Heterogeneous Dynasties and Long-Run Mobility

Jess Benhabib, Alberto Bisin and Ricardo T. Fernholz

Reporter: Shu Hu (ANU)

September 11, 2020

Overview

1 Introduction

1.1 Motivational facts

Empirical:

- Positive correlation between grandparent-child wealth-rank (Boserup 2014)
- Positive correlation between dynastic wealth-ranks across almost 600 years (Barone and Mocetti 2016)

Theoretical:

- Current heterogeneous agents (HA) models cannot well fit long-run (GP-GC) intergenerational mobility described above.

1.2 Economics question and possible directions

Questions:

- How we should extend HA models to capture those features of the long-run intergenerational mobility?

Directions:

- 1. Assume high intergenerational autocorrelation in earnings and/or wealth return;
- 2. Assume persistent heterogeneity in wealth return rate across generations

2 Models

Setups:

- An economy populated by N households, indexed by $i = 1, \cdots, N$
- Households ranked by their wealth
  - $\rho_t(i)$: wealth-rank of household i at time t
  - Define ranked wealth processes $w_{(1)} \geqslant \cdots \geqslant w_{(N)}$ by $w_{(\rho_t(i))}(t) = w_i(t)$,
  - $\rho_t(i) < \rho_t(j)$ iff $w_i(t) > w_j(t)$ or $w_i(t) = w_j(t)$ and $i < j$,
  - $w(t) = w_1(t) + \cdots + w_N(t)$: aggregate wealth of economy

2.1 Model 1: rank-based model

For households $i = 1, \cdots, N$, wealth dynamics are given by

$$d \log w_i(t) = \alpha_{\rho_t(i)} dt + \sigma_{\rho_t(i)} dB_i(t) \tag{2.1}$$

where

- B_i: Brownian motion
- $\alpha_1 + \cdots + \alpha_N$: normalise average growth rate of economy to 0.

2.1 Model 1: rank-based model

Banner et al. (2005) prove that eq. 2.1 admits a stationary distribution iff α_k satisfy

$$\alpha_1 + \cdots + \alpha_k < 0, for\ k < N \tag{2.2}$$

Proposition 1

Consider a rank-based model (eq. 2.1) that satisfies (eq. 2.2) and

$$\sigma_{k+1}^2 - \sigma_k^2 = \sigma_k^2 - \sigma_{k-1}^2 \tag{2.3}$$

for all $k = 2, \cdots, N-1$. The ranked wealth processes satisfy

$$\mathbb{E}[\log w_{(k)}(t) - \log w_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(\alpha_1 + \cdots + \alpha_k)} \tag{2.4}$$

for all $k = 1, \cdots, N-1$.

2.2 Model 2: simple HA consumption model (Benhabib 2019 & 2011)

- Assume each household has CRRA preferences and a joy-of-giving bequest motive
- In equilibrium, intergenerational wealth dynamics for each household is

$$w_i(t+1) = \lambda(r_{i,t})w_i(t) + \beta(r_{i,t}, y_{i,t}) \tag{2.5}$$

  - $y_{i,t}, r_{i,t}$: labor income and wealth return for household i in generation t,
  - $w_i(t)$: wealth holdings of household i in generation t.
  - functions λ and β: average wealth return and average labor income, adjusted for equilibrium household behavior.

2.3 Approximate model 1 using model 2

- Assume function $\pi_t(k)$ identify index i of the k-th ranked household at time t s.t. $\pi_t(k) = i$ iff $\rho_t(i) = k$.

- For each rank $k = 1, \cdots, N$, rank-based approximation of eq. 2.5 is eq. 2.1 with new defined α_k and σ_k^2 eq. 2.6

$$\alpha_k = \mathbb{E}[\log(w_{\pi_t(k)}(t+1)/w(t+1)) - \log(w_{\pi_t(k)}(t)/w(t)))] \tag{2.6-a}$$

$$\sigma_k^2 = Var[\log(w_{\pi_t(k)}(t+1)/w(t+1)) - \log(w_{\pi_t(k)}(t)/w(t)))] \tag{2.6-b}$$

  - α_k and σ_k: average and and variance of growth rate of wealth relative to the aggregate at each rank k of the distribution.

2.3 Approximate model 1 using model 2

At stationary distribution of standard model, the rank-based relative growth rate parameters α_k satisfy, for each rank $k = 1, \cdots, N$,

$$\alpha_k = \mathbb{E}[\log(w_{\pi_t(k)}(t+1)/w_{\pi_t(k)}(t))]$$

$$= \mathbb{E}[\lambda(r_{\pi_t(k),t}) + \beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})/w_{\pi_t(k)}(t)] \qquad (2.7)$$

- expected value of aggregate wealth w satisfies $\mathbb{E}[\log w(t+1)] = \mathbb{E}[\log w(t)]$ by stationarity.

2.3 approximate model 1 using model 2

From eq. 2.7, we have a decomposition

$$\alpha_k = \mathbb{E}[\log(\lambda(r_{\pi_t(k),t})(1 + \frac{\beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})}{\lambda(r_{\pi_t(k),t})w_{\pi_t(k)}(t)}))]$$

$$= \mathbb{E}[\log(\lambda(r_{\pi_t(k),t}))] + \mathbb{E}[\log(1 + \frac{\beta(r_{\pi_t(k),t}, y_{\pi_t(k),t})}{\lambda(r_{\pi_t(k),t})w_{\pi_t(k)}(t)})] \quad (2.8)$$

Proposition 2

If standard model eq. 2.5 is stationary, then its rank-based approximation defined by eq. 2.1 and eq. 2.6 is also stationary.

2.3 Model 3: persistently heterogeneous rank-based model

For each household, wealth dynamics are given by

$$d \log w_i(t) = (\gamma_i + \hat{\alpha}_{\rho_t(i)})dt + \sigma_{\rho_t(i)} dB_i(t) \tag{2.9}$$

- $\gamma_i \in \{\gamma_l, \gamma_h\}$ with $\gamma_h > \gamma_l$,
  - n high-type households with $\gamma_i = \gamma_h$,
  - $N - n$ low-type households with $\gamma_i = \gamma_l$,
- $\sum_{k=1}^{N} \hat{\alpha}_k + \sum_{i=1}^{N} \gamma_i = \sum_{k=1}^{N} \hat{\alpha}_k + (N-n)\gamma_l + n\gamma_h = 0$

2.3 Model 3: persistently heterogeneous rank-based model

Ichiba et al. (2011) show that this model admits a stationary distribution iff it satisfy the following stability condition:

$$\sum_{k=1}^{m} \hat{\alpha}_k + \tilde{m}\gamma_h + (m - \tilde{m})\gamma_l < 0 \tag{2.10}$$

for all $m = 1, \cdots, N-1$; $\tilde{m} = \min(m, n)$.

2.4 Theoretical characterization of asymptotic wealth-rank model eq. 2.1 for mobility

Define occupation times $\xi_{i,k}$ for all i, k, as the fraction of time household i occupies rank k, $\xi_{i,k} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}\{\rho_t(i) = k\}dt$.

- $\sum_{i=1}^N \xi_{i,k} = \sum_{k=1}^N \xi_{i,k} = 1$.

Proposition 2.3

Occupation times $\xi_{i,k}$ in standard rank-based model eq. 2.1 satisfy

$$\xi_{i,k} = \frac{1}{N}, a.s., for\ all\ i, k \tag{2.11}$$

For each household i, the asymptotic wealth-rank satisfies

$$\lim_{\tau\to\infty} \mathbb{E}[\rho_{t+\tau}(i)] = \frac{N+1}{2} \tag{2.12}$$

2.5 Theoretical characterization of asymptotic wealth-rank model eq. 2.9 for mobility

Define occupantion times for low-type and high-type households.

- define low-type household occupation times $\xi_{l,k}$ s.t. $\xi_{l,k} = \xi_{i,k}$, for all ranks $k = 1, \cdots, N$;
- define the high-type occupation times $\xi_{h,k}$ s.t. $\xi_{h,k} = \xi_{j,k}$, for all ranks $k = 1, \cdots, N$.

The low- and high-type occupation times $\xi_{l,k}$ and $\xi_{h,k}$ must satisfy

$$(N - n)\xi_{l,k} + n\xi_{h,k} = 1 \tag{2.13}$$

for all $k = 1, \cdots, N$.

2.5 Theoretical characterization of asymptotic wealth-rank model eq. 2.9 for mobility

Proposition 2.4

Consider a persistently heterogeneous rank-based model eq. 2.9 that satisfies eq. 2.3 and eq. 2.10, then the low- and high-type occupation times $\xi_{l,k}$ and $\xi_{h,k}$ satisfy

$$0 < \xi_{l,1} < \xi_{l,2} < \cdots < \xi_{l,N} < \frac{1}{N-n}, a.s., \tag{2.14}$$

and

$$\frac{1}{n} > \xi_{h,1} > \xi_{h,2} > \cdots > \xi_{h,N} > 0, a.s. \tag{2.15}$$

- It implies that $\xi_{l,1} < \xi_{h,1}$ and $\xi_{h,N} < \xi_{l,N}$ since $\xi_{i,1} + \cdots + \xi_{i,N} = 1$.

2.5 Theoretical characterization of asymptotic wealth-rank model eq. 2.9 for mobility

Theorem 2.5

Consider a persistently heterogeneous rank-based model eq. 2.9 that satisfies eq. 2.3 and eq. 2.10, then

$$\lim_{\tau\to\infty} \mathbb{E}[\rho_{t+\tau}(i)] < \lim_{\tau\to\infty} \mathbb{E}[\rho_{t+\tau}(j)] \text{ \textit{iff} } \rho_t(i) < \rho_t(j) \tag{2.16}$$

for all households $i, j = 1, \cdots, N$.

- expectations are taken w.r.t. the stationary distribution unconditional on the types of households i and j.

3 Calibrations and Results

3.1 Calibration 1: approximated rank-based model

Aim

- construct eq. 2.1 with rank-based parameters α_k and σ_k defined as eq. 2.6.

Procedures:

- **step 1**: parametrize eq. 2.1 (follow Benhabib et al., 2011 & 2019)
- **step 2**: simulate this parameterization of standard model with 2000 generations and 10000 households.
- **step 3**: use simulated results and Fernholz (2017)'s econometric procedure to estimate the relative growth-rate parameters α_k

3.1 Calibration 1: approximated rank-based model

For **step 1**,

- set household lifespan T equal to 45 years
- growth rate of labor earnings equal to 0.01
- preference parameters η, ψ, χ are set to $0.04, 2$, and 0.25
- estate tax and capital income tax, b and ζ are 0.2 and 0.15
- model lifetime labor income $y_{i,t}$ by using a six-state Markov chain calibrated to US SCF with mean and variance of 6.4 and 16.1, and 1 unit is 10000 dollars
- represent the idiosyncratic lifetime return on wealth $r_{i,t}$ by a 4-state Markov chain with mean and variance matching Fagereng et al. (2020) for Norwegian

3.2 Results for calibration 1

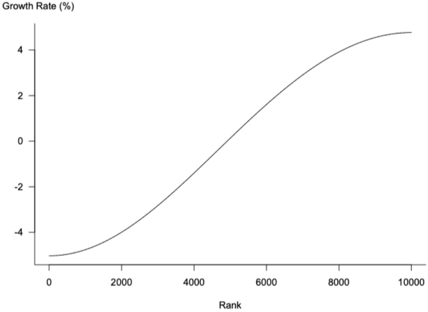

Figure 1: Annualized estimated parameters α_k for the approximated rank-based model.

3.2 Results for calibration 1

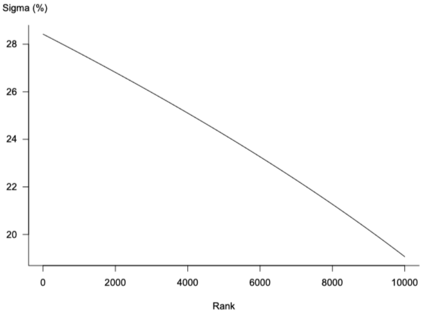

Figure 2: Annualized estimated parameters σ_k for the approximated rank-based model.

3.2 Results for calibration 1

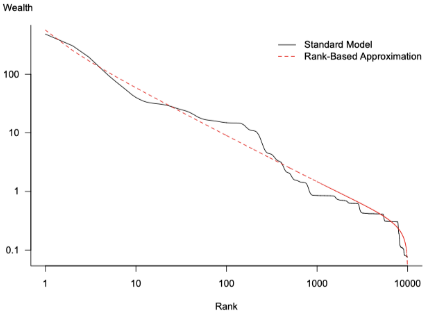

Figure 3: Log-log plot of wealth versus rank for the Standard model (average from 2,000 simulations) and its rank-based approximation.

3.3 Calibration 2: persistently heterogeneous rank-based model

Aim:

- calibrate eq. 2.9 s.t. it maintains approximately the same realistic stationary wealth distribution as approximated rank-based model

Assume:

- 3000 households are high-type, with $\gamma_h = 0.02$
- remaining 7000 low-type have $\gamma_l \approx -0.0086$.
- they use the same estimated σ_k from calibration 1 here.

3.3 Calibration 2: persistently heterogeneous rank-based model

Calibration for $\hat{\alpha}_k$ should be more delicate.

- we cannot use α_k from calibration 1 here because the persistently heterogeneous parameters γ_i from eq. 2.9 lead to a more skewed stationary distribution than in the model eq. 2.1.

Methodology:

- adjust the estimated rank-based relative growth rate parameters α_k in fig1 s.t. $\hat{\alpha}_k \neq \alpha_k$, to maintain a similar stationary distribution for the two rank-based models.

3.3 Calibration 2: persistently heterogeneous rank-based model

Consider the rank-based approximation eq. 2.1 of heterogeneous rank-based model eq. 2.9, where α_k are defined in eq. 2.6.

Relative growth rates α_k' for the rank-based approximation are given by

$$\alpha_k' = \hat{\alpha}_k + (N-n)\xi_{l,k}\gamma_l + n\xi_{h,k}\gamma_h \tag{3.1}$$

for all $k = 1, \cdots, N$.

- Simulation is the only way to get $\hat{\alpha}_k$.

3.3 Calibration 2: persistently heterogeneous rank-based model

Procedure of the simulation: generate estimates of the paras $\hat{\alpha}_k$ from the model eq. 2.9 s.t. α'_k is approximately equal α_k, for each rank k.

- **step 1**: use eq. 3.1 to guess values of paras $\hat{\alpha}_k$ s.t. $\alpha'_k - \alpha_k \approx 0$, for all $k = 1, \cdots, N$,
- **step 2**: simulate the persistently heterogeneous rank-based model with these parameters $\hat{\alpha}_k$ to
  - generate estimates of the rank-based approximation paras α'_k
  - calculate the standard deviation of $\alpha'_k - \alpha_k$ (error term),
- **step 3**: once the error term is calculated, we incrementally alter the values of $\hat{\alpha}_k$ by setting each equal to $x\hat{\alpha}_k$

3.3 Calibration 2: persistently heterogeneous rank-based model

Procedures (continue)

- **step 4**: re-estimate the parameters α'_k and again calculate the sum of squared values $\alpha'_k - \alpha_k$.
- **step 5**: evaluate whether squared error with parameter values $x\hat{\alpha}_k$ is smaller.
  - if so, then keep the new paras and repeat the **steps 1-4** by altering the new paras in the same way;
  - if not, then consider a different value of x in **step 3** and repeat the **step 4-5**.
- **step 6**: this procedure repeats until the sum of squared values $\alpha'_k - \alpha_k$ is larger for the paras $x\hat{\alpha}_k$, for both $x = 1.001$ and $x = 0.999$.

3.4 Results for calibration 3

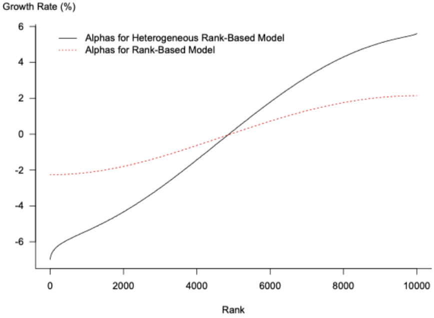

Figure 4: Annualized estimated parameters $\hat{\alpha}_k$ for the persistently heterogeneous rank-based model, and annualized estimated parameters α_k for the approximated rank-based model.

3.5 Model 4 and calibration 3: auto-correlated returns model

Aim

- introduce autocorrelated returns into the standard model eq. 2.5 to capture imperfect social mobility as in Benhabib (2011 & 2019)
- investigate how such returns impact long-run mobility

Wealth returns follow a highly persistent AR-1 process with

$$\log(1 + r_{i,t+1}) = \theta \log(1 + r_{i,t}) + \epsilon_{i,t} \tag{3.2}$$

- $\epsilon_{i,t}$: normally distributed with mean and standard deviation equal to 0.0375 and 0.025;
- persistence parameter θ set to 0.9.
- labor earnings $\log y_{i,t}$ IID and drawn from a normal distribution with mean equal to 0.85 and sd equal to 1.416.

3.6 Results: table 1

	Data	Standard Model	Approximated Rank-Based Model	Persist. Heter. Rank-Based Model	Auto-Correlated Returns Model
Wealth Distribution					
Top 1%	33.6%	33.1%	32.6%	35.1%	98.5%
Top 1-5%	26.7%	26.4%	17.1%	16.6%	0.6%
Top 5-10%	11.1%	6.3%	9.4%	9.1%	0.3%
Top 10-20%	12.0%	8.0%	11.0%	10.6%	0.3%
Top 20-40%	11.2%	10.5%	12.9%	12.4%	0.2%
Top 40-60%	4.5%	7.7%	8.2%	7.9%	0.1%
Bottom 60-100%	-0.1%	8.0%	8.7%	8.3%	0.1%
Wealth-Rank Correlations					
Parent-Child Rank Coeff.	0.191	0.177	0.187	0.218	0.252
Grandparent-Child Rank Coeff.	0.116	0.015	-0.004	0.089	0.147
Long-Run Persistence Coeff.	0.105	0.001	0.000	0.116	0.022

Table 1: Upper part: Average wealth shares from 1,000 simulations of the different models - data from the Survey of Consumer Finances. Lower part: Average coefficients from regressions of child rank on parent rank and grandparent rank from 1,000 simulations of the different model - data from Danish wealth holdings for three generations in Boserup et al. (2014). Average coefficient from regressions of household rank in generation t on household rank in generation $t-23$ (585 years) from 1,000 simulations of the different models - data from estimates of very long-run (585 years) dynastic wealth holdings in Florence, Italy, in Barone and Mocetti (2016).

3.6 Results: table 2

	Top 1%	Top 5%	Bottom 50%	Bottom 25%
High-Type Households	86.6%	72.7%	17.5%	12.0%
Low-Type Households	13.4%	27.3%	82.5%	88.0%

Table 2: Average composition of the top 1%, top 5%, bottom 50%, and bottom 25% of households from 1,000 simulations of the heterogeneous rank-based model.

3.6 Results: figure 5 & 6

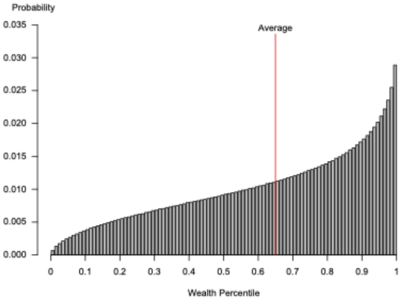

Figure 5: Average high-type household occupation times for different percentiles of the wealth distribution from 1,000 simulations of the heterogeneous rank-based model.

3.6 Results: figure 5 & 6

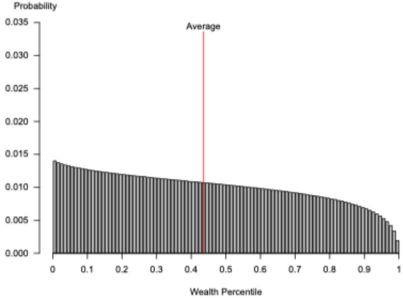

Figure 6: Average low-type household occupation times for different percentiles of the wealth distribution from 1,000 simulations of the heterogeneous rank-based model.

3.6 Results: figure 7

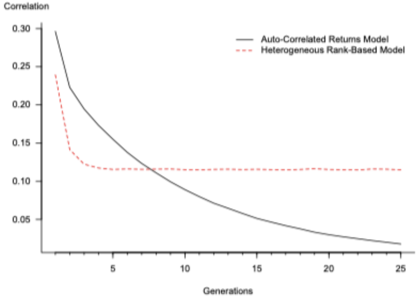

Figure 7: Rank correlations across multiple generations from 1,000 simulations of the heterogeneous rank-based and auto-correlated returns models.