) when calculating $\bar{a}_x(t)$ at older ages. This finding is key to understand how the contributions of longevity and financial risks are moderated differently across ages.

Figure 4 depicts the contribution of financial ($\dot{\delta}(t)D_x^c(t)$) and longevity ($\bar{\rho}(t)H_x^p(t)$) components (SHOULD WE CALL THEM RISKS??) to changes in $\bar{a}_x(t)$ calculated at ages 65, 70 and 75. Positive/negative values contribute to increase/decrease in $\bar{a}_x(t)$. Both, longevity and financial component have played an important role on long-term development of $\bar{a}_x(t)$. During some specific periods of time (1900-1920 and 1940-1980), the financial component has contributed to reductions the value of $\bar{a}_x(t)$. However, from the 1980s onwards, the financial component has increased $\bar{a}_x(t)$. The intensity of these positive contributions has fluctuated over time, with the highest contribution being observed during the decade of the

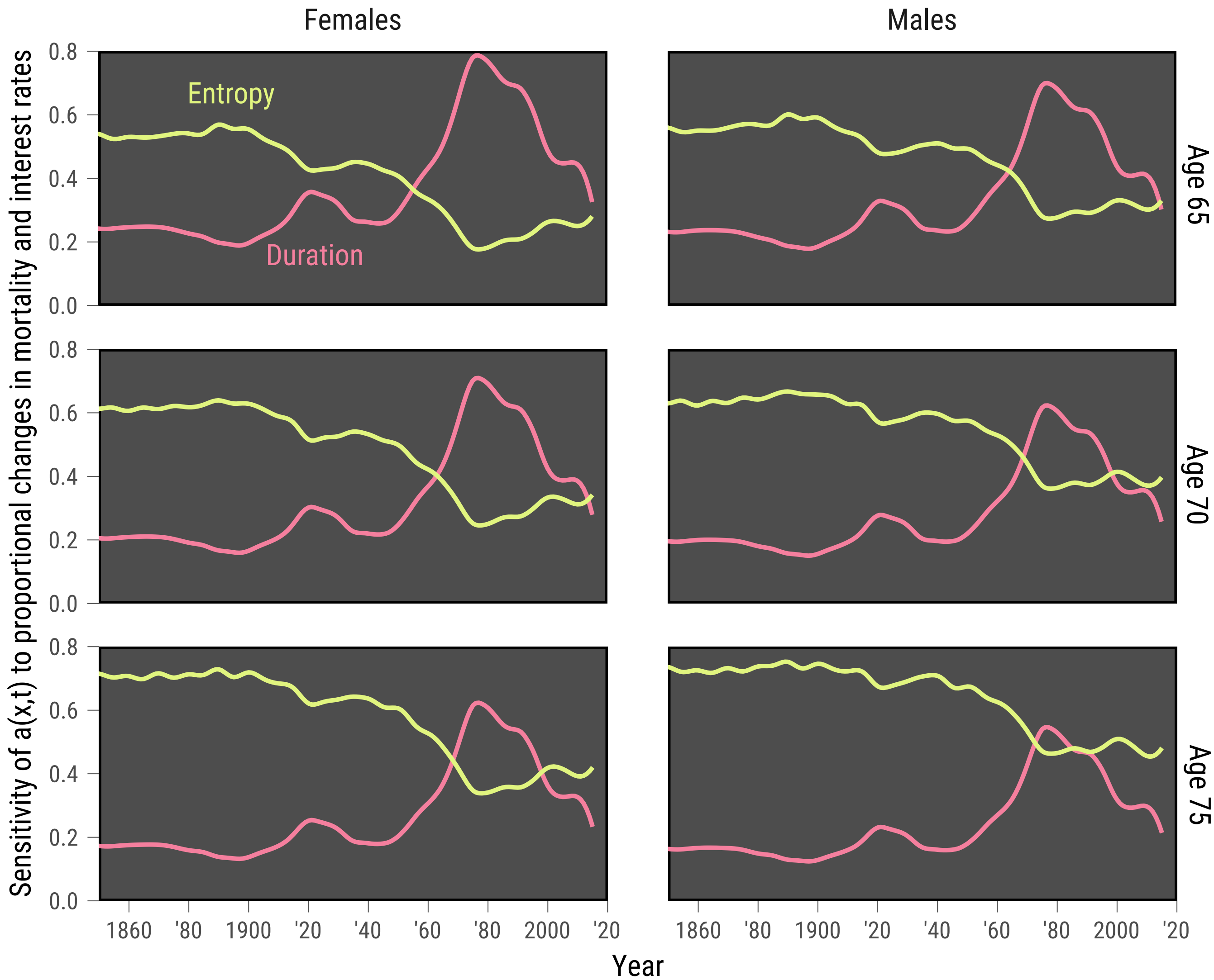

Figure 3: *Entropy and modified duration assuming proportional changes calculated at ages 65, 70 and 75. Both sexes, 1841-2018.*

2000s. The longevity component, on the other hand, has most of the time contributed to increases in $\bar{a}_x(t)$. From the 1980s, we observe steadily contributions of the longevity component to positive changes in $\bar{a}_x(t)$. This trend has changed recently due to mortality deterioration. It is worth noting that longevity contributions to changes in $\bar{a}_x(t)$ are more pronounced in males than in females.

From Figure 4 we observe When looking at $\bar{a}_x(t)$ calculated at different ages, we observe that the longevity component takes more relevance at older ages. For example, we observe that in recent years the longevity component is responsible for a greater contribution of changes in $\bar{a}_{75}(t)$ than for changes in $\bar{a}_{65}(t)$. This pattern is magnified for males. All in all, Figure 4 provides evidence of the contribution of financial and longevity components to changes in the long-term development of life annuity factors.

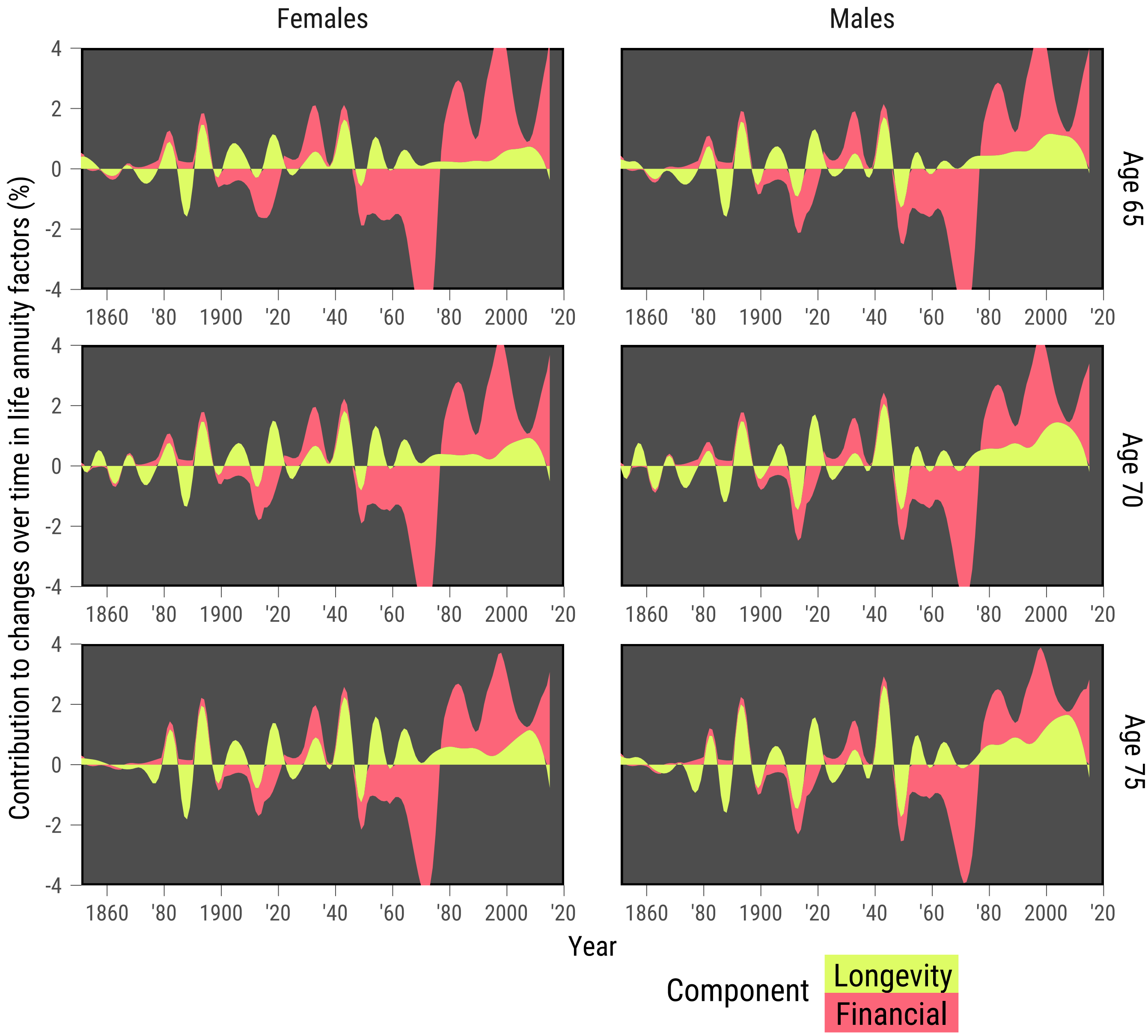

Figure 4: *Decomposition of changes over time in life annuity factors calculated at ages 65, 70 and 75. Both sexes, 1841-2018.*

6 Projection of the decomposition of changes in annuities

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A Appendix

A.1 Entropy with constant changes in $\mu(x+s,t)$

To measure constant changes we make $\mu(s,t)+\gamma$, then

$$\begin{aligned}
\bar{a}_x(t) &= \int_0^\infty v(s,t)e^{-\int_0^s[\mu(x+y,t)+\gamma]dy}ds \\
&= \int_0^\infty v(s,t)e^{-\int_0^s \mu(x+y,t)dy}e^{-\gamma s}ds \\
&= \int_0^\infty v(s,t)\,{}_sp_x(t)e^{-\gamma s}ds
\end{aligned} \tag{23}$$

We expand $e^{-\gamma s}$ to $1-\gamma s+\frac{\gamma^2}{2}s^2+...$, so that

$$\bar{a}_x(t) = \int_0^\infty {}_sp_x(t)v(s,t)[1-\gamma s+\frac{\gamma^2}{2}s^2+...]ds \tag{24}$$

We take the derivative $\bar{a}_x(t)$ with respect to γ and evaluate $\gamma=0$

$$\begin{aligned}
H_x^c(t) &= \frac{1}{\bar{a}_x(t)}\frac{\partial \bar{a}_x(t)}{\partial \gamma}\bigg|_{\gamma=0} \\
&= -\frac{\int_0^\infty s\,{}_sp_x(t)v(s,t)ds}{\bar{a}_x(t)} \\
&= \frac{h_x^c(t)}{\bar{a}_x(t)},
\end{aligned} \tag{25}$$

where $h_x^c(t) = -\int_0^\infty s\,{}_sp_x(t)v(s,t)ds$

A.2 Alternative expression for $H_x^p(t)$

$$
\begin{aligned}
H_x^p(t) &= -\frac{\int_0^\infty {}_sp_x(t)\ln[{}_sp_x(t)]e^{-\int_0^s \delta(y,t)dy}ds}{\int_0^\infty {}_sp_x(t)e^{-\int_0^s \delta(y,t)dy}ds} \\
&= \frac{\int_0^\infty {}_sp_x(t)v(s,t)\int_0^s \mu(x+y,t)dy\,ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t)\int_s^\infty {}_yp_x(t)v(y,t)dy\,ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t){}_sp_x(t)v(s,t)\int_s^\infty \frac{{}_yp_x(t)v(y,t)}{{}_sp_x(t)v(s,t)}dy\,ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t){}_sp_x(t)v(s,t)\bar{a}_{x+s}(t)ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t){}_sE_x(t)\bar{a}_{x+s}(t)ds}{\bar{a}_x(t)} \\
&= \frac{h_x^p(t)}{\bar{a}_x(t)},
\end{aligned}
\tag{26}
$$

where $h_x^p(t) = \int_0^\infty \mu(x+s,t){}_sE_x(t)\bar{a}_{x+s}(t)ds$.

A.3 Duration with constant changes in $\delta(s,t)$

To measure constant changes we make $\delta(s,t)+\gamma$, then

$$
\begin{aligned}
\bar{a}_x(t) &= \int_0^\infty {}_sp_x(t)e^{-\int_0^s[\delta(y,t)+\gamma]dy}ds \\
&= \int_0^\infty {}_sp_x(t)e^{-\int_0^s \delta(y,t)dy}e^{-\gamma s}ds \\
&= \int_0^\infty {}_sp_x(t)v(s,t)e^{-\gamma s}ds
\end{aligned}
\tag{27}
$$

We expand $e^{-\gamma s}$ to $1-\gamma s+\frac{\gamma^2}{2}s^2+...$, so that

$$
\bar{a}_x(t) = \int_0^\infty {}_sp_x(t)v(s,t)[1-\gamma s+\frac{\gamma^2}{2}s^2+...]ds
\tag{28}
$$

We take the derivative $\bar{a}_x(t)$ with respect to γ and evaluate $\gamma = 0$

$$
\begin{aligned}
D_x^c(t) &= -\frac{1}{\bar{a}_x(t)}\frac{\partial \bar{a}_x(t)}{\partial \gamma}\bigg|_{\gamma=0} \\
&= \frac{\int_0^\infty s\,{}_sp_x(t)v(s,t)ds}{\bar{a}_x(t)} \\
&= \frac{d_x^c(t)}{\bar{a}_x(t)},
\end{aligned}
\tag{29}
$$

where $d_x^c(t) = \int_0^\infty s {}_s p_x(t) v(s,t) ds$

A.4 Duration with proportional changes in $\delta(s,t)$

To calculate duration with proportional changes in $\delta(s,t)$, we assume that γ is a small number such that $\delta(s,t)(1+\gamma)$ and $v(s,t) = e^{-\int_0^s \delta(y,t)(1+\gamma)dy}$.

$$
\begin{aligned}
\bar{a}_x(t) &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)(1+\gamma)dy} ds \\
&= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)dy} e^{-\int_0^s \delta(y,t)\gamma dy} ds \\
&= \int_0^\infty {}_s p_x(t) v(s,t) v(s,t)^\gamma ds
\end{aligned} \tag{30}
$$

We expand $v(s,t)^\gamma$ to $1 + \ln(v(s,t))\gamma + \ln(v(s,t))^2 \frac{\gamma^2}{2} + ...$, so that

$$
\bar{a}_x(t) = \int_0^\infty {}_s p_x(t) s(y,t)[1 + \ln(v(s,t))\gamma + \ln(v(s,t))^2 \frac{\gamma^2}{2} + ...] ds \tag{31}
$$

To calculate the duration $D_x^p(t)$ we take the derivate of the expression above with respect to γ and make $\gamma = 0$

$$
\begin{aligned}
D_x^p(t) &= -\frac{1}{\bar{a}_x(t)} \frac{\partial \bar{a}_x(t)}{\partial \gamma}\Big|_{\gamma=0} \\
&= -\frac{\int_0^\infty {}_s p_x(t) v(s,t) \ln(v(s,t)) ds}{\bar{a}_x(t)}
\end{aligned} \tag{32}
$$

Equation 32 can be re-expressed as

$$
\begin{aligned}
D_x^p(t) &= -\frac{\int_0^\infty {}_s p_x(t) v(s,t) \ln(v(s,t)) ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty {}_s p_x(t) v(s,t) \int_0^s \delta(y,t) dy ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) \int_s^\infty {}_y p_x(t) v(y,t) dy ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) {}_s p_x(t) v(s,t) \bar{a}_{x+s}(t) ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) {}_s E_x(t) \bar{a}_{x+s}(t) ds}{\bar{a}_x(t)} \\
&= \frac{d_x^p(t)}{\bar{a}_x(t)}.
\end{aligned} \tag{33}
$$

where $d_x^p(t) = \int_0^\infty \delta(s,t) {}_s E_x(t) \bar{a}_{x+s}(t) ds$.