13 Proof on Uniqueness of the Markov Matrix Q's stationary distribution-2nd Attempt

The proof is in the part 2.3.

1 Conjecture

1.1 Preparation

Assumption 1

Firm-level productivity evolves as a **Markov Chain** on the state space $\Phi = \{\varphi^s\}_{s=1,\cdots,S}$ with transition matrix

$$P = \begin{pmatrix} a+b & c & 0 & \cdots & \cdots & 0 & 0 \\ a & b & c & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b+c \end{pmatrix} \tag{1}$$

- where
  - $0 < a < 1$;
  - $0 < b < 1$;
  - $0 < c < 1$;
  - $a + b + c = 1$.

Corollary 1

Define $\hat{\mu}_t = \frac{\mu_t}{M}$ for any t.

With a **continuum of firms**, or equivalently, when **aggregate uncertainty is absent**, $\epsilon_{t+1} = 0$:

$$\hat{\mu}_{t+1} = (\tilde{P}_t)'(\hat{\mu}_t + G) \tag{2}$$

- where
  - $\tilde{P}_t$: the transition matrix P,
    - where the first $\tilde{s}(\mu_t) - 1$ rows are replaced by 0s;
      - where $\tilde{s}(\mu_t)$ is the threshold of the entry and exit rule when the variance-covariance of the ϵ_{t+1} is 0 (**???**).

1.2 Formal Expression of the conjecture

Given Assumption 1 and if the potential entrants' productivity distribution is
$G = (G(1), G(2), \cdots, G(S))$

- Where (This means G can be any discrete distribution with S values)
  - $0 < G(s) \leq 1$; (**Important**)
  - $\sum_{s=1}^{S} G(s) = 1$.

Then as $S \to \infty$ the stationary productivity distribution **uniquely** converges point-wise to a **Zipf's distribution**, μ^*, which is given by

$$\mu^* = \mu^* Q \tag{3}$$

- Where
  - Q: the transition matrix P,
    - where the first $s^* - 1$ rows are replaced by G's.

2 Proof of Conjecture's Uniqueness

2.1 Theoretical Preparation (Recall notes on 'Finite Markov Chains')

2.1.1 Irreducibility and aperiodicity

They are central concepts of modern Markov chain theory.

Def. (Two states communicate)

Let P be a fixed stochastic matrix.

Two states x and y are said to communicate with each other if there exist positive integers j and k such that

$$P^j(x, y) > 0 \; and \; P^k(y, x) > 0 \tag{4}$$

which means precisely that

1. state x can be reached eventually from state y,
2. state y can be reached eventually from state x.

Def. (Irreducibility)

The stochastic matrix P is called **irreducible** if all states communicate, that is, x and y communicate for **all** (x, y) in $S \times S$.

Remark

Irreducibility is important in terms of long run outcomes. e.g., poverty is a life sentence in the second graph but not the first.

Def. (Period)

The period of a state x is the greatest common divisor of the set of integers

$$D(x) = \{j \geq 1 | P^j(x, x) > 0\} \tag{5}$$

Def. (Aperiodicity: formal)

A stochastic matrix is called aperiodic if the period of every state is 1, and periodic otherwise.

2.1.2 Stationary distributions

As seen in (5), we can shift probabilities forward one unit of time via post multiplication by P.

Some distributions are invariant under this updating process, and such distributions are called stationary, or invariant.

Def. (Stationary, Invariant)

A distribution ψ^* on S is called stationary for P if

$$\psi^* = \psi^* P \tag{6}$$

Remarks

1. From (14), we can get $\psi^* = \psi^* P^t$ for all t.

which tells us an **important fact**:

If the distribution of X_0 is a stationary distribution, then X_t will have this same distribution for all t.

2. Hence, stationary distributions have a natural interpretation as stochastic steady states.
3. Mathematically, a stationary distribution is a fixed point of P when P is thought of as the map $P : \psi \to \psi$ (**A little mistake in the lecture notes**) from (row) vectors to (row) vectors.

Thm.1 (Stochastic matrix vs. Stationary distribution; Existence)

Assume that the state space S is finite if not more assumptions are required.

Every stochastic matrix P has at least one stationary distribution.

Proof. (See my note 4)

Thm.2 (Markov Chain Convergence Theorem)

If P is both aperiodic and irreducible, then

1. P has exactly one stationary distribution ψ^*;
2. For any initial distribution ψ_0, we have

$$||\psi_0 P^t - \psi^*|| \to 0 \; as \; t \to \infty \tag{7}$$

Proof.

Thm.3 (Sufficient condition for Aperiodicity and Irreducibility)

One easy sufficient condition for aperiodicity and irreducibility is that every element P is strictly positive. (How to prove?)

2.1.3 Calculation when only one stationary distribution exists (Used for calculating stationary distribution)

If we restrict attention to the case where only one stationary distribution exists, then one option for finding it is to try to solve the linear system

$$\psi(I_n - P) = 0 \; for \; \psi \tag{8}$$

where I_n is the $n \times n$ identity.

Since the **zero vector** solves this equation (**John's question in the meeting**), we need to (**Answer**) **impose the restriction** that the **solution must be a probability distribution** (But **Why?**).

2.2 Analysis

We want to prove that Q only has one stationary distribution, so we should consider the existence first.

Existence

Since Q is a markov matrix, by **Theorem 1**, we know that Q has at least one stationary distribution.

Uniqueness

1. If we can prove Q is both irreducible and aperiodic, then by **Theorem 2**, we have
   - Q only has one stationary distribution;
   - A method to calculate the stationary distribution μ^*.
2. To prove Q is both irreducible and aperiodic, by **Theorem 3**, we can try to prove that every element in Q is positive.
   - But elements in Q are not all positive, so we cannot go this way.
3. To prove Q is both irreducible and aperiodic, we can also refer to their definition.
   - To prove that Q is irreducible, we should prove that for $Q^n = (q_{ij}^{(n)})_{S\times S}$
   $$\exists \, n \; s.t. \, q_{ij}^{(n)} > 0, \forall i, j \in \{1, \cdots, S\} \tag{9}$$
   .
   - Once we prove that Q is irreducible, then to prove that Q is aperiodic, we only need to show one aperiodic state.
     - An irreducible Markov Matrix with one aperiodic state implies that all states are aperiodic (**To Be Checked**).

2.3 Formal Proof

Proof.existence

Since P is a stochastic matrix, by def. of Markov matrix, we have

1. $P \geq 0$;
2. $1 \cdot P = 1$.

And for $G = (G(1), G(2), G(3), \cdots, G(S))$, by its definition, we have

1. $G \geq 0$;

2. $1 \cdot G = 1$.

By definition of the matrix Q, since Q is the transition matrix P, where the first $(s^* - 1)$ rows are replaced by Gs, we have

1. $Q \geq 0$;
2. $1 \cdot Q = 1$.

By definition, the matrix Q is also a Markov matrix.

By **Theorem 1**, the Markov matrix Q has at least one stationary distribution.

Proof.uniqueness

In order to prove the uniqueness of the stationary distribution of the matrix Q, we need to prove its irreducibility and aperiodicity first.

Proof.irreducibililty

To prove Q is irreducible, by def. of irreducibility we need to show that for all $(x, y) \in \{1, \cdots, S\} \times \{1, \cdots, S\}$, there exist positive integers n such that

1. $Q^n(x, y) > 0$.

Without loss of generosity, let's assume that $x \leq y$, then the definition of irreducibility says for all $(x, y) \in \{1, \cdots, S\} \times \{1, \cdots, S\}$, there exists an positive integer n such that

1. $Q^n(x, y) > 0$ and;
2. $Q^n(y, x) > 0$.

Since $Q^{(y-x)}(x, y)$ can be represented as

$$Q^{(y-x)}(x, y) = \begin{cases} \sum_{s=1}^{S} Q^{(y-x-1)}(x, s)Q(s, y) = \sum_{s=1}^{S}[\sum_{i=1}^{S} Q^{(y-x-2)}(x, i)Q(i, s)]Q(s, y) & if\ x < y \\ Q(x, y) & if\ x = y \end{cases} \tag{10}$$

We notice that there exists one term $Q(x, x+1)Q(x+1, x+2)\cdots Q(y-1, y)$ in the expansional expression of $Q^{(y-x)}(x, y)$ when $x < y$; and $Q^{(y-x)}(x, y) = Q(x, y)$ when $x = y$.

Similarly, there exists one term $Q(y, y-1)Q(y-1, y-2)\cdots Q(x+1, x)$ involved in the expansional expression of $Q^{(y-x)}(y, x)$ when $y > x$; and $Q^{(y-x)}(x, y) = Q(x, y)$ when $x = y$.

Let $n = \{(y-x)|(x, y) \in \{1, \cdots, S\} \times \{1, \cdots, S\}\ and\ x \leq y\} = S - 1$.

Since other possible terms in the above expansional expressions are all non-negative, if we can prove that

1. $Q(s,s+1)>0, \forall s \in \{1,\cdots,S-1\}$;
2. $Q(s,s-1)>0, \forall s \in \{2,\cdots,S\}$.
3. $Q(s,s)>0, \forall s \in \{1,\cdots,S\}$.

Then we can prove that for all $(x,y) \in \{(x,y) \in \{1,\cdots,S\} \times \{1,\cdots,S\} | x \leq y\}$, and $n=S-1$, we have

1. $Q^n(x,y)>0$ and;
2. $Q^n(y,x)>0$.

By the definition of irreducibility, we know that the Markov matrix Q is irreducible.

Let's check whether the following inqualities are true for all $s^* \in \{1,\cdots,S\}$ by consider 2 cases.

1. $Q(s,s+1)>0, \forall s \in \{1,\cdots,S-1\}$;
2. $Q(s,s-1)>0, \forall s \in \{2,\cdots,S\}$.
3. $Q(s,s)>0, \forall s \in \{1,\cdots,S\}$.

Case 1: $s^*=1$.

In this case, we have $Q=P$. By definition, we have

1. $Q(s,s+1)=P(s,s+1)=c, \forall s \in \{1,\cdots,S-1\}$;
2. $Q(s,s-1)=P(s,s-1)=a, \forall s \in \{2,\cdots,S\}$.
3. $Q(s,s)=P(s,s)=b, \forall z \in \{2,\cdots,S-1\}$;
4. $Q(1,1)=P(1,1)=a+b$;
5. $Q(S,S)=P(S,S)=b+c$.

By the definition of the matrix P, we have $a>0$, $b>0$ and $c>0$, so we obtain

1. $Q(s,s+1)=P(s,s+1)=c>0, \forall s \in \{1,\cdots,S-1\}$;
2. $Q(s,s-1)=P(s,s-1)=a>0, \forall s \in \{2,\cdots,S\}$.
3. $Q(s,s)=P(s,s)=b>0, \forall s \in \{2,\cdots,S-1\}$;
4. $Q(1,1)=P(1,1)=a+b>0$;
5. $Q(S,S)=P(S,S)=b+c>0$.

That is, when $s^*=1$, the following inequalities hold:

1. $Q(s,s+1)>0, \forall s \in \{1,\cdots,S-1\}$;
2. $Q(s,s-1)>0, \forall s \in \{2,\cdots,S\}$.
3. $Q(s,s)>0, \forall s \in \{1,\cdots,S\}$.

Case 2: $s^* \in \{2,\ldots,S-1\}$.

In this case, by definition of the matrix Q, for $s \in \{s^*, s^*+1,\cdots,S\}$, Q's row s are the same as in P.

By definition, we still have

1. $Q(s,s+1)=P(s,s+1)=c>0, \forall s \in \{s^*,\cdots,S-1\}$;
2. $Q(s,s-1)=P(s,s-1)=a>0, \forall s \in \{s^*,\cdots,S\}$;
3. $Q(s,s)=P(s,s)=b>0, \forall s \in \{s^*,\cdots,S-1\}$;

4. $Q(S, S) = P(S, S) = b + c > 0$.

Since for $s \in \{1, 2, \cdots, s^* - 1\}$, Q's row s are replaced by G, by definition of G, we have

1. $Q(s, s+1) = G(s+1) > 0, \forall s \in \{1, 2, \cdots, s^* - 1\}$;
2. $Q(s, s-1) = G(s-1) > 0, \forall s \in \{2, 3, \cdots, s^* - 1\}$.
3. $Q(s, s) = G(s) > 0, \forall s \in \{1, \cdots, s^* - 1\}$.

Therefore, when $s^* \in \{2, \ldots, S - 1\}$, we still have

1. $Q(s, s+1) > 0, \forall s \in \{1, 2, \cdots, S - 1\}$;
2. $Q(s, s-1) > 0, \forall s \in \{2, 3, \cdots, S\}$;
3. $Q(s, s) > 0, \forall s \in \{1, \cdots, S\}$.

Case 3: $s^* = S$

In this case, by definition of the matrix Q, for $s = S$, Q's row s are the same as in P, that is,

1. $Q(S, S-1) = P(S, S-1) = a > 0$;
2. $Q(S, S) = P(S, S) = b + c > 0$.

Since for $s \in \{1, 2, \cdots, S - 1\}$, Q's row s are replaced by G, by definition of G, we have

1. $Q(s, s+1) = G(s+1) > 0, \forall s \in \{1, 2, \cdots, S - 1\}$;
2. $Q(s, s-1) = G(s-1) > 0, \forall s \in \{2, 3, \cdots, S - 1\}$.
3. $Q(s, s) = G(s) > 0, \forall s \in \{1, \cdots, S - 1\}$.

Therefore, when $s^* = S$, we still have

1. $Q(s, s+1) > 0, \forall s \in \{1, 2, \cdots, S - 1\}$;
2. $Q(s, s-1) > 0, \forall s \in \{2, 3, \cdots, S\}$;
3. $Q(s, s) > 0, \forall s \in \{1, \cdots, S\}$.

Overall, for all $s^* \in \{1, 2, \ldots, S\}$, we have

1. $Q(s, s+1) > 0, \forall s \in \{1, \cdots, S - 1\}$;
2. $Q(s, s-1) > 0, \forall s \in \{2, \cdots, S\}$.
3. $Q(s, s) > 0, \forall s \in \{1, \cdots, S\}$.

which is what we want to prove.

By the expansional expression of $Q^{(y-x)}(x, y)$ and $Q^{(y-x)}(y, x)$, we can conclude that for all $(x, y) \in \{(x, y) \in \{1, \cdots, S\} \times \{1, \cdots, S\} | x \leq y\}$, and $, n = \{(y - x) | (x, y) \in \{1, \cdots, S\} \times \{1, \cdots, S\} \; and \; x \leq y\} = S - 1$, we have

1. $Q^n(x, y) > 0$ and;
2. $Q^n(y, x) > 0$.

Therefore, by definition of Markov matrix's irreducibility, the Markov matrix Q is irreducible.

Proof.aperiodicity

In order to prove Q's aperiodicity, we need to show that for all $s \in \{1, \cdots, S\}$, we have

$$gcd\{t : Q^t(s,s) > 0\} = 1 \tag{11}$$

- where
  - gcd: the greatest common divisor.

That is, for all $s \in \{1, \cdots, S\}$, we have

$$Q(s,s) > 0 \tag{12}$$

We need to consider 2 cases by different values of s^*.

Case 1: $s^* = 1$**.**

By definition of Q, $Q = P$, so we have

1. $Q(s,s) = b, \forall s \in \{2, \cdots, S-1\}$
2. $Q(s,s) = a + b, \forall s = 1$;
3. $Q(s,s) = b + c, \forall s = S$.

Since $a > 0$, $c > 0$ and $b > 0$, we have

1. $Q(s,s) = b > 0, \forall s \in \{2, \cdots, S-1\}$
2. $Q(s,s) = a + b > 0, \forall s = 1$;
3. $Q(s,s) = b + c > 0, \forall s = S$.

When $s^* = 1$, we have

1. $Q(s,s) > 0, \forall s \in \{1, \cdots, S\}$.

That is, for all states $s \in \{1, 2, \cdots, S-1, S\}$, when $s^* = 1$, we have

$$gcd\{t : Q^t(s,s) > 0\} = 1 \tag{13}$$

By definition of the aperiodicity of a Markov matrix, we know that when $s^* = 1$, the Markov matrix Q is **aperiodic**.

Case 2: $s^* \in \{2, \ldots, S-1\}$**.**

For $s \in \{s^*, \cdots, S\}$, rows s in Q are the same as in P, so we still have

1. $Q(s,s) = b > 0, \forall s \in \{s^*, \cdots, S-1\}$;
2. $Q(s,s) = b + c > 0, \forall s = S$.

For $s \in \{1, \cdots, s^* - 1\}$, rows s in Q are replaced by G, so we have

1. $Q(s,s) = G(s) > 0, \forall s \in \{1, \cdots, s^* - 1\}$;

When $s^* \in \{2, \cdots, S-1\}$, we still have

1. $Q(s,s) > 0, \forall s \in \{1, \cdots, S\}$;

That is, for states $s \in \{1, 2, \cdots, S-1, S\}$, when $s^* \in \{2, \ldots, S-1\}$, we also have

$$gcd\{t : Q^t(s,s) > 0\} = 1 \tag{14}$$

By definition of the aperiodicity of a Markov matrix, we know that the Markov matrix Q is **aperiodic**, when $s^* \in \{2, \ldots, S-1\}$.

Case 3: $s^* = S$.

For $s = S$, rows s in Q are the same as in P, so we still have

1. $Q(S, S) = P(S, S) = b + c > 0$;
2. $Q(S, S-1) = P(S, S-1) = a > 0$.

For $s \in \{1, \cdots, S-1\}$, rows s in Q are replaced by G, so we have

1. $Q(s, s) = G(s) > 0, \forall s \in \{1, \cdots, S-1\}$;

When $s^* = S$, we still have

1. $Q(s, s) > 0, \forall s \in \{1, \cdots, S\}$;

That is, for states $s \in \{1, 2, \cdots, S-1, S\}$, when $s^* = S$, we also have

$$gcd\{t : Q^t(s, s) > 0\} = 1 \tag{15}$$

By definition of the aperiodicity of a Markov matrix, we know that the Markov matrix Q is **aperiodic**, when $s^* = S$.

By the above three cases, we have

Overall, for $s^* \in \{1, \cdots, S\}$, we have

1. $Q(s, s) > 0, \forall s \in \{1, \cdots, S\}$;

That is, for states $s \in \{1, 2, \cdots, S-1, S\}$, when $s^* \in \{1, \cdots, S\}$, we also have

$$gcd\{t : Q^t(s, s) > 0\} = 1 \tag{16}$$

By definition of the aperiodicity of a Markov matrix, we know that the Markov matrix Q is **aperiodic**, for all $s^* \in \{1, \cdots, S\}$.

Proof.uniqueness

Since the Markov matrix Q is irreducible and aperiodic, by **Theorem 2**, we can conclude that

1. Q has exactly one stationary distribution μ^*;
2. For any initial distribution μ_0, we have

$$\|\mu_0 Q^t - \mu^*\| \to 0 \; as \; t \to \infty \tag{17}$$