10 Large firm dynamics and the business cycle

(**?**)==(**What it is?**)

(**??**)==(**Why it is?**)

(**???**)==(**What and Why it is?**)

(**TBC**)==(**To Be Checked**)

0 Abstrat and intro

0.1 Abstract

1. Large firm dynamics (**What it is?**) drive the business cycle (**Fluctuation in Economics**)?
    - They can answer this question by developing a quantitative theory (**?**) of aggregate fluctuations (**?**) caused by firm-level disturbances (**?**) alone.
        - They show that a standard heterogeneous firm dynamics setup (**?**) already contains in it a theory of the business cycle (**?**), without appealing to aggregate shocks (**?**).
        - They offer an analytical characterization of the law of motion (**?**) of the aggregate state (?) in this class of models (**?**)-the firm size distribution (**?**)- and show that aggregate output and productivity dynamics display:
            - persistence
            - volatility
            - Time-varying second moments.
        - They explore
            - the key role of moments of the firm size distribution
                - the role of large firm dynamics
            - in shaping aggregating fluctuations, theoretically, quantitatively and in the data.

0.2 Intro

1. Aggregate prices and quantities exhibit persistent dynamics and time-varying volatility.
    - Business cycle theories have typically resorted to exogenous aggregate shocks in order to generate such features of aggregate fluctuations (**e.g., IS-MP_PC model?**)
    - A recent literature has instead proposed that the origins of business cycles may be traced back to micro-level disturbances (**Based on individuals' choices?**) .
    - But we **lack a framework** (**place where they can work on**) that enables a **systematic**

evaluation of the link between the **micro-level decisions driving firm growth, decline and churning** and the **persistence and volatility of macro-level outcomes**.

2. This paper seeks to evaluate the impact of large firm dynamics (**?**) on aggregate fluctuations.
    - Building on a **standard firm dynamics setup** (**What is it?**), they develop a quantitative theory of aggregate fluctuations arising from firm-level shocks **alone**.
    - They derive an analytical characterization of the law of motion of the firm size distribution (**Law of motion, its characterization?**) (the aggregate state variable in this class of models).
    - They also show that resulting aggregate output and productivity dynamics are endogenously
        - persistent,
        - volatile
        - Exhibit time-varying second moments.
    - And then they explore the **key role of moments of the firm size distribution** (**What are they?**) (and in particular, the role of large firms dynamics (**?What roles**)) in shaping aggregte fluctuations, theoretically, quantitatively and in the data.
    - Their results imply that
        - Large firm dynamics include sizeable movements in aggregates and account for 30% of aggregate fluctuations (**To be checked**).
3. Their setup follows Hopenhayn's (1992) industry dynamics framework (**?**).
    - Firms differ in their idiosyncratic productivity level,
        - which is assumed to follow a discrete Markovian process (**What? Why**).
    - Incumbents
        - have access to a decreasing returns to scale technology using labor as the only input (**What? Why**).
        - Produce a unique good in a perfectly competitive market (**What? Why**).
        - face an operating fixed cost in each period which, in turn, generates endogenous exit (**What? Why**).
        - As previous incumbents exit the market, they are replaced by new entrants (**What? Why**).
4. The crucial difference relative to Hopenhayn (1992) is that: they do not reply on the traditional **"continuum of firms"** assumption (**?**) in order to characterise the law of motion for the firm size distribution.
    - They characterize the law of motion for any finite number of firms (**What, why?**)
5. Their **first theoretical results** shows that, generically,
    - the firm size distribution is time-varying in a stochastic fashion (**What is it?**).
        - As is well know (**?**), this distribution is the aggregate state variable in this class of models (**?**).
    - An immediate implication of the findings is
        - Aggregate productivity, aggregate output and factor prices are themselves stochastic (**Why?**).
    - In a nutshell, they show that

        - the standard workhorse model in the firm dynamics literature (once assumption regarding a continuum of firm is dropped (**How to understand it?**)) already features aggregate fluctuations (**How?**).
6. Then, they specialize their model to the case of random growth dynamics at the firm level.
    - Given their focus on large firm dynamics, the evidence put forth by Hall (1987) (**What is that?**) in favor of **Gibrat's law** (**?**) for large firms makes this a natural baseline to consider (**What baseline?**).
    - With this assumption in place, their **second main theoretical contribution** is to solve analytically for the dynamics of aggregate productivity (**?**), up to the contribution of entry an exit (**?**).
7. Their **third theoretical result** is to show: the steady-state firm size distribution is Pareto distributed.
    - They discuss the role (**What are they? Why they?**) of random growth, entry and exit and decreasing returns to scale in generating this result.
    - The upshot of this is that our model can endogenously deliver a first-order distributional feature of the data (**?**):
        - the **co-existence** of **a large number of small firms** and **a small, but non-negligible, number of very large firms** (**What? Why**), orders of magnitude larger than the average firm in the economy (**What? Why**).
8. Their **fourth theoretical results** sheds light on the micro origins of aggregate persistence, volatility and time-varying uncertainty.
    - Leveraging on our characterization of the law of motion of the aggregate productivity (**What? Why**), we are able to show, analytically, that
        - persistence in **aggregate output in increasing with firm-level productivity persistence** and **with the share of economic activity accounted by large firms** (**What? Why**);
        - **aggregate volatility** decays only slowly with the **number of firms in the economy**, and that this rate of decay is generically **a function** of the **size distributions of incumbents and entrants**, as well as the **degree of decreasing return** (**What? Why**);
        - **aggregate volatility dynamics** are endogenously driven by the **evolution of the cross-sectional dispersion of firm sizes** (**What? Why**).
9. Taken together, their theoretical results also deliver a simple economic intuition: **why large firm dynamics may drive the business cycle?** (**What? Why**)
    - Answering this question requires answering:
        - Following an idiosyncratic shock to a very large firm (e.g. G.M.), why its competitors (e.g. Ford) do not increase their scale and gain market share (**What is an idiosyncratic shock? Why**)
            - If this were to happen, production would merely be reallocated from G.M. to Ford, rather than reduced in the aggregate (**Make sense, but why?**)
                - If the shock is purely idiosyncratic to G.M., then demand for auto-mobiles would be unaffected while primary input prices (e.g., wages in Detroit) should decline as G.M. sheds workers.
        - Their quantitative results imply that this reallocation effect is second order

(**Second order?**):

- Under a **Pareto firm size distribution** (**?**), the **productivity gap** between G.M. and Ford is large enough (**?**) such that:
  - the shock to G.M. does have aggregate consequences (**?**) and auto-worker wages do decline (**?**) but
  - Ford is unproductive enough relative to G.M. such that, under decreasing returns to scale, the amount of reallocation is limited (**What? Why**).

10. Then they explore the **quantitative implications** of their **setup**.
    - Due to their characterization of aggregate state dynamics (**?**), their **numerical strategy** is substantially less computational intensive than that traditionally used when solving for heterogeneous agents' models (**What? Why**).
      - This allows us to (**Why?**)
        - solve the model featuring **a very large number of firms** and
        - thus **match the firm size distribution accurately**.
11. Their **first set of quantitative results** shows that **the standard model firm dynamics with no aggregate shocks** (**?**) is able to **generate sizable fluctuations in aggregates**.
    - aggregate output (aggregate productivity) fluctuations amount to 30% (24%, respectively) of that observed in the data.
      - These fluctuations have their origins in large firm dynamics (**???**)
    - In particular, they show how **fluctuations at the upper end of the firm size distribution** (induced by shocks to very large firms) lead to movements in aggregates (**To be checked**).
    - They supplement this analysis by showing that the same correlation holds true empirically:
      - aggregate output and productivity fluctuations in the data coincide with movements in the tail of the firm size distribution (**To be checked**).
12. Then, they focus on the origins of time-varying aggregate volatility (**?**).
    - Consistently with their analytical characterization (**?**), theiry quantitative results show that
      - the **evolution of aggregate volatility** is determined by the **evolution of the cross-sectional dispersion** in the firm size distribution (**???**).
      - (Unlike the extant literature) The latter is the **endogenous outcome of firm-level idiosyncratic shocks** and **not the result of exogenous aggregate second moment shocks** (**???**).
    - They compare these results against the data and find consistent patterns:
      - **Aggregate volatility is high** whenever **cross-sectional dispersion is high** (**???**).
13. This paper relates to two distinct literatures:
    - an emerging literature on the **micro-origins of aggregate fluctuations**;
      - Gabaix's (2011) seminal work introduces the "Granular Hypothesis":

        - Whenever the firm size distribution is fat tailed, **idiosyncratic shocks average out at a slow enough rate** that it is possible for these to translate into aggregate fluctuations (**??? To be checked**).
        - Relative to Gabaix (2011), their main contribution is to ground the granular hypothesis in a well specified firm dynamics setup (**? To be checked**):
            - In their setting, **firms' entry, exit and size decisions** reflect **optimal forward-looking choice** (**???**), given firm-specific productivity processes and (aggregate) factor prices (**???**).
            - This allows them to both (**???**)
                - generalize the existent theoretical results and
                - Quantify their importance.
    - The recent contribution of di Giovanni, Levchenko and Mejean (2014) provides an **empirical benchmark** to this literature and, in particular, to their quantification exercises discussed above (**?**).
        - Working with census data for France, they estimate the contribution of **firm specific volatility** to **aggregate sales growth volatility**.
        - Proposed paper's quantitative results show that
            - the magnitude of aggregate fluctuations implied by their firm dynamics environment account for about 38% of this contribution.
- the more estabilished **firm dynamics literature** that follows from the seminal contribution of Hopenhayn (1992).
    - Some papers in this literature have explicitly studied **aggregate fluctuations** in a **firm dynamics framework** (**?**) (Campbell and Fisher (2004), Lee and Mukoyama (2008, 2015), Clementi and Palazzo (2016) and Bilbiie et al. (2012)).
    - A more recent strand of this literature has focused on the **time-varying nature of aggregate volatility** and its **link with the cross-sectional distribution of firms** (**?**) (e.g., Bloom et al, 2018).
    - Invariably, in this literature, they show that its **standard workhorse model** already contains in it a **theory of aggregate fluctuations and time-varying aggregate volatility**.
        - **Standard workhorse model**: once the **assumption regarding a continuum of firms** is dropped and the **firm size distribution is fat tailed** (**???**).
        - They show this both theoretically and quantitatively in an otherwise transparent and well understood setup (**To be checked**).
            - They eschew the myriad of frictions that Hopenhayn's (1992) framework has been able to support (**?**).
                - Such frictions are capital adjustment costs, labor market frictions, credit constraints or limited substitution possibilities across goods (**???**).
            - They can do this because their focus is on large firm dynamics which are arguably less encumbered by such frictions (**???**).

1 Model Setup

1.0 Why these setups

1. As mentioned in 0.2 Intro, this paper analyze **a standard firm dynamics setup (Hopenhayn, 1992)** with **a finite** but **possibly large number of firms**. (???)
2. They show how to **solve for** and **characterize** the **evolution of the firm size distribution without relying on the usual law of large numbers assumption** (**TBC**)
3. They prove that
    - In this setting, the **firm size distribution** does not converge to **a stationary distribution** (**???**), but instead **fluctuates stochastically around it**.
    - **Aggregate prices** and **quantities** are **not constant over time** (**???**)as the continuum assumption in Hopenhayn (1992) does not apply (**???**).
        - This continuum assumption repeatedly invoked by the subsequent literature.
            - **Q: if this assumption does not hold, then what we can do? Why?**
        - To do this (**?**), they started by describing the economic environment.
            - As is standard in this class of models, this involves specifying **a firm-level productivity process**, **the incumbents' problem** and **the entrants' problem**.

1.1 Setups in this model

1. Firms differ in their productivity level, which is assumed to follow a discrete Markovian process. (**??**)
2. Incumbents
    - have access to a **decreasing returns to scale** technology using labor as the only input (**LR; ??**)
    - produce a **unique** good in a **perfectly competitive market** (**??**).
    - face an **operating cost** at each period, which in turn generates **endogenous exit** (**??**).
3. There are also a **large** (but **finite**) **number** of **potential entrants** that **differ** in their **productivity**.
    - To operate next period, potential entrants have to pay an **entry cost** (**???**).
4. The economy is closed by specifying a **labor supply function** that **increases with the wage** (**??**).

1.2 Mathematical descriptions of these setups

1.2.1 Productivity process

1. We assume a **finite** but **potentially large number** of **idiosyncratic** productivity levels.
2. The **productivity space** is described by a S-tuple $\Phi = \{\varphi^1, \cdots, \varphi^S\}$ with $\varphi > 1$ such that $\varphi^1 < \cdots < \varphi^S$ (**State space???**).

3. The **idiosyncratic state-space** is evenly distributed in logs (**???**), where φ is the log step between two productivity levels: $\frac{\varphi^{s+1}}{\varphi^{s}} = \varphi$.
4. A firm is in state (or productivity state) s when its idiosyncratic productivity is equal to φ^s.
5. A **firm's productivity level** is assumed to follow a **monotone Markov chain** with **a transition matrix** P (**???**).
6. Denote $F(\cdot|\varphi^s)$ as the **conditional distribution of the next period's idiosyncratic productivity** $\varphi^{s'}$, given the **current period's idiosyncratic productivity** φ^s (**??**).

1.2.2 Incumbents' problem

1. The **only aggregate state variable** of this model is the **distribution of firms** on the set Φ (**???**).
   - Denote this distribution by a $(S \times 1)$ **vector** μ_t **giving the number of firms** at each productivity level s at time t. (**???**)
     - For current setup description, they abstract from explicit time t notation, but will return to it when they characterize the law of motion of the aggregate state. (**???**)
2. Given an **aggregate state** μ and an **idiosyncratic productivity level** φ^s, the incumbent solves the following static profit maximization problem:
$$\pi^*(\mu, \varphi^s) = \max_n \{\varphi^s n^\alpha - w(\mu)n - c_f\} \tag{1}$$
   - where (**??? Production function; range of α?**)
     - n: labor input;
     - $w(\mu)$: the wage for a given aggregate state μ;
     - c_f: the operating cost to be paid in unit of output every period.
   - Easy to show that π^* is increasing in φ^s and decreasing in w for a given aggregate state μ. (**TBC**)
3. The output of a firm is then (**??? How to get this expression?**)
$$y(\mu, \varphi^s) = (\varphi^s)^{\frac{1}{1-\alpha}} (\frac{\alpha}{w(\mu)})^{\frac{\alpha}{1-\alpha}} \tag{2}$$
   - In following, the **size of a firm** will refer to its **output level** if not otherwise specified.
4. The timing of decisions for incumbents is standard:
   - The incumbent first draws its idiosyncratic productivity φ^s at the beginning of the period, pays the operating cost c_f and then hires labor to produce. (**???At the same time?**)
   - It then decides whether
     - to exit at the end of the period or
     - to continue as an incumbent the next period.
5. The present discounted value of being an incumbent for a given aggregate state μ and idiosyncratic productivity level φ^s, denoted by $V(\mu, \varphi^s)$, can be defined by the Bellman equation: (**???Why 0 in the max operator?**)
$$V(\mu, \varphi^s) = \pi^*(\mu, \varphi^s) + \max\{0, \beta \int_{\mu' \in \Lambda} \sum_{\varphi^{s'} \in \Phi} V(\mu', \varphi^{s'}) F(\varphi^{s'}|\varphi^s) \Gamma(d\mu'|\mu)\} \tag{3}$$
   - where

- β: discount factor;
- $\Gamma(\cdot|\mu)$: the conditional distribution of μ', tomorrow's aggregate state;
- $F(\cdot|\varphi^s)$: the conditional distribution of tomorrow's idiosyncratic producitivty for a given today's idiosyncratic productivity of the incumbent;
- Λ: the set of $(S \times 1)$-vectors whose elements are non-negative.

- The second term on the RHS of the value function above encodes an endogenous exit decision (**??**).
  - As is standard in this framework, this decision is defined by a threshold level of idiosyncratic productivity, given an aggregate state.
  - Formally, since the instantaneous profit is increasing in the idiosyncratic producitivity level, there is a unique index $s^*(\mu)$ for each aggregate state μ, such that: (**??? What if $\varphi^s = \varphi^{s^*(\mu)}$?**)
    - for $\varphi^s \geq \varphi^{s^*(\mu)}$, the incumbent firm continues to operate next period and
    - for $\varphi^s \leq \varphi^{s^*(\mu)}$, the incumbent firm decides to exit next period.

1.2.3 Entrants' problem

1. There is an **exogenously given**, **constant** and **finite number** of prospective entrants M.
2. Each potential entrant has access to a **signal** about their **potential productivity next period**, (**? And/ or**) **should they decide to enter today** (**A signal of two things?**).
   - The entrants' signals (**???**) are distributed according
   $$G = (G_q)_{q \in [1,\ldots,S]} \tag{4}$$
     - which is a discrete distribution over Φ.
   - MG_q gives the number of potential entrants for each signal level φ^q.
     - This is because there is a total of M potential entrants every period.
3. To enter the market, they have to pay a **sunk entry cost** which, in turn, leads to an endogenous entry decision which is again characterized by a threshold level of initial signals. (**???**)
   - If a potential entrant decides to **pay the entry cost** c_e, then she will **produce next period with a productivity level drawn from** $F(\cdot|\varphi^q)$.
     - Given this, we can define the **value of a potential entrant** with **signal** φ^q **for a given aggregate state** μ as $V^e(\mu, \varphi^q)$ (**How to get this?**):
     $$V^e(\mu, \varphi^q) = \beta \int_{\mu' \in \Lambda} \sum_{\varphi^{q'} \in \Phi} V(\mu', \varphi^{q'}) F(\varphi^{q'}|\varphi^q) \Gamma(d\mu'|\mu) \tag{5}$$
     - Prospective entrants **pay the entry cost** and **produce next period** if the above value is greater or equal to the entry cost c_e. (**Make sense**)
       - As in the incumbent's exit decision, this now introduces a threshold level of signal, $e^*(\mu)$, for a givenn aggregate state μ such that (**???**)
         - for $\varphi^q \geq \varphi^{e^*(\mu)}$, the potential entrant starts operating next period;
         - for $\varphi^q \leq \varphi^{e^*(\mu)}$, the potential entrant decides not to start operating next period.

- For simplicity, unless otherwise stated, we assume that the entry cost is set to 0: $c_e = 0$, which in turn implies that $\varphi^{e^*(\mu)} = \varphi^{s^*(\mu)}$. (**TBC, ??**)

1.2.4 Labor market and aggregation

1. Assume that the **supply of labor** at a given wage w is given by $L^s(w) = Mw^\gamma$ with $\gamma > 0$. (**???**)
   - That is, for a given wage level, the labor supply function is a linear function of M, the number of potential entrants.
   - This assumption is necessary because in following, they will be interested in **characterizing the behaviour of aggregate quantities and prices** (**? TBC**), as they **let M increase**.
     - Note that if **total labor supply** were to be kept **fixed**, **increasing M** would lead to an **increase in aggregate demand for labor**.(**???**)
     - Therefore, the wage would increase mechanically (**???**).
     - They therefore make this assumption to abstract from this mechanical effecct of increasing M on the equilibrium wage. (**??? Explain why this assumption and this setup?**)
2. To find equilibrium wages, they derive **aggregate labor demand** in this economy.
   - To derive aggregate labor demand, we must note that (**??? Why Cobb-Douglas Form**)
     - If Y_t is aggregate output, i.e., the sum of all individual incumbents' output, then
       $$Y_t = A_t^{1-\alpha}(L_t^d)^\alpha \tag{6}$$
       - where
         - $A_t^{1-\alpha}$: aggregate TFP gross of the contribution of the fixed and entry costs (**???**);
           - Henceforth, it will be convenient to differentiate between aggregate TFP, $A_t^{1-\alpha}$, and the term A_t itself (**??**)
           - With some abuse of language (**???**), they refer to the latter (i.e., A_t) as **aggregate productivity**, which is given by
             $$A_t = \sum_{i=1}^{N_t} (\varphi^{s_{i,t}})^{\frac{1}{1-\alpha}} \tag{7}$$
             - where
               - $\varphi^{s_{i,t}}$: the productivity level at date t of the i^{th} firm among the N_t operating firms at data t (**???**).
             - This can be rewritten by aggregating over all firms that leave the same productivity level: (**???**)
               $$A_t = \sum_{s=1}^{S} \mu_{s,t}(\varphi^s)^{\frac{1}{1-\alpha}} = B'\mu_t \tag{8}$$
               - where
                 - B: the $(S \times 1)$ vector of parameters $((\varphi^1)^{\frac{1}{1-\alpha}}, \cdots, (\varphi^S)^{\frac{1}{1-\alpha}})$. (**???**)

- $\mu_{s,t}$: ? the number of operating firms in state s at date t. (**???**)
  - As discussed above (**???**), the distribution of firms μ_t across the discrete state space $\Phi = \{\varphi^1, \cdots, \varphi^S\}$ is a $(S \times 1)$ vector equal to $(\mu_{1,t}, \cdots, \mu_{S,t})$ such that $\mu_{s,t}$ is equal to the number of operating firms in state s at date t.
- L_t^d: the aggregate labor demand, the sum of all incumbents' labor demand in period t;
  - By the same argument, easy to show that aggregate labor demand is given by (**??? TBC**)

$$L^d(w_t) = \left(\frac{\alpha A_t^{1-\alpha}}{w_t}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

- Note that the model behaves as a one factor model with aggregate TFP, $A_t^{1-\alpha}$.

3. The market clearing condition then equates labor supply and labor demand, i.e., $L^s(w_t) = L^d(w_t)$.
   - Given date t productivity distribution μ_t, they can solve for the equilibrium wage to get:

$$w_t = \left(\alpha^{\frac{1}{1-\alpha}} \frac{B' \mu_t}{M}\right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}} \tag{10}$$

     - This last equation leads to the following expression for aggregate output (**??? Tautology**)

$$Y_t = A_t^{1-\alpha} L_t^{\alpha} \tag{11}$$

   - From these expressions, note that the **wage** and **aggregate output** is fully pinned down by the **distribution** μ_t. (**???**)
     - Given a current-period distribution of firms across productivity levels, we can solve for all equilibrium quantities and prices.
4. Finally note that if, as we will show below, **idiosyncratic shocks to large firms** do lead to **variation of μ_t over time**, then this will induce **variations in the equilibrium wage** as instructed by (10), which, in turn, will lead to **reallocation of economic activity across firms** (**???**).
   - To see this, define the **elasticity** of **output of a given firm** i to an **idiosyncratic productivity change in another firm** j as: (**??? TBC**)

$$\frac{\partial \log y_i}{\partial \log \varphi^{s_{j,t}}} = \frac{\partial \log y_i}{\partial \log w_t} \frac{\partial \log w_t}{\partial \log \varphi^{s_{j,t}}} = -\frac{\alpha}{1-\alpha} \frac{\partial \log w_t}{\partial \log \varphi^{s_{j,t}}} \tag{12}$$

   - Intuitively, the steeper decreasing returns to scale are, the lower this elasticity is.
     - The steeper decreasing returns to scale: the steeper the decline in the marginal productivity of firm i as it expands, following a decline in wages. (**???**)
   - By the above argument, for a **given wage response elasticity to large firm shocks**, the strength of this **countervailing reallocation effect** is lower, the smaller α is. (**???**)
   - They are left to understand the behavior of the **second components** of this elasticity.
     - That is, the response of wages to idiosyncratic shocks.

        - This, in turn, implies (**???**) understanding how the aggregate state, the firm-size distribution, evolves over time as a function of idiosyncratic shocks.
            - They will address it in the section 2. (**??? TBC**)

2 Aggregate state dynamics and uncertainty: General results

2.0 Map to this section

1. First show how to **characterize** the **law of motion for the productivity distribution**, the **aggregate state** in this economy. (**??? TBC**)
2. Prove that, generically, the **distribution of firms across productivity levels** is **time-varying** in a **stochastic** fashion. (**??? TBC**)
    - An immediate **implication** of this result is that **aggregate productivity** A_t and **aggregate prices** are themselves **stochastic** as **they are simply a function of this distribution**. (**??? TBC**)
3. Show that the **characterization of the stationary firm productivity distribution** offered in Hopenhayn (1992) is **nested** in our model when we **take uncertainty to 0**. (**??? TBC**)

2.1 Law of motion of the productivity distribution

1. In a setting with a continuum of firms, Hopenhayn (1992) shows that by appealing to a law of large numbers, the law of motion for the producitivity distribution is in fact deterministic.
    - In the current setting, with a finite number of incumbents, a similar argument cannot be made.
    - They now show how to characterize the law of motion for the productivity distribution when they move away from the continuum case.
2. In order to build intuition for their general results below, they start by exploring a **simple example** where, for simplicity, they **ignore entry and exit of firms**.
    - Assume that
        - there are only three levels of productivity ($S = 3$) and four firms.
        - At time period t, these firms produce with the intermediate level of productivity (the productivity level chosen that period).
        - these firms have an equal probability of $\frac{1}{4}$ of going up or down in the productivity ladder.
        - the probability of staying at the same intermediate level is $\frac{1}{2}$.
            - That is, the second row of the transition matrix P in this simple example is given by $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (**??? How to construct the P, why second row? How about the first and the third rows?**).
    - **Analysis**

- First note that, if instead of four firms, we had assumed a continuum of firms, the law of numbers would hold such that, at $t+1$, there would be **exactly** $\frac{1}{4}$ of the (mass of) firms at the highest level of productivity, $\frac{1}{2}$ would remain at the intermediate level and $\frac{1}{4}$ would transit to the lowest level of productivity (**??? TBC; according to the probability**).
- Since the number of firms is finite, this is not the case here.
  - e.g., a distribution of firms, such as the on presented in the bottom-right panel of figure 1 is possible with a positive probability (**??? A discrete case**).
    - Of course, many other arrangements would also be possible outcomes.
  - Thus, in this example, the number of firms in each productivity bin at $t+1$ follows a multinomial distribution with a number of trials of 4 and an event probability vector $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})'$ (**???**).
- In this simple example, **all firms are assumed to have the same productivity level at time** t.
  - It is easy to extend this example to any initial arrangement of firms over productivity bins.
    - This is because, for any initial number of firms over productivity level, the distribution of these firms across productivity levels next period follows a **multinomial** (**???**).
    - Therefore, the total number of firms in each productivity level next period, is simply a **sum of multinomials**, i.e., the result of transitions from all initial productivity bins (**?**).

3. More generally, for S **productivity levels, and an (endogenous) finite number of incumbents**, N_t, **making optimal employment and production decisions** and **accounting for entry and exit decisions**, the following theorem holds:

Theorem 1 (Law of motion)

The number of firms at each productivity level at $t+1$, given by the $(S \times 1)$ vector μ_{t+1}, conditional on the current vector μ_t, follows a sum of multinomial distributions and can be expressed as (**???**)

$$\mu_{t+1} = m(\mu_t) + \epsilon_{t+1} \tag{13}$$

- where
  - ϵ_{t+1}: a random vector with mean 0 and a variance-covariance matrix $\sum(\mu_t)$ (**???**)

$$\sum(\mu_t) = \sum_{s=s^*(\mu_t)}^{S} (MG_s + \mu_{s,t})W_s \tag{14}$$

  - $m(\mu) = (P_t^*)'(\mu_t + MG)'$
    - Where
      - P_t^*: the transition matrix P with the first $(s^*(\mu_t) - 1)$ rows replaced by 0s,
      - G: the entrants' signals distribution;
      - M: the number of prospective entrants;

            - $W_s = diag(P_{s,.}) - P'_{s,.}P_s$ (**???**);
                - where
                    - $P_{s,.}$: denotes the s-row of the transition matrix P.

Proof. (??? TBC)

4. After taking into account the dynamics of incumbent firms and entry/exit decisions, the law of motion ((14)) of the aggregate state is remarkably simple:
    - Tomorrow's distribution is an affine functiion of today's distribution up to a stochastic term, ϵ_{t+1}, that reshuffles firms across productivity levels.
        - law of motion of the aggregate state: the distribution of firms over productivity levels (**? TBC**).
5. In the simple example without entry and exit, given the state transition probabilities, they should for example observe that
    - on average, the number of firms remaining at the intermediate level of productivity is twice that of those transiting to the highest level of productivity. (**??? According to our assumptions?**)
        - This is precisely what the affine part ($m(\mu_t)$**??? TBC**) of (14) captures:
            - The term $m(\mu_t)$ reflects these typical transitions, which are a functiion of matrix P alone.
    - With a finite number of firms, in any given period, there will be **stochastic deviations** from these typical transitions as we discuss above. (**???**)
        - In the theorem, this is reflected in the "reshuffling shock" term, ϵ_{t+1}, that enters in the law of motion given by (14). (**???**)
        - **How important this reshuffling shock** is **for the evolution of the firm distribution** is dictated by the **variance-covariance matrix** $\sum(\mu_t)$, which, in turn, is a **function** of the **transition matrix** P, the **current firm distribution** μ_t, and, in the general case with entry and exit, the **signal distribution available to potential entrants**. (**???**)

2.2 Steady-state equilibrium and the stationary distribution

1. The above characterization, especially (5), is instructive of the differences of the current setup relative to a standard Hopenhayn economy (**?**).
    - The latter corresponds to the case where **all of the relevant firm dynamics** are encapsulated by the affine term $m(\mu)$. (**What is the result of this paper's case?**)
        - In particular, it is immediate to verify that, under a continuum of firms, (**??? TBC**)
            - the **variance-covariance matrix** in **Theorem 1** is equal to **0**;
            - the **aggregate state** μ_t becomes **non-stochastic**.
2. The following corollary to **Theorem 1** shows that the **deterministic dynamics of the productivity distribution** under **no aggregate uncertainty** are similar to the one in Hopenhayn (1992) framework.

Corollary 1 (From Theorem 1)

Define $\hat{\mu}_t = \frac{\mu_t}{M}$ for any t.

With a continuum of firms, or equivalently, when aggregate uncertainty is absent, $\epsilon_{t+1} = 0$:

$$\hat{\mu}_{t+1} = (\tilde{P}_t)'(\hat{\mu}_t + G) \tag{15}$$

- where
  - $\tilde{P}_t$: the transition matrix P,
    - where the first $\tilde{s}(\mu_t) - 1$ rows are replaced by 0s, (**??? Why replaced by 0s and the other not replaced by 0s**)
      - where $\tilde{s}(\mu_t)$ is the threshold of the entry and exit rule when the variance-covariance of the ϵ_{t+1} is 0 (**???**).

Proof.

From Theorem 1, by taking $\mathbb{V}ar[\epsilon_{t+1}] = 0$ and dividing both sides by M, we get this corollary.

3. Under this special case,
   - the law of motion for the distribution of firms across productivity levels is deterministic and (**make sense**)
   - its evolution is given by (15).
4. An immediate consequence of this corollary is that, **under appropriate conditions on the transition matrix** P (**??? So what conditions?**), this **law of motion** converges to a **self-reproducing distribution**.
   - This defines the deterministic steady-state equilibrium of our model
     - where the wage is (**?????? How to understand this sentence?**)
       - constant and
       - conditional on a value for firm-level productivity
         - the value and policy functions that solve the firms' problem are constant (**???**).
   - Following Hopenhayn's (1992), this paper dub the **distribution of firm-productivity levels**, which obtains at the steady-state, as the **stationary distribution**, defined as (**How to understand it???**)

$$\hat{\mu} = (I - \tilde{P}')^{-1}\tilde{P}'G \tag{16}$$

     - where
       - $\tilde{P}$: the transition matrix P,
         - where the first $s^*(\hat{\mu}) - 1$ rows are now replaced by 0s to account for equilibrium entry and exit dynamics (**??? Why replaced by 0s and the other not replaced by 0s**);
           - s^*: the steady-state value of the entry/exit thesholds.
     - Note that this distribution does not depend on time t (**???**).

- Despite the **presence of the idiosyncratic shocks**, the **mass of firms at each productivity level** is **constant**.
  - The presence of idiosyncratic shocks imply firms transiting across **productivity states** and **eventual exit**.
5. Taking stock (**???**), this paper have derived a law of motion for any finite number of firms (**Theorem 1**) and shown that, generally, the distribution of firms across productivity levels is time-varying in a stochastic fashion (Also **Theorem 1**).
   - Corollary 1 implies that, in the continuum case, the **distribution** converges to a **stationary object** and, as a result, there are **no aggregate fluctuations**.

3 Aggregate state dynamics under Gibrat's Law

3.0 Maps to this section

1. In this section, they analyze a special case of the Markovian process driving firm-level productivity: **random growth dynamics**.
2. With this assumption (special case mentioned above) in place, they then solve for the **law of motion of aggregate productivity** up to the **policy functiion** on firm entry and exit (**???**).
   - By solving for this law of motion, they are then able to characterize how **aggregate fluctuations**, **aggregate persistence** and **time-varying aggregate volatility** arise as an **endogenous feature** of **equilibrium firm dynamics**.
3. They start by specializing the general Markovian process driving the evolution of firm-level productivity to the case of random growth (**3.1**).
   - After exploring the firm-level implications of this assumption, they revisit the steady-state results described in the the previous section (**3.2**).

3.1 Special Case of random growth: driving the evolution of firm-level productivity

Assumption 1 (A restriction on general Markov process)

Firm-level productivity evolves as a **Markov Chain** on the state space $\Phi = \{\varphi^s\}_{s=1,\cdots,S}$ with transition matrix

$$P = \begin{pmatrix} a+b & c & 0 & \cdots & \cdots & 0 & 0 \\ a & b & c & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b+c \end{pmatrix} \tag{17}$$

1. This a restriction on the general Markov process P in section 1.
   - It provides

        - a **parsimonious parametrization** (**???**) for the evolution of firm-level productivity
        - by only considering, for **each productivity level**, (**??? TBC**)
            - the **probability of improving**, c,
            - the **probability of declining**, a, and
            - their complement, $b = 1 - a - c$, the **probability of remaining at the same productivity level**.
    - This process also embeds the assumption that (**How to understand the barriers??? Is it saying the probability of 0s and nonzeros.**)
        - there are **reflecting barriers** (**?**) in productivity, both at the top and at the bottom, inducing a **well-defined maximum and minimum level** (**?**) for firm-level productivity.
    - This simple parametrization will be key in obtaining the **closed-form results** (**?**) below.
2. The Markovian process defined in Assumption 1 has been first introduced by Champernowne (1953) and Simon (1955) and studied extensively in Cordoba (2008).
    - For completeness (**??**), this paper also summarize the properties (**properties of what?**) proved in the latter.

Properties 1 [Cordoba 2008]

For a given firm i at time t with productivity level $\varphi^{s_{i,t}}$ with $s_{i,t} \neq 1, S$ (**how to understand it???**) that follows the Markovian process in Assumption 1 (**How to understand the whole sentence?**), then we have the following:

- The **conditonal expected growth rate** and **conditional variance of firm-level productivity** are given by (**???**)

$$\mathbb{E}[\frac{\varphi^{s_{i,t+1}} - \varphi^{s_{i,t}}}{\varphi^{s_{i,t}}}|\varphi^{s_{i,t}}] = a(\varphi^{-1} - 1) + c(\varphi - 1) \tag{18}$$

$$\mathbb{V}ar[\frac{\varphi^{s_{i,t+1}} - \varphi^{s_{i,t}}}{\varphi^{s_{i,t}}}|\varphi^{s_{i,t}}] = \sigma_e^2$$

    - where σ_e^2 is a constant. (**What is φ and σ_e???**)
    - Both the conditions expected growth rate and the conditional variance are independent of i's productivity level, $\varphi^{s_{i,t}}$. (**Because $\varphi^{s_{i,t}}$ does not go into the expression?**)
- As $t \to \infty$, the probability of firm i having productivity level φ^s is (**???**)

$$\mathbb{P}(\varphi^{s,t} = \varphi^s) \to_{t \to \infty} K(\varphi^s)^{-\delta} \tag{19}$$

    - where
        - $\delta = \frac{\log(a/c)}{\log \varphi}$; (**?**)
        - K: a normalization constant.
    - Therefore, the stationary distribution of the Markovian Process in Assumptioin 1 is Pareto with tail index $\delta = \frac{\log(a/c)}{\log \varphi}$. (**Pareto distribution? ??? TBC**)

3. In short, Cordoba (2008) shows that the **Markov process in Assumption 1** is a convenient way to obtaint **Gibrat's law** (**???**) on a **discrete state space**.
    - In particular, Cordoba (2008) shows that **whenever** (**???**) firm-level productivity follows this process, its conditional expected growth rate and its conditional variance are

independent of the current level (**which is the result of the part 1 of the property 1??**)

4. The above assumption yields a tractable way of handling firm dynamics over time. (**The cardinality of the state space is S?**)
    - At several points of the analysis below, this paper will also be interested in understanding how the economy behaves with an **ever larger number of firms**.
        - This raises the question of whether the **maximum possible level of firm-level productivity** should be **kept fixed**. (**???**)
            - If this was the case, and given the **tight link between size and productivity** (**?**) implied by this paper's model, **increasing the number of firms** would imply **a constant absolute size of the large firms** (**??? TBC**).
            - As di Giovanni and Levchenko (2012) show this (**the conclusion above**) is counter-factual (**??? TBC**):
                - in cross-country data, whenever the **size of the economy increases** (**# of large firms increases???**), the absolute size of the top 10 firms in the economy increases.
                - To accord with this evidence, in the following assumptions, this paper **allow the maximum productivity-level to increase with the number of firms**.

Assumption 2

Assume that $\varphi^S = ZN^{\frac{1}{\delta}}$, for some constant Z.

5. This assumption restricts the **rate** at which the **maximum-level of productivity** scales with the **number of firms** (**? Z?**).
    - To understand **why this is a natural restriction to impose**,
        - First note that the **stationary distribution of the Markovian process in Assumption 1** discussed above is also the **cross-sectional distribution of a sample of firms of size N** (**???**).
            - Since the former (**?**) is power-law distributed (**???**), so is the latter (**???**).
        - Second, from Newman (2005), the **expectation** of the **maximum value of a sample N of random variables** drawn from **a power law distribution with tail index δ** is proportional to $N^{\frac{1}{\delta}}$ (**???**).
            - Under **Assumption 2**, for **any sample of size N** following the Markovian process in **Assumption 1**, the **stationary distribution of this sample** is **Pareto distributed with a constant tail index δ. 2** (**??? Pareto distribution? Why**)

3.2 Steady-state equilibrium characterization

1. With the above two assumptions in place, this paper now provide a **detailed description of the steady-state equilibrium** (**???**).

- They start by characterising further the **stationary distribution** in **Corollary 1**, which they are now able to solve in **closed-form** (**???**).
- Then, they present a **full solution of the firm's problem** by deriving the **policy and value functions** in the **steady-state**. (**TBC**)

2. First, in **Corollary 2**, this paper study the limiting case
   - when the **number of firms goes to infinity** under
     - **Assumption 1** and
     - for $S \to \infty$ (**??? continuum case of productivity levels?**)
   - proof of this corollary is in two steps:
     - first, solve **closed form** (**???**) for the **stationary distribution**, given a **maximum level of productivity** φ^S under **Assumption 1**;
     - Second, take the **limit of this distribution** when the **number of productivity bins,** S**, goes to infinity**.

Corollary 2

Assume **Assumption 1** holds.

If the **potential entrants' productivity distribution is Pareto (i.e.,** $G_s = K_e(\varphi^s)^{-\delta_e}$**)**, then as $S \to \infty$, the **stationary productivity distribution**(**???**) **converges point-wise to** (**??? A sequence of function converges to a function**):

$$\hat{\mu}_s = K_1(\frac{\varphi^s}{\varphi^{\bar{s}^*}})^{-\delta} + K_2(\frac{\varphi^s}{\varphi^{\bar{s}^*}})^{-\delta_e} \ for\ s \geq \bar{s}^* \tag{20}$$

- where
  - $\bar{s}^*$: the steady-state entry/exit thesholds for $S \to \infty$;
  - $\delta = \frac{\log(a/c)}{\log \varphi}$ (**??? Same as above, but ???**);
  - K_1, K_2: constants, independent of s.

Proof. (TBC)

3. The stationary productivity distribution for surviving firms (i.e. for $s \geq \bar{s}^*$), is a mixture of two Pareto distributions:
   - the stationary distribution of the Markovian process, given by **Assumptiion 1** with tail index δ, (**the first term of RHS in (20); also why we need property 1?**) and
   - The potential entrant distribution with tail index δ_e.
4. The first of these distributions (**1st one in 3. above**) is a consequence of **Gibrat's law** and **a lower bound on the size distribution** (**??? TBC**).
   - This works in a similar way to the existent **random growth literature** (**e.g.?**).
   - In the context of this paper's model, this **lower bound friction** (**Refer to the lower bound on the size distribution?**) results from **optimal entry and exit decisions by firms**. (**make sense by the following example,**)
     - Every period, there is a number of firms whose productivity draws are low enough to induce to exit.

            - These (firms) are replaced by low-productivity entrants inducing brunching around the exit/entry threshold, $\hat{s}^*$, as in Luttmer (2007, 2010, 2012).
                - Unlike Luttmer, entrants in this paper can **enter at every productivity level**, according to a Pareto distribution (**??? TBC**).
                    - This leads to the second term in the productivity distribution (20). (**??**)
5. While the **corollary 2** characterises the **stationary firm-level productivity distribution**, it is immediate to apply these results to the **firm size distribution**. (**???, TBC**)
    - This is because the **firm size distribution maps one to one to** μ_t. (**???**)
        - To see this, recall that the output of a firm with productivity level φ^s is given by:

$$y_s = (\varphi^s)^{\frac{1}{1-\alpha}} (\frac{\alpha}{\bar{w}})^{\frac{\alpha}{1-\alpha}} \tag{21}$$

            - where
                - $\bar{w}$: the limit of the steady-state value of the wage when S goes to infinity.
        - Therefore, in the steady-state, the **number of firms** of size y_s is given by μ_s (**??**).
    - **Corollary 2** implies that
        - For **sufficiently large firms**, the tail of the firm size distribution is Pareto distributed with tail index given by $\min\{\delta(1-\alpha), \delta_e(1-\alpha)\} = (1-\alpha)\min\{\delta, \delta_e\}$ (**???**).
            - For **high productivity levels (i.e., for large s)**, the **tail of the productivity distribution** is given by the **smaller tail index**, i.e., the **fattest-tail** (**?**) distribution among the two.
        - The **discrepancy** (**???**) between the **firm (output) size distribution** and the **firm productivity distribution** is governed (**?**) by the **degree of returns to scale** α (**???**).
            - The **higher** the **degree of returns to scale**, the lower (**???**)
                - the ratio between the productivity distribution tail, δ, and
                - the firm size distribution tail $\delta(1-\alpha)$.
                    - $\delta(1-\alpha)$ is an **observable quantity** since it is the **tail of the firm size distribution** (for **sufficient large firms** and as soon as $\delta_e < \delta$ (**?? Why we need these?**)) (**??? Tautology?**)
                    - They are using this feature ($\delta(1-\alpha)$ **is an observable quantity???**) in the **calibration of the quantitative section** (**Section 4?**) of this paper.
6. The result above (**Corollary 2**) characterizes the **firm productivity** (**distribution???**) and **firm size distribution** in the steady-state, thus pinning down the **aggregate state** (**?**) up to $\bar{s}^*$, the **firm's policy function** (**?**).
    - Then, they will solve for **this policy function** along with the **associated value function for incumbents**, thus presenting **a full characterisation of the steady-state equilibrium**.

Proportion 1 (?? TBC)

Under **Assumption 1**, when $S \to \infty$ and φ is small enough, then in the stationary equilibrium,

1. the **value function of a firm facing productivity level φ^s and wage $\bar{w}$** is equal to

$$V(\mu, \varphi^s) = \frac{-c_f}{1-\beta}[1-\beta r_2^{[s-\bar{s}^*+1]^+}] + \frac{1-\alpha}{1-\rho\beta}(\frac{\alpha}{\bar{w}})^{\frac{\alpha}{1-\alpha}}(\varphi^{\frac{1}{1-\alpha}})^s[1-\rho\beta(\frac{r_2}{\varphi^{\frac{1}{1-\alpha}}})^{[s-\bar{s}^*+1]^+}] \quad (22)$$

   - where
     - $[x]^+ = Max(x, 0)$ (**??**);
     - r_2: a constant defined in the appendix bounded above by 1 (**??**).
2. the policy function is characterized by the threshold (**??**)

$$\bar{s}^* = [(1-\alpha)\log(\frac{c_f(1-r_2)(1-\rho\beta)}{\rho(1-\beta)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(1-r_2\varphi^{\frac{-1}{1-\alpha}})})(\log\varphi)^{-1} + \alpha(\log\bar{w})(\log\varphi)^{-1}] \quad (23)$$

   - where
     - $[x]$: the ceiling function, i.e., the least succeeding integer of x (**??**).

Proof. (TBC)

7. For a sketch of the proof, note that
   - Under **Assumption 1**, the **next period's value of (surviving) incumbents** can only take **one of three values** (**??? TBC, why not 4 values?**), depending on their **idiosyncratic productivity realisation** (**???**).
     - This implies that
       - **solving for the value function** is equivalent to **solving a second-order difference equation** (**???**).
     - The constant r_2 in the proposition is the **relevant solution of this equation** (**???**).
8. Intuitively, the value function of an incumbent is simply the present value of instaneous profits adjusted by exit risk.
   - Note that
     - The first term of the value function reflects the present discounted value of fixed operating costs. (**??? TBC**)
     - The second term reflects the present discounted value of variable profits. (**???TBC**)
   - In turn, each of these terms are expressed as the product of (**???TBC**)
     - the **present discounted value** that would obtain in **a world without exit**, and
     - an **adjustment for the exit risk**, encoded in the square brackets terms.
       - To understand the latter, note that
         - The **exit probability** of **an incumbent currently in a high idiosyncratic productivity bin** is **low**.
           - In this case, the term in square brackets is close to 1.
             - i.e., the exit-risk adjustment of future profits is small.
         - The converse holds for a low productivity incumbent.
9. The second result (policy function) in the proposition 1 gives the policy function for entry and exit decisions.
   - **A large operating fixed cost,** c_f, or **a large equilibrium wage,** $\bar{w}$, **increase** the **value of the entry/exit threshold,** $\bar{s}^*$. (**???TBC**)

- Intuitively, either will **reduce the value of being an incumbent** and thus rendering it **more likely that a firm exits** or **a potential entrant declines to enter**.
- As we will see below in section 3.4, much of this intuition carries through in a special case where the aggregate state is time-varying (**???**).

3.3 A complete analytical characterization for an Economy without Entry and Exit

1. Having described the steady-state equilibrium, they are now interested in characterizing the **dynamics of aggregate productivity** and **output** under **aggregate uncertainty**.
    - Start by analysing the simpler case of **an economy without entry and exit**.
        - This allows us to derive
            - the **law of motion** (**theorem 1**) of the **aggregate state** analytically and;
            - as a byproduct, **closed form-expressions** for **aggregate output dynamics** which exhibit **persistence** and **time-varying volatility**.
                - Their key results will generalize to the case with entry and exit (**3.4**).
2. Economies without entry and exit are a special case of the setup introduced in section 1
    - when (**?? Why this setup make sense for an economy without entry and exit?**)
        - fixed operating costs, c_f, are 0 and;
        - entry costs, c_e, are large enough.
    - In this case, firms (incumbents) always (**??**)
        - have a **positive present discounted value of profits**, irrespective of **their current idiosyncratic productivity draw** and;
        - **never choose to exit** (**because of the positive present discounted value of profits**).
    - For a **large enough entry cost**, **no potential entrant** will (**?? TBC**)
        - choose to start producing either, irrespective of the **current aggregate state and idiosyncratic productivity signal** (**negative present discounted value of profits**).
    - Therefore, the **total number of firms** is fixed at N, (**?? According to the reasoning above**)
        - which is now a **parameter of the model** rather than **an endogenous variable to be solved for**.
3. Without entry and exit, this paper start by noting that the **law of motion for the aggregate state**, i.e., the **productivity distribution**, is a special case of **Theorem 1** where Equation (13) is

$$\mu_{t+1} = P' \mu_t + \epsilon_{t+1} \tag{24}$$

    - where
        - P: the **transition matrix for firm-level productivity**,
        - ϵ_{t+1}: a random vector with mean zero and variance-covariance matrix $\sum(\mu_t) = \sum_{s=1}^{S} \mu_{s,t} W_s$

- $W_s = diag(P_{s,.}) - P'_{s,.}P_{s,.}$
  - $P_{s,.}$: the s^{th}-row of the transition matrix P.
- Without endogenous exit, relative to Theorem 1,
  - the transtion matrix across productivity states, P is no longer time-varying and,
  - under no entry, they (**the transition matrix???**) **no longer have to keep track of** the **contributions of entrants** to the **law of motion of the aggregate state**.
  - The variance of the ϵ_{t+1} is still time-varying as it remains a function of the lagged realization of the productivity distribution.

4. As (24) shows, the law of motion of the aggregate state μ_t is stochastic.
   - i.e., μ_t is a random vector.
   - The following proposition (**proposition 2**) describes the (unconditional) behaviour of this random vector.
     - Thus the stochastic properties of the aggregate variable of the model with no entry/exit.

Proposition 2

For the no entry and exit case and under **Assumption 1 (special case)**,

1. the unconditional mean of μ_t is $\mu = (\mu_1, \cdots, \mu_s, \cdots, \mu_S)'$ and is given by

$$\mu_s = \mathbb{E}[\mu_{s,t}] = N\frac{1-\varphi^{-\delta}}{\varphi^{-\delta}(1-(\varphi^S)^{-\delta})}(\varphi^s)^{-\delta} \tag{25}$$

   - where $\delta = \frac{\log(a/c)}{\log(\varphi)}$.

2. The **unconditional variance-covariance matrix** of μ_t is

$$\mathbb{V}ar[\mu_t] = \sum_{k=0}^{\infty}(P')^k(\sum_{s=1}^{S}\mu_s W_s)P^k \tag{26}$$

   - where
     - P: the transition matrix for firm-level productivity,
     - $W_s = diag(P_{s,.}) - P'_{s,.}P_{s,.}$,
       - $P_{s,.}$: denotes the s^{th}-row of the transition matrix P in **Assumption 1**.

Proof. (??? TBC)

5. This proposition shows that the **aggregate state** μ_t, fluctuates around its **mean**.
   - In turn, this mean is the stationary distribution of firm productivity μ,
     - which is Pareto distributed with tail index δ. (**???**)
       - It is simply a particular case of **Corollary 2** for the case with no entry and exit. (**???**)
   - The second part of the proposition shows that **deviations of μ_t from** μ are governed by the **variance-covariance matrix,** $\mathbb{V}ar[\mu_t]$.
     - which
       - takes the conditional variance-covariance matrix of μ_t and

        - adjust it by the **persistence** (**??? how to understand the persistence?**) in the law of motion of μ_t, as given by the transition matrix P.
6. From the expressions for aggregate TFP, (8) and the equilibrium wage, (10), it is immediate that A_t is a sufficient stastistic (**?**) for the relative price of labor in the model (**???**).
    - By deriving the law of motion for A_t, we are therefore able to characterize the **law of motion** for **aggregate prices**, and **output** (**???**).

Theorem 2

Under **Assumption 1**, then

1. **Aggregate productivity dynamics** is given by

$$A_{t+1} = \rho A_t + O_t^A + \sigma_t \varepsilon_{t+1} \tag{27}$$

   and

$$\sigma_t^2 = \varrho D_t + O_t^\sigma \tag{28}$$

2. **Aggregate output dynamics** (in percentage deviation from its steady-state value, $\hat{Y}_t$) is given by

$$\hat{Y}_{t+1} = \rho \hat{Y}_t + \kappa \hat{O}_t^A + \psi \frac{\sigma_t}{A} \varepsilon_{t+1} \tag{29}$$

    - where
        - $\mathbb{E}[\varepsilon_{t+1}] = 0$;
        - $\mathbb{V}ar[\varepsilon_{t+1}] = 1$.
        - Parameters
            - $\rho = a\varphi^{\frac{-1}{1-\alpha}} + b + c\varphi^{\frac{1}{1-\alpha}}$,
            - $\varrho = a\varphi^{\frac{-2}{1-\alpha}} + b + c\varphi^{\frac{2}{1-\alpha}} - \rho^2$.
        - $D_t = \sum_{s=1}^S ((\varphi^s)^{\frac{1}{1-\alpha}})^2 \mu_{s,t}$.
        - O_t^A and O_t^σ: a correction for the upper and lower reflecting barriers in the idiosyncratic state space.(**???**)
        - A: the steady-state value of the aggregate productivity A_t.
        - κ: a constant defined in the Appendix, (**???**)
        - $\psi = (1 - \frac{\alpha}{\gamma(1-\alpha)+1})$ is such that $\hat{Y}_t = \psi \hat{A}_t$, (**???**)
            - where $\hat{A}_t$ (resp. $\hat{O}_t^A$) is the percentage deviation from steady state of A_t (resp. O_t^A).

Proof. (??? TBC)

7. **Theorem 2** provides a full description of aggregate productivity and aggregate output dynamics in this paper's model.
    - The first part of the theorem states that
        - The aggregate productivity tomorrow is the sum of
            - ρA_t, the expected aggregate productivity (**???**) and

            - $\sigma_t \varepsilon_{t+1}$, a mean zero aggregate productivity shock with time-varying volatility as instructed by the term D_t. (**???**)
            - O_t^A, a correction term (**???**)
                - arising from having imposed bounds on the state-space.
                - It vanishes as the state-space bounds increase;
                - This paper relegate its precise functional form and further discussion of this term to the appendix.
    - The second part of the theorem states that
        - Given the **law of motion for the aggregate productivity**, it is straightforward to characterize the **law of motion for equilibrium output**. (**??**)
            - The **dynamic properties of A_t**, **persistence** and **time-varying volatility**, carry though to the **percentage deviation of aggregate output from its steady-state value, $\hat{Y}_t$**.
                - This is because: as in the aggregate, this model behave as a one factor RBC(**?**) model (**???**).
8. **Theorem 2** implies that
    - **Aggregate productivity** and **aggregate output** are **persistent** and **exhibit time-varying volatility**. (**???**)
        - Intuitively, since there are **no aggregate shocks in this model**, **aggregate persistence** (**?**) simply reflects **firm-level persistence in productivity**.
    - To understand the time-varying nature of aggregate volatility, it is easiest to consider an extreme case with only two firms.
        - In that case, **aggregate productivity** would simply reflect the **weighted sum of the two firms' productivity**.
        - The **volatility of the aggregate productivity** would then depend on **(the square of) the relative size of these two firms**,
            - which will be **time varying** as they are subject to **independent shocks**.
        - The term D_t in **Theorem 2** gives the generalization of this intuitive argument for a **large but finite** number of firms.(**??**)
9. To better understand this result, and building on the expressions in **Theorem 2**, this paper now details (**??? Propostion 3?**)
    - how **persistence in aggregates** depend on **micro-level parameters**.
        - To understand the **persistence of aggregate output,** ρ, note that
            - At the steady-state, ρ satisfies $\mathbb{E}_t[y_{i,t+1}|y_{i,t+1}] = \rho y_[i,t]$ for a given firm i (**???**)
                - i.e., aggregate persistence is nothing but **firm-level persistence** (**???**).
            - Building on this, the following proposition further characterizes how **aggregate persistence** depends on **a paremters governing firm-level dynamics**.
    - Then **(firm-level) origins of time-varying volatility in aggregates**.

Proposition 3

Let $\delta = \frac{\log \frac{a}{c}}{\log \varphi}$ be the tail index of the stationary productivity distribution as in **Corollary 2**.(**???** **TBC**)

If $\delta(1-\alpha) \geq 1$, then the persistence of the aggregate output ρ satisfies the following properties: (**???**)

1. Holding δ constant, aggregate persistence is increasing in firm-level persistence:
$$\frac{\partial \rho}{\partial b} \geq 0 \tag{30}$$
2. Holding b constant, aggregate persistence is decreasing in the **tail index of the stationary productivity distribution**
$$\frac{\partial \rho}{\partial \delta} \leq 0 \tag{31}$$
3. If the productivity distribution is Zipf (**?**), aggregate productivity dynamics contain a unit root
$$if\ \delta = \frac{1}{1-\alpha},\ \rho = 1 \tag{32}$$

Proof. (??? TBC)

10. To interpret the **condition** under which the **proposition** is valid, recall that $\delta(1-\alpha)$ gives the **tail index** of the **stationary firm size distribution**.
    - The proposition applies to **Pareto distributions** that are **(weakly) thinner than Zipf** (**???**).
    - According to (i),
        - the **persistence of the aggregate state** (and hence aggregate productivity, wages and output) is **increasing in** the **probability,** b (**???**),
            - that firms do not change their productivity from one time period to the other. (**???**)
            - Intuitively, the higher is firm-level productivity persistence, the more persistent are aggregates. (**???**)
    - According to (ii) in the proposition 3,
        - **aggregate persistence** will **decrease** with the **tail index of the stationary firm-level productivity distribution**. (**???**)
            - To understand this, note that this **tail index** is given by $\frac{a}{c}$. (**???**)
            - The **thinner the tail**, the **larger this ratio** is (**???**)
                - and thus, the **larger** is the **relative probability of a firm having a lower productivity tomorrow**.
            - This therefore induces **stronger mean reversion** in **productivity (and size)** at the **firm level** which, in turn, leads to **lower aggregate persistence** (**???**).
            - Thus, a fatter tail in the size distribution implies heightened aggregate persistence. (**??**)
    - In the limiting case where the **stationary size distribution** is given by **Zipf's law** ($\delta(1-\alpha) = 1$ in case (iii)), **aggregate persistence** is equal to 1. (**???**)
        - That is, Zipf's law implies **unit-root type dynamics in aggregates** (**???**).

11. They are now interested in understanding how **aggregate volatility (and its evolution)** (?) depend on the **parameters driving the micro-dynamics** (?).
    - To do this, they find it convenient to first write the **expression for the conditional volatility of $\hat{Y}_{t+1}$** as (**???**)

$$\mathbb{V}ar_t[\hat{Y}_{t+1}] = \psi\frac{\sigma_t^2}{A^2} = \psi\varrho\frac{D}{A^2}\frac{D_t}{D} + \psi\frac{O^\sigma}{A^2}\frac{O_t^\sigma}{O^\sigma} \tag{33}$$

        - where
            - A and D are the steady-state counterparts of A_t and D_t.
        - Interpretation of these objects,
            - $D_t = \sum_{s=1}^{S}((\varphi^s)^{\frac{1}{1-\alpha}})^2\mu_{s,t}$ is proportional to the **second moment of the firm size distribution at time** t, which is a well-defined measure of dispersion. (**???**)
            - D is proportional to the steady-state dispersion in firm size.
            - Finally, note that at the steady-state, ϱ satisfies $\varrho = \mathbb{V}ar[\frac{y_{i,t+1}}{y_{i,t}}]$.
                - That is, ϱ is given by the variance of firm-level output growth at the steady-state.
            - Thus, **aggregate volatility** is simply **firm-level volatility** scaled by the **current level of dispersion** in the economy. (**???**)
                - As the latter varies over time, so does aggregate volatility.
    - Note also taht the expression above implies that the **unconditional expectation of conditional variance** can be written as

$$\mathbb{E}\frac{\sigma_t^2}{A^2} = \varphi\varrho\frac{D}{A^2} + \varphi\frac{O^\sigma}{A^2} \tag{34}$$

    - With these two objects in place, the following proposition characterizes how **aggregate volatility** and **its dynamics** (?) depend on the **primitives of the model** (**???**).

Proposition 4

Let $\delta = \frac{\log\frac{a}{c}}{\log\varphi}$ be the tail index of the stationary productivity distribution as in **Corollary 2**.(**??? TBC**), then

1. Under **Assumption 2** and if $1 < \delta(1-\alpha) < 2$, the **unconditional expectation of the variance of aggregate output** satisfies

$$\mathbb{E}[\frac{\sigma_t^2}{A^2}] \sim_{N\to\infty} \frac{\varrho G_1}{N^{2-\frac{2}{\delta(1-\alpha)}}} \tag{35}$$

    - where
        - G_1: a function of model parameters but independent of N.
2. The **dynamics of conditional variance of aggregate output** depends on the **dispersion of firm size**:

$$\frac{\partial\mathbb{V}ar_t[\hat{Y}_{t+1}]}{\partial D_t} = \frac{\psi\varrho}{A^2} \geq 0 \tag{36}$$

Proof. (??? TBC)

12. Part (i) of **proposition 4** characterizes the **average level of volatility of aggregate output** in this model. (**?**)
    - It builds on the results that, under **random growth dynamics for firm-level productivity**, (**???**)
        - the **stationary incumbent size distribution** is **Pareto distributed**.
    - The **assumption on** $\delta(1-\alpha)$ ensures that
        - this **distribution is sufficiently fat tailed** (**??? TBC**).
    - The $\sim_{N\to\infty}$ notation means that
        - in expectation, the **conditonal variance of the aggregate growth rate scales with** N, **the number of firms**, at a rate that is equal to the rate of the expression on the right hand side. (**???**)
13. The Key conclusion of the first part of **Proposition 4** is therefore that,
    - for $1 < \delta(1-\alpha) < 2$, the **variance of aggregate output** scales at a **slower rate than** $\frac{1}{N}$.
        - Recall that the latter (**?** $\frac{1}{N}$**?**) would be the **rate of decay** implied by a **shock-diversification argument** relying on **standard central limit arguments**.
        - This is **not** the case when the **firm size distribution is fat tailed**, as it is here.
        - Rather, as the proportion makes clear, the **rate of decay of aggregate volatility** depends on the **tail index of the size distribution of firms**.
            - The **closer** is this **distribution to Zipf's law**, the **slower** is the **rate of the decay**. (**???**)
14. This proposition thus generalized the main result (**???**) in Gabaix (2011) to an environment where (**???**)
    - **firm dynamics** are the **result of optimal size decisions**, given the **idiosyncratic productivity process** characterized by the **Markovian process** in **Assumptioin 1** and
    - the **Pareto distribution of firm sizes** is an **equilibrium outcome** consistent with **optimal firm decisions**.
15. Part (ii) of **propostion 4** shows that
    - the **evolution of aggregate volatility over time** mirrors that (**evolution?**) of D_t. (**???**)
        - the evolution of aggregate volatility over time is the **conditional variance of aggregate output**.
    - As dicussed above, D_t is proportional to the **second moment of the firm size distribution at time** t.
    - Thus, whenever the **firm size distribution at time** t is **more dispersed** than the **stationary distribution (**$D_t > D$**)**, **aggregate volatility** is **higher**.
16. The second part of the proposition is therefore related to a literature looking at the **connectiion between micro and macro uncertainty** (Bloom et al, 2018 and Kehrig, 2015).
    - **Consistent** with the results of this literature, the proposition yields a **direct, positive, link** between the **two levels of uncertainty**.
    - **Unlike** this literature, this link between the **cross-sectional dispersion of micro units** and **conditional aggregate volatility** is **endogenous** and **emerges without resorting to exogenous aggregate shocks influencing the first and second moments of**

firms' growth. (**??**)

3.4 Aggregate persistence and volatility in Economies with Entry and Exit

1. This paper now extend the characterization of aggregate persistence and volatility to economies with endogenous, forward-looking, firm entry and exit decisions.
    - Relative to the no entry and exit case discussed above, the **law of motion** for the **firm productivity distribution** now depends on the **current realization of productivity signals across entrants** and **a potentially time-varying entry and exit threshold** (**???**).
        - The latter (**? threshold?**) implies that, without further assumptions, they can no longer solve for the **full solution of the model** (**???**),
            - which now **includes a policy function** describing how **firms' entry and exit decisions** depends on the **aggregate state**.
        - Despite this, they are able to show that
            - their **analytical expressions** for the **persistence and volatility of aggregate output generalise** to the case with entry and exit.
            - In particular, whenever the entrant productivity signal distribution is thinner tailed than the productivity distribution of incumbents, they will show that
                - the **contribution of entry and exit to aggregate volatiltiy** is **second-order** (**??? how to understand second-order?**).
2. To show this, they follow the same steps (**?**) in 3.3.
    - Recalling the general law of motion for the aggregate state variable in **Theorem 1**,

$$\mu_{t+1} = (P_t^*)'(\mu_t + MG) + \epsilon_{t+1} \tag{37}$$

        - It makes clear that the law of motion of the aggregate state depends on
            - the distribution of productivity signals for potential entrants, G and
            - the current (endogenous) threshold for entry and exit decisions, $s^*(\mu_t)$.
        - This is because
            - P_t^*, the ***transition matrix of firms across productivity states**, also encodes **entry and exit transitions**.
    - Specializing this setup to the Gibrat's law case (i.e. **Assumption 1**), it is then straightforward to show that
        - this general law of motion of the aggregate state implies the following dynamics for aggregate productivity:

$$A_{t+1} = \rho A_t + \rho E_t(\varphi) + O_t^A + \sigma_t \varepsilon_{t+1} \tag{38}$$

        and

$$\sigma_t^2 = \varrho D_t + \varrho E_t(\varphi^2) + O_t^\sigma \tag{39}$$

        - Relative to **Theorem 2**, the contribution of entry and exit appears in the net entry terms, E_t.

- These terms give the difference between the entry and exit contributions for both the level and the volatility of aggregate productivity with

$$E_t(x) = (M \sum_{s=s^*(\mu_t)}^{S} G_s(x^s)^{\frac{1}{1-\alpha}}) - ((x^{s^*(\mu_t)-1})^{\frac{1}{1-\alpha}} \mu_{s^*(\mu)-1,t}) \quad (40)$$

  - Where
    - x is either φ or φ^2.
  - Intuitively,
    - the first term of the expression gives the contribution of today's entrants,
    - Second term gives that of exciters.
  - As a result, **aggregate productivity of incumbents tomorrow,** A_{t+1}, now depends on $\rho E_t(\varphi)$**, the expected aggregate productivity of today's net entrants**, conditional on their **survival** (ρ **is the survival rate?**).

3. It is worth noting that entry and exit decisions do not alter the fact that, as in 3.3,
   - **aggregate productivity** is **persistent** and exhibits **time-varying volatility**.
     - Furthermore, given **this law of motion for aggregate productivity**, it is immediate to show that
       - the law of motion for aggregate output inherits these same properties

$$\hat{Y}_{t+1} = \rho \hat{Y}_t \rho \kappa_1 \hat{E}_t(\varphi) + \kappa_2 \hat{O_t^A} + \psi \frac{\sigma_t}{T} \epsilon_{t+1} \quad (41)$$

  - where
    - κ_1 and κ_2: constants defined in the appendix (**???**).
    - ρ: the same parameter as that defined in **Theorem 2**, for the no entry and exit case.
  - It follows that
    - the **persistene of aggregate productivity and output**, depends on **deep parameters governing firm-level dynamics** in the same way as in **Proposition 3**.

4. Turning to the **volatility of aggregate output**, they are not able to generalise **Proposition 4** to the case of entry and exit as follows:

Proposition 5

Let $\delta = \frac{\log \frac{a}{c}}{\log \varphi}$ be the **tail index of the stationary productivity distribution** as in **Corollary 2**. (**??? TBC**), and let δ_e be the **tail index associated with the productivity distribution of potential entrants**, then

1. Under **Assumption 2** and if $1 < \delta(1-\alpha) < 2$, the **unconditional expectation of the aggregate variance** satisfies

$$\mathbb{E}[\frac{\sigma_t^2}{A^2}] \sim_{M\to\infty} \frac{\varrho G_1}{N^{2-\frac{2}{\delta(1-\alpha)}}} + \frac{\varrho G_2}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}} \quad (42)$$

   - where

- G_1 and G_2: a function of model parameters but independent of N and M, (**respectively?**).

2. The **dynamics of conditional variance of aggregate output** depends on the **dispersion of firm size**:

$$\frac{\partial \mathbb{V}ar_t[\hat{Y}_{t+1}]}{\partial D_t} = \frac{\psi \varrho}{A^2} \geq 0 \tag{43}$$

Proof. (??? TBC)

5. Relative to the simpler case without entry and exit, they now need to **take a stand on the distribution of productivity signals across entrants** (???).
   - They
     - follow the earlier approach in **Corollary 2** and
     - assumes it to be **Pareto distributed with tail parameter** δ_e.
   - With this assumption in place, the proposition implies that (**???**)
     - given for the **no entry and exit** case, the **characterization of aggregate volatility** carrries through to the **current, more general setup**.
6. Part (ii) of the **proposition 5** remains unchanged, while part (i) of the proposition shows that
   - the **variance of aggregate output fluctuations** still declines at a slower rate than $\frac{1}{N}$. (**???**)
     - Relative to the no entry and exit case, this **rate of decay** now depends on the **tail indexes of the size distributions** of **both incumbents** $\delta(1-\alpha)$ **and entrants** $\delta_e(1-\alpha)$.
       - In particular, whenever the size distribution of entrants (i.e., whenever $\delta < \delta_e$), the **first term in the expression** for aggregate volatility dominates (**??**).
       - In this case, asymptotically, the **rate of decay of aggregate volatility** will be a function of the **tail index of incumbents**, at the first order (**? 1st-order**), and they recover the result in **proposition 4**.
         - Intuitively, whenever the **size distribution of incumbents** is close to **Zipf's law** (i.e., $\delta(1-\alpha)$ is close to 1, **?**) and **probability of observing very large entrants** is **small**, **aggregate volatility** depends on **large incumbent firms** alone.
           - This implies that,despite the fact that **they cannot solve for entry dynamic explicitly**,
             - they can still **describe the behavior of aggregate volatility**,
               - As the **contribution of entry and exit** is **second order** (**???**) and it suffices to **track the dynamics of large incumbents** (**???**).
           - Further, as they will argue below, this is likely to be the empirically relevant regime, as both conditions (fat-tailed incumbents and relatively thinner tailed entrants) are met in the data (**???**) .
   - Taken together, these results imply that (**??**)

- the voluminous literature building on the framework of Hopenhayn (1992) has **overlooked the potentially non-neglible aggregate dynamics** implied by the model, **even when the number of firms entertained is large**.

3.5 Policy and value functions: a special case

1. As discussed above, with endogenous entry and exit decisions, and without further assumptions, it is not possible to make further headway analytically. (**??**)
    - This is because
        - **entry and exit decisions** are **forward looking** and
        - **firms** therefore need to forecast the **future productivity distribution** or equivalently, **future wages**.
2. Despite the fact that, as shown in the section 3.4,their results regarding **aggregate persistence and volatility** do not depend on the **extensive margin**,
    - it is nevertheless useful to understand **entry and exit decisions** on **two counts** (**What & Why useful???**):
        - First, the **behavior of the extensive margin** following **large firm shocks** is of independent **interest**.
        - Second, a **quantitative evaluation of the general model** requires a solution to these **forward-looking decisions**.
    - Technically, the **key challenge** in solving for value and policy functions in this model is that
        - These are **nonlinear functions of the aggregate state variable** μ_t.
            - **while not infinite dimensional** (as in Hopenhayn, 1992), this is **still a large dimensional object** which they **cannot solve for analytically**.
    - Their **proposed solution**, following most the literature on heterogeneous agents, is to **reduce the dimensionality of the state-space**.
3. Unlike most of the literature, this paper does know the **form that the law of motion for A_t takes**. (**???**)
    - This per se (**?**), does not solve their problem (**???**)
        - since E_t and σ_t are still functions of μ_t. (**???**)
    - Mathematically, this means that A_t is not a recursive map: (**???**)
        - A_t maps to **past values of itself** but **also to other moments of the productivity distribution**.
    - A_t becomes a **recursive map** if they make the following assumption (**Assumption 3**): (**???**)

Assumption 3

Assume that, when **forming expectations about future wages**, firms take $\frac{E_t}{A_t}$, $\frac{O_t^A}{A_t}$ and $\frac{\sigma_t}{A_t}$ to

- be **constant** and
- equal to their **stationary equilibrium counterparts**, $\frac{E}{A}$, $\frac{O^A}{A}$ and $\frac{\sigma}{A}$, respectively,

such that the **optimisation problem of the firm** is now

$$V(A_t, \varphi^s) = \pi^*(A_t, \varphi^s) + \max\{0, \beta\mathbb{E}[\hat{V}(A_{t+1}, \varphi^{s'})|A_t, \varphi^s]\} \tag{44}$$

Subject to the **perceived law of motion**

$$A_{t+1} = \rho A_t + \rho\frac{E}{A}A_t + \frac{O^A}{A}A_t + \frac{\sigma}{A}A_t\epsilon_{t+1} \tag{45}$$

4. Under **Assumption 3**, the firm perceives the **moments**, $\frac{E_t}{A_t}$, $\frac{O_t^A}{A_t}$ and $\frac{\sigma_t}{A_t}$, to be **fixed** at their **stationary equilibrium counterparts** such that (45) then follows from (38).
   - Intuitively, **Assumption 3** implies that, (**???**)
     - When **forecasting future wages**, **firms ignore**
       - the **time-varying contribution** of **net entry to aggregate productivity**,
       - the **time-varying nature** of **aggregate volatility**.
     - This in turn implies that (**???**)
       - the **perceived expectation of future wage** depends only on **current productivity** A_t .
         - To see this explicitly, (**???**)
           - note from **Lemma 3** in Appendix B.B9, the **perceived expectation of tomorrow's wage to some power** ξ is

$$\mathbb{E}_t[w_{t+1}^{\xi}] = (\alpha^{\frac{1}{1-\alpha}}\frac{A_t}{M})^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}}\mathbb{E}_t[(\frac{A_{t+1}}{A_t})^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}}] \tag{46}$$
$$= (\alpha^{\frac{1}{1-\alpha}}\frac{A_t}{M})^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}}\mathbb{E}_t[(\rho + \rho\frac{E}{A} + \frac{O^A}{A} + \frac{\sigma}{A}\epsilon_{t+1})^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}}]$$
$$under\ Assumption\ 3$$

           - Therefore, the **conditional expectation (a nonlinear function) of aggregate productivity growth (i.e., the expectation from the RHS of the above equation)** simplifies to a **constant** (**???**).
           - Thus, the **expectation of tomorrow's wage (to some power)** (**LHS**) is solely a **function of** A_t **today**.
     - These admittedly **strong assumptions** allow them to make progress analytically be rendering **the wage forecasting problem tractable**.
       - In particular, this implies that
         - the **perceived aggregate state** in the **value and policy functions** is now A_t (**???**) and delivers **closed form epxressions** for **these objects**, (**???**)
           - which they summarize in the following proposition (**Proposition 6**):

Proposition 6

- Under **Assumptions 1 and 3**,
- when $S \to \infty$,
- given A_t (**perceived aggregate state?**) and **idiosyncratic productivity level** $\varphi^s \geq \varphi^{s^*(A_t)}$

the **value of an incumbent firm** is

$$\hat{V}(A_t,\varphi^s)=\frac{-c_f}{1-\beta}[1-\frac{a\beta}{\tilde{r_2}(\frac{1}{\beta}-\frac{1}{\tilde{\beta}})+a}\tilde{r_2}^{s-s^*(A_t)+1}] \tag{47}$$

$$+\frac{1-\alpha}{1-\rho\tilde{\beta}\alpha}(\frac{\alpha}{w_t})^{\frac{\alpha}{1-\alpha}}(\varphi^{\frac{1}{1-\alpha}})^s[1+\frac{-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\tilde{\beta}\alpha+\rho\beta-\rho\tilde{\beta}\alpha}{\beta(\tilde{r_2}\frac{1}{\beta}-\frac{1}{\tilde{\beta}})+a)}(\varphi^{\frac{1}{1-\alpha}})(\frac{\tilde{r_2}}{\varphi^{\frac{1}{1-\alpha}}})^{s-s^*(A_t)+1}]$$

with the **corresponding policy function** is given by

$$s^*(A_t)=[(1-\alpha)\log(\frac{c_f(1-\frac{a}{\tilde{r_2}(\frac{1}{\beta}-\frac{1}{\tilde{\beta}})+a}\frac{\beta}{\tilde{\beta}_2}\tilde{r_2})(1-\rho\tilde{\beta}\alpha)}{\rho(1-\beta)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(1+\frac{-\beta a\varphi^{\frac{-1}{1-\alpha}}\tilde{\beta}_\alpha-\tilde{\beta}_\alpha\beta}{\beta(\tilde{r_2}(\frac{1}{\beta}-\frac{1}{\tilde{\beta}})+a)}\frac{\tilde{r_2}}{\tilde{\beta}_2})})(\log\varphi)^{-1} \tag{48}$$

$$+\alpha(\log w_t)(\log\varphi)^{-1}]$$

- where
    - $[x]$: the **ceiling function**, i.e., the **least succeeding integer of x,** (**???**) and
    - $\tilde{\beta_\alpha},\tilde{\beta_2}$ and $\tilde{r_2}$: constants defined in the appendix (**???**).

Proof. (TBC ???)

5. The proof of **Proposition 6** follows a **guess-and-verify** strategy.
    - It is easiest to understand this solution to the firms' problem by first noting the similarities with the **steady-state solution** given in **Proposition 1**.
        - In particular, note that the **value and policy functions assume the same form as before**.
            - **Value function**:
                - Its **first** term reflects the **present value** of **fixed operating costs**,
                - Its **second** term reflects the **expected present discounted value** of **profits**.
            - **Policy function**: (**???**)
                - Its **first** term indicates the **contribution of fixed cost considerations** for the **entry and exit deicisions**,
                - Its **second** term reflect **variable profits**.
        - Indeed, it is easy to show that, (**???**)
            - if **wages were fixed across time**, the terms $\tilde{\beta_\alpha}=\tilde{\beta_2}=\beta$ such that the **value and policy functions** in this proposition would exactly **coincide with** those given in the **steady-state equilibrium**. (**???**)
                - Wages were fixed across time: there was **no aggregate uncertainty**. (**?**)
6. The **solution** for **both the value and policy functions** now **reflect the time-varying nature** of **aggregate risk**.
    - That is, even when holding **firm-level productivity** fixed, the **value** and **entry and exit decisions of a firm** are **time-varying** and depend on the **current realisation of wages** (**???**).

- In the **value function**, the **aggregate risk** appears in **two intuitive ways**:
  - **First**, under **aggregate uncertainty**, the **threshold for entry and exit** will now be **time-varying**, as instructed by the policy function (**??**).
    - As such, the terms **adjusting firm value for exit risk**, now depend on the **current value of this threshold**, (**???**)
      - which in turn, depends on the **current realization of wages**.
  - **Second**, the **expected present discounted value of profits** is also **time-varying** and depends on the **current realization of wages**.
- The **policy function** also reflects **aggregate risk**. (**???**)
  - Thus, for the **same value of firm-level productivity**,
    - if the **current realization of wage is high**, the **threshold** for **entry and exit increases**.
  - This intuitive:
    - As shown above, **aggregate productivity** is **persistent** and (**???**)
    - therefore, following a **positive shock** that **increases the current value of wages**,
      - firms expect the **cost of variable inputs** to **remain high** in the future.
    - This implies that the **expected present value of profits** is now **lower**,
      - which implies that **even relatively high productivity firms** might now find it optimal to exit (or not enter) (**???**).

4 Quantitative results

4.0 Map to this section

1. In this section, they present the quantitative implications of this model.
   - They solve the model under the particular case of firm-level random productivity growth (**Assumption 1**)
     - which they discussed in previous section.
   - They first calibrate the steady-state solution of the model to match firm-level moments (**???**).
     - Based on this calibration, they they use the law of motion of the aggregate productivity and **Assumption 3** to solve numerically for the firms' policy function.
     - Using this numerical solution, they quantitatively assess the performance of the model with respect to standard business cycle statistics and inspect the mechanism rendering firm-level idiosyncratic shocks into aggregate fluctuations.
     - They then quantitatively explore the role of large firms in shaping the business cycle.
   - Throughout, they show empirical evidence that is consistent with their mechanism.

4.1 Steady-state equilibrium

1. They choose to calibrate the production units in this model to firm-level data.
2. They then assume
    - random productivity growth at the firm level.
        - I.e., they follow **Assumption 1** in the previous section.
3. To obtain data counterparts for these and further moments discussed below, they use publicly available tabulations of firm size and firm size by age from the Business Dynamics Statistics data.
4. According to this model, they can read off the tail of the productivity distribution of incumbents from its empirical counterpart by using the relation $\delta(1-\alpha) = 1.097$ and their assumed value for α.
5. They are left with calibrating the remaining parameters of the firm-level productivity process.
6. They now turn to the mapping between model and data counterparts of these tw targets.
7. Turning to the choice of their target for firm-level productivity volatility, it is useful to first understand the mapping between firm-level productivity $\varphi^{s_{i,t}}$ and its empirical counterpart.
8. They are interested in accurately matching the characteristics of large firms.
9. Filled (black) circles give the size distribution derived from the Business Dynamics Statistics (BDS) from the US Census.
10. The model does well in matching the firm size distribution:
11. Turning to heterogeneity in productivity, and in particualr, to how productive large firms are in this model, their calibration implies that the interequartile ratio in firm-level producitivity is 1.29.

4.2 Business cycle implications

1. They now solve the model outside the steady-state equilibrium and provide a qualification of its performance as a theory of the business cycle.

4.2.1 Numerical strategy

1. The characterization of the law of motion of the aggregate productivity in **Theorem 2** and its generalisation to the case with entry and exit (summarised in **Theorem 3** in online Appendix?), are key to the numerical strategy.
2. The numerical strategy similar in spirit to
    - the Krusell-Smith approach in that agents only take in account a reduced set of moments of the underlying high-dimensional state variable.

4.2.2 Business cycle statistics

1. Using the calibration in Table 2 and the numerical algorithm describes in the Appendix D.D2, they compute the business cycle statistics.
2. The standard deviation of aggregate output in the model is $0.55\%, 30\%$ of the annula volatility of HP-filtered real GDP in the data.
3. Crucially, in their model, these aggregate TFP dynamics are not the result of an exogenous

"aggregate" shock.
4. There are two benchmarks against which to compare this number
5.

4.2.3 Robustness checks

1. They have just seen that their baselin calibration implies non-negligible aggregate fluctuations.
    - In next section, and leveraging from the same baseline calibration, they will explore the time-varying nature of aggregate volatility and its dependence on the evolution of the firm size distribution.
        - As instructed in 3.3, 3.4.
    - Before proceeding, it is perhaps useful to pause and consider the role of key parameters of the model when quantifying both the level and time-varying nature of aggregate volatility in this model.
        - This leads them to consider alternative calibration strategies,
            - which they present here as robustness checks.

4.2.4 Inspecting the mechanism

4.3 Large firm dynamics over the Business Cycle

4.4 Distributional dynamics and the business cycle

5 Conclusion

1. A small number of firms accounts for a substantial share of aggregate economic activity.
    - This opens the possibility of doing away with aggregate shocks, instead tracing back the origins of aggregate fluctuations to large firm dynamics.
    - They build a quantitative firm dynamics model in which they cast this hypothesis.
2. The first part of their analysis characterizes, analytically, the law of motion of the firm size distribution and shows that the implied aggregate output and productivity dynamics are persistent, volatile and exhibit time-varying second moments.
3. In the second part of the paper, they explore quantitatively and in the data, the role of the firm size distribution (and in particular, that of large firm dynamics) in shaping aggregate fluctuations.
    - Taken together, their results imply that a large fraction of aggregate dynamics can be rationlized by large firm dynamics.
4. The results in this paper suggest at least two fruitful ways of extending their analysis:
    -