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Source: *Econometrica*, Vol. 51, No. 5 (Sep., 1983), pp. 1485-1504
Published by: The Econometric Society
Stable URL: https://www.jstor.org/stable/1912286

Econometrica, Vol. 51, No. 5 (September, 1983)

A DIFFICULTY WITH THE OPTIMUM QUANTITY OF MONEY

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A general equilibrium model with money and finitely many immortal consumers is studied. Consumers hold money for self-insurance against random fluctuations. Money may earn interest. Equilibria exist if the rate of interest is sufficiently small. Equilibria may not exist if the rate of interest is too close to some consumer's rate of pure time preference. It follows that Pareto optimality may not be guaranteed by paying interest on money.

1. INTRODUCTION

IN THIS PAPER, I correct an error made in a previous paper [**2**] on Milton Friedman's theory of the optimum quantity of money (Friedman [**5**]). In my previous paper, I studied a rigorous version of Friedman's model. I proved that no equilibrium exists if the rate of interest paid on money equals consumers' rate of pure time preference. I claimed that equilibria exist for all interest rates less than the rate of time preference and I conjectured that these equilibria could be made arbitrarily close to Pareto optimal by making the interest rate be sufficiently close to the rate of pure time preference. I interpreted this conjecture as supporting Friedman's theory. It turns out this this conjecture is false and in fact that my existence theorem was incorrect.[2] Equilibria may not exist if the interest rate is too close to the rate of time preference. In the present paper, I give an example which demonstrates this last fact. Also, I correct my existence theorem and prove that monetary equilibria exist if the interest rate is sufficiently close to zero. The equilibrium existence theorem also requires that each consumer's pure rate of time preference be sufficiently close to zero and that his non-interest income be bounded away from zero.

Before proceeding, I explain Friedman's work and my past and present results in more detail. Friedman assumed that all consumers have the same rate of pure time preference and argued that a social optimum would be achieved if money earned interest at this rate. He reasoned that if the interest rate were less than the rate of time preference, individuals would make socially wasteful efforts to economize money balances.

The model of my previous paper follows fairly closely Friedman's literary description of a model. I follow him in making consumers immortal. My model differs from his mainly in that I exclude the transaction motive for holding money and include only the precautionary motive.

In my previous paper, I interpreted Friedman's social optimum to be a Pareto optimum and asked if paying interest on money would lead to Pareto optimality. Suppose that consumers all have the same rate of time preference ρ, as Friedman

[1] I am grateful to anonymous referees and to Martin Hellwig for helpful comments.

[2] I discovered the error myself, but it was also pointed out to me by Martin Hellwig.

assumed they do. Under normal circumstances, a monetary equilibrium is not Pareto optimal if the interest rate r is less than ρ, for consumers use money to insure themselves against fluctuations in their income and needs, and it is costly to hold enough to achieve perfect insurance. It is natural to conclude that if r equaled ρ, consumers would hold enough money to insure themselves perfectly. However, it turns out that because consumers are immortal, they would need an infinite amount of money to insure themselves perfectly if their needs and endowments were truly random. For this reason, there exists no monetary equilibrium when $r = \rho$, except in special cases where fluctuations in consumers' income and expenditure are cyclic.

In my previous paper, I proved this last fact and went on to conjecture that if r were sufficiently close to ρ, consumers would accumulate enough money so as to be nearly perfectly self-insuring and the monetary equilibrium would be nearly Pareto optimal. That is, money could substitute for complete Arrow–Debreu markets nearly perfectly.

This idea is incorrect. An example given in the present paper shows that equilibria do not necessarily exist if the interest rate is too close to ρ.

It is not hard to understand intuitively why equilibria may fail to exist when the interest rate is too close to the rate of pure time preference. If the interest rate is close to the rate of time preference, consumers find it advantageous to accumulate very large money balances for the purposes of self-insurance. This desire for money balances forces the prices of goods to be very low. Low prices in turn mean that consumers' non-interest income is low. But the consumers face a fixed expenditure each period which does not depend on the price level. This expenditure is a lump-sum tax paid to the government to finance the interest payments on money. If consumers' non-interest income is low, they must hold large money balances simply in order to earn enough money to pay the tax. This need for money balances in effect reduces the balances available for self-insurance. If the interest rate is too high, the demand for real balances for self-insurance exceeds the quantity available no matter how low prices may be, and so equilibrium is impossible.

This existence problem can be contrasted with the so-called Hahn problem in monetary theory. (See Hahn [**6**].) The Hahn problem is to construct a general equilibrium model in which money has value. That problem is solved in this paper by including a precautionary motive for holding money. I assure that this motive is sufficiently strong by assuming that incomes fluctuate randomly, that consumers are risk-averse and that rates of pure time preference are small. The existence problem discussed here is that the precautionary demand for money may be insatiable. It may be impossible to satisfy it with any level of aggregate real money balances. I eliminate this insatiability problem by assuming that the interest rate is low and by assuming that non-interest incomes are bounded away from zero. The role of the latter assumption is explained in Section 3.

The insatiability problem is not new to the literature. Friedman assumed that the marginal usefulness of money became negative for large real balances (Friedman [**5**, pp. 17–18]). A similar assumption appears in Brock's work on the optimum quantity of money (Brock, [**3**, **4**]). Brock considers a one consumer optimal growth model in which money appears in the utility function. He points out if the utility for money were insatiable, then no equilibrium would exist if the interest rate equaled the rate of pure time preference (Brock [**3**, pp. 764–765; **4**, pp. 143–144]). I show that the satiation assumption would not be satisfied in a model with immortal consumers who have a precautionary motive for holding money.

It might be thought that the difficulty I find in Friedman's theory is an artifact of the immortality of consumers. However, in a model with mortal consumers, such as a model with overlapping generations, the theory of the optimum quantity of money encounters other difficulties. A mortal individual cannot achieve perfect self-insurance by using asset balances. Therefore, no model with overlapping generations, money, and uninsurable risk could have Pareto optimal equilibria.

Friedman never speaks of Pareto optimality. When he says that a real rate of interest is optimum, he seems to mean that it maximizes the utility of the average consumer. Of course, in the model I consider, there might exist an interest rate which was optimal in this sense or which maximized some other social welfare function. But it would be of little interest to know that such an optimal rate existed. In any model with policy instruments and a social welfare function, it is usually the case that the instruments have optimal levels. The optimal interest rate would not necessarily be close to the rate of pure time preference, even in an overlapping generations model.

One could, of course, eliminate the problem of insatiability simply by assuming that money has some negative attributes which would discourage people from holding more than a certain amount of it. Friedman suggests that people with lots of money would need to hire guards to protect it. It is hard for me to imagine that money in a bank account would require such protection or would have any negative properties at all. But even if one were to accept such an assumption, it is not clear that it would save Friedman's theory. The negative attributes of money would indeed inhibit people from accumulating money indefinitely. But for this very reason, no equilibrium at any interest rate would be Pareto optimal. No consumer would achieve perfect self-insurance. Also, it is not clear that the interest rate which maximized average consumer utility would bear any definite relation to the rate of pure time preference.

One can see this difficulty by introducing uncertainty into a deterministic model expressing Friedman's story. The essence of his model and argument may be paraphrased as follows. Let the one period utility function of a consumer be $u(x_t) + v(M_t)$, where x_t is consumption during period t and M_t is money held at the end of period t. Suppose that x_t is a number. The first order conditions of the consumer's maximization problem are

$$\frac{du(x_t)}{dx} - \frac{dv(M_t)}{dM} = (1 + r_t)(1 + \rho)^{-1}\frac{du(x_{t+1})}{dx},$$

for all t, where r_t is the real interest rate on money and ρ is the rate of pure time preference. Suppose that $r_t = \rho$ and that the consumer is in a steady state with $x_t = x_{t+1}$. Then the above equation becomes $dv(M_t)/dM = 0$. This equation determines M_t if one assumes as Friedman did that $dv(M)/dM$ is strictly decreasing and crosses zero continuously. Since money can be produced at no social cost, it makes sense to set M at the level at which $dv(M)/dM = 0$ and so to set r_t equal to ρ.

If x_{t+1} were a random variable, as in the model of this paper, then the above equation would become

$$\frac{du(x_t)}{dx} - \frac{dv(M_t)}{dM} = (1 + r_t)(1 + \rho)^{-1} E_t\left(\frac{du(x_{t+1})}{dx} \right),$$

where E_t is the expectation conditional on information held at time t. This equation does not yield any immediate conclusion about the appropriate value of r_t. One might try to develop some intuition by supposing that r_t equaled ρ and that $E_t x_{t+1} = x_t$. If du/dx were convex, $E_t(du(x_{t+1})/dx)$ would exceed $du(x_t)/dx$, which means that $dv(M_t)/dM$ would be negative and not zero, as it should be.

Considerations of this sort give me the hunch that if $dv(M)/dM$ obeyed Friedman's assumptions, then the value of r_t which maximized average consumer utility would exceed ρ. However, I have not been able to verify this hunch.

The organization of the paper is as follows. In the next section, I describe the model. In Section 3, I state the correct equilibrium existence theorem, which asserts that equilibria exist if the interest rate is zero or nearly zero. In Section 4, I give the example which proves that equilibria do not exist if the interest rate is too close to consumers' pure rate of time preference. Sections 5 and 6 prove the existence theorem. Some of the technical results proved in Section 5 are used in Section 4. Section 6 is the only section that would be difficult to follow without having my previous paper in hand.

There have recently been two other papers which formalize the theory of the optimum quantity of money. Benhabib and Bull [1] consider a model with a transactions demand for money and no precautionary demand. In their model, the optimum quantity theory works. Hellwig [7] considers a model very similar to the model of this paper except that goods as well as money may serve as a store of value. He turns up the same difficulty with the optimum quantity theory as I do here. He emphasizes that randomly varying taxes may overcome this difficulty.

2. THE MODEL

I here review the model of my previous paper. The model is of a pure trade economy in which endowments and utility functions fluctuate. There are finitely many immortal consumers. Money is used to even out fluctuations in income and needs.

All randomness is generated by a single exogenous stochastic process

$\{s_t\}_{t=-\infty}^{\infty}$, which is an ergodic stationary Markov chain with no transitory states. It is assumed that the s_t are distributed according to the unique stationary distribution of the chain. All consumers observe the process s_t, so that they have completely symmetric information.

In dealing with the process $\{s_t\}$, I use notation different from that in my previous paper. An economy of notation is achieved by writing functions of the variables $(\ldots, s_{t-1}, s_t)$ as functions of the entire infinite sequence $\mathbf{s} = (\ldots, s_t, s_{t+1}, \ldots)$ which are measurable with respect to s_n, for $n \leq t$. More formally, s_t varies over a finite set S, and $\Sigma = \prod_{t=-\infty}^{\infty} S$.[3] An element of Σ is denoted by $\boldsymbol{s}$, and s_t is the tth component of $\boldsymbol{s}$. The probability associated with the process s_t is thought of as defined on the complete σ-field in Σ generated by the random variables s_t, for all t. $\mathscr{S}_t$ denotes the complete σ-field in Σ generated by the random variables s_n, for $n \leqq t$.

There are I consumers and L goods. The *endowment* of consumer i, for $i = 1, \ldots, I$, is determined by the function $\omega_i : S \rightarrow R_+^L$, where R_+^L is the set of nonnegative vectors in L-dimensional Euclidean space. The endowment vector in period t is $\omega_i(s_t)$. The *utility* of consumer i is determined by a function $u_i : R_+^L \times S \rightarrow R$. The utility derived from bundle x consumed in period t is $u_i(x, s_t)$.

A *consumption program* is a sequence of functions $\boldsymbol{x} = (x_1, x_2, \ldots)$, where $x_t : \Sigma \rightarrow R_+^L$ is measurable with respect to $\mathscr{S}_t$, for all t. I assume that the x_t are uniformly bounded.

Consumer i discounts future utility by the factor δ_i, where $0 < \delta_i < 1$, so that he assigns to the program $\boldsymbol{x}$ the utility $U_i(\boldsymbol{x}) \equiv E\sum_{t=1}^{\infty} \delta_i^{t-1} u_i(x_t(\boldsymbol{s}), s_t)$, where E denotes expectation. His rate of pure time preference is $\delta_i^{-1} - 1$. I do not assume that all consumers have the same rate of time preference.

An *allocation* for the economy is of the form $(\boldsymbol{x}_i)_{i=1}^{I}$, where each $\boldsymbol{x}_i$ is a consumption program. The allocation is *feasible* if $\sum_i x_{it}(\boldsymbol{s}) = \sum_i \omega_i(s_t)$ with probability one.

A feasible allocation $(\boldsymbol{x}_i)$ is said to be *Pareto optimal* if there exists no other feasible allocation $(\boldsymbol{y}_i)$ such that $U_i(\boldsymbol{y}_i) \geqq U_i(\boldsymbol{x}_i)$, for all i, with strict inequality for some i.

A *price system* is a sequence of functions $p = (p_1, p_2, \ldots)$, where $p_t : \Sigma \rightarrow R_+^L$ is measurable with respect to $\mathscr{S}_t$.

Every consumer starts with a *money* balance $M_{i0} : \Sigma \rightarrow [0, \infty]$, which is measurable with respect to $\mathscr{S}_0$. $M_{i0}(\boldsymbol{s})$ is the balance held at the end of period zero. The M_{i0} are part of the data of the model. I assume that $\sum_i M_{i0}(\boldsymbol{s}) = 1$. That is, the initial stock of money is one. Money held from the end of one period to the beginning of the next earns interest at the rate $r \geqq 0$. The number r is not market determined but is one of the givens of the model. The interest is paid by the government and is financed by a constant lump-sum tax. The tax paid each period by consumer i is $r\tau_i$, where $\sum_j \tau_j = 1$ and $\tau_j > 0$, all j. Notice that the total tax collections and interest payments are equal, so that the stock of money is

[3] In my previous paper, the set S was denoted by A.

always one. The money balance of consumer i at the end of period t is $M_{it}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{s}) = (1+r)^t M_{i0}(\boldsymbol{s}) + \sum_{n=1}^{t}(1+r)^{t-n}[p_n(\boldsymbol{s})\cdot(\omega_i(s_n) - x_n(\boldsymbol{s})) - r\tau_i]$ when the price system is $\boldsymbol{p}$ and he uses consumption program $\boldsymbol{x}$.

The *budget set* of consumer i is $\beta_i(\boldsymbol{p}) = \{\boldsymbol{x} \mid \boldsymbol{x}$ is a consumption program and $M_{it}(\boldsymbol{p},\boldsymbol{x},\boldsymbol{s}) \geqq 0$ almost surely, for all $t\}$. The *demand* set of consumer i is $\xi_i(p) = \{\boldsymbol{x} \in \beta_i(\boldsymbol{p}) \mid U_i(\boldsymbol{x}) \geqq U_i(\boldsymbol{y})$, for all $\boldsymbol{y} \in \beta_i(\boldsymbol{p})\}$.

A *monetary equilibrium* consists of $((\boldsymbol{x}_i), \boldsymbol{p})$ such that (i) $(\boldsymbol{x}_i)$ is a feasible allocation, (ii) $\boldsymbol{x}_i \in \xi_i(\boldsymbol{p})$, for all i, (iii) $\boldsymbol{p}$ is a price system, and (iv) there exists $b > 0$ such that $b < p_{tk}(\boldsymbol{s}) < b^{-1}$ with probability one, for all t and k. The last condition says that the price level is bounded over time and bounded away from zero. It follows that the value of money is bounded and bounded away from zero, so that the long-run average real interest rate is r.

3. EXISTENCE THEOREM

In this section, I state a correct version of the existence theorem for monetary equilibrium which I gave in my previous paper. First of all, I list the assumptions.

ASSUMPTION 3.1: $\{s_n\}$ is a stationary Markov chain.

ASSUMPTION 3.2: $\{s_n\}$ is ergodic and has no transient states.

That is, for any states s and s', the probability that $s_n = s'$ given that $s_0 = s$ is positive for some $n > 0$.

ASSUMPTION 3.3: $\omega_i(s) \neq 0$, for every i and s.

ASSUMPTION 3.4: $\sum_i \omega_{ik}(s) \geqq I$, for every s and k.

ASSUMPTION 3.5: $u_i(\cdot, s)$ is everywhere twice continuously differentiable, for all i and s.[4]

ASSUMPTION 3.6: The matrix $(\partial^2 u_i(x,s)/\partial x_j \partial x_k)$ is negative definite and $\partial u_i(x,s)/\partial x_k > 0$, for all i, s, and k and for every $x \in R_+^L$.

ASSUMPTION 3.7: M_{i0} is deterministic and is a positive number, for all i. Also $\sum M_{i0} = 1$, and $\tau_i = M_{i0}$, for all i.

ASSUMPTION 3.8: There is γ such that $0 < \gamma < 1$ and prob$[\omega_{ik}(s_2) \leqq \gamma$, for $k = 1, \ldots, L \mid s_1] > 0$, for every $s_1 \in S$.

Before stating the last assumption, I introduce some key bounds. Let $\bar{\omega} \in R_+^L$

[4] According to the standard mathematical definition of differentiability, Assumption 3.5 implies that all the first and second partial derivatives of $u_i(x,s)$ are finite for x on the boundary of R_t^L.

be such that $\sum_i \omega_{ik}(s) < \bar{\omega}_k$, for all k and s. By Assumptions (3.5) and (3.6), there exist $\underline{q}$ and $\bar{q}$ in R_+^L such that $0 < \underline{q}_k < \partial u_i(x,s)/\partial x_k < \bar{q}_k$, for all s and for all $x \in R_+^L$ such that $x \leqq \bar{\omega}$.

ASSUMPTION 3.9: There exists $Q \in R^L$ such that $Q_k > 0$, for all k, and the following are true. For all i and s, $(\partial u_i(x,s)/\partial x_k) > Q_k$ whenever x is such that $x_k \leqq \gamma \underline{q}_k^{-1} \sum_j \bar{q}_j$, for all k. Also for every i and s, $(\partial u_i(x,s)/\partial x_k) < Q_k$, if $x \in R_+^L$ is such that $x_j \leq 1$, for all j and $x_k = 1$.

The above assumptions are exactly the same as Assumptions 1–9 of my previous paper, except that I have strengthened the third assumption. (Before I assumed that for each i, $\omega_i(s) \neq 0$ for some s rather than for all s.) Another change made from the previous paper is to denote the tax paid by consumer i as $r\tau_i$ rather than τ_i. The existence theorem is as follows.

THEOREM 3.10: *There exists* $\underline{\delta} < 1$ *such that if* $\underline{\delta} \leqq \delta_i < 1$, *for all i, then there exists* $\bar{r}$ *such that* $0 < \bar{r} < \min_i(\delta_i^{-1} - 1)$ *and a monetary equilibrium exists for every r such that* $0 \leqq r \leqq \bar{r}$.

Key assumptions are (3.3), (3.8), and (3.9). Assumptions (3.8) and (3.9) guarantee that there is enough fluctuation in incomes relative to needs so that consumers will want to arrange insurance somehow. The $\bar{\delta}$ in the theorem guarantees that consumers are sufficiently interested in future utility to want to use money for self-insurance. Assumption (3.3) guarantees that consumer incomes are bounded away from zero. This lower bound together with the bound $\bar{r}$ avoid the insatiability problem brought out by the example of the next section.

If there were no lower bound on income, then there might exist no equilibrium at any positive interest rate. In order to understand why this is so, suppose that the income of no consumer were bounded away from zero. If the interest rate were positive, the taxes $r\tau_i$ would be positive. The income of any consumer i would sometimes fall below $r\tau_i$. In fact, his income might stay below $r\tau_i$ for an arbitrarily large number of periods. The only way he could be sure to have nonnegative balances at all times would be to hold balances so large that the interest income earned on them would cover the taxes. Hence, no consumer i would ever hold balances less than τ_i. These minimum balances add up to the total money supply, since the taxes are just large enough to pay the interest on the stock of money. It follows that no money would be left over for precautionary balances and the demand for such balances could never be satisfied.

4. AN EXAMPLE

In this section, I give an example in which all consumers have the same rate of pure time preference and no monetary equilibrium exists if the interest rate is too close to this rate of time preference. Also, no monetary equilibrium is Pareto optimal.

THE EXAMPLE: There is one kind of consumption good and there are two consumers, indexed by 1 and 2. The random variables, s_t, are independently and identically distributed. s_t takes on two values, a and b, each with probability 1/2. The utility function of each consumer is $u(x) = \log(x + 1)$. Notice that utility is deterministic. The rate of pure time preference of each consumer is 0.1. The initial endowment of consumer i at time t is $\omega_i(s_t)$, where ω_i is defined as follows. $\omega_1(a) = \omega_2(b) = 1/4$. $\omega_1(b) = \omega_2(a) = 1\frac{3}{4}$. Notice that the total supply of the good is constant, since $\omega_1(s) + \omega_2(s) = 2$, for all s. There is one unit of money in the economy. The tax paid by each consumer each period is $r/2$, where r is the interest rate. I do not specify the initial money balances, M_{i0}. They may be anything. They may even be random.

It is easy to check that the above example satisfies the assumptions of Theorem 3.10, so that a monetary equilibrium exists if the interest rate r is zero or close enough to zero.

It is possible to associate with any monetary equilibrium marginal utilities of money for the individual consumers. The marginal utility of money for consumer i is a sequence of functions $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \ldots)$, where $\lambda_{it} : \Sigma \to (0, \infty)$ is measurable with respect to $\mathscr{S}_t$. I show in Section 5 that these marginal utilities exist under assumptions satisfied by the above example.

Throughout the rest of this section, $((\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{p})$ denotes a monetary equilibrium with interest rate r, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ denote the associated marginal utilities of money, and $M_{it}(\boldsymbol{s})$ denotes the equilibrium money balance of consumer i at the end of period t.

It follows from (5.2) and (5.3) below that the following are true.

(4.1) For all t, for $i = 1, 2$ and for almost every $\boldsymbol{s}$,

$$\lambda_{it}(\boldsymbol{s}) p_t(\boldsymbol{s}) \geqq \frac{du_i(x_{it}(\boldsymbol{s}), s_t)}{dx} = (x_{it}(\boldsymbol{s}) + 1)^{-1},$$

with equality if $x_{it}(\boldsymbol{s}) > 0$.

(4.2) For all t, for $i = 1, 2$ and for almost every $\boldsymbol{s}$,

$$\lambda_{it}(\boldsymbol{s}) \geqq (1 + r)(1.1)^{-1} E[\lambda_{i,t+1} | \mathscr{S}_t](\boldsymbol{s}),$$

with equality if $M_{it}(\boldsymbol{s}) > \underline{M}_{it}(\boldsymbol{s})$,

where $\underline{M}_{it}(\boldsymbol{s})$ denotes the minimum balance of consumer i at the end of period t. $\underline{M}_{it}(\boldsymbol{s})$ is the smallest balance consumer i can hold and be sure that he will never be obliged to hold negative balances. The $\underline{M}_{it}$ are defined in Section 5.

I now show that no monetary equilibrium is Pareto optimal if $0 \leq r < 0.1$. Suppose that $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is Pareto optimal and that $0 \leq r < 0.1$. Since the utility functions are concave and the set of feasible allocations is convex, it follows that

$(\boldsymbol{x}_1, \boldsymbol{x}_2)$ maximizes a social welfare function of the form $a_1 U(\bar{\boldsymbol{x}}_1) + a_2 U(\bar{\boldsymbol{x}}_2)$, where $(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2)$ varies over feasible allocations, where $a_1 > 0$ and $a_2 > 0$, and where $U(\boldsymbol{x}) = E\sum_{t=1}^{\infty}(1.1)^{1-t}\log(x_t(\boldsymbol{s}) + 1)$. Since $\omega_1(s_t) + \omega_2(s_t)$ does not depend on s_t, there is $P > 0$ such that

$$a_i \frac{d}{dx}\log(x_{it}(\boldsymbol{s}) + 1) \leqq P,$$

with equality if $x_{it}(\boldsymbol{s}) > 0$, for all t and i and with probability one. Hence, $x_{1t}(\boldsymbol{s})$ and $x_{2t}(\boldsymbol{s})$ do not depend on $\boldsymbol{s}$ and t and may be written as x_1 and x_2, respectively. Since $x_1 + x_2 = 2$, either x_1 or x_2 is at least one. Suppose that $x_1 \geq 1$.

I now show that for some t,

$$\text{(4.3)} \qquad \lambda_{1t}(\boldsymbol{s}) > (1 + r)(1.1)^{-1}E[\lambda_{1,t+1} \mid \mathscr{S}_t](\boldsymbol{s}),$$

with positive probability. If (4.3) were false, then by (4.2),

$$\lambda_{1t}(\boldsymbol{s}) = (1 + r)(1.1)^{-1}E[\lambda_{1,t+1} \mid \mathscr{S}_t](\boldsymbol{s})$$

almost surely, for all t, so that

$$\lim_{t\to\infty} E\lambda_{1t} = \lim_{t\to\infty} (1.1)^{t-1}(1 + r)^{1-t}E\lambda_{11} = \infty.$$

This is impossible, for the λ_{1t} are bounded from above. In order to see that the λ_{1t} are bounded, recall that by the definition of a monetary equilibrium, there exists $\underline{p} > 0$ such that $p_t(\boldsymbol{s}) \geqq \underline{p}$ with probability one, for all t. Also, since $x_1 > 0$, (4.1) holds with equality. Hence $\lambda_{1t}(s) = (p_t(\boldsymbol{s}))^{-1}(x_1 + 1)^{-1} \leqq \underline{p}^{-1}$. This completes the proof of (4.3).

I now show that (4.3) leads to a contradiction. By (4.3), there is t and a set $A \in \mathscr{S}_t$ such that $\text{Prob}\,A > 0$ and $\lambda_{1t}(\boldsymbol{s}) > (1 + r)(1.1)^{-1}E[\lambda_{1,t+1}(\boldsymbol{s}) \mid \mathscr{S}_t](\boldsymbol{s})$ if $\boldsymbol{s} \in A$. By (4.2), $M_{1t}(\boldsymbol{s}) = \underline{M}_{1t}(\boldsymbol{s})$, for $\boldsymbol{s} \in A$. The probability that $\boldsymbol{s} \in A$ and $s_{t+1} = a$ is $\frac{1}{2}\text{Prob}\,A > 0$. But if $\boldsymbol{s} \in A$ and $s_{t+1} = a$, then

$$\begin{aligned} M_{1,t+1}(\boldsymbol{s}) &= (1 + r)\underline{M}_{1t}(\boldsymbol{s}) - \frac{r}{2} + p_{t+1}(\boldsymbol{s}) \cdot \left(\frac{1}{4} - x_1\right) \\ &\leqq (1 + r)\underline{M}_{1t}(\boldsymbol{s}) - \frac{r}{2} - \frac{3}{4}\, p_{t+1}(\boldsymbol{s}) \\ &< (1 + r)\underline{M}_{1t}(\boldsymbol{s}) - \frac{r}{2} = M_{1,t+1}(\boldsymbol{s}), \end{aligned}$$

which is impossible. This proves that the allocation $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is not Pareto optimal.

I next show that there is $\underline{r}$ such that $0 < \underline{r} < 0.1$ and no monetary equilibrium exists with interest rate $r > \underline{r}$. This assertion together with what has just been proved will demonstrate that no monetary equilibrium is Pareto optimal. It will also prove that no monetary equilibrium exists with an interest rate too close to the pure rate of time preference 0.1.

First of all, I prove that

(4.4) there exists no monetary equilibrium if $r > 0.1$.

Inequality (4.2) implies that

$$E\lambda_{it} \leqq (1.1)(1+r)^{-1}E\lambda_{i,t-1} \leqq \cdots \leqq (1.1)^{t-1}(1+r)^{1-t}E\lambda_{il}.$$

Hence, $\lim_{t\to\infty}E\lambda_{it} = 0$ if $r > 0.1$. This contradicts the fact that the λ_{it} are bounded away from zero. In order to see that the λ_{it} are bounded away from zero, recall that by the definition of a monetary equilibrium there exists $\bar{p} > 0$ such that $p_t(s) \leq \bar{p}$ with probability one, for all t. Also, since $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is feasible, $x_{i2}(\boldsymbol{s}) \leqq 2$ with probability one for $i = 1,2$ and for all t. Hence by (4.1),

(4.5) $\lambda_{it}(\boldsymbol{s}) \geqq (3\bar{p})^{-1}$ with probability one, for $i = 1,2$ and for all t.

I now prove that there exists $\underline{r}$ such that $0 < \underline{r} < 0.1$ and no monetary equilibrium exists if $0.1 \geqq r \geqq \underline{r}$.

The idea of the argument is as follows. By inequality (4.5), consumers' marginal utilities of money are bounded away from zero. Some consumer must at some time have a marginal utility of money very near to the lowest level it ever reaches. If this is so and if r is close to 0.1, then it follows that the consumer's marginal utility of money must remain near its minimum level for a long time afterward. (This assertion follows from inequality (4.2).) If the consumer's marginal utility of money is nearly constant over a long period of time, then he does not protect himself against a run of bad luck by increasing his marginal utility of money and so buying less. (Bad luck occurs if the consumer's endowment is only $1/4$.) In fact, I show that the consumer will with positive probability eventually hold negative money balances. This contradicts the definition of a monetary equilibrium.

I now turn to the formal proof. Let $\underline{\lambda} = \min_i \inf_t(\operatorname{ess\,inf} \lambda_{it})$, where $\operatorname{ess\,inf} \lambda_{it} = \sup\{c > 0 \mid \operatorname{prob}[\lambda_{it}(\boldsymbol{s}) < c] = 0\}$. By 4.5, $\underline{\lambda} > 0$.

I first show that

(4.6) $p_t(\boldsymbol{s}) \leqq (2\underline{\lambda})^{-1}$ almost surely, for all t.

Since $x_{1t}(\boldsymbol{s}) + x_{2t}(\boldsymbol{s}) = 2$ almost surely, for all t, it follows that for each t and almost every $\boldsymbol{s}$, there exists i such that $x_{it}(\boldsymbol{s}) \geqq 1$. By inequality (4.1), for this i one has $\underline{\lambda}p_t(\boldsymbol{s}) \leqq \lambda_{it}p_t(\boldsymbol{s}) = (1 + x_{it}(\boldsymbol{s}))^{-1} \leqq 1/2$. This proves (4.6).

I next prove that

(4.7) $M_{it}(\boldsymbol{s}) \geqq 1/2 - (8r\underline{\lambda})^{-1}$ almost surely, for $i = 1,2$ for all t.

If the state s_{t+1} is bad for consumer i, then his endowment is $\omega_{i,t+1}(\boldsymbol{s}) = 1/4$, so that by inequality (4.6) he can earn at most $(8\underline{\lambda})^{-1}$ by selling his endowment. Hence, if s_{t+1} is bad for consumer i, then, $M_{i,t+1}(\boldsymbol{s}) \leqq (1+r)M_{i,t}(\boldsymbol{s}) - r/2 + (8\underline{\lambda})^{-1}$. But then, $M_{i,t+1}(\boldsymbol{s}) < M_{i,t}(\boldsymbol{s})$, unless $M_{it}(\boldsymbol{s}) \geqq 1/2 - (8r\underline{\lambda})^{-1}$. In fact, if $M_{it}(\boldsymbol{s}) < 1/2 - (8r\underline{\lambda})^{-1}$, then $M_{i,t+K}(\boldsymbol{s}) < 0$ if K is sufficiently large and if $s_{t+1}, \ldots, s_{t+k}$ are all bad for consumer i. But $s_{t+1}, \ldots, s_{t+K}$ may all be bad for i with positive probability. Since $M_{i,t+K}(\boldsymbol{s})$ is almost surely nonnegative, a contradiction occurs unless (4.7) is true.

I now choose an event and a time period for which the marginality utility of money of one of the consumers is very close to $\underline{\lambda}$. Choose t such that for $i = 1$ or 2, $\text{prob}\{s \mid \lambda_{it}(s) < \underline{\lambda}(1+\epsilon)\} > 0$, where $\epsilon > 0$ will be determined below. Without loss of generality, I may assume that $i = 1$ and $t = 1$. Let

$$(4.8) \qquad \Sigma_1 = \{s \mid \lambda_{11}(s) < \underline{\lambda}(1+\epsilon)\}.$$

By assumption $\text{prob}\,\Sigma_1 > 0$.

Let δ be a small positive number. δ will be determined below. I now show that the ϵ of (4.8) and $\underline{r}$ may be chosen so that

$$(4.9) \qquad \lambda_{it}(s) \leqq (1+\delta)\underline{\lambda} \qquad \text{for almost every} \quad s \in \Sigma_1 \quad \text{if} \quad 1 \leqq t \leqq 41.$$

Inequalities (4.2) and (4.8) together imply that

$$(4.10) \qquad \underline{\lambda}(1+\epsilon) \geqq \lambda_{11}(s) \geqq \left(\frac{1+r}{1.1}\right)^{t-1}\left[(1/2)^{t-1}\lambda_{1t}(s) + \left(1-(1/2)^{t-1}\right)\underline{\lambda}\right],$$

for almost every $s \in \Sigma_1$ and for $t > 1$. The second inequality above follows from the fact that conditional on the history $(\ldots, s_0, s_1)$ there are 2^t possible values of λ_{it}, each occurring with equal probability and all at least as large as $\underline{\lambda}$.

A rearrangement of (4.10) yields

$$\lambda_{1t}(s) \leqq \underline{\lambda} + \underline{\lambda}2^{t-1}\left[\left(\frac{1.1}{1+r}\right)^{t-1}(1+\epsilon) - 1\right]$$

$$\leqq \underline{\lambda} + \underline{\lambda}2^{40}\left[\left(\frac{1.1}{1+r}\right)^{40}(1+\epsilon) - 1\right], \qquad \text{if} \quad 1 < t \leqq 41.$$

Clearly, for given δ, the right hand side is less than $(1+\delta)\underline{\lambda}$, provided that $\underline{r}$ is sufficiently close to 0.1 and ϵ is sufficiently small. This completes the proof of (4.9).

I now show that

$$(4.11) \qquad p_t(s)x_{1t}(s) > 3(8\underline{\lambda})^{-1},$$

for $t = 1, \ldots, 41$ and for almost every $s \in \Sigma_1$, provided that δ is sufficiently small and (4.9) is true.

In order to prove (4.11), I express the equilibrium consumption of the consumers and the price of the consumption good as functions of the marginal utilities of money. I drop the variables t and s, for the moment, so that x_i is the consumption of consumer i, λ_i is his marginal utility of money and p is the price of the consumption good, all at one moment of time and in one state of the world. These variables satisfy the following relations.

$$(x_1+1)^{-1} \leqq \lambda_1 p, \qquad \text{with equality if} \quad x_1 > 0,$$

$$(x_2+1)^{-1} \leqq \lambda_2 p, \qquad \text{with equality if} \quad x_2 > 0, \quad \text{and}$$

$$x_1 + x_2 = 2.$$

Solving these relations, I obtain that $px_1 = (2/3)\lambda_1^{-1}$, if $\lambda_1 \leqq (1/3)\lambda_2$ and $px_1 = (4\lambda_1\lambda_2)^{-1}(3\lambda_2 - \lambda_1)$, if $(1/3)\lambda_2 \leqq \lambda_1 \leqq 3\lambda_2$. px_1 is a nonincreasing function of λ_1 and a nondecreasing function of λ_2. Therefore, if $\lambda_1 \leqq (1+\delta)\underline{\lambda}$ and $\lambda_2 \geqq \underline{\lambda}$, it follows that

$$px_1 \geqslant \left(4(1+\delta)\underline{\lambda}^2\right)^{-1}(3\underline{\lambda} - (1+\delta)\underline{\lambda})$$

$$= (1/4)(1+\delta)^{-1}(2-\delta)\underline{\lambda}^{-1} > 3(8\underline{\lambda})^{-1},$$

provided that δ is sufficiently small. By the definition of $\underline{\lambda}$, $\lambda_{2t}(s) \geqq \underline{\lambda}$, for all t and almost every s. By (4.9), $\lambda_{1t}(s) \leqq (1+\delta)\underline{\lambda}$, for almost every $s \in \Sigma_1$ and for $t = 1, \ldots, 41$. Therefore (4.11) is true.

I now assume that δ is so small that (4.11) is true. Also, I assume that $\underline{r}$ is so close to 0.1 and ϵ is so small that (4.9) is true. I also assume that $\underline{r} > 1/20$. This determines $\underline{r}$ and ϵ. Notice that $\underline{r}$ does not depend on $\underline{\lambda}$ or on the particular equilibrium in any way. $\underline{r}$ is a true *a priori* bound.

I now derive a contradiction. Let $\Sigma_1' = \{s \in \Sigma_1 \mid s_2 = \cdots = s_{41} = a\}$. Clearly, prob $\Sigma_1' > 0$. I will show that $M_{1,41}(s) < (1/2) - (8r\underline{\lambda})^{-1}$ if $s \in \Sigma_1'$. This contradicts (4.7).

By (4.7), $M_{2,1}(s) \geqq (1/2) - (8r\underline{\lambda})^{-1}$ almost surely, so that $M_{1,1}(s) \leqq 1/2 + (8r\underline{\lambda})^{-1}$ almost surely. If $s \in \Sigma_1'$ and $1 \leqq t \leqq 41$, then $\omega_{1t}(s) = 1/4$, so that by (4.6), $p_t(s)\cdot\omega_{1t}(s) \leq (8\underline{\lambda})^{-1}$ almost surely. Also by (4.11), $p_t(s)x_{1t}(s) \geqq 3(8\underline{\lambda})^{-1}$ almost surely. Hence,

$$M_{1,2}(s) = (1+r)M_{1,1}(s) - (1/2)r - p_1(s)x_1(s) + p_1(s)\omega_{11}(s)$$

$$\leqq (1+r)\left(1/2 + (8r\underline{\lambda})^{-1}\right) - (1/2)r - 3(8\underline{\lambda})^{-1} + (8\underline{\lambda})^{-1}$$

$$\leqq 1/2 + (8r\underline{\lambda})^{-1} - (8\underline{\lambda})^{-1} \qquad \text{almost surely.}$$

Continuing by induction on t, one obtains

$$M_{1,41}(s) \leqq 1/2 + (8r\underline{\lambda})^{-1} - 40(8\underline{\lambda})^{-1}$$

$$< 1/2 - (8r\underline{\lambda})^{-1} \qquad \text{almost surely for} \quad s \in \Sigma_1'.$$

The second inequality here follows from the fact that $r \geqq \underline{r} > 1/20$. Since inequality (4.7) has been contradicted, there exists no equilibrium if $0.1 \geqq r \geqq \underline{r}$.

5. THE MARGINAL UTILITIES OF MONEY

I here prove that in a monetary equilibrium consumers have well-defined marginal utilities of money and minimum money balances. I assume that Assumptions 3.1, 3.5, and 3.6 apply. Let $((\boldsymbol{x}_i), \boldsymbol{p})$ be a monetary equilibrium with interest rate r and suppose that $r < \delta_i^{-1} - 1$, for all i.

I first define T-period horizon marginal utilities λ_{it}^T. The definition is by induction, on T. For $T = 0$, I let

$$(5.1) \qquad \lambda_{it}^{0}(s) = \min\left\{ a \mid \frac{\partial u_i(x_{it}(s), s_t)}{\partial x_k} \leqq a p_{tk}(s), \text{ for } k = 1, \ldots, L \right\},$$

where $t = 1, 2, \ldots$.

By Assumptions 3.5 and 3.6, λ_{it}^{0} is well-defined. Given λ_{it}^{T}, let $\lambda_{it}^{T+1}(s) = \max\{\lambda_{it}^{0}(s), (1+r)(\delta_i)E[\lambda_{i,t+1}^{T} \mid \mathscr{S}_t](s)\}$. It should be clear that $\lambda_{it}^{T+1}(s) \geqq \lambda_{it}^{T}(s)$ almost surely, for all i, t, and T.

I now show that the $\lambda_{it}^{T}(s)$ are essentially bounded, uniformly in t and T. That is, there exists $b > 0$ such that $\lambda_{it}^{T}(s) \leqq b$ almost surely, for all t and T. By the definition of λ_{it}^{T} and since $(1+r)\delta_i < 1$, it is sufficient to show that the λ_{it}^{0} are essentially bounded. To see that this is so, observe that since the u_i are continuously differentiable and the feasible allocation (x_i) is essentially bounded, it follows that the $(\partial u_i(x_{it}(s), s_t)/\partial x_k)$ are essentially bounded. Also, by the definition of a monetary equilibrium, the prices $p_{tk}(s)$ are essentially bounded away from zero. These facts together imply that the λ_{it}^{0} are essentially bounded.

It now follows that the limits $\lambda_{it}(s) = \lim_{T\to\infty}\lambda_{it}^{T}(s)$ are well-defined and finite almost surely. The λ_{it} are the desired marginal utilities of money.

I now define the minimum money balances. I define T-period horizon minimum balances and let T go to infinity. Again, the definition is by induction on T. Let $\underline{M}_{it}^{0}(s) = 0$. Suppose that $\underline{M}_{it}^{T}(s)$ has been defined. Let

$$\underline{M}_{it}^{T+1}(s) = (1+r)^{-1}\max\left\{0, \tau_i r - \operatorname{ess\,inf}_t\left[p_{t+1}\cdot\omega_i - \underline{M}_{i,t+1}^{T}\right](s)\right\}.$$

I now define the symbol "$\operatorname{ess\,inf}_t$". The random variable following $\operatorname{ess\,inf}_t$ above is measurable with respect to $\mathscr{S}_{t+1}$. Let $f: \Sigma \to R$ be any random variable measurable with respect to $\mathscr{S}_{t+1}$. For each $a \in S$, let $f_a(s) = E[f \mid \mathscr{S}_t$ and $s_{t+1} = a](s)$. Let $\operatorname{ess\,inf}_t f(\bar{s}) = \min\{f_a(s) \mid a \in S, \text{prob}[s_{t+1} = a \mid s_t = \bar{s}_t] > 0\}$ and $s_n = \bar{s}_n$, $n \leq t\}$. Assumption 3.1 implies that $\operatorname{ess\,inf}_t f$ is well defined. Clearly, $f(s) \geqq (\operatorname{ess\,inf}_t f)(s)$ almost surely. Also, if $g: \Sigma \to R$ is $\mathscr{S}_t$-measurable and such that $f(s) \geqq g(s)$ almost surely, then $g(s) \leqq \operatorname{ess\,inf}_t f(s)$ almost surely.

If a consumer has at least τ_i units of money, he can avoid ever violating his budget constraint simply by never buying anything and using his income to pay his taxes. Therefore, $\underline{M}_{it}^{T}(s) \leqq \tau_i$ with probability one. It should be clear that $\underline{M}_{it}^{T+1}(s) \geqq \underline{M}_{it}^{T}(s)$ with probability one, for all T. Hence, the limits $\underline{M}_{it}(s) = \lim_{T\to\infty}\underline{M}_{it}^{T}(s)$ are well-defined. The $\underline{M}_{it}$ are the desired minimum balances.

It is not hard to see that the following are true.

$$(5.2) \qquad \frac{\partial u_i(x_{it}(s), s_t)}{\partial x_k} \leqq \lambda_{it}(s) p_{tk}(s)$$

with equality if $x_{itk}(s) > 0$, with probability one and for all i, k, and t.

$$(5.3) \qquad \lambda_{it}(s) \geqq (1+r)\delta_i E\left[\lambda_{i,t+1} \mid \mathscr{S}_t\right](s)$$

with equality if $M_{it}(p, x_i, s) > \underline{M}_{it}(s)$, with probability one and for all i and t.

If either of the above conditions were false, it would be possible to define a program in the consumer's budget set which would give him higher expected utility. This would contradict the definition of a monetary equilibrium.

6. PROOF OF EXISTENCE THEOREM

I here show how the proof of the existence theorem in my previous paper (Theorem 1) must be modified so as to give a correct proof of Theorem 3.10 stated above. In proving the existence theorem in my previous paper, I proceeded as follows. I truncated the economy, eliminating all periods after period T. I changed the model further by giving each consumer one unit of utility for each unit of money held at the end of period T. I then proved that the resulting finite horizon economy had a monetary equilibrium. The key steps were to prove that prices in these finite horizon equilibria were bounded and bounded away from zero, uniformly in T. I then allowed T to go to infinity and applied a Cantor diagonal argument to obtain a monetary equilibrium in the limit.

The error is in the proof that prices are bounded away from zero. This assertion is Lemma 8 of the previous paper. Before stating this lemma correctly, I recall some notation from that paper. By Assumption 3.7, the initial money stock of each consumer is fixed at the end of period zero and is independent of the history of $\{s_t\}$ up to time zero. Therefore, all allocations, prices and so on can be written as functions of the history of the process from period 1. Formally, a history is a finite sequence $a_1, \dots, a_t$ of elements of S such that prob$[s_1 = a_1, \dots, s_t = a_t] > 0$. I denote a history by $a_1, \dots, a_t$ in order to distinguish it from the finite sequence of random variables $s_1, \dots, s_t$.

The correct statement of Lemma 8 (of the previous paper) is as follows.

LEMMA 6.1: *Let $\delta_1, \dots, \delta_I$ be fixed and such that $0 < \delta_i < 1$, for all i. There exist $\bar{r} > 0$, $\underline{p} \in R^L_+$ and $\bar{\lambda} > 0$ such that $\underline{p}_k > 0$, for all k, and the following are true. Let $((x_i), p)$ be a T-period monetary equilibrium with interest rate r, where $0 \leqq r \leqq \bar{r}$. Let (λ_i) be the associated vector of marginal utilities of money. Then, $p_t(a_1, \dots, a_t) \geqq \underline{p}$ and $\lambda_{it}(a_1, \dots, a_t) \leqq \bar{\lambda}$, for all histories $a_1, \dots, a_t$ and for all t.*

Before proving this lemma, I recall some more notation and some results from the previous paper. $\alpha_{it}(a_1, \dots, a_t)$ denotes the marginal utility of expenditure of consumer i in period t, which is distinct from the marginal utility of money. α_{it} is the marginal utility λ^0_{it} defined in (5.1). That is,

$$\alpha_{it}(a_1, \dots, a_t) = \min\left\{ \alpha \geq 0 \,\middle|\, \frac{\partial u_i(x_{it}(a_1, \dots, a_t), a_t)}{\partial x_k} \right.$$
$$\left. \leqq \alpha p_{tk}(a_1, \dots, a_t), a \text{ for all } k \right\},$$

where $((x_i), p)$ is as in the lemma.

Let $\bar{\omega}$, $\underline{q}$, and $\bar{q}$ be the vectors defined just before Assumption 3.9. Since $0 \leq x_{it}(a_1, \ldots, a_t) \leqq \bar{\omega}$, it follows that

$$(6.2) \qquad \lambda_{it}(a_1, \ldots, a_t) p_{tk}(a_1, \ldots, a_t) \geqq \frac{\partial u_i(x_{it}(a_1, \ldots, a_t), a_t)}{\partial x_k} > \underline{q}_k,$$

for all i, k, and t and for all histories $a_1, \ldots, a_t$, where (λ_i) is as in the above lemma.

It is also true that

$$(6.3) \qquad p_t(a_1, \ldots, a_t) \leqq \max_i \alpha_{it}^{-1}(a_1, \ldots, a_t)\bar{q},$$

for all $a_1, \ldots, a_t$ and for all t. This statement follows from Lemma 1 of the previous paper.

I use the following result, which follows from Lemma 2 of the previous paper.

$$(6.4) \qquad \max_i \alpha_{it}(a_1, \ldots, a_t) < b \min_i \alpha_{it}(a_1, \ldots, a_t),$$

for all t and $a_1, \ldots, a_t$, where $b = \max_k \underline{q}_k^{-1} \bar{q}_k$.

I also use the minimum money balances, discussed in the previous section of this paper. $\underline{M}_{it}(a_1, \ldots, a_t)$ denotes the minimum money balance of consumer i in the T-period equilibrium $((x_i), p)$. $\underline{M}_{it}(a_1, \ldots, a_t)$ is the minimum money balance at the end of the period t when the history of the state of the exogenous stochastic process is $a_1, \ldots, a_t$. Observe that $\underline{M}_{iT}(a_1, \ldots, a_T) = 0$, for all $a_1, \ldots, a_T$.

I use the following facts, which correspond to formulas 28–30 of the previous paper. I have simply corrected these formulas by taking account of the minimum money balances.

$$(6.5) \qquad \lambda_{iT}(a_1, \ldots, a_T) = \max(\alpha_{iT}(a_1, \ldots, a_T), 1).$$

If $t < T$, then

$$\lambda_{it}(a_1, \ldots, a_t) = \max\{\alpha_{it}(a_1, \ldots, a_t), \delta_i(1+r)E[\lambda_{i,t+1}(a_1, \ldots, a_t, s_{t+1}) \mid s_t = a_t]\}.$$

$$(6.6) \qquad \lambda_{iT}(a_1, \ldots, a_T) > 1 \quad \text{only if} \quad M_{iT}(p, x_i; a_1, \ldots, a_T) = 0.$$

If $t < T$, then

$$\lambda_{it}(a_1, \ldots, a_t) > \delta_i(1+r)E[\lambda_{i,t+1}(a_1, \ldots, a_t, s_{t+1}) \mid s_t = a_t]$$

only if $M_{it}(p, x_i; a_1, \ldots, a_t) = \underline{M}_{it}(a_1, \ldots, a_t)$, where $M_{it}(p, x_i; a_1, \ldots, a_t)$ denotes the money balance of consumer i at the end of period t in the T-period equilibrium $((x_i), p)$.

(6.7) For all t, $\lambda_{it}(a_1, \ldots, a_t) > \alpha_{it}(a_1, \ldots, a_t)$ only if $x_{it}(a_1, \ldots, a_t) = 0$.

PROOF OF LEMMA 6.1: It is sufficient to find $\bar{\lambda}$ as in the Lemma, for by (6.2) I may let $\underline{p} = (\bar{\lambda})^{-1}\underline{q}$.

Let $\hat{r} > 0$ be such that $(1+\hat{r})^{-1} > \max_i \delta_i$. Let K be a positive integer such that

$$(6.8) \qquad \min_i (\delta_i(1+\hat{r}))^{-K} b^{-1} > 1.$$

Let

$$(6.9) \qquad \bar{\lambda} = b + b^2(\bar{q}\cdot\bar{\omega}) \sum_{k=1}^{K} (1+\hat{r})^{k-1} \max_i \tau_i^{-1}.$$

By Assumption 3.7, $\tau_i > 0$, for all i, so that $\bar{\lambda} < \infty$. Let

$$(6.10) \qquad \epsilon = \min_{i,s} \underline{q}\cdot\omega_i(s).$$

Since $\omega_i(s) \neq 0$, for all i and s and since $\underline{q}_k > 0$, for all k, it follows that $\epsilon > 0$.

Let $\bar{r}$ be such that $0 < \bar{r} \leq \hat{r}$ and so small that

$$(6.11) \qquad \bar{r} \sum_{k=1}^{K} (1+\bar{r})^{k-1} < \epsilon\big(b^2(\bar{q}\cdot\bar{\omega})\big)^{-1}.$$

I claim that $\bar{r}$ and $\bar{\lambda}$ satisfy the conditions of the lemma. Let $((x_i), p)$ and (λ_i) be as in the lemma. I must show that

$$(6.12) \qquad \lambda_{it}(a_1, \ldots, a_t) \leqq \bar{\lambda}, \quad \text{for all } i, \text{ for all histories } a_1, \ldots, a_t \text{ and for all } t.$$

I prove (6.12) by backwards induction on t. First of all, (6.12) is true for $t = T$. In order to see that this is so, fix $a_1, \ldots, a_T$ and let i be such that $M_{iT}(p, x_i; a_1, \ldots, a_T) > 0$. By (6.5) and (6.6), $\alpha_{iT}(a_1, \ldots, a_T) \leqq \lambda_{iT}(a_1, \ldots, a_T) = 1$. Hence, (6.4) implies that $\alpha_{jT}(a_1, \ldots, a_T) \leqq b$, for all j. But then by (6.5), $\lambda_{jT}(a_1, \ldots, a_T) \leqq \max(b, 1) = b$, for all j. Finally by (6.9), $b \leqq \bar{\lambda}$. This proves (6.12) for $t = T$.

Suppose by induction that (6.12) is true for $t+1, \ldots, T$. I now show that

$$(6.13) \qquad \underline{M}_{i,t+n}(p, x_i; a_1, \ldots, a_{t+n}) \leqq \max\Big(0, \tau_i - r^{-1}\epsilon(\bar{\lambda})^{-1}\Big),$$

for $n = 0, 1, \ldots, T-t$ and for all i and $a_1, \ldots, a_{t+n}$. I repeat the argument used to prove inequality (4.7) of the present paper. Since $\lambda_{i,t+n}(a_1, \ldots, a_{t+n}) \leqq \bar{\lambda}$, for all i, it follows from (6.2) that $p_{t+n}(a_1, \ldots, a_{t+n}) \geqq (\bar{\lambda})^{-1}\underline{q}$. It follows that a lower bound on consumer i's income in any of the periods $t+1, \ldots, T$ is $rM_i + \epsilon(\bar{\lambda})^{-1}$, where M_i is his money balance at the end of the previous period and ϵ is as in (6.10). His tax payments are $r\tau_i$. Clearly, if $rM_i + \epsilon(\bar{\lambda})^{-1} \geqq r\tau_i$, then he can keep his money holdings positive indefinitely simply by never spending money on consumption. It follows that the smallest nonnegative number M_i

satisfying this inequality is an upper bound on $\underline{M}_{i,t+n}$. This number is the right hand side of inequality (6.13).

I now prove that $\lambda_{it}(a_1, \ldots, a_t) \leqq \bar{\lambda}$, for all i, t, and $a_1, \ldots, a_t$. Suppose that $\lambda_{it}(a_1, \ldots, a_t) > \bar{\lambda}$, for some i. Without loss of generality, I may assume that $i = 1$, so that

$$(6.14) \qquad \lambda_{1t}(a_1, \ldots, a_t) > \bar{\lambda}.$$

I prove that (6.14) implies the following.

(6.15) There exist i and a history $a_{t+1}, \ldots, a_{t+N}$ following a_t such that

$$\lambda_{i,t+n}(a_1, \ldots, a_{t+n}) \geqq (\delta_i(1+r))^{-n} b^{-1}\bar{\lambda} \text{ and}$$

$$M_{i,t+n}(p, x_i; a_1, \ldots, a_{t+n}) \geqq \tau_i - \bar{\lambda}^{-1} b^2 (\bar{q} \cdot \bar{\omega}) \left(\sum_{k=1}^{n} (1+r)^{k-1} \right),$$

for $n = 0, \ldots, N$, where $N = \min(K, T-t)$.

This statement leads to a contradiction. First of all, suppose that $N = T - t$. Then, (6.9) and (6.15) imply that $M_{iT}(p, x_i; a_1, \ldots, a_T) > 0$. But then by (6.6), $\lambda_{iT}(a_1, \ldots, a_T) = 1$. However by (6.9) and (6.15), $\lambda_{iT}(a_1, \ldots, a_T) \geqq (\delta_i(1 + r))^{t-T} b^{-1}\bar{\lambda} \geqq b^{-1}\bar{\lambda} > 1$, which is a contradiction.

Suppose that $N = K$. Then, (6.8) and (6.15) imply that $\lambda_{i,t+K}(a_1, \ldots, a_{t+K}) \geqq (\delta_i(1+r))^{-K} b^{-1}\bar{\lambda} > \bar{\lambda}$, which contradicts the induction hypothesis. This proves that (6.15) leads to a contradiction and hence that (6.14) is impossible. Hence, the induction step in the proof of (6.12) will be completed once (6.15) is proved.

I now prove (6.15). Let i be such that $M_{i,t}(p, x_i; a_1, \ldots, a_t) \geqq \tau_i$, where $a_1, \ldots, a_t$ are as in (6.14). Such an i exists by the assumption that $\sum_{i=1}^{I} M_{i0} = \sum_{i=1}^{I} \tau_i = 1$ (Assumption (3.7).

I first show that $\lambda_{it}(a_1, \ldots, a_t) \geqq b^{-1}\bar{\lambda}$. Observe that

$$\begin{aligned} \bar{\lambda} &< \lambda_{1t}(a_1, \ldots, a_t) \\ &= \max\left(\alpha_{1t}(a_1, \ldots, a_t), \delta_1(1+r) E\left[\lambda_{1,t+1}(a_1, \ldots, a_t, s_{t+1}) \mid s_t = a_t\right]\right) \\ &\leqq \max\left(\alpha_{1t}(a_1, \ldots, a_t), \delta_1(1+r)\bar{\lambda}\right) = \alpha_{1t}(a_1, \ldots, a_t). \end{aligned}$$

The second inequality follows from the induction hypothesis on t (regarding (6.12)). Hence by (6.4) and (6.5),

$$\lambda_{it}(a_1, \ldots, a_t) \geqq \alpha_{it}(a_1, \ldots, a_t) \geqq b^{-1} \alpha_{1t}(a_1, \ldots, a_t) > b^{-1}\bar{\lambda}.$$

I have now proved that the inequalities of (6.15) are satisfied for $t = 0$.

I now prove by induction on n that $a_{t+1}, \ldots, a_{t+N}$ exist as in (6.15). Suppose that the conditions of (6.15) are satisfied for n no larger than some nonnegative integer; call it n again. I may suppose that $n < N$. Then,

$$M_{i,t+n}(p, x_i; a_1, \ldots, a_{t+n}) \geqq \tau_i - \bar{\lambda}^{-1} b^2(\bar{q} \cdot \bar{\omega})\left(\sum_{k=1}^{n} (1+r)^{k-1}\right)$$

$$> \max\left(0, \tau_i - r^{-1}\epsilon(\bar{\lambda})^{-1}\right).$$

The last inequality follows from (6.9) and (6.11). Hence by (6.13) and (6.6),

$$\lambda_{i,t+n}(a_1, \ldots, a_{t+n})$$

$$= \delta_i(1+r)E\left[\lambda_{i,t+n+1}(a_1, \ldots, a_{t+n}, s_{t+n+1}) \mid s_{t+n} = a_{t+n}\right],$$

so that for some a_{t+n+1},

$$\lambda_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) \geqq (\delta_i(1+r))^{-1} \lambda_{i,t+n}(a_1, \ldots, a_{t+n})$$

$$\geqq (\delta_i(1+r))^{-(n+1)} b^{-1} \bar{\lambda}.$$

The last inequality follows from the induction hypothesis on n.

I now show that

$$M_{i,t+n+1}(p, x_i; a_1, \ldots, a_{t+n+1}) \geqq \tau_i - \bar{\lambda}^{-1} b^2(\bar{q} \cdot \bar{\omega})\left(\sum_{k=1}^{n+1} (1+r)^{k-1}\right).$$

If

$$\alpha_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) < \lambda_{i,t+n+1}(a_1, \ldots, a_{t+n+1}),$$

then by (6.7), $x_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) = 0$, so that

$$M_{i,t+n+1}(p, x_i; a_1, \ldots, a_{t+n+1})$$

$$\geqq (1+r) M_{i,t+n}(p, x_i; a_1, \ldots, a_{t+n}) - r\tau_i$$

$$\geqq (1+r)\left[\tau_i - \bar{\lambda}^{-1} b^2(\bar{q} \cdot \bar{\omega})\left(\sum_{k=1}^{n} (1+r)^{k-1}\right)\right] - r\tau_i$$

$$\geqq \tau_i - \bar{\lambda}^{-1} b^2(q \cdot \bar{\omega})\left(\sum_{k=1}^{n+1} (1+r)^{k-1}\right).$$

The second inequality follows from the induction hypothesis on n.

Suppose now that

$$\alpha_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) = \lambda_{i,t+n+1}(a_1, \ldots, a_{t+n+1}).$$

Then, by the choice of a_{t+n+1}

$$\alpha_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) > b^{-1}\bar{\lambda}.$$

It follows from (6.4) that

$$\min_j \alpha_{j,t+n+1}(a_1, \ldots, a_{t+n+1}) > b^{-2}\bar{\lambda},$$

so that by (6.3)

$$p_{t+n+1}(a_1, \ldots, a_{t+n+1}) \leqq b^2\bar{\lambda}^{-1}\bar{q}.$$

Hence,

$$p_{t+n+1}(a_1, \ldots, a_{t+n}) \cdot x_{i,t+n+1}(a_1, \ldots, a_{t+n+1}) \leqq b^2\bar{\lambda}^{-1}(\bar{q} \cdot \bar{\omega}).$$

It follows that

$$\begin{aligned} &M_{i,t+n+1}(p, x_i; a_1, \ldots, a_{t+n+1}) \\ &\quad \geqq (1+r)M_{i,t+n}(p, x_i; a_1, \ldots, a_{t+n}) - r\tau_i - b^2\bar{\lambda}^{-1}(\bar{q} \cdot \bar{\omega}) \\ &\quad \geqq (1+r)\left[\tau_i - \bar{\lambda}^{-1}b^2(\bar{q} \cdot \bar{\omega}) \sum_{k=1}^{n} (1+r)^{k-1}\right] - r\tau_i - b^2\bar{\lambda}^{-1}(\bar{q} \cdot \bar{\omega}) \\ &\quad = \tau_i - \bar{\lambda}^{-1}b^2(\bar{q} \cdot \bar{\omega}) \sum_{k=1}^{n+1} (1+r)^{k-1}. \end{aligned}$$

This completes the proof that the two inequalities of (6.15) are satisfied for $n + 1$, and so completes the induction step in the proof of (6.15). *Q.E.D.*

This completes the correction of the proof of the existence theorem in my previous paper. The other arguments and results in the paper are true, provided that certain easy adjustments are made in order to include minimum money balances. For instance, inequality 9 of that paper should read

$$\lambda_{in}(a_1, \ldots, a_n) > \delta_i(1+r)E\left[\lambda_{i,n+1}(a_1, \ldots, a_n, s_{n+1}) \mid s_n = a_n\right] \text{ only if}$$

$$M_{in}(p, x_i; a_1, \ldots, a_n) = \underline{M}_{in}(p; a_1, \ldots, a_n),$$

where $\underline{M}_{in}(p; a_1, \ldots, a_n)$ is consumer i's minimum money balance when the price system is p.

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Manuscript received September, 1980; final revision received December, 1982.

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