Unravelling the contribution of financial and longevity risks to changes over time in life annuities

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life annuities

interest or mortality?

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Duration

Duration: sensitivity of $\bar{a}_x(t)$ to constant changes in δ

$$D_x(t) = -\frac{\frac{\partial \bar{a}_x(t)}{\partial \delta}}{\bar{a}_x(t)}$$

DURATION: SENSITIVITY OF $\bar{a}_x(t)$ TO CONSTANT CHANGES IN δ

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How annuities respond to **changes in interest rates**? (Milevsky, 2013; Charupat, Kamstra, Milevsky, 2015)

What about changes in mortality?

Entropy

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ENTROPY OF LIFE ANNUITY

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They showed that

$$H_x(t) = \frac{\int_0^\infty \mu(x+s,t)_s p_x(t) v(s,t) \bar{a}_x(t) ds}{\bar{a}_x(t)}.$$

CHANGES OVER TIME IN $\mu(x, t)$ AND $\delta(s, t)$

Rate of mortality improvement

$$\rho(x, t) = -\frac{\frac{\partial \mu(x,t)}{\partial t}}{\mu(x,t)}.$$

Change in the term-structure of interest rates

$$\varphi(s, t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)}.$$

Changes over time in $\bar{a}_x(t)$

Derivative of $\bar{a}_x(t)$ with respect to time t:

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Relative derivative of $\bar{a}_x(t)$:

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Decomposing $\acute{\bar{a}}_x(t)$

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Age and term attribution

AGE AND TERM DECOMPOSITION

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where

- $\bar{\rho}_x(t; x_{i-1}, x_i)$: weighted average improvement in the age group $[x_{i-1}, x_i)$,
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Analogous to term-attribution (Daul, Sharp and Sørensen, 2012) and key-durations (Ho, 1992) in fixed income.

Cause of death decomposition

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where,

- $H_x^i(t)$:the cause-specific entropy,
- $\bar{\rho}_x^i(t)$: average rate of mortality improvement of cause i.

Assuming a single interest rate $\delta(t)$

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It suffices to use the **modified duration** ($D_x(t)$) and **entropy** ($H_x(t)$) together with $\dot{\delta}(t)$ and $\bar{\rho}(t)$ to determine the contribution of **financial and longevity risks** to changes over time in **life annuities**.

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No assumptions about the functional form of δ and μ (entirely data-driven).

DECOMPOSITION OF $\acute{\bar{a}}_x(t)$

Decomposition	Financial component	Longevity component
General	$\bar{\varphi}(t)D_x(t)$	$\bar{\rho}(t)H_x(t)$
Age-Term	$\sum_{j=1}^{m} \bar{\varphi}(t; t_{j-1}, t_j)D_x(t; t_{j-1}, t_j)$	$\sum_{i=1}^{n} \bar{\rho}_x^i(t)H_x^i(t)$
Cause of death	$\bar{\varphi}(t)D_x(t)$	$\sum_{i=1}^{n} \bar{\rho}_x(t; x_{i-1}, x_i)H_x^p(t; x_{i-1}, x_i)$
Single δ	$\dot{\delta}(t)D_x(t)$	$\bar{\rho}(t)H_x(t)$

Changes in life annuities in the UK

Historical data

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HISTORICAL CONTRIBUTIONS OF MORTALITY AND INTEREST RATES

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Yield curve

- UK government bonds, also known as Gilts (Bank of England, 2021),
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Causes of death

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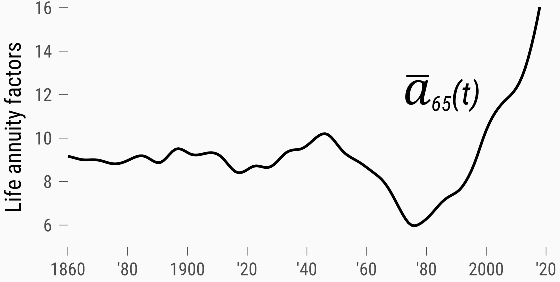
- 2001-2016.

Life annuity factors

LIFE ANNUITY FACTORS AT AGE 65. MALES, 1841-2018



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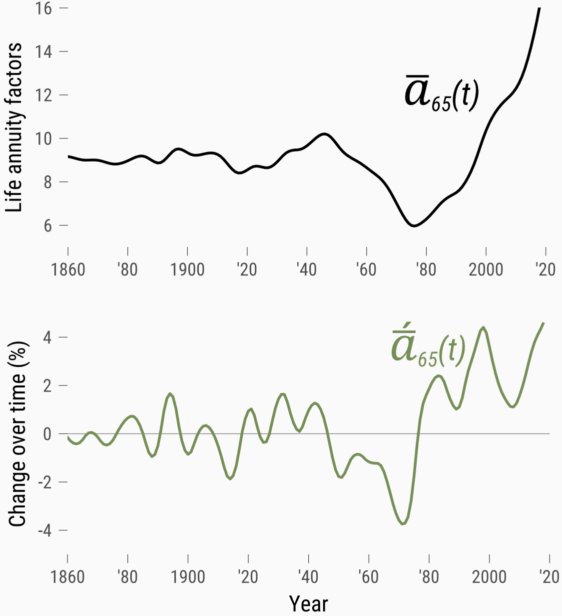
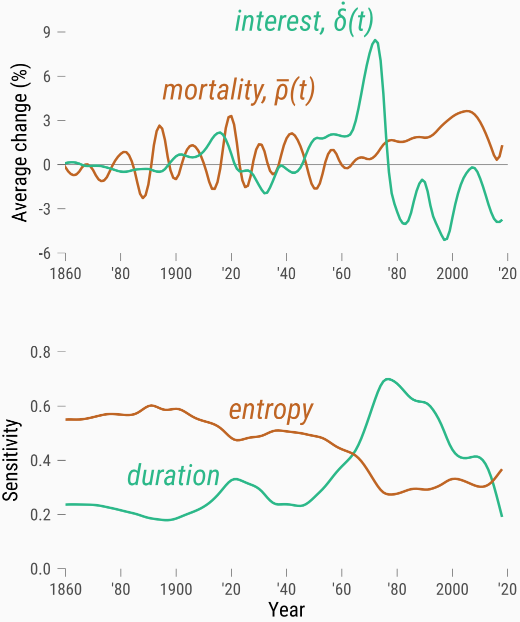


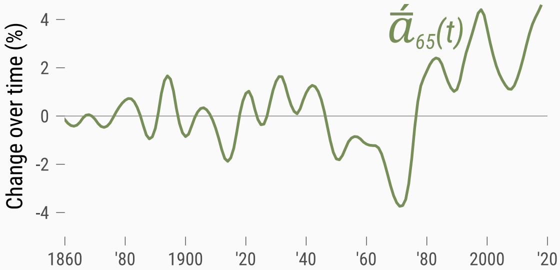
Changes over time and sensitivities

CHANGES OVER TIME AND SENSITIVITIES. MALES, 1841-2018



Assuming a single $\delta(t)$

DECOMPOSITION OF $\acute{\bar{a}}_x(t)$ AT AGE 65. MALES, 1841-2018



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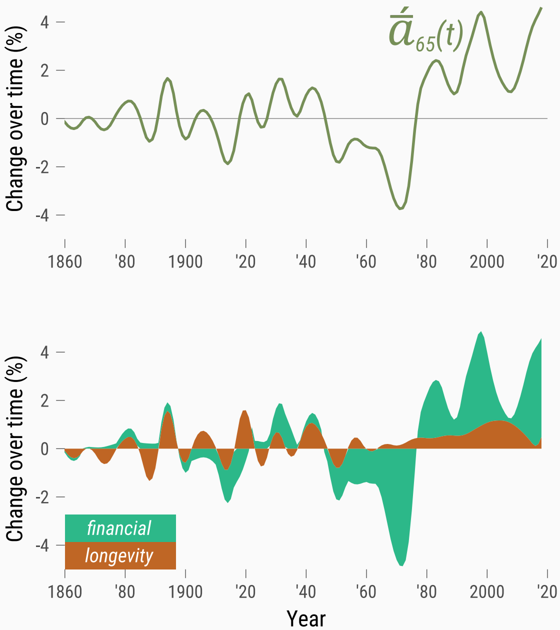
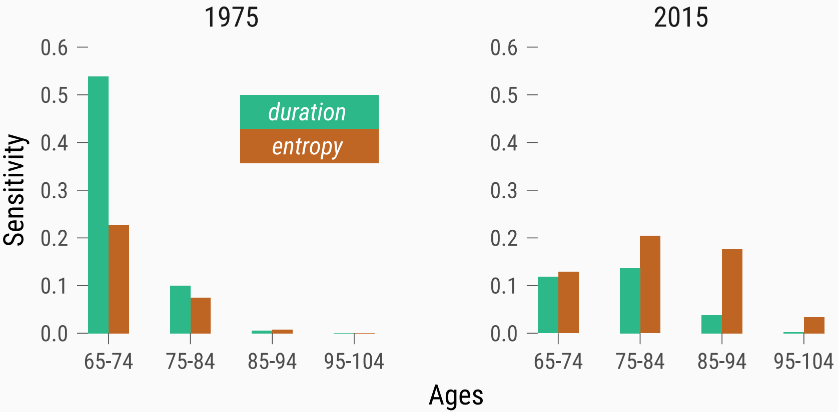


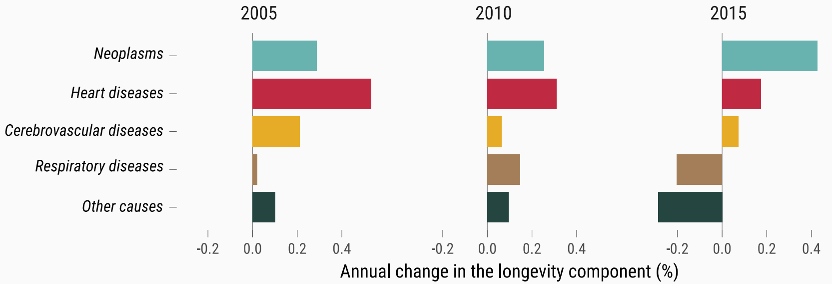
Age and term attribution

AGE AND TERM ATTRIBUTION. MALES, 1970-2018



Causes of death

CAUSE OF DEATH ATTRIBUTION



To sum up

Bringing both perspectives together

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Historical developments in the UK

- **Longevity risk** has, most of the time, contributed to **increase** in $\bar{a}_x(t)$, but during some periods it has been **masked by high financial risk**.

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- **At higher ages** (i.e. age 75 or older ages):
  - The sensitivity of $\bar{a}_x(t)$ to μ is higher,
  - Policies aiming at **increasing retirement ages entail higher longevity risk** (e.g. Denmark, Alvarez et al (2021)).

Next steps

What about the future?

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UNRAVELLING THE CONTRIBUTION OF FINANCIAL AND LONGEVITY RISKS TO CHANGES OVER TIME IN LIFE ANNUITIES

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Rate of mortality improvement

$$\rho(x,t) = -\frac{\frac{\mu(x,t)}{\partial t}}{\mu(x,t)} = -\frac{\dot{\mu}(x,t)}{\mu(x,t)}. \tag{1}$$

Change in interest rates over time

$$\varphi(s,t) = -\frac{\frac{\delta(s,t)}{\partial t}}{\delta(s,t)} = -\frac{\dot{\delta}(s,t)}{\delta(s,t)}. \tag{2}$$

DECOMPOSITION

Entropy

$$H_x^p(t) = \frac{\int_0^\infty \mu(x+s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} \tag{3}$$

Duration

$$D_x^p(t) = \frac{\int_0^\infty \delta(s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} \tag{4}$$

Time derivative of $\bar{a}_x(t)$

$$\begin{aligned}\dot{\bar{a}}_x(t) &= \int_0^\infty \rho(s,t)\mu(s,t)_s|\bar{a}_x(t)ds + \int_0^\infty \varphi(s,t)\delta(s,t)_s|\bar{a}_x(t)ds \\ &= \int_0^\infty \rho(s,t)_sM_x(t)ds + \int_0^\infty \varphi(s,t)_sW_x(t)ds,\end{aligned}$$

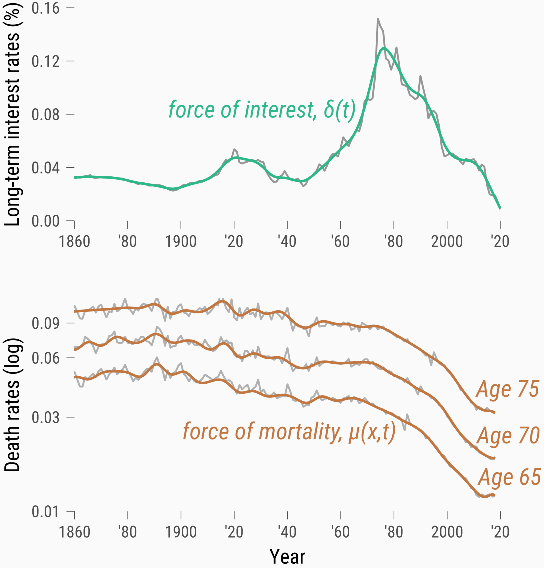
where ${}_sM_x(t) = \mu(s,t)_s|\bar{a}_x(t)$ and ${}_sW_x(t) = \delta(s,t)_s|\bar{a}_x(t)$.

Relative derivative of $\bar{a}_x(t)$

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)} = \underbrace{\bar{\rho}(t)H_x^p(t)}_{\text{longevity component}} + \underbrace{\bar{\varphi}(t)D_x^p(t),}_{\text{financial component}}$$

where $\bar{\rho}_x(t) = \frac{\int_0^\infty \rho(x+s,t)_sM_x(t)ds}{\int_0^\infty {}_sM_x(t)ds}$ and $\bar{\varphi}(t) = \frac{\int_0^\infty \varphi(s,t)_sW_x(t)ds}{\int_0^\infty {}_sW_x(t)ds}$

INTEREST AND MORTALITY RATES FOR MALES IN THE UK, 1841-2018



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