Unraveling the time dynamics of life annuities

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Abstract

Actuaries and risk managers aim to mitigate the impact of interest rate fluctuations on portfolio values. Similarly, demographers have long measured the effects of mortality changes on longevity metrics like life expectancy. This paper integrates these perspectives by developing a new decomposition method to quantify the contributions of mortality and interest rate changes to life annuity factors. Our intuitive formulations enable actuaries and risk managers to assess stochastic changes in financial and longevity risks within their life annuity portfolios.

To demonstrate our method, we analyze the historical development of life annuity prices using financial and mortality data from the United Kingdom since 1841. Our findings reveal a significant interplay between longevity and financial risk, with high financial risk sometimes obscuring the impact of longevity.

We develop differential equations to determine the sources of change in reserves. Such equations clearly show that contributions of mortality improvements and changes in the term structure of interest.

The differential equations developed in this paper are complementary to the traditional Thiele equation to identify the sources of change in life annuities.

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1 Introduction

There is a prevalent concern among pension funds and insurance companies about how simultaneous fluctuations in interest and mortality affect their life annuity portfolios. This concern is emphasized under the current setting that high-income countries are facing: rapid changes in interest rates, and mortality improvements (Djeundje et al., 2022) that have been reversed by the Covid-19 epidemic (Aburto et al., 2022). There are two fundamental uncertainties about the time dynamics of life annuities. First, there is uncertainty about the development of cohort mortality and interest rates themselves. The second uncertainty relates to the sensitivity of life annuities (and corresponding reserves, capital charges, etc.) to the simultaneous change in mortality and interest rates. In other words, how mortality and interest change over time and how life annuities react to them.

The first source of uncertainty has been studied extensively and a wide range of models to forecast mortality and interest rates have been proposed under many different perspectives[1] (Cairns et al., 2019, 2017; Kallestrup-Lamb et al., 2020) (ADD REFERENCES TO MORTALITY MODELS). The second source of uncertainty has been examined to a lesser extend. Most of the studies regarding interest-rate sensitivity are related to immunization theory, while for mortality, there is a limited number of studies that have addressed this issue. An important contribution in this regard is recently done by Rabitti and Borgonovo (2020). They used model life tables for the United States in 1990 and for the United Kingdom (years 1990-1994) and assumed different levels of constant interest rates to show that, for these life tables, the sensitivity to mortality is higher when interest rate is low (i.e. 0.5%), whereas the financial component becomes the key risk driven when both sensitives are taken into account.

The analysis of Rabitti and Borgonovo (2020) makes strides in providing a joint perspective of the sensitivities to changes in mortality and interest rates. However, it is unclear whether their results hold for different time periods or with real data for the term-structure of interest rates[2]. It is also unclear how rapid the sensitives change over time and what are the sources of such changes (e.g. driving causes of death). Furthermore, differences in age-specific mortality improvements and differentials in term-specific attributions terms of the yield curve might produce contrasting effects in life annuities. At present, there is no single tool to disentangle the effect of i) simultaneous fluctuations in mortality and interest, and ii) the sensitivity to such fluctuations in life annuities, reserves and related risk measures.

In this article, we introduce continuous-time formulations to address this issue. We quantify the simultaneous contribution of changes in mortality and interest to changes over time in life annuities. To illustrate our equations, we first look at the long-term development of life annuity factors using financial and demographic data from the United Kingdom from 1841 to 2016. Next, we dig into age-specific and term-specific attributions and ravel out the causes of death that drive the longevity component. Finally, we extend our equations to changes in reserves and develop the intuition about how such equations can be used in stochastic risk management. Our formulations are neat, intuitive and can be easily used by actuaries and risk managers to assess the financial and longevity risks embedded in their life annuities' portfolios (using their own, real data). From the actuarial perspective, our formulations are useful in devising strategies for better risk management by levering on well-known results from mathematical demography and immunization.

[1] It is important to highlight that forecasting (both for mortality and interest rates) is out of the scope of this paper. Instead, the focus is on the development of mathematical expressions to determine the sources of change in life annuities.

[2] Specifically, because their analysis is limited to historical life tables from 30 years ago (i.e. US 1990 and the UK 1990-1994) and with the synthetic assumption of constant interest rates.

We have organized the remainder of the article as follows. First, we describe useful results from the actuarial and demographic literature on mortality and interest rate sensitivities in Section 2. Next, we derive the equations to unravel the sources of changes over time in life annuities in Section 3. Then, in Section 4, we illustrate our equations with historical data from the United Kingdom. In Section 5, we extend our equations to annuity reserves and risk management. Lastly, we conclude with a discussion about the applicability of the equations developed in this paper. The code to replicate the results and all the deriviations introduced in this article are available in the following open-source repository: *link available upon publication.*

2 Sensitivity to interest and mortality changes

Duration

Duration (hereafter denoted by D) is a fundamental quantity used to measure the effect of interest rates in any financial instrument. It is defined as the price sensitivity of a life annuity (or bond, fixed income, or any other financial instrument) to changes in the force of interest δ (Charupat et al., 2016; Milevsky, 2013). There are different types of duration and according to their interpretation, they can be used in different applications. For example, Macaulay duration estimates how many years it will take for an investor to be repaid the bond's price by its total cash flows. Modified duration measures the price change in a bond given a 1% change in interest rates. Mathematically, duration is the first-order derivative of the financial quantity with respect to δ, whereas the second-order derivative is given by the convexity. Both, duration and convexity are used in interest-rate immunization (Courtois et al., 2007; Fisher and Weil, 1971; Redington and Clarke, 1951; Santomero and Babbel, 1997; Shiu, 1990) where portfolios are hedged against fluctuations in interest rates.

Entropy

The entropy (hereafter denoted by H) is a measure developed by demographers and population biologists to quantify the sensitivity of life expectancy to changes in the force of mortality (Demetrius, 1974; Goldman and Lord, 1986; Keyfitz, 1977; Leser, 1955; Vaupel, 1986), such that higher the values of the entropy; the higher the sensitivity of life expectancy is to mortality rates. It has been shown that the entropy increases or decreases depending of the ages where mortality improvements take place (Aburto et al., 2019). The entropy is also been used by demographers as an indicator of how lifespans vary within populations (Aburto et al., 2020; Fernandez and Beltrán-Sánchez, 2015; Vaupel et al., 2011).

An important contribution in understanding the time dynamics of life expectancy via the entropy is done by Vaupel and Canudas-Romo (2003). They show that the time derivative of life expectancy ($e(0,t)$) with respect to time t can be expressed in terms of the general pace of mortality improvement (ρ_e) and the entropy ($H_e(t)$), such that:

$$\frac{\partial e(0,t)}{\partial t} = \rho_e(t) H_e(t) e(0,t). \tag{1}$$

Equation (1) clearly disentangles the changes over time in mortality and the sensitivity of life expectancy to these changes. This Equation allows further decomposition of age-specific and cause-specific contributions to mortality improvement and entropy.

Haberman et al. (2011) extended the concept of entropy to life annuities. In this context, the entropy measures the sensitivity of a life annuity to proportional changes in the force of mortality.

The entropy of can be seen as an indicator longevity risk (Rabitti and Borgonovo, 2020). Alvarez et al. (2021) shows that it can be used to examine socio-economic disparities in pension systems.

There is a growing line of research that has used the entropy[3] in the context of mortality-immunization[4]. In particular, Li and Hardy (2011); Tsai and Chung (2013); Tsai and Jiang (2011); Wang et al. (2010) derive discrete formulas for life annuity entropies and convexities assuming constant and proportional changes in the force of mortality, μ. Mortality-immunization has been applied to different types of life insurance and annuity products (Levantesi and Menzietti, 2018; Li and Luo, 2012a,b; Luciano et al., 2015; Wong et al., 2015).

Bringing both perspectives together

Research regarding the sensitivity to mortality changes (mostly developed by demographers) and on sensitivity to interest rates (mostly developed by actuaries and financial managers) have remained largely disconnected[5]. Paradoxically, both strands of research direct efforts at the same objective of study: gain a better understanding of the inherent sources of change in life contingent quantities.

Recently, Lin and Tsai (2020) made strides on this regard by deriving discrete formulas to calculate sensitivity of life annuities (and whole life insurances) to simultaneous changes mortality and interest rates. They introduced a synthetic variable called 'the force of mortality-interest', which results from the addition of the force of mortality and interest ($\mu^* = \mu + \delta$). While the application of Lin and Tsai (2020) is interesting since it combines sensitivity in mortality and interest rates, it is unlikely that μ and δ change at the same pace over time.

In this article we bring together the aforementioned results from the actuarial and demographic literature to the analysis of the time dynamics of life annuities. In the following section, we derive equations to this aim and describe their applications.

3 Time dynamics of life annuities

3.1 Preliminaries

Let us introduce some notation about time variables used to portray the dynamics of life annuities. Let (x) denote an insured life aged x, where $x > 0$. The future lifetime of (x) is denoted by random variable S_x. Therefore, $x + S_x$ is the random variable representing the lifetime of (x). Time $S = s$ is the time variable associated to the *development of the policy*.

The actuarial value of the policy associated to life (x) is determined using information about the corresponding economic-demographic environment where the life annuity is written and provided. Let T be the random variable representing the time when such information of *the economic-demographic environment* is generated. We assume that this information is generated continuously, such that T is continuous.

Time t and s are both time variables, however, they are not necessary the same and it is important to highlight their differences.

Information about eonomic-demographic enviroment used to evaluate life annuities are collected at time t. However, it could be that there is a lag between the time where the actuarial assumptions

[3] with the name of mortality durations but it is essentially the same formulation as in Haberman et al. (2011)

[4] similar to interest-immunization, mortality-immunization is defined as a set of strategies to ensure that the value of a portfolio will be little affected in response to changes in mortality rates

[5] with exception of Alvarez et al. (2021); Haberman et al. (2011); Rabitti and Borgonovo (2020) that made clear the links between both strands of research

are set (at time t) and the time of development of the policy (denoted by time s)[6]. In some cases it could be $t = s$, however, we decide to make a clear distinction between these two time variables with the purpose of unraveling the sources of change in life annuities[7]. For this reason, all of the quantities in this paper are indexed by t.

Let $I_{s,t} = \mathbb{1}_{\{S_x > s,t\}}$ be the indicator of survival of life (x) to time s with information generated at time t. Its expected value is $\mathbb{E}[I_{s,t}] = {}_s p_x(t)$ is the time t period survival probability from age x to age $x+s$. Probability ${}_s p_x(t)$ can be expressed as ${}_s p_x(t) = e^{-\int_0^s \mu(x+y,t)dy}$, where $\mu(x,t)$ is the force of mortality at age x in time t.

Let $\delta(s,t)$ be the time t forward force of interest at maturity s. This measure is a general form to express the term-structure of interest rates. Quantity $v(s,t) = e^{-\int_0^s \delta(y,t)dy}$ is the time t discount factor, for a cash-flow payable at maturity s.

The derivative with respect to time t is denoted by adding a point on top of the function of interest. For example, time derivatives for the forces of mortality and interest are expressed as:

$$\dot{\mu}(x,t) \equiv \frac{\partial \mu(x,t)}{\partial t}, \tag{2}$$

and

$$\dot{\delta}(s,t) \equiv \frac{\partial \delta(s,t)}{\partial t}. \tag{3}$$

The rate of mortality improvement at age x (or progress in reducing mortality over t) is defined as

$$\rho(x,t) = -\frac{\frac{\partial \mu(x,t)}{\partial t}}{\mu(x,t)} = -\frac{\dot{\mu}(x,t)}{\mu(x,t)}. \tag{4}$$

Similarly, the relative change over time in the forward force of interest at maturity s is captured by

$$\varphi(s,t) = -\frac{\frac{\partial \delta(s,t)}{\partial t}}{\delta(s,t)} = -\frac{\dot{\delta}(s,t)}{\delta(s,t)}. \tag{5}$$

The actuarial present value of a continuous life annuity at age x evaluated at time t is given by

$$\bar{a}_x(t) = \int_0^\infty {}_s p_x(t) v(s,t) ds = \int_0^\infty {}_s E_x(t) ds, \tag{6}$$

where ${}_s E_x(t) = {}_s p_x(t) v(s,t)$.

A life annuity deferred s years starting to be paid at age $x+s$ is expressed as

$$_{s|}\bar{a}_x(t) = {}_s E_x(t) \bar{a}_{x+s}(t). \tag{7}$$

[6]In with-profits contracts, time t can refer to the first-order assumptions, that differs from the information of the development of the policy (second-order basis)

[7]In Section 5.1 the link between t and s is analyzed explicitly in terms of the Thiele differential equation for a life annuity policy

3.2 Two types of changes: constant and proportional to μ and δ

We are interested in measuring changes in $\bar{a}_x(t)$ with respect to the time variable t. To achieve this aim, we first describe how $\bar{a}_x(t)$ reacts to changes in the forces of mortality and interest. As mentioned in Section 2, the entropy of a life annuity is a sensitive measure that captures changes in $\bar{a}_x(t)$ with respect to $\mu(x,t)$. The entropy of a life annuity issued for a life (x) and evaluated at time t is represented by $H_x(t)$. This quantity is denoted in general terms as:

$$H_x(t) = \frac{1}{\bar{a}_x(t)} \cdot \frac{\partial \bar{a}_x(t)}{\partial \mu(x,t)}. \tag{8}$$

Analogously, duration captures the sensitivity of $\bar{a}_x(t)$ to changes in interest rates. It is defined as the relative derivative of the annuity factor with respect to changes in the force of interest (Milevsky, 2012a,b):

$$D_x(t) = \frac{1}{\bar{a}_x(t)} \cdot \frac{\partial \bar{a}_x(t)}{\partial \delta(s,t)}. \tag{9}$$

Greater values for $H_x(t)$ and $D_x(t)$ indicate that $\bar{a}_x(t)$ is highly sensitive to changes in $\mu(x,t)$ and $\delta(s,t)$ respectively. The entropy and the duration can be determined by assuming the relationship between changes in in $\mu(x,t)$ and $\delta(s,t)$ relate to $\bar{a}_x(t)$. All formulations in the following sections are developing assuming constant (or parallel) and proportional movements in these quantities.

Changes in $\bar{a}_x(t)$ with respect to $\mu(x,t)$

The entropy of a life annuity is denoted by $H_x^c(t)$ when changes in $\mu(x,t)$ are held constant at all ages and by $H_x^p(t)$ when changes are performed proportional. Based on the results developed by Tsai and Chung (2013) and Lin and Tsai (2020), when $\mu(x,t)$ is changed constantly to $\mu(x,t)+\gamma$ such that γ is a small number (see proof in Appendix), the entropy of $\bar{a}_x(t)$ becomes

$$H_x^c(t) = -\frac{\int_0^\infty s\,{}_sp_x(t)v(s,t)ds}{\bar{a}_x(t)} = \frac{h_x^c(t)}{\bar{a}_x(t)}, \tag{10}$$

where $h_x^c(t) = -\int_0^\infty s\,{}_sp_x(t)v(s,t)ds$. The term $h_x^c(t)$ is expressed in absolute (monetary) terms, whereas the entropy $H_x^c(t)$ is dimensionless because it does not depend on the absolute value of $\bar{a}_x(t)$.

Haberman et al. (2011) and Tsai and Chung (2013) show that when changes in $\mu(x,t)$ are assumed to be proportional to a small number γ such that $\mu(x,t)(1+\gamma)$, the entropy of $\bar{a}_x(t)$ becomes

$$H_x^p(t) = -\frac{\int_0^\infty {}_sp_x(t)\ln[{}_sp_x(t)]v(s,t)ds}{\int_0^\infty {}_sp_x(t)v(s,t)ds}. \tag{11}$$

Alternatively, we show (see proof in Section A.2 of the Appendix) that Equation (11) can be expressed as

$$H_x^p(t) = \frac{\int_0^\infty \mu(x+s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} = \frac{h_x^p(t)}{\bar{a}_x(t)}, \tag{12}$$

where $h^p_x(t) = \int_0^\infty \mu(x+s,t)_s|\bar{a}_x(t)ds$. Analogous to the case where changes are assumed to be constant, quantities $h^p_x(t)$ and $H^p_x(t)$ are expressed in absolute and relative terms respectively. The formulations shown in this section are closely related to the ones developed in the mortality-immunization literature (Lin and Tsai, 2020; Tsai and Chung, 2013). We extended them to the continuous case.

Changes in $\bar{a}_x(t)$ with respect to $\delta(s,t)$

Similar to the entropy, changes in $\bar{a}_x(t)$ with respect to $\delta(s,t)$ can be assumed to be either constant or proportional. For the former case where changes are constant (or parallel), duration is expressed as:

$$\begin{aligned} D^c_x(t) &= -\frac{\int_0^\infty s\,{}_sp_x(t)v(s,t)ds}{\bar{a}_x(t)} \\ &= \frac{d^c_x(t)}{\bar{a}_x(t)}, \end{aligned} \tag{13}$$

where $d^c_x(t) = -\int_0^\infty s\,{}_sp_x(t)v(s,t)ds$. Thus, assuming constant changes in $\delta(s,t)$ results into common types of duration known in finance as *monetary duration*, $d^c_x(t)$, and *modified duration*, $D^c_x(t)$ (see Milevsky (2012a) and Tsai and Chung (2013) for further details). It is worth noting that Equations (10) and (13) are identical such that $d^c_x(t) = h^c_x(t)$. Interestingly, this indicates that constant (i.e. parallel) changes in the force of mortality have essentially the same effect as parallel changes in the force of interest.

Assuming proportional changes in $\delta(s,t)$ (see proof in the Appendix) results in:

$$D^p_x(t) = -\frac{\int_0^\infty {}_sp_x(t)v(s,t)\ln(v(s,t))ds}{\bar{a}_x(t)}. \tag{14}$$

Note that when assuming constant force of mortality such that $v(s,t) = e^{-\delta(t)s}$, we have that $D^p_x(t) = \delta(t)D^c_x(t)$,

$$\begin{aligned} D^p_x(t) &= -\frac{\int_0^\infty {}_sp_x(t)v(s,t)\ln(v(s,t))ds}{\bar{a}_x(t)} \\ &= -\delta(t)\frac{\int_0^\infty s\,{}_sp_x(t)e^{-\delta(t)s}ds}{\bar{a}_x(t)} \\ &= \delta(t)D^c_x(t). \end{aligned} \tag{15}$$

Equation (14) can also be re-expressed as:

$$\begin{aligned} D^p_x(t) &= \frac{\int_0^\infty \delta(s,t)_s|\bar{a}_x(t)ds}{\bar{a}_x(t)} \\ &= \frac{d^p_x(t)}{\bar{a}_x(t)}. \end{aligned} \tag{16}$$

where $d^p_x(t) = \int_0^\infty \delta(s,t)_s|\bar{a}_x(t)ds$.

3.3 Time derivative of $\bar{a}_x(t)$

We now compute the derivative of $\bar{a}_x(t)$ with respect to the time variable t, $\dot{\bar{a}}_x(t) = \frac{\partial \bar{a}_x(t)}{\partial t}$, such that

$$\dot{\bar{a}}_x(t) = \int_0^\infty {}_s\dot{p}_x(t)v(s,t)ds + \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds. \tag{17}$$

To develop a closed-form solution for Equation (17) we consider the general case where ${}_sp_x(t) = e^{-\int_0^s \mu(x+y,t)dy}$ and $v(s,t) = e^{-\int_0^s \delta(s,t)dy}$. We analyse separately each of the two terms in the right side of Equation (17). Let us first focus on the first term:

$$\begin{aligned}
\int_0^\infty {}_s\dot{p}_x(t)v(s,t)ds &= \int_0^\infty v(s,t)e^{-\int_0^s \dot{\mu}(x+y,t)dy}ds \\
&= -\int_0^\infty v(s,t){}_sp_x(t)\int_0^s \dot{\mu}(x+y,t)dyds \\
&= -\int_0^\infty \dot{\mu}(x+s,t)\int_s^\infty v(y,t){}_yp_x(t)dyds \\
&= -\int_0^\infty \dot{\mu}(x+s,t){}_sE_x(t)\bar{a}_{x+s}(t)ds \\
&= \int_0^\infty \rho(x+s,t)\mu(x+s,t){}_s|\bar{a}_x(t)ds.
\end{aligned} \tag{18}$$

The second part equals:

$$\begin{aligned}
\int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds &= \int_0^\infty {}_sp_x(t)e^{-\int_0^s \dot{\delta}(y,t)dy}ds \\
&= -\int_0^\infty {}_sp_x(t)v(s,t)\int_0^s \dot{\delta}(y,t)dyds \\
&= -\int_0^\infty \dot{\delta}(s,t)\int_s^\infty {}_yp_x(t)v(y,t)dyds \\
&= \int_0^\infty \varphi(s,t)\delta(s,t){}_s|\bar{a}_x(t)ds.
\end{aligned} \tag{19}$$

Thus, $\dot{\bar{a}}_x(t)$ can be expressed as

$$\begin{aligned}
\dot{\bar{a}}_x(t) &= \int_0^\infty \rho(x+s,t)\mu(x+s,t){}_s|\bar{a}_x(t)ds + \int_0^\infty \varphi(s,t)\delta(s,t){}_s|\bar{a}_x(t)ds \\
&= \int_0^\infty \rho(x+s,t){}_sM_x(t)ds + \int_0^\infty \varphi(s,t){}_sW_x(t)ds,
\end{aligned} \tag{20}$$

where ${}_sM_x(t) = \mu(x+s,t){}_s|\bar{a}_x(t)$ and ${}_sW_x(t) = \delta(s,t){}_s|\bar{a}_x(t)$. Dividing Equation (20) by $\bar{a}_x(t)$ yields to:

$$\acute{\bar{a}}_x(t) = \frac{\dot{\bar{a}}_x(t)}{\bar{a}_x(t)} = \underbrace{\bar{\rho}_x(t) H_x^p(t)}_{\text{longevity component}} + \underbrace{\bar{\varphi}(t) D_x^p(t),}_{\text{financial component}} \tag{21}$$

where $\bar{\rho}_x(t) = \frac{\int_0^\infty \rho(x+s,t)_s M_x(t) ds}{\int_0^\infty {}_s M_x(t) ds}$ and $\bar{\varphi}(t) = \frac{\int_0^\infty \varphi(s,t)_s W_x(t) ds}{\int_0^\infty {}_s W_x(t) ds}$ are the weighted-average changes in mortality and interest rates respectively. Functions $\bar{\rho}_x(t)$ and $\bar{\varphi}(t)$ capture the stochastic change in the forces of mortality and interest whereas $H_x^p(t)$ and $D_x^p(t)$ capture the sensitivity due to changes in μ and δ. In other words, Equation (21) entails that changes over time in annuity factors are driven by $\bar{\rho}_x(t)$ and $\bar{\varphi}(t)$, which are modulated by $H_x^p(t)$ and $D_x^p(t)$ respectively[8].

3.4 Age and term contributions to changes in $\bar{a}_x(t)$

Equation (21) gives a general way of understanding the sensitivity of the value of an annuity to overall changes in mortality and the term structure of interest rates. However, we may also be interested in understanding the contribution of particular age groups or specific terms of the yield curve to changes in the life annuity portfolio. This is in the spirit of the perfomance attribution exercises that are common in fixed income (see, e.g., Daul et al. (2012)).

Let us define n age groups $[x_{i-1}, x_i)$, $i = 1, \ldots, n$, with $x_0 = x$ and $x_n = \infty$, and m term groups $[t_{j-1}, t_j)$, $j = 1, \ldots, m$, with $t_0 = 0$ and $t_m = \infty$. Equation (20) can be decomposed by age and term groups as follows:

$$\acute{\bar{a}}_x(t) = \sum_{i=1}^{n} \int_{x_{i-1}-x}^{x_i - x} \rho(x+s,t)_s M_x(t) ds + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \varphi(s,t)_s W_x(t) ds. \tag{22}$$

We can also decompose the entropy $h_x^p(t)$ into n age groups as follows:

$$\begin{aligned} h_x^p(t) &= \sum_{i=1}^{n} \int_{x_{i-1}-x}^{x_i - x} \mu(x+s,t)_s|\bar{a}_x(t) ds \\ &= \sum_{i=1}^{n} h_x^p(t; x_{i_1}, x_i), \end{aligned} \tag{23}$$

with $h_x^p(t; x_{i-1}, x_i) = \int_{x_{i-1}-x}^{x_i - x} \mu(x+s,t)_s|\bar{a}_x(t) ds$, denoting the dollar entropy associated to proportional changes in the force of mortality between ages x_{i-1} and x_i and $H_x^p(t; x_{i-1}, x_i) = \frac{h_x^p(t; x_{i-1}, x_i)}{\bar{a}_x(t)}$, the corresponding dimensionless entropy.

Similarly, we can decompose the dollar duration $d^p(t)$ into m term groups as follows:

$$\begin{aligned} d_x^p(t) &= \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \delta(s,t)_s|\bar{a}_x(t) ds \\ &= \sum_{j=1}^{m} d_x^p(t; t_{j-1}, t_j), \end{aligned} \tag{24}$$

[8] Equation (21) can be seen as an extension of the Equation (1) developed by Vaupel and Canudas-Romo (2003) to measure changes over time in life expectancy

with $d^p_x(t;t_{j-1},t_j) = \int_{t_{j-1}}^{t_j} \delta(s,t)_s|\bar{a}_x(t)ds$, denoting the dollar duration associated to proportional changes in term structure between terms t_{j-1} and t_j and $D^p_x(t;t_{j-1},t_j) = \frac{d^p_x(t;t_{j-1},t_j)}{\bar{a}_x(t)}$, the corresponding dimensionless duration. These age-specific entropies and term-specific durations are akin to the key q-durations (Li and Luo, 2012a) and key rate durations (Ho, 1992), which measure the sensitivity of financial instruments to changes in specific portions of the life table and term structure of interest rates.

Combining Equations (22), (23), (24):

$$\acute{\bar{a}}_x(t) = \sum_{i=1}^{n} \bar{\rho}_x(t;x_{i-1},x_i) H^p_x(t;x_{i-1},x_i) + \sum_{j=1}^{m} \bar{\varphi}(t;t_{j-1},t_j) D^p_x(t;t_{j-1},t_j), \tag{25}$$

where

$$\bar{\rho}_x(t;x_{i-1},x_i) = \frac{\int_{x_{i-1}-x}^{x_i-x} \rho(x+s,t)_s M_x(t)ds}{\int_{x_{i-1}-x}^{x_i-x} {}_sM_x(t)ds}$$

and

$$\bar{\varphi}(t;t_{j-1},t_j) = \frac{\int_{t_{j-1}}^{t_j} \varphi(s,t)_s W_x(t)ds}{\int_{t_{j-1}}^{t_j} {}_sW_x(t)ds}$$

are, respectively, the weighted average improvement rates in the age group $[x_{i-1}, x_i)$ and the weighted average pace of change in the force of interest for the term group $[t_{j-1}, t_j)$.

3.5 Cause of death decomposition

Depending on the population, different causes of death can contribute substantially to the dynamics of the life annuity portfolio (Kallestrup-Lamb et al., 2020; Lin and Cox, 2005). A contemporary issue related to this pattern is the case of the excess mortality due to Covid-19 observed worldwide during the years 2020 and 2021 (Aburto et al., 2022). To get insights about how causes of death shape the time dynamics of life annuities, the longevity component of Equation (21) can be easily decomposed into cause-specific contributions.

Let ${}_sp^i_x(t) = e^{-\int_0^s \mu^i(x+y,t)dy}$ be the probability of surviving cause i from age x to age $x+s$ at time t, and $\mu^i(x+y,t)$ be the associated hazard. For $i = 1,\ldots,n$ independent causes of death, we have that ${}_sp_x(t) = \prod_{i=1}^n {}_sp^i_x(t)$. In this context, a whole life annuity can be expressed as $\bar{a}_x(t) = \int_0^\infty {}_sp^1_x(t)\cdots{}_sp^n_x(t)v(s,t)ds$. Deriving $\bar{a}_x(t)$ with respect to time t yields to

$$\begin{aligned}
\dot{\bar{a}}_x(t) &= \int_0^\infty {}_s\dot{p}^1_x(t)\cdots{}_sp^n_x(t)v(s,t)ds + \cdots + \int_0^\infty {}_sp^1_x(t)\cdots{}_s\dot{p}^n_x(t)v(s,t)ds + \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds \\
&= \int_0^\infty {}_s\acute{p}^1_x(t){}_sp^1_x(t)\cdots{}_sp^n_x(t)v(s,t)ds + \cdots + \int_0^\infty {}_sp^1_x(t)\cdots{}_s\acute{p}^n_x(t){}_sp^n_x(t)v(s,t)ds + \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds \\
&= \int_0^\infty {}_s\acute{p}^1_x(t){}_sp_x(t)v(s,t)ds + \cdots + \int_0^\infty {}_s\acute{p}^n_x(t){}_sp_x(t)v(s,t)ds + \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds \\
&= \sum_{i=1}^n \int_0^\infty {}_s\acute{p}^i_x(t){}_sp_x(t)v(s,t)ds + \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds.
\end{aligned} \tag{26}$$

Following the same procedure of Equation (18), each component $\int_0^\infty {}_s\acute{p}^i_x(t)_s p_x(t)v(s,t)ds$ reduces to $\int_0^\infty \rho^i(x+s,t)_s M^i_x(t)ds$, where ${}_sM^i_x(t) = \mu^i(x+s,t)_s|\bar{a}_x(t)$. The last component of Equation (26) equals $\int_0^\infty \varphi(s,t)_s W_x(t)ds$ as in Equation (20). Thus, Equation (26) equals

$$\dot{\bar{a}}_x(t) = \sum_{i=1}^{n} \int_0^\infty \rho^i(x+s,t)_s M^i_x(t)ds + \int_0^\infty \varphi(s,t)_s W_x(t)ds. \tag{27}$$

Dividing by $\bar{a}_x(t)$ we have

$$\acute{\bar{a}}_x(t) = \sum_{i=1}^{n} \bar{\rho}^i_x(t) H^i_x(t) + \bar{\varphi}(t) D^P_x(t), \tag{28}$$

where $H^i_x(t) = \frac{\int_0^\infty {}_sM^i_x(t)ds}{\bar{a}_x(t)}$ is the cause-specific entropy and $\bar{\rho}^i_x(t) = \frac{\int_0^\infty \rho^i_x(x+s,t)_s M^i_x(t)ds}{\int_0^\infty {}_sM^i_x(t)ds}$ is the average rate of mortality improvement of cause i.

3.6 Assuming constant force of interest

It is common that life annuities are computed by using a single interest rate, which represents the long-term expected return. Thus, $\acute{\bar{a}}_x(t)$ can be re-expressed by assuming a single interest rate $\delta(t)$ that varies over time t, such that $v(s,t) = e^{-\delta(t)s}$. This assumption affects the second part of Equation (17) as:

$$\begin{aligned} \int_0^\infty {}_sp_x(t)\dot{v}(s,t)ds &= \int_0^\infty {}_sp_x(t)\frac{\partial\left[e^{-\delta(t)s}\right]}{\partial t}ds \\ &= -\dot{\delta}(t)\int_0^\infty s_s p_x(t)e^{-\delta(t)s}ds \\ &= \dot{\delta}(t)d^c_x(t). \end{aligned} \tag{29}$$

Substituting Equations (18) and (29) in Equation (17) results into:

$$\acute{\bar{a}}_x(t) = \bar{\rho}(t)H^p_x(t) + \dot{\delta}(t)D^c_x(t). \tag{30}$$

Given that $D^p_x(t) = \delta(t)D^c_x(t)$ (see Equation (15)), it is straight-froward to show that in the case of a single interest rate, Equations (21) and (30) are equivalent. Thus, it suffices to use the entropy ($H^p_x(t)$) and the modified duration ($D^c_x(t)$) together with $\bar{\rho}(t)$ and $\dot{\delta}(t)$ to determine the contribution of financial and longevity risks to changes over time in life annuities.

3.7 Recap of formulations

Throughout Section 3, we developed equations that can be used to determine the sources of change over time in life annuities. Overall, these equations comprise a financial component and a longevity component. Depending on the application, one can get insights about the age-specific and term-specific attribution of these components as well as cause-specific contributions. One can also make assumptions about the yield rate used in the calculations of life annuity portfolios. A summary of the equations introduced in this section is shown in the Table 1.

Description	**Financial component**	**Longevity component**
General representation	$\bar{\varphi}(t)D_x(t)$	$\bar{\rho}(t)H_x(t)$
Age-Term attributions	$\sum_{j=1}^{m} \bar{\varphi}(t;t_{j-1},t_j)D_x(t;t_{j-1},t_j)$	$\sum_{i=1}^{n} \bar{\rho}_x(t;x_{i-1},x_i)H_x^p(t;x_{i-1},x_i)$
Causes of death	$\bar{\varphi}(t)D_x(t)$	$\sum_{i=1}^{n} \bar{\rho}_x^i(t)H_x^i(t)$
Single interest rate	$\dot{\delta}(t)D_x(t)$	$\bar{\rho}(t)H_x(t)$

Table 1: Summary of expressions of the financial and longevity components used to determine the sources of change $\acute{a}_x(t)$.

One of the main advantages of our framework is that the financial and longevity components are additive. Thus, the expressions in Table 1 can be easily combined to get insights about the sources of change in $\acute{a}_x(t)$. For example, it can be the case that the actuary analysing fluctuations in the pension fund is interested in determining the term-specific contribution to the financial component as well as the causes of death that drive the longevity component. To this aim, the actuary can apply the equation $\acute{a}_x(t) = \sum_{j=1}^{m} \bar{\varphi}(t;t_{j-1},t_j)D_x(t;t_{j-1},t_j) + \sum_{i=1}^{n} \bar{\rho}_x^i(t)H_x^i(t)$ to examine such dynamics. In the following section, we illustrate how our equations can be used using empirical data.

4 Historical changes in life annuities in the United Kingdom

In this section we illustrate the formulations introduced in Section 3 by examining the sources of change in the long-term development of life annuities in the United Kingdom. We use two centuries of financial and demographic data, from 1841 to 2018. Age-specific mortality rates come from the (Human Mortality Database, 2020), whereas long-term interest rates are represented by the yield on 2.5% consols up to 2015, then by the yield of 20 year maturity bills (Bank of England, 2021 INSERT CITE IN .BIB FILE). We smoothed mortality and interest rates using splines (Camarda et al., 2012; Green and Silverman, 1993) to obtain continuous estimates of the forces of mortality and interest respectively (Figure 1). Using our estimates of μ and δ, we construct time series for life annuities at different ages and for both sexes. In this section we first focus on life annuity factors calculated for males at age 65, which represent a cut-off age for retirement in several populations.

Figure 1 about here

The upper panel of Figure 2 depicts values for $\bar{a}_x(t)$ from 1841 to 2018 for males at age 65. We observe that during the second half of the 19th century and up to the decade of 1940s, $\bar{a}_{65}(t)$ remained at similar levels with small fluctuations. Thereafter, $\bar{a}_{65}(t)$ exhibited a sharp decline up to the decade of 1980s, followed by an increasing pattern that has remained until recent years. The lower panel of Figure 2 show the relative derivative of $\bar{a}_{65}(t)$ with respect to time ($\acute{\bar{a}}_{65}(t)$).

It is clear that $\acute{\bar{a}}_{65}(t)$ captures well the changes over time in life annuity; positive values indicate that $\bar{a}_{65}(t)$ goes up and negative values indicate a decrease in $\bar{a}_{65}(t)$. In the following sections we analyse the sources of change in the time trend of $\acute{\bar{a}}_{65}(t)$ by making use of the equations developed in Section 3.

Figure 2 about here

4.1 Contribution of financial and longevity components to changes in $\bar{a}_x(t)$

As shown in the formulas developed in Section 3, changes over time in $\bar{a}_x(t)$ can be explained in terms of longevity and financial components. Each component depends on the time fluctuations of $\mu(x,t)$ and $\delta(s,t)$ (captured by $\bar{\rho}(t)$ and $\dot{\delta}(t)$ respectively) which are modulated by the sensitivity of $\bar{a}_x(t)$ to changes in the forces of mortality and interest (entropy, $H_x^p(t)$ and modified duration, $D_x^c(t)$). Figure 3 depicts the components responsible of fluctuations over time in $\bar{a}_x(t)$.

Figure 3 about here

The upper panel of Figure 3 shows that between 1860 and 1940, there were some periods of mortality improvements and deterioration. However, as of the 1980s, $\bar{\rho}(t)$ increased steadily indicating an average mortality improvement between 2% to 3%. In recent years, $\bar{\rho}(t)$ trended downwards, which coincides with the mortality deterioration currently prevailing in the United Kingdom (Djeundje et al., 2022). Regarding changes over time in the force of interest, we observe that $\dot{\delta}(t)$ has also remained at similar levels for most of the observation period with some fluctuations between 1930 and 1950. Since the 1960s, $\dot{\delta}(t)$ went up rapidly reaching the highest observed increase (of around 8% in the decade of 1970s). Thereafter, $\dot{\delta}(t)$ declined faster and moved towards negative values entailing a decline in interest rates, which is still ongoing until recent time. Trends in $\bar{\rho}(t)$ and $\dot{\delta}(t)$ are responsible for the time trend we observe in $\bar{a}_x(t)$, which is modulated by the entropy, $H_x^p(t)$, and modifed duration, $D_x^c(t)$.

The lower panel of Figure 3 shows that during most of the observation period, $H_x^p(t)$ was higher than $D_x^c(t)$. This indicates that $\bar{a}_x(t)$ has been more sensitive to changes in mortality than to changes in interest rates. However, these sensitivities have changed over time. The entropy show a decreasing trend that prevailed until to the 1980s. Modified duration, on the other hand, remained somewhat constant over time but as of the 1950s, it trended upwards reaching its maximum in the 1980s (of about 0.8). Thereafter, it decline again. It is worth noting that during some specific years, $H_x^p(t)$ and $D_x^c(t)$ depict the same values (i.e. in the 1960s). This means that during these years, $\bar{a}_x(t)$ was equally sensitive to changes in mortality and interest rates.

Using all the measures shown in Figure 3, we next calculate the contributions of financial ($\dot{\delta}(t)D_x^c(t)$) and longevity risks ($\bar{\rho}(t)H_x^p(t)$) to changes over time in $\bar{a}_{65}(t)$. The upper panel of Figure 4 shows $\acute{\bar{a}}_{65}$[9] and the lower panel shows the financial and longevity components. Positive/negative values contribute to increase/decrease in $\bar{a}_x(t)$. From Figure 4 it is clear that both, longevity and financial risks have played an important role in the long-term development of $\bar{a}_x(t)$.

Figure 4 about here

[9]This is the same as Figure 2. We show it here so the reader can make sense of the decomposition

We can distinguish among four periods where life annuity factors fluctuated in response to specific changes in financial and longevity risks. From 1860 to 1945, stable annuity factors modulated by contributions of longevity and financial components to the time trend. In a second period, from 1945 to 1970, we observe decreasing annuity factors mainly driven by the financial component. Around 1970, interest rates reached the highest point, which translated into a negative contribution of the financial component. In the next period, from 1970 to 2015, the time trend changed. Annuities went up due to positive contributions of both components. In the last period, from 2015 onwards, $\bar{a}_x(t)$ continue to increase with the longevity component contributing less to the trend. This reflects the deceleration of mortality improvements occurring in the United Kingdom and many other populations Djeundje et al. (2022).

4.2 Attribution across age-groups and terms

Next, we illustrate how the longevity and financial components in Figure 4 can be further decomposed in age-term specific attributions. The yield curve used in this analysis was constructed using government bonds in the UK (commonly known as gilts) issued at different maturities. Data on gilts was retreived from the Bank of England (2021) and it covers the years 1970-2020.

Figure 5 depicts the contribution of different age groups to the longevity component (i.e. ages 65-74,75-84, 85-94, 95-104) and terms in the yield curve (i.e. years 0-9, 10-19, 20-29, 30-39) from 1970 to 2018. At first glance, Figure 5 reveals that the major contributions comes from ages 65-74 and interest rates with terms below 20 years.

Figure 5 about here

In Figure 6, we make a zoom into the age-term sensitivities for the years 1975 and 2015. The left panel shows that in 1975, durations and entropies for the 65-74 age group (i.e. terms 0-9 years) were much higher than in the other age-term groups. This indicates that annuities were highly sensitive to changes in mortality and interest in these ages-terms. However, in 2015 the picture changes, such that the entropy is much higher than the duration, but also the sensitivity to mortality improvements is more higher at ages between 75-94.

These findings are in stark contrast to the analysis carried out by Rabitti and Borgonovo (2020), who argue that duration is the key driver in the total sensitivity. Indeed, Figure 6 clearly shows that there is an interplay between interest and mortality and that both factors contribute substantially to the change in life annuities.

Figure 6 about here

4.3 Cause of death contribution

Finally, we dig into the causes of death that drive the longevity component. To this aim we obtain cause-specific death rates from the Human Cause-of-Death Database (HCD, INSERT REFERENCE IN BIB FILE) for the United Kingdom for the period 2001-2016. The causes are categorized using the ICD-10 international classification of diseases. For illustration purposes, we group are six major causes of death used in the analysis: neoplasms (ICD-10 codes C00-D48), hearth diseases (ICD 10 codes I00-I52), cerebrovascular diseases (ICD-10 codes G45, I60-I69), respiratory diseases (ICD-10 codes J00-J22, J30-J98, U04) and other causes.

Figure 7 about here

Figure 7 depicts the cause-specific contributions to the longevity component for the years 2005, 2010 and 2015. Positive values indicate an increase in the longevity component while negative values indicate a reduction. In 2005, the main cause of death driving life annuity changes was heart diseases, followed by neoplasms and cardiovascular diseases and minor (but positive contributions) from respiratory diseases. This configuration changed substantially over time. In 2015, neoplasms were the main contribution to positive changes followed by heart diseases, whereas respiratory diseases and other causes of death contributed negatively to the change in longevity. EXPLAIN WHY IS IMPORTANT TO KNOW THESE SOURCES AND LINK THEM TO SOME STUDIES.

5 Time dynamics of reserves

Let a probability space $\{\Omega, \mathscr{F}, \mathbb{P}\}$ Let $\mathbf{F}$=$\{\mathscr{F}_t\}$ a family of sub-sigmaalbegras. $\mathscr{F}_t$ represents the information of economic-demographic environment available at time t.

Recalling that $I_{s,t} = \mathbb{1}_{\{S_x>s,t\}}$ and $\mathbb{E}[I_{s,t}] = {}_sp_x(t)$.

We define a payment process $dB_t(s) = b(s)I_{s,t}ds$ where $b(s)$ is the continuous rate of benefit payments over the development of the policy.

The policy value ${}_s\mathcal{V}_x(t) = \dfrac{1}{v(s,t)} \int_s^\infty v(y,t)dB_t(y)$

Mortality and interest rates are not known at time zero, hence they are stochastic. With this in mind, and stating that the pension fund must hold, at all time, the financial value of the benefit, the prospective reserve is expressed as:

$$ {}_sV_x(t) = \mathbb{E}[{}_s\mathcal{V}_x(t) \cdot v(s,t)|\mathscr{F}_t] = \mathbb{E}[\int_s^\infty v(y,t)dB_t(y)|\mathscr{F}_t] \tag{31} $$

$$ {}_sV_x(t) = \int_s^\infty v(y,t)_y p_x(t)b(y)dy \tag{32} $$

Same as XXX we differentiate XX with respect to t

$$ {}_s\dot{V}_x(t) = \int_s^\infty \dot{v}(y,t)_y p_x(t)b(y)dy + \int_s^\infty v(y,t)_y\dot{p}_x(t)b(y)dy \tag{33} $$

$$ \begin{aligned} {}_s\dot{V}_x(t) &= \int_s^\infty \varphi(y,t)\delta(y,t)_yV_x(t)dy + \int_s^\infty \rho(x+y,t)\mu(x+y,t)_yV_x(t)dy \\ &= \int_s^\infty \varphi(y,t)_yW_x^V(t)dy + \int_s^\infty \rho(x+y,t)_yM_x^V(t)dy \end{aligned} \tag{34} $$

where ${}_yM_x^V(t) = \mu(x+y,t)_yV_x(t)$ is the change in provision due to mortality and ${}_yW_x^V(t) = \delta(y,t)_yV_x(t)$ is the interest accrued. Quantities ${}_yM_x^V(t)$ and ${}_yW_x^V(t)$ denote the changes in the reserve over the time frame of the policy, that is $[s,\infty)$, whereas $\rho(x+y,t)$ and $\varphi(y,t)$ denote the changes at time t in the demographic-economic environment where the reserve is evaluated. The differences between changes in s and t are highlighted in the next section.

Similar to the case of the single annuity factor section 3, we are interested on finding an expression for the relative change of the provision ${}_s\acute{V}_x(t)$. Dividing Equation (34) by ${}_sV_x(t)$ yields to

$$
{}_s\acute{V}_x(t) = \underbrace{{}_s\bar{\rho}_x(t) \cdot {}_sH_x^V(t)}_{\text{longevity component}} + \underbrace{{}_s\bar{\varphi}(t) \cdot {}_sD_x^V(t)}_{\text{financial component}}, \tag{35}
$$

where ${}_s\bar{\rho}_x(t) = \frac{\int_s^\infty \rho(x+y,t)_y M_x^V(t)dy}{\int_s^\infty {}_yM_x^V(t)dy}$ is the average change in mortality, and ${}_s\bar{\varphi}(t) = \frac{\int_s^\infty \varphi(y,t)_y W_x^V(t)dy}{\int_s^\infty {}_yW_x(t)dy}$ is the average change term-structure of interest rates. Sensitivities of the reserve to changes in mortality and interest are respectively ${}_sH_x^V(t) = \dfrac{\int_s^\infty {}_yM_x^V(t)dy}{{}_sV_x(t)}$ and ${}_sD_x^V(t) = \dfrac{\int_s^\infty {}_yW_y^V(t)dy}{{}_sV_x(t)}$.

Equation (35) decomposes the changes over time in reserve, where the effect of mortality improvements is modulated by the entropy and changes in the term structure of interest rates is modulated by duration. Each of the terms in Equation (35) can be further decomposed in age-term attributions and cause-specific contributions using the expressions from Table 1.

5.1 Time dynamics over s and over t

Quantities μ and δ are not known at time zero, therefore, the actuarial valuation of the reserve ${}_sV_x(t)$ is often done by the incorporation of assumptions (and models) that provide a reasonable depiction of the *economic-demographic environment* where the policies operate. Equations (34) and (35) capture the time dynamics of the reserve with respect to changes over time in the assumptions used for the actuarial valuation. These changes are indexed by time variable t.

On the other hand, changes in the reserve ${}_sV_x(t)$ over the *time horizon where the policy is in force* are described by the well-known Thiele's differential equation. Such changes are indexed by time variable s.

Equation (34) and the Thiele's differential are complementary as they depict the dynamics of the reserve at times t and s. At the same time, there is a close relationship between these two differential equations, which will be explored next.

The Thiele's differential equation for the reserve ${}_sV_x(t)$ is found by differentiating Equation (32) with respect to s:

$$
\begin{aligned}
\frac{\partial_s V_x(t)}{\partial s} &= \mu(x+s,t)_sV_x(t) + \delta(s,t)_sV_x(t) - b(s) \\
&= {}_sM_x^V(t) + {}_sW_x^V(t) - b(s)
\end{aligned} \tag{36}
$$

Then, ${}_sM_x^V(t) = \mu(x+s,t)_sV(t)$ is the attribution to mortality of the policy holder due to reaching age $(x+s)$, ${}_sW_x^V(t) = \delta(s,t)_sV(t)$ is the interest earned on the amount of the reserve ${}_sV(t)$, and $b(s)$ is the rate of benefit payments. All these quantities are evaluated at time t.

In ${}_s\dot{V}_x(t)$, the attributions ${}_sM_x^V(t)$ and ${}_sW_x^V(t)$ are modulated by the changes in mortality and interest at t, and integrated over the remaining time-in-force of the policy $[s,\infty)$. This means that ${}_s\dot{V}_x(t)$ solely quantifies the time-t development in the actuarial assumptions when the reserve is evaluated at time s. This relationship is more clear by differentiating ${}_s\dot{V}_x(t)$ with respect to s:

$$
\frac{\partial_s \dot{V}_x(t)}{\partial s} = \rho(x+s,t)_sM_x^V(t) + \varphi(s,t)_sW_x^V(t) \tag{37}
$$

Equation (37) captures the change in the provision in two time dimensions: changes in the assumptions at time t, and at the development of the reserve at time s. Equation (37) expands the traditional Thiele's differential equation by disentangling the sources of variation in financial and longevity components via $\rho(x+s,t)$ and $\varphi(s,t)$. This equation together with Equation (34) has imporant applications in accounting, reserving, project development and risk management. Some of this applications and further developments are described in the following section.

It is worth noting that the rate of payment $b(s)$ does not appear in Equation (37) as in the payment processes assumed in this section, the benefits change solely over the development of the contract i.e. $[s,\infty)$, but do not depend on changes assumptions about the economic-demographic environment, i.e. changing at time t. This is reasonable as the benefit payments are adjusted over the length of the contract. However, this assumption can be relaxed and the case for using $b(s,t)$ can be done.

Furthermore, it could be that time t and s are the same. However in most cases this is not necessary true. In practice, a life insurance or a pension fund could perform continuous evaluations of the actuarial provision based on market-based yield curves where financial information is continuously updated, so then $t=s$ holds. The demographic statistics on the other hand are usually published with some lags and estimates of μ are done in broader time intervals, so the experience the development of the contract at time s hardly matched the same pace of update in demographic assumptions at time t. In addition, one can think about the usefulness of separating time t and s in scandinavian-type of life insurance products where it is common to distinguish between deterministic *first-order* actuarial basis, which are prudently set by the appointed actuary, the and *second-order* actuarial basis, which is usually based on models.

5.1.1 Risk management applications

The equations developed in Section 5 allow for the scrutiny of risk drivers in the development of reserves, which is in line with modern risk management approaches.

For instance, the current version of Solvency II regulations entail that the company should hold capital requirements for longevity risk corresponding to a shock of an immediate reduction of 20% in mortality rates at all ages. The effect of this shock can be easily translated in factor $\bar{\rho}=0.2$ and used to determine the impact in the reserve $\acute{V}_t$. Further scenario-based shocks of mortality and interest rates can be done to ${}_s\acute{V}_x(t) = {}_s\bar{\rho}_x(t)_sH_x^V(t) + {}_s\bar{\varphi}(t)_sD_x^V(t)$, where ${}_s\bar{\rho}_x(t)$ and $\bar{\varphi}$ are stressed. By knowing the duration and entropy, the practitioner can make use of this tool to assess the impact of insurance and financial risks in their life annuity portfolio.

Along the same lines, ${}_s\bar{\rho}_x(t)$ and ${}_s\bar{\varphi}(t)$ can be modeled as stochastic processes adapted to some filtration $\mathbf{G}=\{\mathcal{G}_t\}_{t\geq 0}$ such that ${}_s\rho_x^{\mathbf{G}}(t) = \mathbb{E}[{}_s\bar{\rho}_x(t)|\mathcal{G}_t]$ and ${}_s\varphi^{\mathbf{G}}(t) = \mathbb{E}[{}_s\bar{\varphi}(t)|\mathcal{G}_t]$. Thus, the source of randomness in these stochastic processes can be explicitly modeled using forecasting models to produce stochastic simulations allowing for the calculation of common risk metrics (e.g. VaR, Expected Shortfall, etc.). Allowing for stochastic simulations of ${}_s\rho_x^{\mathbf{G}}(t)$ and ${}_s\varphi^{\mathbf{G}}(t)$ is particularly useful in Asset-Liability modeling where the impact of stochastic shocks in the reserves is translated into corresponding management actions that mitigate associated risks.

The risk scrutiny can go further by analyzing the age-specific and term-specific stochastic variation in ${}_s\rho_x^{\mathbf{G}}(t)$ and ${}_s\varphi^{\mathbf{G}}(t)$ by adapting the formulas in Table 1. Such formulas allow for further segregation by sub-population, risk factors or any other cause of death.

6 Concluding remarks

In this article we developed differential equations to unravel the sources of change in life annuities and associated reserves. Such equations are neat, intuitive and easy to implement with real data. They represent an expansion of the mathematical toolkit used in pension and life insurance.

The main advantage of our differential equations is that they account for the simultaneous variation in mortality and interest rates. Indeed our main differential equation shows that changes over time in life annuities equal changes in mortality and interest rates modulated by their sensitivities to each of them (i.e. entropies and durations).

Actuaries and risk managers often monitor duration and entropies separately neglecting the effect that one indicator has on the other one. In reality, mortality and interest rates change simultaneously over time and durations-entropies vary accordingly as they are adapted to the new mortality-interest environment. While there is no concluding evidence about the correlation between mortality and interest rates, it is true they both affect the value of life annuity products and reserves. This simultaneous behaviour and change in sensitivities to mortality-interest is explicitly accounted in the differential equations developed here.

Another key advantage of the differential equations developed here is that they are model-free, and can be used as a general expressions of the time dynamics of life annuities. This entails a lack of assumptions about the development of demographic-economic environment where life annuities are evaluated. Instead, one key application of these equations is to monitor the development of sources of change in life annuity portfolios using real data. To show this, we use more than 100 years of data from the UK, where our equations are applied to obtain data-driven insights of the development of life annuity portfolios at different granular levels (e.g. age-term specific contributions and causes-of-death).

The main focus of this article is unraveling the time dynamics of life annuities and associated reserves. However, similar formulations to the ones introduced here can be developed for other life contingent products, where lump sums, expenses and more sophisticated payment processes are considered. The general framework introduced here easily allow for future developments.

Another possible development is in the expansion of our equations to a multi-state Markov framework where intensities other than mortality are modeled. This will naturally result in expression to state-specific sensitivities (i.e. entropies) where age-term variations in the state-space intensities can also accounted for.

We do forsee futher applications of the differential equations developed in this paper, both for the actuarial practice and for academic purposes. Whatever the applications of upcoming developments, they will establish their utility in the future.

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7 Figures

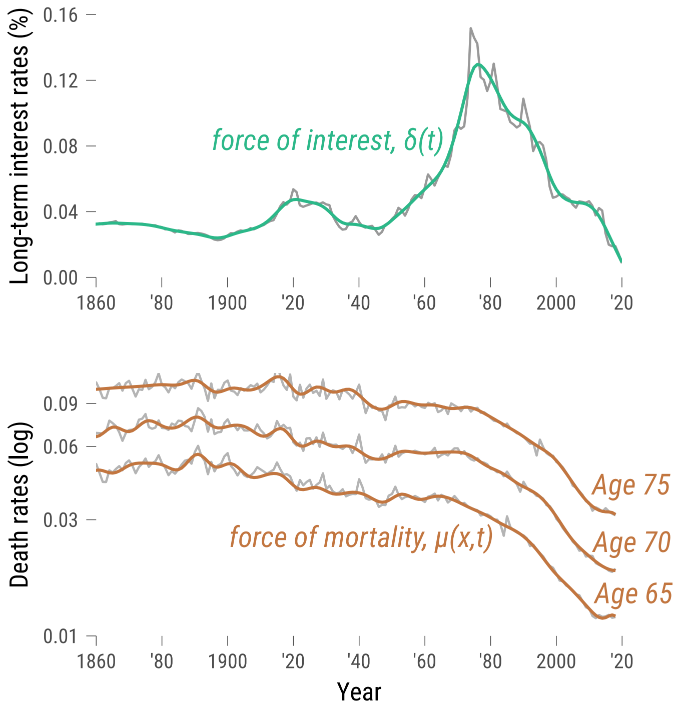

Figure 1: Trends in interest and mortality rates calculated at ages 65, 70 and 75. Males, 1860-2018.

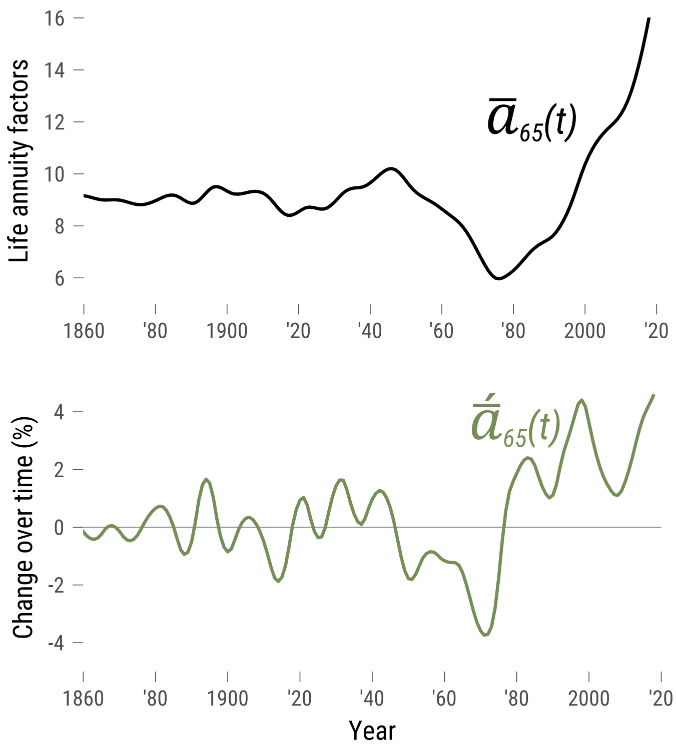

Figure 2: Trends in life annuity factors and relative change in $\bar{a}_x(t)$ calculated at age 65. Males, 1860-2018.

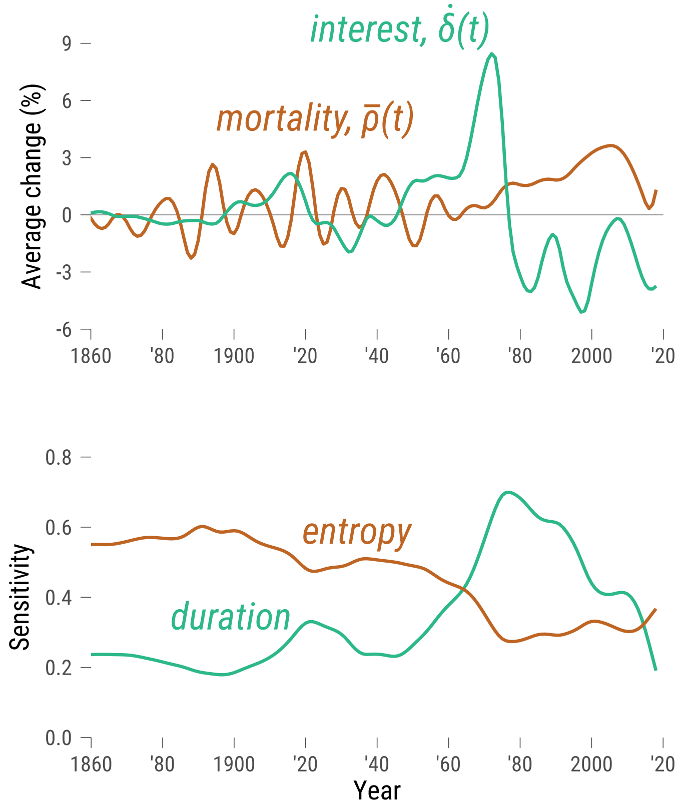

Figure 3: Upper panel shows the average mortality improvement ($\bar{\rho}(t)$), and change in interest rates ($\dot{\delta}(t)$). Lower panel depicts the entropy (H) and modified duration (D). Males at age 65, 1860-2018.

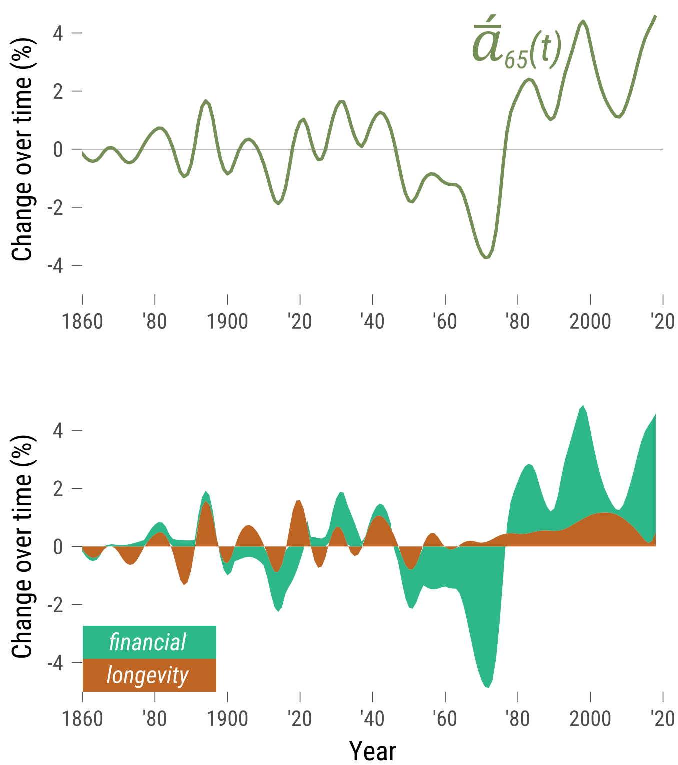

Figure 4: Decomposition of changes over time in life annuity factors calculated at ages 65. Males, 1860-2018.

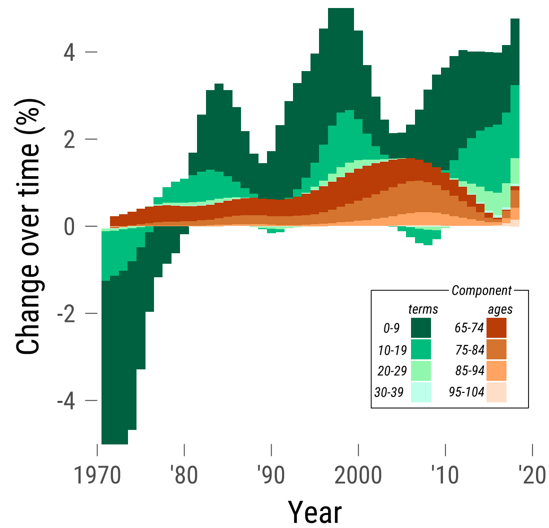

Figure 5: Age and term attributions to changes over time in life annuity factors calculated at ages 65. Males, 1970-2018.

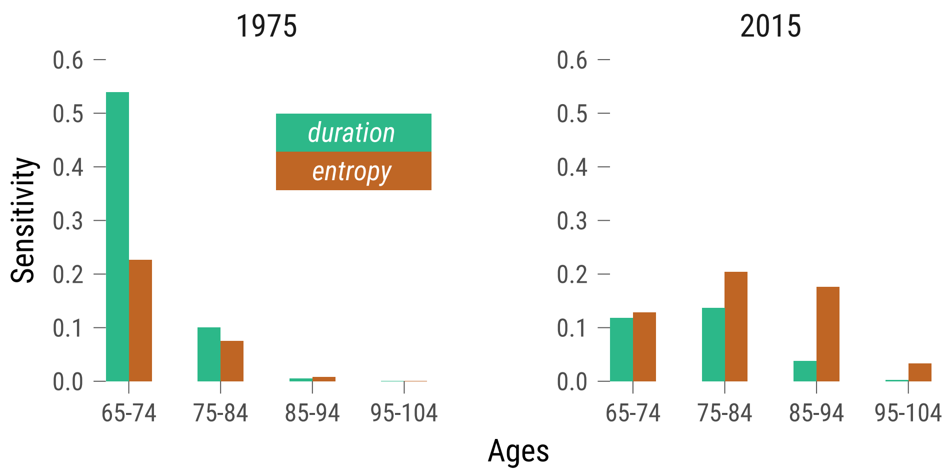

Figure 6: Entropy and duration by age groups. Males, 1975 and 2015.

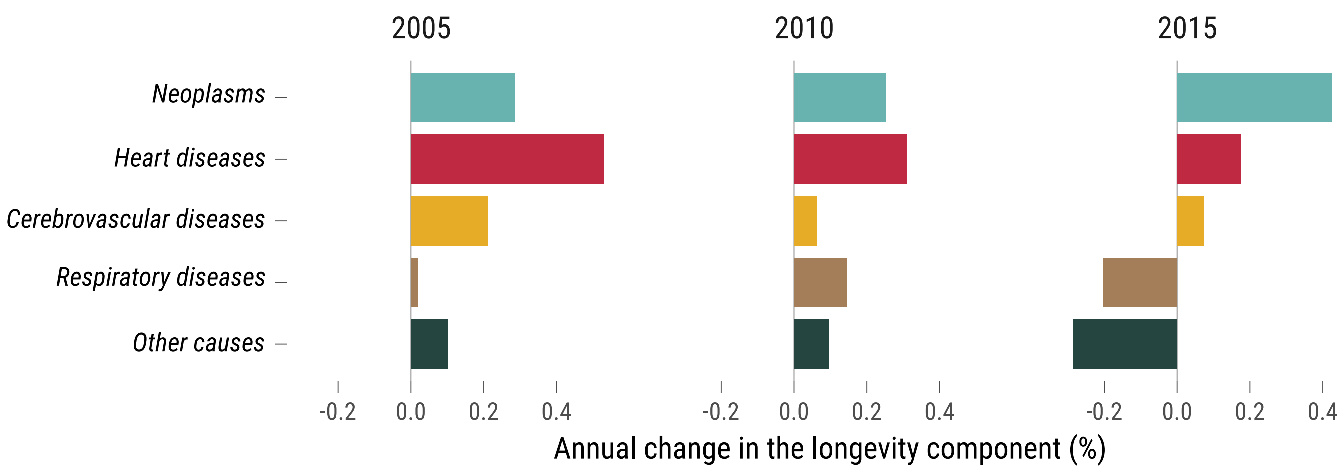

Figure 7: Causes of death contributions to the changes in the longevity component. Males, 2005, 2010 and 2015.

A Appendix

A.1 Entropy with constant changes in $\mu(x+s,t)$

To measure constant changes we make $\mu(s,t)+\gamma$, then

$$
\begin{aligned}
\bar{a}_x(t) &= \int_0^\infty v(s,t)e^{-\int_0^s [\mu(x+y,t)+\gamma]dy} ds \\
&= \int_0^\infty v(s,t)e^{-\int_0^s \mu(x+y,t)dy} e^{-\gamma s} ds \\
&= \int_0^\infty v(s,t)_s p_x(t) e^{-\gamma s} ds
\end{aligned} \tag{38}
$$

We expand $e^{-\gamma s}$ to $1 - \gamma s + \frac{\gamma^2}{2}s^2 + ...$, so that

$$
\bar{a}_x(t) = \int_0^\infty {}_s p_x(t) v(s,t)[1 - \gamma s + \frac{\gamma^2}{2}s^2 + ...]ds \tag{39}
$$

We take the derivative $\bar{a}_x(t)$ with respect to γ and evaluate $\gamma = 0$

$$
\begin{aligned}
H_x^c(t) &= \frac{1}{\bar{a}_x(t)} \frac{\partial \bar{a}_x(t)}{\partial \gamma}\bigg|_{\gamma=0} \\
&= -\frac{\int_0^\infty s\, {}_s p_x(t) v(s,t) ds}{\bar{a}_x(t)} \\
&= \frac{h_x^c(t)}{\bar{a}_x(t)},
\end{aligned} \tag{40}
$$

where $h_x^c(t) = -\int_0^\infty s\, {}_s p_x(t) v(s,t) ds$

A.2 Alternative expression for $H_x^p(t)$

$$
\begin{aligned}
H_x^p(t) &= -\frac{\int_0^\infty {}_s p_x(t) \ln[{}_s p_x(t)] e^{-\int_0^s \delta(y,t)dy} ds}{\int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)dy} ds} \\
&= \frac{\int_0^\infty {}_s p_x(t) v(s,t) \int_0^s \mu(x+y,t) dy\, ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t) \int_s^\infty {}_y p_x(t) v(y,t) dy\, ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t)_s p_x(t) v(s,t) \int_s^\infty \frac{{}_y p_x(t) v(y,t)}{{}_s p_x(t) v(s,t)} dy\, ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t)_s p_x(t) v(s,t) \bar{a}_{x+s}(t) ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \mu(x+s,t)_s | \bar{a}_x(t) ds}{\bar{a}_x(t)} \\
&= \frac{h_x^p(t)}{\bar{a}_x(t)},
\end{aligned} \tag{41}
$$

where $h_x^p(t) = \int_0^\infty \mu(x+s,t)_s | \bar{a}_x(t) ds$.

A.3 Duration with constant changes in $\delta(s,t)$

To measure constant changes we make $\delta(s,t)+\gamma$, then

$$\begin{aligned} \bar{a}_x(t) &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s [\delta(y,t)+\gamma]dy} ds \\ &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)dy} e^{-\gamma s} ds \\ &= \int_0^\infty {}_s p_x(t) v(s,t) e^{-\gamma s} ds \end{aligned} \tag{42}$$

We expand $e^{-\gamma s}$ to $1-\gamma s+\frac{\gamma^2}{2}s^2+...$, so that

$$\bar{a}_x(t) = \int_0^\infty {}_s p_x(t) v(s,t)[1-\gamma s+\frac{\gamma^2}{2}s^2+...]ds \tag{43}$$

We take the derivative $\bar{a}_x(t)$ with respect to γ and evaluate $\gamma=0$

$$\begin{aligned} D_x^c(t) &= -\frac{1}{\bar{a}_x(t)}\frac{\partial \bar{a}_x(t)}{\partial \gamma}\bigg|_{\gamma=0} \\ &= \frac{\int_0^\infty s\,{}_s p_x(t) v(s,t) ds}{\bar{a}_x(t)} \\ &= \frac{d_x^c(t)}{\bar{a}_x(t)}, \end{aligned} \tag{44}$$

where $d_x^c(t) = \int_0^\infty s\,{}_s p_x(t) v(s,t) ds$

A.4 Duration with proportional changes in $\delta(s,t)$

To calculate duration with proportional changes in $\delta(s,t)$, we assume that γ is a small number such that $\delta(s,t)(1+\gamma)$ and $v(s,t)=e^{-\int_0^s \delta(y,t)(1+\gamma)dy}$.

$$\begin{aligned} \bar{a}_x(t) &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)(1+\gamma)dy} ds \\ &= \int_0^\infty {}_s p_x(t) e^{-\int_0^s \delta(y,t)dy} e^{-\int_0^s \delta(y,t)\gamma dy} ds \\ &= \int_0^\infty {}_s p_x(t) v(s,t) v(s,t)^\gamma ds \end{aligned} \tag{45}$$

We expand $v(s,t)^\gamma$ to $1+\ln(v(s,t))\gamma+\ln(v(s,t))^2\frac{\gamma^2}{2}+...$, so that

$$\bar{a}_x(t) = \int_0^\infty {}_s p_x(t) s(y,t)[1+\ln(v(s,t))\gamma+\ln(v(s,t))^2\frac{\gamma^2}{2}+...]ds \tag{46}$$

To calculate the duration $D_x^p(t)$ we take the derivate of the expression above with respect to γ and make $\gamma = 0$

$$
\begin{aligned}
D_x^p(t) &= -\frac{1}{\bar{a}_x(t)} \frac{\partial \bar{a}_x(t)}{\partial \gamma}\bigg|_{\gamma=0} \\
&= -\frac{\int_0^\infty {}_s p_x(t) v(s,t) \ln(v(s,t)) ds}{\bar{a}_x(t)}
\end{aligned}
\tag{47}
$$

Equation 47 can be re-expressed as

$$
\begin{aligned}
D_x^p(t) &= -\frac{\int_0^\infty {}_s p_x(t) v(s,t) \ln(v(s,t)) ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty {}_s p_x(t) v(s,t) \int_0^s \delta(y,t) dy ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) \int_s^\infty {}_y p_x(t) v(y,t) dy ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) {}_s p_x(t) v(s,t) \bar{a}_{x+s}(t) ds}{\bar{a}_x(t)} \\
&= \frac{\int_0^\infty \delta(s,t) {}_s|\bar{a}_x(t) ds}{\bar{a}_x(t)} \\
&= \frac{d_x^p(t)}{\bar{a}_x(t)}.
\end{aligned}
\tag{48}
$$

where $d_x^p(t) = \int_0^\infty \delta(s,t) {}_s|\bar{a}_x(t) ds$.