Notes_detailed_draft

Shu Hu

1 intro

1.1 abstract

- recent empirical work shows
  - positive correlation between grandparent-child wealth-rank
    - even after controlling for parent-child wealth-rank
  - positive correlation between dynastic wealth-ranks across almost 600 years!
- this paper shows
  - a simple heterogeneous agents model with idiosyncratic returns to wealth (B 2011 & 2019)
    - generates a realistic wealth distribution
    - fails to capture these long-run patterns of wealth mobility
  - an extension of basic model by incorporating persistent heterogeneity in returns to wealth
    - able to simultaneously match the
      - wealth distribution
      - short-run wealth mobility
        - Q: what is the SR wealth mobility?
      - long-run wealth mobility

1.2 recent heterogeneous modeling of consumption of consumption-saving decision

- Q: what is the optimal consumption-saving model with heterogeneous agent?
- successfully identify the main drivers of wealth inequality
  - Q: what main drivers of wealth inequality?
  - wealth inequality is documented by fat-tailed distributions of wealth (data vs model)
  - main drivers of wealth inequality
    - stochastic labor earnings
    - idiosyncratic returns to wealth
- fit well with inter-generational social mobility of wealth (But why reasonably well?)
  - Q: what is inter-generational social mobility of wealth

    - it produces a realistic parent-child wealth-rank correlation (Benhabib 2019)

1.3 current HA models not well fit long-run (GP-GC) inter-generational mobility

- evidence on long-run (GP-GC) inter-generational mobility
- description of unwell fitting: cannot generate
    - a large enough coefficient for grandparent-child wealth-rank
    - a large enough correlation for dynastic wealth-ranks over very long time periods

1.4 issues of extending the models by postulating intergenerational autocorrelation in earnings and in the wealth return rate

- evidence favor the non-existence of independent direct causal effects across generations, beyond parent-child effects
- large intergenerational autocorrelations (e.g. of wealth returns) ought to generate even fatter tails in the wealth distributions than observations
- extending the model along two lines above requires postulating very long intergenerational autocorrelation in earnings and in wealth return rates to capture long-run persistence
    - this is unplausible and not parsimonious as an explanation

1.5 evidence documenting rank-wealth correlations across generations (LR) and its interpretation in literature

- a positive correlation between grandparent-child wealth rank (Boserup 2014)

  it exists after controlling for parent-child wealth rank
    - data: 3 generations of Danish wealth data
    - issues
        - parent and grandparent wealth are correlated
        - possible measure errors
    - solution: use 2SLS to identify direct grandparent effects
    - interpretation: indirect effects, "social status" across generations
    - interpretation issue: grandparent effects not necessarily go through parents
- a possible direct causal effect of grandparent-child interactions (Braun and Stuhler 2018)
    - identification strategy: quasi-exogenous variation in the time of grandparents' death during WW2
    - findings
        - no effects of direct contacts between grandparents and grandchildren
        - grandparent effects operating through indirect mechanisms

- no evidence for an independent influence of grandparents once they condition on status of both parents (Warren and Hauser 1997)
    - data: Wisconsin Longitudinal Study
- striking evidence on long-run dynastic wealth-rank correlation
    - high persistence of wealth across 5 generations (Clark and Cummins 2015)
        - data: rare surnames in England and Wales between 1858 and 2012
            - issue: less amenable to statistical inference
        - interpretation: wealth, education, or occupational status are transmitted via an underlying and unobserved latent factor
            - this factor is a representation of abilities, preferences, dynastic network connections, or other relevant characteristics
            - hard to identify whether and how this latent factor is affected by the movement and by historical and institutional dimensions (Mare 2011; Braun and Stuhler 2018)
    - significant positive wealth elasticities, occupational persistence (Barone and Mocetti 2016)
        - data: families in Florence between 1427 and 2011

1.6 this paper study a simple HA model and extend it to introduce persistent heterogeneity in wealth return rate across generations

- persistent heterogeneity in wealth return rate across generations: allow households in some dynasties to have wealth grow faster on average than households in other dynasties
- this form of persistent heterogeneity
    - not take a stand on the precise interpretation of it
    - can be seen as a formalization of a latent factor representation of various dynastic characteristics suggested in the literature

2 model: wealth dynamics

2.0 intro

- this section develop theory for analysing LR persistence in rank-wealth correlation
- study rank-based models of wealth dynamics
    - models where wealth growth rates depend on wealth-rank rather than wealth level
    - this model convenient for analysis because they
        - allow for an analytic characterization of asymptotic wealth-ranks and
        - approximate standard heterogeneous agents model
- structure of section 2
    - 2.1 introduce a standard rank-based model and relates it to heterogeneous agents models

- 2.2 introduce persistent heterogeneity in wealth return rates across generations into standard rank-based model
- 2.3 derives theoretical results about LR persistence of wealth-rank correlations

2.1 rank-based model

- setups
  - an economy populated by N households, indexed by $i = 1, \cdots, N$
  - rank households by their wealth
    - $\rho_t(i)$: wealth-rank of household i at time t
    - define ranked wealth processes $w_{(1)} \geq \cdots \geq w_{(N)}$ by $w_{(\rho_t(i))}(t) = w_i(t)$
    - $\rho_t(i) < \rho_t(j)$ iff $w_i(t) > w_j(t)$ or $w_i(t) = w_j(t)$ and $i < j$
    - $w(t) = w_1(t) + \cdots + w_N(t)$: aggregate wealth of economy
  - wealth dynamics
    - equation 2.1

$$d \log w_i(t) = \alpha_{\rho_t(i)} dt + \sigma_{\rho_t(i)} dB_i(t) \tag{2.1}$$

    - B_i: Brownian motion
      - $\alpha_1 + \cdots + \alpha_N$: normalise average growth rate of economy to 0
    - eq 2.1 admits a stationary distribution iff α_k satisfy eq 2.2

$$\alpha_1 + \cdots + \alpha_k < 0, for k < N \tag{2.2}$$

- eq 2.2 suffices to guarantee that no household in the top ranks grows faster than in the lower ranks
- cause it break away from the average population wealth
- this condition is consistent with wealth return which are constant or even increasing in wealth in a standard heterogeneous agent model of wealth dynamics, as shown in 2.2
- prop. 2.1
- consider a rank-based model (eq. 2.1) that satisfies (eq. 2.2) and also eq. 2.3

$$\sigma_{k+1}^2 - \sigma_k^2 = \sigma_k^2 - \sigma_{k-1}^2 \tag{2.3}$$

for all $k = 2, \cdots, N-1.$

- the ranked wealth processes satisfy eq. 2.4

$$\mathbb{E}[\log w_{(k)}(t) - \log w_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(\alpha_1 + \cdots + \alpha_k)} \tag{2.4}$$

for all $k = 1, \cdots, N-1$

- expectation is taken w.r.t. stationary distribution
- it follows wealth inequality at stationary distribution
- is increasing in the volatility parameters σ_k
- decreasing in the sums of relative growth parameters α_k

- rank-based model as approximation

  introduce a simple heterogeneous agents consumption-saving model, aligned with Benhabib (2019 & 2011), show it can be formally mapped into an approximated rank-based model in eq. 2.1 section 3 shows an appropriate calibration of this model indeed approximates the heterogeneous agents consumption-saving model well

  - basics of HA model
    - each household has CRRA preferences and a joy-of-giving bequest motive
  - in equilibrium, intergenerational wealth dynamics for each household is eq. 2.5

$$w_i(t+1) = \lambda(r_{i,t})w_i(t) + \beta(r_{i,t}, y_{i,t}) \tag{2.5}$$

- $y_{i,t}, r_{i,t}$: labor income and wealth return for household i in generation t
  - $w_i(t)$: wealth holdings of household i in generation t
- functions λ and β: average wealth return and average labor income, adjusted for equilibrium household behavior
- they are the same for all households
  - regard eq. 2.5 as standard model
  - rank-based approximation of eq. 2.5 is eq. 2.1 with new defined eq. 2.6

$$\begin{aligned}\alpha_k &= \mathbb{E}[\log(w_{\pi_t(k)}(t+1)/w(t+1)) - \log(w_{\pi_t(k)}(t)/w(t)))] \\ \sigma_k^2 &= Var[\log(w_{\pi_t(k)}(t+1)/w(t+1)) - \log(w_{\pi_t(k)}(t)/w(t)))]\end{aligned} \tag{2.6}$$

for each rank $k = 1, \cdots, N$.

- α_k and σ_k: average and and variance of growth rate of wealth relative to the aggregate at each rank k of the distribution

- assume function $\pi_t(k)$ identify index i of the k-th ranked household at time t s.t.
  - $\pi_t(k) = i$ iff $\rho_t(i) = k$
- relative growth rate parameters α_k represent the main link between rank-based model (eq. 2.1) and standard model (eq. 2.5)
  - at stationary distribution of standard model, the rank-based relative growth rate parameters α_k satisfy, for each rank $k = 1, \cdots, N$, eq. 2.7

$$\begin{aligned}&= \mathbb{E}[\log(w_{\pi_t(k)}(t+1)/w_{\pi_t(k)}(t))] \\ &= \mathbb{E}[\lambda(r_{\pi_t(k),t}) + \beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})/w_{\pi_t(k)}(t)]\end{aligned} \tag{2.7}$$

    - expected value of aggregate wealth w satisfies $\mathbb{E}[\log w(t+1)] = \mathbb{E}[\log w(t)]$ by stationarity.
  - from eq. 2.7, we have eq. 2.8

$$\begin{aligned}\alpha_k &= \mathbb{E}[\log(\lambda(r_{\pi_t(k),t})(1 + \frac{\beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})}{\lambda(r_{\pi_t(k),t})w_{\pi_t(k)}(t)}))] \\ &= \mathbb{E}[\log(\lambda(r_{\pi_t(k),t}))] + \mathbb{E}[\log(1 + \frac{\beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})}{\lambda(r_{\pi_t(k),t})w_{\pi_t(k)}(t)})]\end{aligned} \tag{2.8}$$

- it yields a simple decomposition of the rank-based relative growth rates α_k into the two main parametric functions characterizing the model, β and λ
    - 1st term: log return to wealth adjusted for equilibrium behavior, \lambda, at rank k in the wealth distribution
- 2nd term: average log ratio of labor income adjusted for equilibrium household behavior, β, to capital income λw, at rank k in the distribution
- decomposition eq. 2.8 enables us to map the stability condition for rank-based models eq. 2.2 into a condition in terms of β and λ in standard model eq. 2.5
- in standard model eq. 2.5, wealth returns $r_{i,t}$ and labor income $y_{i,t}$ are both independent of the wealth rank of household i at time t
- first component of eq. 2.8, λ, is independent of wealth rank
- second component of eq. 2.8, $\beta/\lambda w$, is decreasing in wealth rank
- since wealth w is increasing in rank
- in standard model, $\alpha_1 < \alpha_2 < \cdots < \alpha_N$
- because the ratio of labor income to capital income is lower at higher ranks in the wealth distribution
- a negative relationship between wealth and returns is not required to satisfy the stability condition 2.2
- prop 2.2
- if standard 2.5 is stationary, then its rank-based approximation defined by eq. 2.1 and eq. 2.6 is also stationary.

2.2 persistently heterogeneous rank-based model

- a form of persistent heterogeneity in the average growth rates of households in rank-based model eq. 2.1
    - for each household , wealth dynamics are given by eq. 2.9

$$d\log w_i(t) = (\gamma_i + \hat{\alpha}_{\rho_t(i)})dt + \sigma_{\rho_t(i)}dB_i(t) \tag{2.9}$$

- $\gamma_i \in \{\gamma_l, \gamma_h\}$,
- $\gamma_h > \gamma_l$
- assume n of households are high types with $\gamma_i = \gamma_h$
- $N - n$ households are low types with $\gamma_i = \gamma_l$
- keep normalizing average growth rate of wealth to 0
- it requires .. in this economy
- to admit a stationary distribution, persistently heterogeneous setup eq. 2.9 must satisfy a stability condition that generalizes the condition eq. 2.2 for standard rank-based model eq. 2.1 with no heterogeneity
    - this condition states eq. 2.10 (Ichiba et al. 2011)

$$\sum_{k=1}^{m} \hat{\alpha}_k + \tilde{m}\gamma_h + (m-\tilde{m})\gamma_l < 0 \tag{2.10}$$

for all $m = 1, \cdots, N-1$; $\tilde{m} = \min(m, n)$.

- condition 2.10 generates condition eq. 2.2 for standard rank-based model
    - it ensures that no top subset of households grows faster than the aggregate
    - sufficient to guarantee that
        - high-type, high-growth households in the top ranks do not break away from the rest of population
        - this ensures that the average relative growth rate of high-type households when occupying the top m ranks of the wealth distribution is negative

2.3 long-run wealth-rank correlations

provide a theoretical characterization of asymptotic wealth-rank for both the standard rank-based model and model with persistent heterogeneity show that persistent heterogeneity is required to generate long-run wealth-rank correlations

- implications of rank-based model eq. 2.1 for mobility
    - define occupation times $\xi_{i,k}$ for all i, k, as the fraction of time household i occupies rank k, $\xi_{i,k} = \lim_{T\to\infty} \frac{1}{T}\int_0^T 1\{\rho_t(i) = k\}dt$.
        - by definition, the occupation times must add up to 1, s.t. $\sum_{i=1}^{N} \xi_{i,k} = \sum_{k=1}^{N} \xi_{i,k} = 1$.
    - prop. 2.3
        - occupation times $\xi_{i,k}$ in standard rank-based model eq. 2.1 satisfy eq. 2.11

$$\xi_{i,k} = \frac{1}{N}, a.s., for\ all\ i, k \tag{2.11}$$

- furthermore, for each household i, the asymptotic wealth-rank satisfies eq. 2.12

$$\lim_{\gamma\to\infty} \mathbb{E}[\rho_{t+\tau}(i)] = \frac{N+1}{2} \tag{2.12}$$

- this result is a consequence of the fact that all households in model eq. 2.1 display identical expected wealth dynamics
    - they spend equal time in all ranks, eq. 2.11
        - they must on average approach the same rank asymptotically, necessarily the median of the distribution, eq. 2.12
        - the standard rank-based model eq. 2.1 cannot produce LR wealth-rank correlations
- implications of rank-based model with persistent heterogeneity, eq. 2.9
    - if household i is a low-type household with $\gamma_i = \gamma_l$, then, by symmetry, the fraction of time household i spends in each rank k is equal to the fraction of time any other low-type household spends in each rank k
        - define low-type household occupation times $\xi_{l,k} s.t. \xi_{l,k} = \xi_{i,k}$, for all ranks $k = 1, \cdots, N$;
    - if household j is a high-type with $\gamma_j = \gamma_h$, then by similar reasons above

        - define the high-type occupation times $\xi_{h,k}$ s.t. $\xi_{h,k} = \xi_{j,k}$, for all ranks $k = 1, \cdots, N$.
    - because the sum of occupation time across all ranks or individual households must equal 1, it follows that the low- and high-type occupation times $\xi_{l,k}$ and $\xi_{h,k}$ must satisfy eq. 2.13

$$(N-n)\xi_{l,k} + n\xi_{h,k} = 1 \tag{2.13}$$

for all $k = 1, \cdots, N$.

- prop. 2.4
    - consider a persistently heterogeneous rank-based model eq. 2.9 that satisfies eq. 2.3 and eq. 2.10
        - then the low- and high-type occupation times $\xi_{l,k}$ and $\xi_{h,k}$ satisfy eq. 2.14 and eq. 2.15

$$0 < \xi_{l,1} < \xi_{l,2} < \cdots < \xi_{l,N} < \frac{1}{N-n}, a.\,s.\,, \tag{2.14}$$

and

$$\frac{1}{n} > \xi_{h,1} > \xi_{h,2} > \cdots > \xi_{h,N} > 0, a.\,s. \tag{2.15}$$

- since the occupation times for both low- and high-type households satisfy $\xi_{i,1} + \cdots + \xi_{i,N} = 1$, prop. 2.4 implies that $\xi_{l,1} < \xi_{h,1}$ and $\xi_{h,N} < \xi_{l,N}$
    - this means that low-type households spend more time at the lowest ranks of the wealth distribution than high-type households
- next theorem 2.5 uses this result to show that the heterogeneous rank-based model eq. 2.9 will feature persistence in wealth-ranks over infinitely long time horizons

- theorem 2.5
    - consider a persistently heterogeneous rank-based model eq. 2.9 that satisfies eq. 2.3 and eq. 2.10
    - then eq. 2.16

$$\lim_{\gamma\to\infty} \mathbb{E}[\rho_{t+\tau}(i)] < \lim_{\tau\to\infty} \mathbb{E}[\rho_{t+\tau}(j)] \; iff \; \rho_t(i) < \rho_t(j) \tag{2.16}$$

    for all households $i, j = 1, \cdots, N$.
        - expectations are taken w.r.t. the stationary distribution
            - unconditional on the types of households i and j
                - the values of the parameters γ_i and γ_j unknown and not part of the information set
        - it implies that LR asymptotic household rank correlation will be positive in the heterogeneous rank-based model
            - this is because the higher-ranked households today are expected to occupy higher ranks in the future as well

- thm 2.5's result is intuitive
  - since all high-type households are ex-ante identical and high-type households are expected to occupy higher ranks than low-type households
    - expected asymptotic rank of these households is the median of the top n ranks of the wealth distribution,
  - expected asymptotic rank of low-type households is the median of the bottom $N - n$ ranks because of similar reasons
  - without knowing its type, the expected asymptotic rank of some household i is thus a weighted average of the medians of the top n and bottom N-n ranks
    - weights equal to the respective probabilities that household i is a high type and that it is a low type
  - because higher-ranked households are more likely to be high-type households, it follows that the weight on the median of the top n ranks is greater for such high-ranked households
    - hence the expected asymptotic rank is also higher

3 simulations

3.0 a simulation analysis of wealth dynamics

- calibrate each of the models of section 2
  - approximated rank-based model eq. 2.1, calibrated using the standard model eq. 2.5
  - persistently heterogeneous rank-based model eq. 2.9
  - an extension of eq. 2.5 where wealth returns are auto-correlated across generation
- compare their simulated wealth dynamics along various relevant empirical dimensions regarding the wealth distribution and wealth-rank persistence over generations

3.1 calibrations

- approximated rank-based model
  - aim: construct eq. 2.1 using 2.6 to define rank-based parameters α_k and σ_k
  - step 1: parametrize eq. 2.1 (follow B 2011 & 2019, find the interpretations of paras there)
    - set household lifespan T equal to 45 years
    - growth rate of labor earnings equal to 0.01
    - preference parameters η, ψ, χ are set to $0.04, 2,$ and 0.25
    - estate tax and capital income tax, b and ζ are 0.2 and 0.15
    - model lifetime labor income $y_{i,t}$ by using a six-state Markov chain calibrated to US Survey of Consumer Finances
      - mean and variance for $y_{i,t}$ are 6.4 and 16.1
      - 1 unit is $10_0 00$ dollars
    - represent the idiosyncratic lifetime return on wealth $r_{i,t}$ by a 4-state Markov chain

      - mean and variance of these returns approximately match the empirical results of Fagereng et al. (2020) for Norwegian
  - step 2: simulate this parameterization of standard model
    - 2000 generations
    - household number $N = 10_000$
  - step 3: use simulated results and Fernholz (2017)'s econometric procedure to estimate the relative growth-rate parameters \alpha_k
    - use α_k to find values for rank-based variance parameter σ_k satisfying eq. 2.3 that yield a stationary distribution for the model, according to eq. 2.4
  - fig1
    - plots annualized estimated relative growth-rate parameters α_k for the rank based approximation of the standard model
    - shows that these parameters satisfy the stability condition 2.2, with estimated values such that $\alpha_1 < \alpha_2 < \cdots < \alpha_N$.
  - fig2
    - plots annualized estimated variance parameters σ_k for the rank-based approximation of the standard model
  - fig3
    - log-log plot of wealth vs rank for both standard model and its rank-based approximation
    - rank-based approximation generates a smoothed version of the wealth distribution from standard model
- persistently heterogeneous rank-based model
  - aim: calibrate eq. 2.9 s.t. it maintains approximately the same realistic stationary wealth distribution as approximated rank-based model
  - assume
    - 3000 of households are high-type, with $\gamma_h = 0.02$
    - remaining 7000 low-type have $\gamma_l \approx -0.0086$.
  - we can use the same estimated paras values for σ_k from the approximated rank-based model (fig2) for the persistently heterogeneous rank-based model
  - calibration for $\hat{\alpha}_k$ is more delicate
    - why
      - we cannot simply use the estimated values of α_k from the approximated rank-based model (fig1) for the persistently heterogeneous rank-based model
      - since the persistently heterogeneous parameters γ_i from eq. 2.9 lead to a more skewed stationary distribution than in the model eq. 2.1
    - how
      - adjust the estimated rank-based relative growth rate parameters α_k in fig1 s.t. $\hat{\alpha}_k \neq \alpha_k$, to maintain a similar stationary distribution for the two rank-based models
    - consider the rank-based approximation eq. 2.1 of heterogeneous rank-based model eq. 2.9

        - where α_k are defined in eq. 2.6
    - relative growth rate para α'_k for the rank-based approximation are given by eq. 3.1

$$\alpha'_k = \hat{\alpha}_k + (N-n)\xi_{l,k}\gamma_l + n\xi_{h,k}\gamma_h \tag{3.1}$$

for all $k = 1, \cdots, N$.

    - according to prop. 2.1, stationary distributions of rank-based approximation of eq. 2.9 and rank-based model eq. 2.1 will be the same if we choose $\hat{\alpha}_k s.t. \alpha'_k = \alpha_k$, for each rank k
    - solving for $\hat{\alpha}_k$ that achieve this equality is complicated by the fact that
        - we cannot directly solve for the occupation times $\xi_{l,k}$ and $\xi_{h,k}$ in eq. 3.1
            - but instead must rely on simulations of the persistently heterogeneous rank-based model to generate estimates of these parameters
  - use a simple procedure to generate estimates of the paras $\hat{\alpha}_k$ from the model eq. 2.9 s.t. α'_k is approximately equal α_k, for each rank k.
    - use eq. 3.1 to guess values of paras $\hat{\alpha}_k$ s.t. $\alpha'_k - \alpha_k \approx 0$, for all $k = 1, \cdots, N$
    - simulate the persistently heterogeneous rank-based model with these parameters $\hat{\alpha}_k$ to
        - generate estimates of the rank-based approximation paras α'_k
            - calculate the standard deviation of $\alpha'_k - \alpha_k$
        - once the error term is calculated, we incrementally alter the values of $\hat{\alpha}_k$ by setting each equal to $x\hat{\alpha}_k$
        - x is slightly less than or slightly greater than 1
        - re-estimate the parameters α'_k and again calculate the sum of squared values $\alpha'_k - \alpha_k$
    - conditions
        - if squared error with parameter values $x\hat{\alpha}_k$ is smaller (how to define small), then keep the new paras and repeat the procedure by altering the new paras in the same way
            - if not, then consider a different value of x and repeat the procedure
        - this procedure repeats until the sum of squared values $\alpha'_k - \alpha_k$ is larger for the paras $x\hat{\alpha}_k$, for both $x = 1.001$ and $x = 0.999$.
            - fig4 show the estimated result
- auto-correlated returns model
  - aim
    - introduce autocorrelated returns into the standard model eq. 2.5 to capture imperfect social mobility as in B (2011 & 2019)
    - investigate how such returns impact long-run mobility
  - assume
    - wealth returns follow a highly persistent AR-1 process with eq. 3.2

$$\log(1 + r_{i,t+1}) = \theta \log(1 + r_{i,t}) + \epsilon_{i,t} \tag{3.2}$$

        - $\epsilon_{i,t}$: normally distributed with mean and standard deviation equal to 0.0375 and 0.025
            - persistence parameter θ set to 0.9
        - (for symmetry?) labor earnings $\log y_{i,t}$ IID and drawn from a normal distribution with mean equal to 0.85 and sd equal to 1.416.
    - other respects of auto-correlated return model identical to standard model which was used to calibrate the rank-based model

3.2 results

- table 1

	Data	Standard Model	Approximated Rank-Based Model	Persist. Heter. Rank-Based Model	Auto-Correlated Returns Model
Wealth Distribution					
Top 1%	33.6%	33.1%	32.6%	35.1%	98.5%
Top 1-5%	26.7%	26.4%	17.1%	16.6%	0.6%
Top 5-10%	11.1%	6.3%	9.4%	9.1%	0.3%
Top 10-20%	12.0%	8.0%	11.0%	10.6%	0.3%
Top 20-40%	11.2%	10.5%	12.9%	12.4%	0.2%
Top 40-60%	4.5%	7.7%	8.2%	7.9%	0.1%
Bottom 60-100%	-0.1%	8.0%	8.7%	8.3%	0.1%
Wealth-Rank Correlations					
Parent-Child Rank Coeff.	0.191	0.177	0.187	0.218	0.252
Grandparent-Child Rank Coeff.	0.116	0.015	-0.004	0.089	0.147
Long-Run Persistence Coeff.	0.105	0.001	0.000	0.116	0.022

Table 1: Upper part: Average wealth shares from 1,000 simulations of the different models - data from the Survey of Consumer Finances. Lower part: Average coefficients from regressions of child rank on parent rank and grandparent rank from 1,000 simulations of the different model - data from Danish wealth holdings for three generations in Boserup et al. (2014). Average coefficient from regressions of household rank in generation t on household rank in generation $t-23$ (585 years) from 1,000 simulations of the different models - data from estimates of very long-run (585 years) dynastic wealth holdings in Florence, Italy, in Barone and Mocetti (2016).

    - upper parts: wealth shares of different subsets of households in SCF data can be compared with those generated by different calibrated models
        - all models other than auto-correlated returns model are calibrated to match these wealth shares and do relatively well at this (especially for the top 1% wealth share)
    - lower part: parent-child wealth correlations (compared to Boserup et al. 2014)
    - lower part: numerical results on rank coefficient are consistent with theoretical results of section 2
        - prop. 2.3 implies that household wealth ranks will be uncorrelated over very long time periods in rank-based approximation of standard model
        - thm 2.5 implies household wealth ranks will be positively correlated over arbitrarily long time periods in a rank-based model that features persistent heterogeneity
        - furthermore, the auto-correlated return model

            - it is able to
                - approximately match the empirical results of Boserup et al (2014) and
                - generate a significant link between child and grandparent wealth ranks, after controlling for parent wealth rank
            - but fail to match the long-run link between dynastic wealth ranks reported by Barone and Mocetti (2016)
            - generates an implausibly skewed wealth distribution (see last col. of table 1)
    - conclusion
        - only heterogeneous rank-based model can match all aspects of the data simultaneously
            - wealth dist
            - link between child, parent and grandparent wealth ranks
            - positive correlation of dynastic wealth ranks over very long time periods
- table 2:

	Top 1%	Top 5%	Bottom 50%	Bottom 25%
High-Type Households	86.6%	72.7%	17.5%	12.0%
Low-Type Households	13.4%	27.3%	82.5%	88.0%

Table 2: Average composition of the top 1%, top 5%, bottom 50%, and bottom 25% of households from 1,000 simulations of the heterogeneous rank-based model.

useful to study the properties of heterogeneous rank-based model closely

    - composition of top 1% and top 5% wealth-ranked households in terms of low- and high-type households
        - high-type households make up the great majority of the top 1% and 5%
        - still a non-negligible minority (by how much?) of low-type households in these top subsets
    - composition of bottom 50% and bottom 25% ranked households
        - low-type households are more common in top subsets of wealth dist than high-type households are in bottom subsets of wealth dist
        - fraction of low-type households in bottom 25% approximately matches the fraction of high-type households in the top 1%
            - even though the latter is a much smaller and more exclusive subset of the wealth distribution

- table 5 & 6:

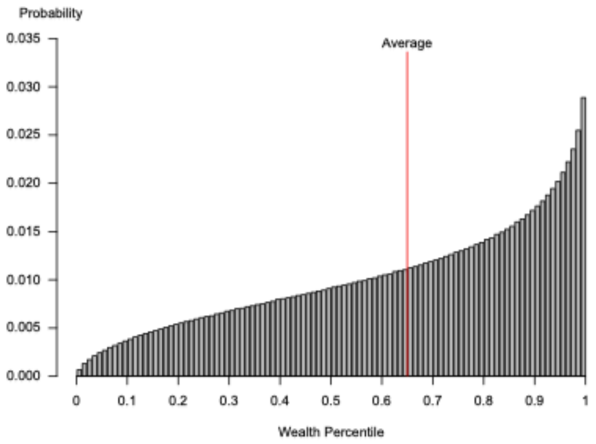

Figure 5: Average high-type household occupation times for different percentiles of the wealth distribution from 1,000 simulations of the heterogeneous rank-based model.

estimated occupation times of different percentiles of the wealth distribution for, respectively, high-type and low-type households

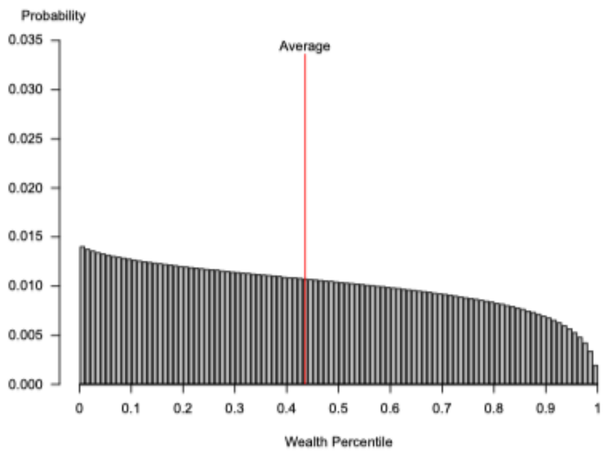

Figure 6: Average low-type household occupation times for different percentiles of the wealth distribution from 1,000 simulations of the heterogeneous rank-based model.

  - estimated occupation times presented in the figures are clearly consistent with the result in prop. 2.4
  - maximum average occupation time for a high-type household in any percentile of the wealth distribution is $1/3000$ \approx 0.033%.
    - 3000 high-type households
  - maximum occupation time for a low-type household in any percentile is $1/7000$ \approx 0.014%.
    - 7000 low-type households

- Fig 7

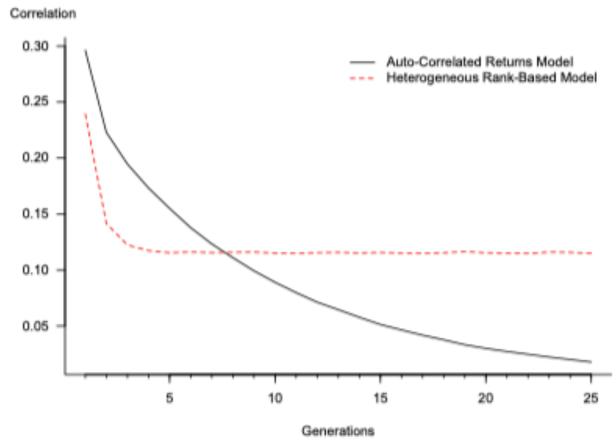

Figure 7: Rank correlations across multiple generations from 1,000 simulations of the heterogeneous rank-based and auto-correlated returns models.

  - correlation between the wealth ranks of households in generation t and generation $t+x$, with values of x ranging from 1 to 25, for both the heterogeneous rank-based and auto-correlated returns models
    - very LR persistence of wealth rank that exists in the heterogeneous rank-based model can be seen most clearly
  - although the auto-correlated returns model is able to generate substantial persistence in rank across one or two generations, the rank correlation in this model quickly declines towards 0 as the generation gap between households increases
  - heterogeneous rank-based model generates a more realistic but smaller persistence in wealth rank across one or tow generations
    - this persistence never falls below 0.1 even as generational gap grows large
    - as predicted by thm 2.5

4 conclusion

1. consider a simple heterogeneous agents model based on Benhabib et al. (2019)
2. show such standard models fail to match recent empirical results regarding LR wealth mobility
   - in particular, this type of model does not generate a positive correlation between grandparent-child wealth rank, after controlling for parent-child wealth rank
   - does not generate a positive correlation between dynamic wealth ranks across very long time periods
3. extend the standard model to include persistent heterogeneity in wealth return
4. show this extended model is able to simultaneously match the wealth distribution, SR-wealth mobility and LR wealth mobility
5. not standing for precise interpretation of form of persistent heterogeneity, it can be seen as a formalization of the latent factor representation of
   - abilities,
   - preferences,

- dynastic network connections,
- occupational persistence or
- other relevant characteristics.