One Conjecture on Firm Size Distribution under Firm Dynamic Setting

Shu HU

RSE, ANU

Advisor: John Stachurski

Overview

1. Introduction
2. Markov Matrix/Chain
3. Aperiodicity, Irreducibility, Stationary Distribution
4. Firm Dynamic (Entry/Exit) Model
5. Carvalho & Grassi's Double Pareto Theory
6. One Conjecture
7. What's next

1.1 Introduction: Power Law Distribution

- The so-called Power Law distribution, or Pareto distribution is very common in both natural and social research.
- For example, the Famous Zipf's Law.

Power Law Distribution

If a random variable X follows Pareto distribution, then its counter cumulative function (survival function) should be given by

$$Pr(X > x) = C \cdot (x)^{-\delta} \tag{1}$$

where

- C is a constant
- $\delta > 0$ is a positive parameter.

1.2 Introduction: Firm Size Distribution

- Research on Firm Size Distribution is important to both theoretical development and practical applications.

Literature 1: Firm Size Distribution

- Analysis on relationships among firm size distribution, Gibrat's Law and Pareto Distribution attract lots of attention (Gibrat, 1931; Champernowne, 1953; Simon, 1955; Córdoba).
- The distribution of U.S. firm sizes closely follows the Pareto distribution (Axtell, 2001, 2011)
- This fact can also be found in many other countries (Graricano and etc., 2016, AER; Mueller and etc., 2017, AER; etc.).

1.3 Introduction: Firm Dynamics and My Question

- However, it is more realistic to analyse the firm size distribution under firm dynamic setting (Hopenhayn, 1992).
- My Economic Question arises from the above thought:
  - How prevalent the Pareto distribution and firm size distribution are in Hopenhayn's entry-exit model?

Literature 2: Firm Dynamics

- Hopenhayn (1992) developed a standard and famous firm dynamics model with continuum number of firms in his paper.
- Carvalho & Grassi (2019) analyze this setup with a finite but possibly large number of firms.

2.1 Markov Matrix

Definition. Markov (Stochastic) Matrix

A **Markov matrix** (or stochastic matrix) is an $n \times n$ square matrix $P = (p_{i,j})$ such that

- Each element of P is nonnegative ($p_{i,j} \geq 0$);
- Each row of P sums to 1, i.e.,

$$p_{i,1} + p_{i,2} + \cdots + p_{i,n} = 1$$

for all $i \in \{1, \ldots, n\}$

2.2 Markov Chain 1

Definition. State Space

Let S be a finite set with n elements $\{x_1, \ldots, x_n\}$, where the set S is called the **state space** and $x_1, \ldots, x_n$ are the **state values**.

Definition. Markov Property

For any date t and any state $y \in S$,

$$\mathcal{P}\{X_{t+1} = y | X_t\} = \mathcal{P}\{X_{t+1} = y | X_t, X_{t-1}, ...\}$$

2.2 Markov Chain 2

Definition. Markov Chain

A **Markov chain** $\{X_t\}$ on S is a sequence of random variables on S (from probability sample space Ω) that have the **Markov property**.

Dynamics of a Markov chain

The dynamics of a Markov chain $\{X_t\}$ are fully determined by the set of values

$$P(x, y) = \mathcal{P}\{X_{t+1} = y | X_t = x\} \ (x, y \in S)$$

where

- $P(x, y)$ is the (x, y) element of the Markov matrix P.

2.3 Markov Matrix & Markov Chain: An Example

Entry/Exit

- Consider a firm who, at any given time t, is either entrant (state 0) or incumbent (state 1).
- Suppose that, over a one month period,
  - An entrant enter the industry with probability 0.5.
  - An incumbent exit the industry with probability 0.5.

2.3 Markov Matrix & Markov Chain: An Example

Applications:

- Sample space: $\Omega = \{entrant, incumbent\}$;
- State space: $S = \{0, 1\}$;
- Markov Chain: $\{X_t\}$

  where

  - X_t is a random variable $X_t : \Omega \to S$:
    - $X_t(entrant) = 0$
    - $X_t(incumbent) = 1$.

Markov Chain in this Example (Continued)

- Transition probabilities: $P(0,1) = 0.5$ and $P(1,0) = 0.5$.
  - In Markov Matrix form

$$P = \begin{pmatrix} P(0,0) & P(0,1) \\ P(1,0) & P(1,1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

3.1 Irreducibility

Definition. Two States Communicate

Let P be a fixed stochastic matrix.

Two states x and y communicate with each other if there exist a positive integer n such that

$$P^n(x, y) > 0 \text{ and } P^n(y, x) > 0$$

Definition. Irreducibility

The stochastic matrix P is called **irreducible** if all its states communicate with each other.

3.2 Aperiodicity

Definition. Period

The period of a state x is the greatest common divisor of the set of integers

$$D(x) = \{j \geq 1 | P^j(x,x) > 0\}$$

Definition. Aperiodicity

A stochastic matrix is called aperiodic if the period of its every state is 1.

3.3 Stationary Distribution 1

Distribution and its Dynamics

Suppose that

- $\{X_t\}$ is a Markov chain with stochastic matrix P.
- the distribution of X_t is known to be ψ_t.

Then the distribution of X_{t+m} is updated by

$$\psi_{t+m} = \psi_t P^m$$

where

- where P^m is the m-th power of P.

3.3 Stationary Distribution 2

Definition. Stationary Distribution

A distribution ψ^* on S is called stationary for P if

$$\psi^* = \psi^* P$$

Proposition 1

Every stochastic matrix P has at least one stationary distribution.

3.3 Stationary Distribution 3

Proposition 2

If a Markov matrix P is both aperiodic and irreducible, then

- P has exactly one stationary distribution ψ^*
- For any initial distribution ψ_0, we have

$$\|\psi_0 P^t - \psi^*\| \to 0 \text{ as } t \to \infty$$

3.4 Aperiodicity, Irreducibility, Stationary Distribution: Example

Employment/Unemployment Transition Example

- States 0 and 1 of P communicate with each other since we can find positive integer $n = 1$ such that

$$P^{(1)}(0,1) = P(0,1) = 0.5 > 0 \textit{ and } P^{(1)}(1,0) = P(1,0) = 0.5 > 0$$

- P is irreducibility since we can find positive integer $n = 1$ such that

$$P^{(1)}(0,1) = 0.5 > 0 \textit{ and } P^{(1)}(1,0) = 0.5 > 0$$

$$P^{(1)}(0,0) = 0.5 > 0 \textit{ and } P^{(1)}(1,1) = 0.5 > 0$$

3.4 Aperiodicity, Irreducibility, Stationary Distribution: Example

Example (Continued)

- For P, the period of state 0 is 1 and the period of the states, since for all positive intergers n, we have

$$P^{(n)}(0,0) > 0 \textit{ and } P^{(n)}(1,1) > 0$$

- P is aperiodic since the period of P's every state is 1.

4.Firm Dynamic (Entry/Exit) Model

Basic Setup

- Firm size state space: $\varphi^s \in \{\varphi^1, \cdots, \varphi^S\} = \Phi$
- Firm size threshold: $\varphi^{s^*} \in \{\varphi^1, \cdots, \varphi^S\} = \Phi$
- Entrants' firm size follows a distribution

$$G = (G(\varphi^1), G(\varphi^2), \cdots, G(\varphi^S))$$

  where
  - $0 < G(\varphi^s) < 1$
  - $\sum_{s=1}^{S} G(\varphi^s) = 1$

4.Firm Dynamic (Entry/Exit) Model

Basic Setup

- Incumbents' firm size distribution evolves following a Markov chain on the firm size state space Φ with transition matrix

$$P = \begin{pmatrix} a+b & c & 0 & \cdots & \cdots & 0 & 0 \\ a & b & c & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b+c \end{pmatrix}$$

  where
  - $0 < a, b, c < 1$,
  - $a + b + c = 1$.

  That is, incumbents' firm size is Pareto distributed.

4.Firm Dynamic (Entry/Exit) Model

Basic Setup

- Law of motion for stationary all firm size distribution between incumbents and entrants:

$$\mathbb{Q}(\varphi^s, = \varphi^{s+1}) = P(\varphi^s, \varphi^{s+1})\mathbb{1}\{\varphi^s \geq \varphi^{s^*}\} + G(\varphi^{s+1})\mathbb{1}\{\varphi^s < \varphi^{s^*}\} \tag{2}$$

  where

  - $\mathbb{Q}$ is the evolution Markov matrix for all firms at the stationary.

- Stationary firm size distribution $\mu^* = (\mu_1^*, \cdots, \mu_S^*)$ is given by

$$\mu^* = \mu^* \mathbb{Q}$$

5. Carvalho & Grassi's Double Pareto Theory

Proposition 3

Suppose that we follow Basic Firm Dynamics Setups.

If the entrant's firm size follows **a Pareto distribution** G, that is,

$$G_s = K_e(\varphi^s)^{-\delta_e} \tag{3}$$

then as $S \to \infty$, the stationary firm size distribution $\mu^* = (\mu_1^*, ..., \mu_S^*)$ will uniquely converges point-wise to a Pareto distribution

$$\sum_{i>s} \mu_i^* = Pr\{\varphi > \varphi^s\} = C \cdot (\varphi^s)^{-\delta} \tag{4}$$

where

- C is a constant
- $\delta > 0$: pinned down by a, c, φ

6.1 One Conjecture

Proposition 4 (My Conjecture)

Suppose that we follow Basic Firm Dynamics Setup.

If the entrant's firm size **follow any distribution** G,

then as $S \to \infty$, the stationary firm size distribution $\mu^* = (\mu_1^*, ..., \mu_S^*)$ will uniquely converges point-wise to a Pareto distribution

$$\sum_{i>s} \mu_i^* = Pr\{\varphi > \varphi^s\} = C \cdot (\varphi^s)^{-\delta} \tag{5}$$

where

- C is a constant
- $\delta > 0$: pinned down by a, c, φ

6.2 Proof of Stationary Distribution's Uniqueness

I have proved the uniqueness of stationary firm size distribution in **Proposition 4**

Basic Idea of the Proof

- **Existence**

  I show that Q is a Markov matrix, regardless of s^* and G. By **Proposition 1**, this distribution exist.

- **Uniqueness**

  I show that Q is **irreducible** and **aperiodic**, regardless of s^* and G. By **Proposition 2**, this distribution is unique.

6.3 Simulation: Goal and Expectation

Goal of the Simulation

- Graphically examine whether the stationary firm size distribution is also Pareto under my conjecture, by log-log plots.
- **Log-log plots**

  The Log–log plot is necessary but insufficient evidence for a power law relationship.

Expectation for the Simulations' Results

- The right-hand tail of double-log stationary firm size distribution will look like a straight line with a negative slope.

6.3 Simulation: Why this Expectation

Reason:

- Recall my conjecture (Proposition 4), if the stationary firm size distribution will converges to a Pareto distribution, then it will have this form
$$\sum_{i>s} \mu_i^* = Pr\{\varphi > \varphi^s\} = C \cdot (\varphi^s)^{-\delta}$$
- Take the log terms for both sides of the above equation, we will get
$$\log \sum_{i>s} \mu_i^* = \log Pr\{\varphi > \varphi^s\} = \log C - \delta \cdot \log(\varphi^s)$$
- Since $\delta > 0$, then $\log \mu_s^*$ will be linear in $\log(\varphi^s)$ with a negative slope $-\delta$ on the right-hand tail of the firm size distribution.

6.3 Simulation: Algorithm

Algorithm

- **Step 1:**

  Generate Q by considers Incumbents' evolution P and 9 different types of Entrants' distribution G.

- **Step 2:**

  Calculate the counter cumulative distribution, $Pr\{\varphi > \varphi^s\}$ of stationary firm size distribution by the iteration method and etc, and plot them against firm size state values φ^s.

- **Step 3:**

  Calculate the log term of the counter cdfs, $\log Pr\{\varphi > \varphi^s\}$, and plot them against the log term of the firm size state values $\log \varphi^s$ for each entrant's type.

6.3 Simulation: Algorithm

Algorithm (Continued)

- **Step 4:**

  Focus on the Right-hand tails of the double-log-term distributions, and plot them.

6.3 Simulation: Value Assignments

Simulation: Value Assignments

Parameters	Value
S	1000
s^*	50
a	0.6129
c	0.3870
h	1.57
φ	1.1
S1	171
S2	86
α	0.7
β	0.2

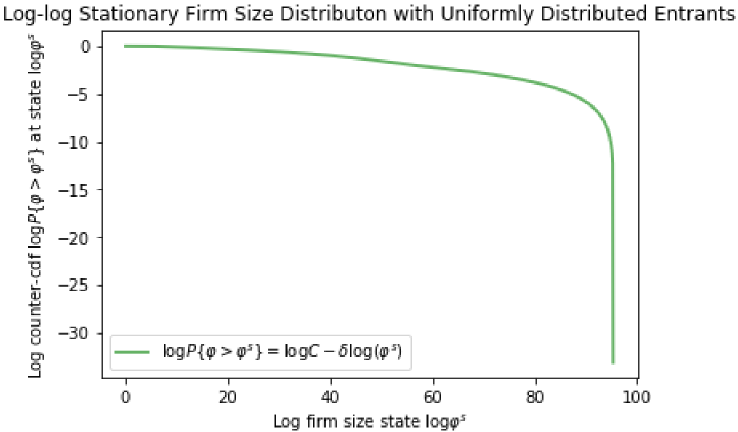
Note: For the simulation, unless I intentionally mention, I will use $(S, s^*, a, c, \varphi, \alpha, \beta) = (1000, 50, 0.6129, 0.3870, 1.1, 0.7, 0.2)$ from the above table.

6.3 Simulation 1

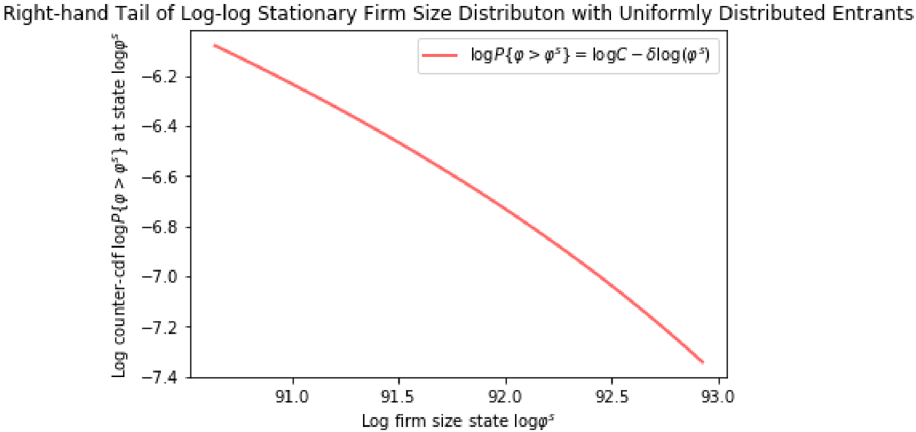
When G is uniformly distributed with pmf

$$G(\varphi^s) = \frac{1}{S}, where\ s \in \{1, 2, ..., S\} \tag{6}$$

6.3 Simulation 1



6.3 Simulation 1



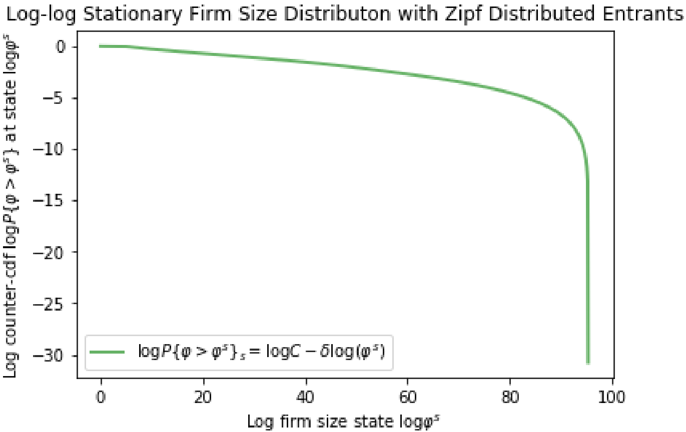
6.3 Simulation 2

When G is is Zipf distributed with pmf

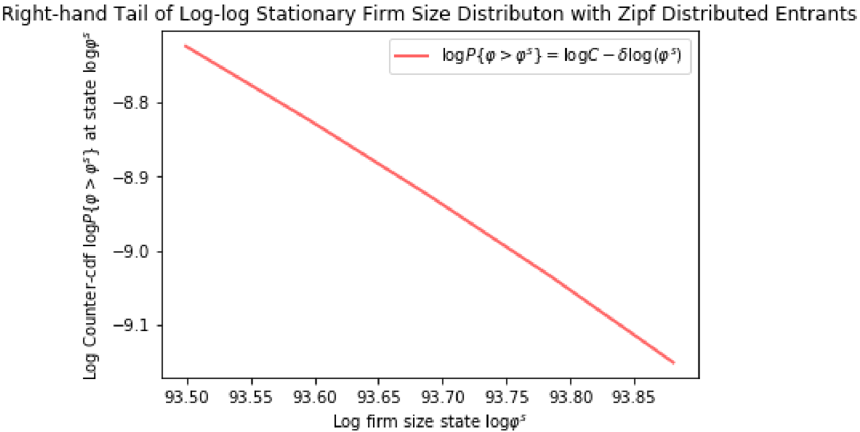
$$G(\varphi^s) = \frac{1}{(s)^a} \frac{1}{H_{S,a}} \tag{7}$$

where

- $H_{S,a} = \sum_{n=1}^{S} (\frac{1}{n^a})$;
- $a \geq 0$


Log-log Stationary Firm Size Distributon with Zipf Distributed Entrants
Log counter-cdf $\log P\{\varphi > \varphi^s\}$ at state $\log\varphi^s$
0
−5
−10
−15
−20
−25
−30
$\log P\{\varphi > \varphi^s\}_s = \log C - \delta \log(\varphi^s)$
0
20
40
60
80
100
Log firm size state $\log\varphi^s$

6.3 Simulation 2



6.3 Simulation 3

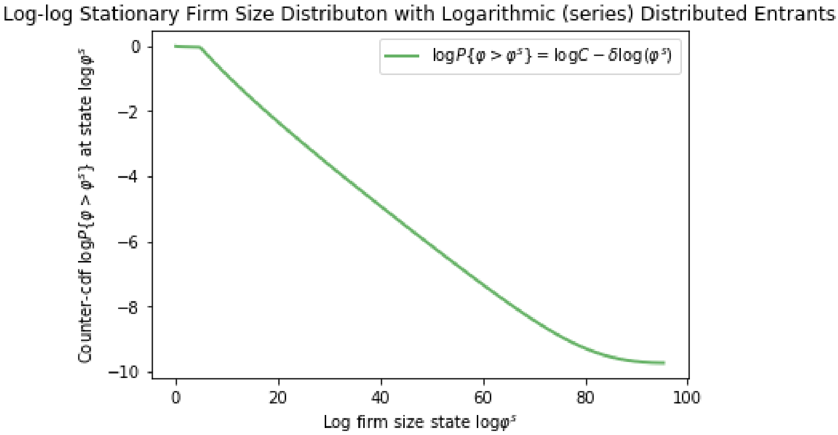
When G is is Logarithmic (series) distributed with pmf

$$G(\varphi^s) = -\frac{p^s}{s\log(1-p)}, where\ s \in \{1, 2, \cdots, S\} \tag{8}$$

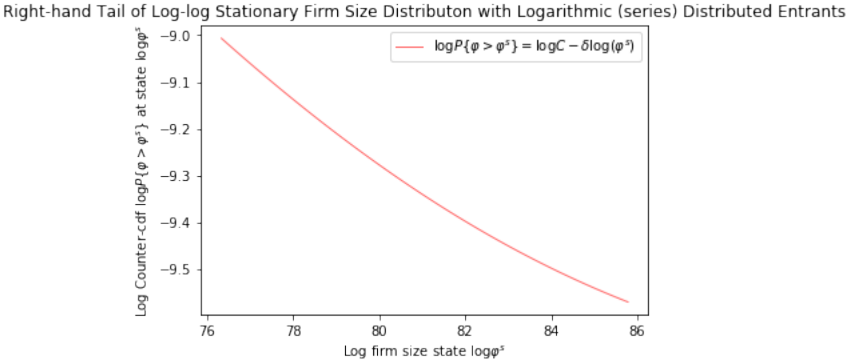
where

- $0 < p < 1$: probability of success in each trial;

6.3 Simulation 3



6.3 Simulation 3



6.3 Simulation 4

When G is is Binomial distributed with pmf,

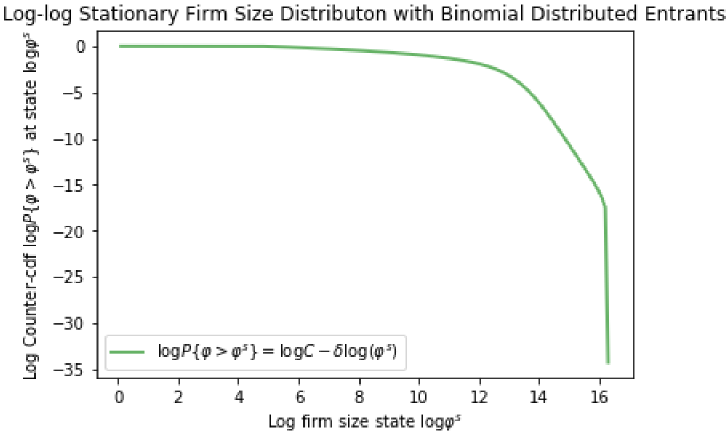
$$G(\varphi^s) = \binom{S-1}{s} p^s (1-p)^{S-1-s} \tag{9}$$

where

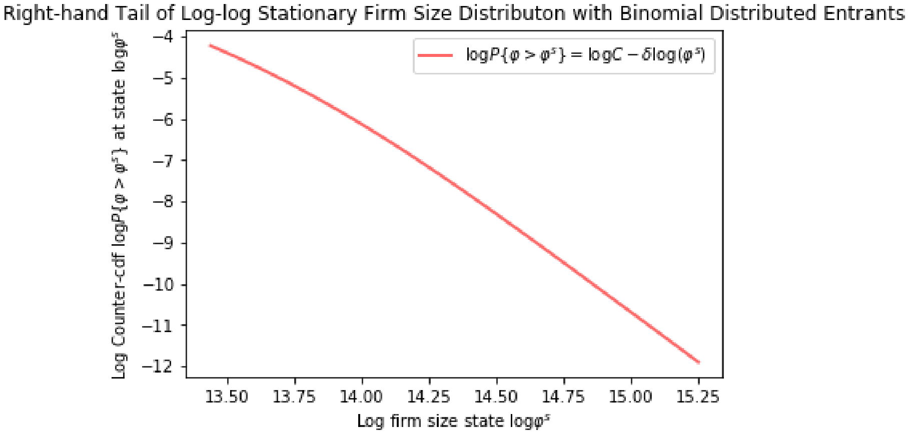
- $s \in \{0, 1, 2, \cdots, S-1\}$
- $0 < p < 1$.
- $\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$.

Notice: Parameters Change

- $S = S1$
- $p = 0.8$


Log-log Stationary Firm Size Distributon with Binomial Distributed Entrants
Log Counter-cdf $\log P\{\varphi > \varphi^s\}$ at state $\log\varphi^s$
0
−5
−10
−15
−20
−25
−30
−35
$\log P\{\varphi > \varphi^s\} = \log C - \delta \log(\varphi^s)$
0
2
4
6
8
10
12
14
16
Log firm size state $\log\varphi^s$

6.3 Simulation 4



When G is Poisson distributed with pmf,

$$G(\varphi^s) = e^{-\lambda}\frac{\lambda^{s-1}}{(s-1)!}, where\ s \in \{1, 2, \cdots, S\} \tag{12}$$

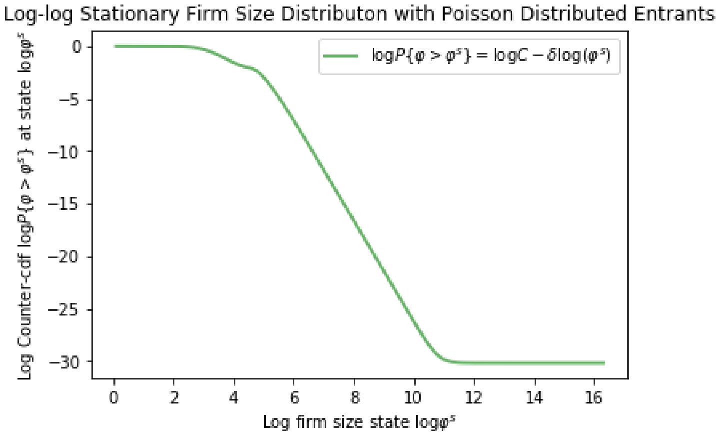
where

- $0 < p < 1$: probability of success in each trial;
- $Sp = \lambda \geq 0$.

Notice: Parameters Change

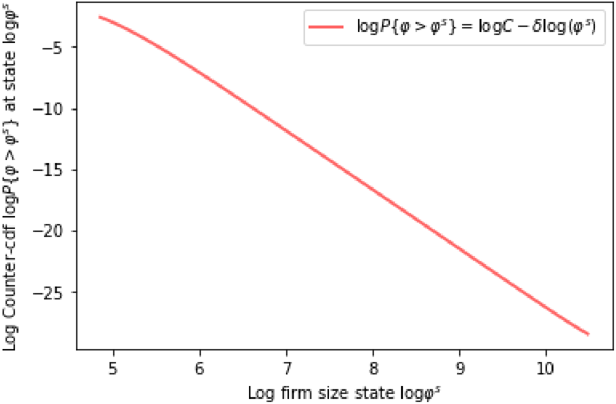
- $S = S1$

6.3 Simulation 5



6.3 Simulation 5

Right-hand Tail of Log-log Stationary Firm Size Distributon with Poisson Distributed Entrants



6.3 Simulation 6

When G is Geometric distributed with pmf,

$$G(\varphi^s) = (1-p)^{s-1}p, \mathit{where}\ s \in \{1, ..., S\} \tag{13}$$

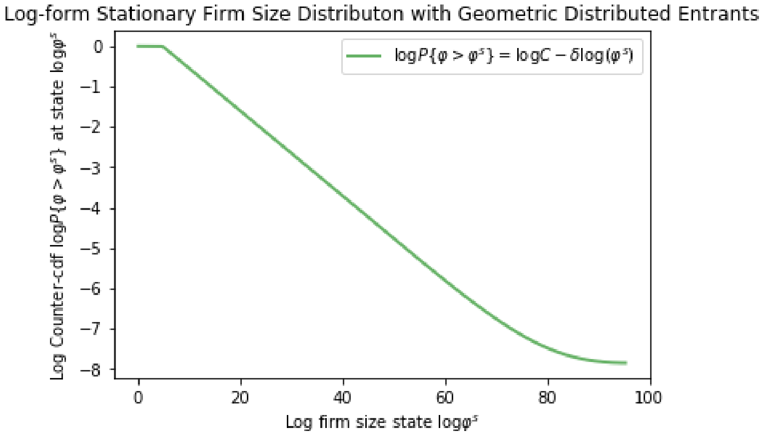
where

- $0 < p < 1$: probability of success in each trial;

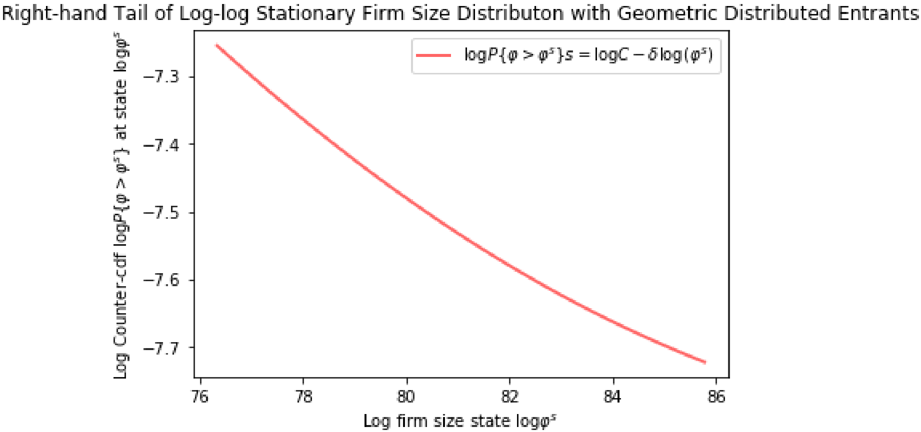
Notice: Parameters Change

- $p = 0.01$

6.3 Simulation 6



6.3 Simulation 6



6.3 Simulation 7

When G is Negative Binomial distributed with pmf,

$$G(\varphi^s) = \frac{\Gamma(s+S)}{\Gamma(S)\Gamma(s+1)} p^S (1-p)^s, \tag{14}$$

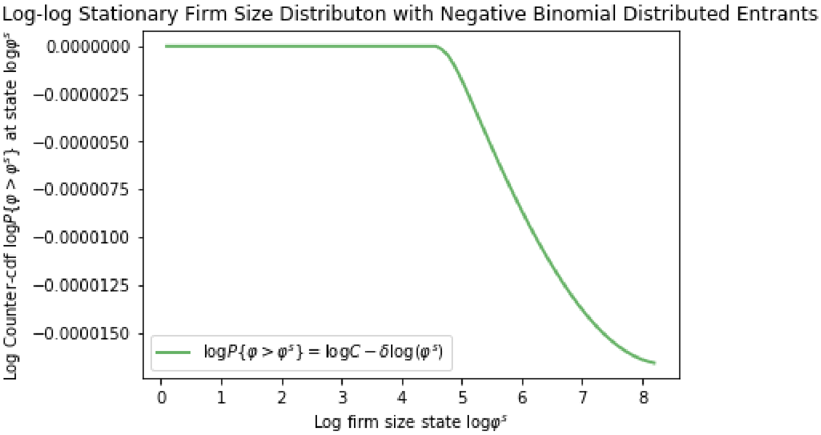
where

- $s \in \{0, 1, 2, \cdots, S-1\}$
- $0 < p < 1$: probability of success in each trial.

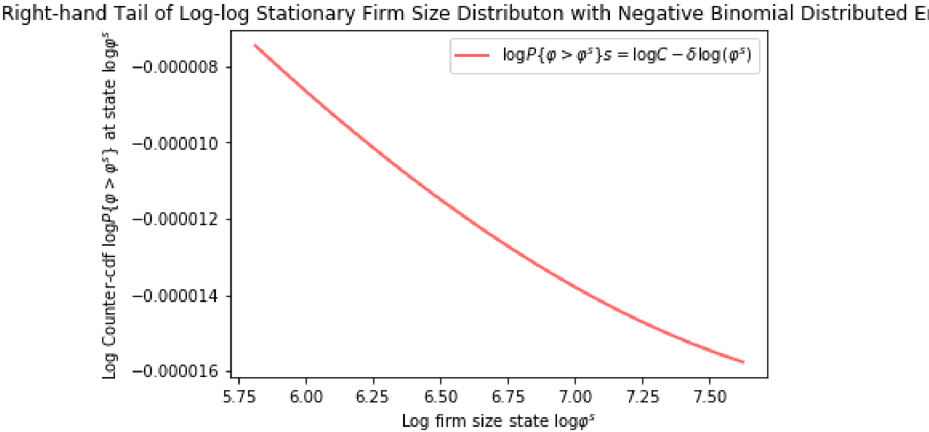
Notice: Parameters Change

- $S = S2$
- $p = 0.5$

6.3 Simulation 7



6.3 Simulation 7



6.3 Simulation 8

When G is Beta-Binomial distributed with pmf,

$$G(\varphi^s) = \binom{S-1}{s} \frac{B(s+\alpha, S-1-s+\beta)}{B(\alpha, \beta)} \tag{15}$$

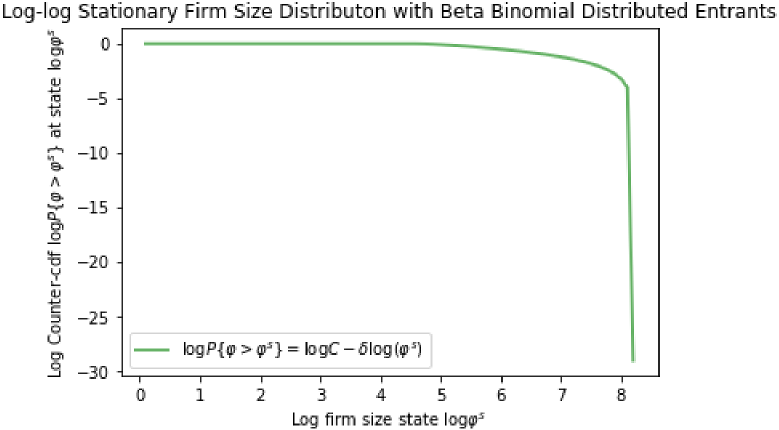
where

- $s \in \{0, 1, 2, \cdots, S-1\}$
- $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,
  - Γ: the Gamma Function,
  - $\Gamma(n) = (n-1)!$,
  - $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$,
  - $\alpha > 0$
  - $\beta > 0$

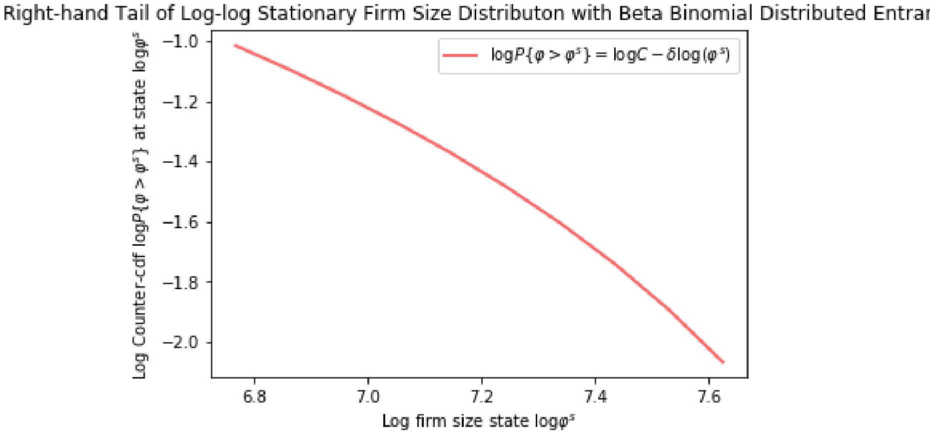
Notice: Parameters Change

- $S = S2$

6.3 Simulation 8



6.3 Simulation 8

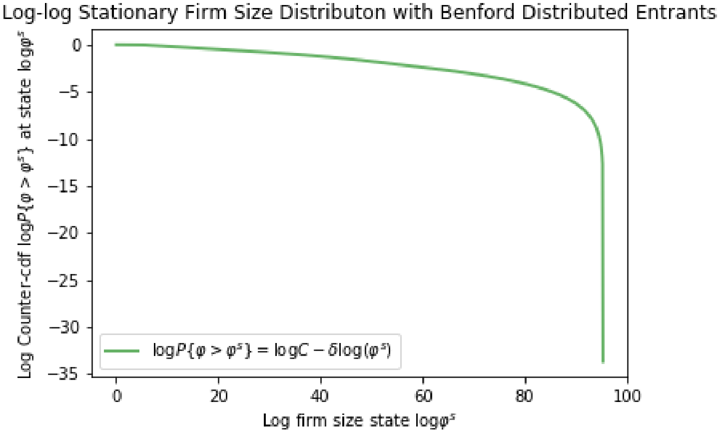


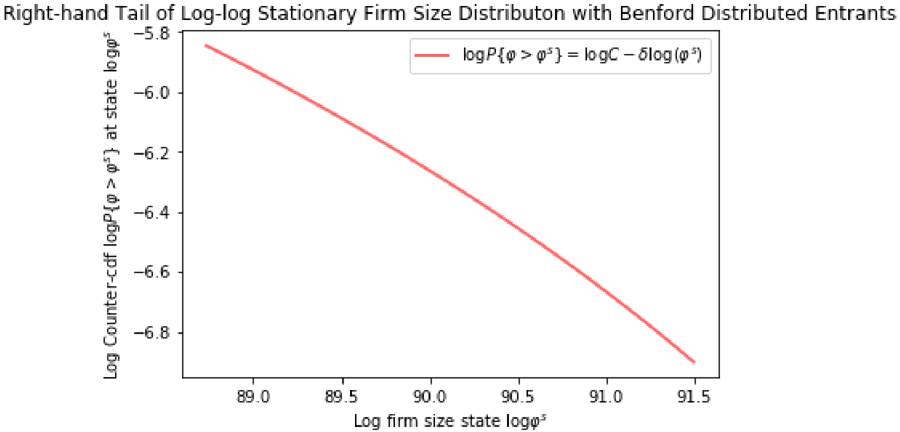
6.3 Simulation 9

When G is Benford distributed with pmf,

$$G(\varphi^s) = \log_{S+1}(1 + \frac{1}{s}), where\ s \in \{1, \cdots, S\} \tag{16}$$

6.3 Simulation 9




Right-hand Tail of Log-log Stationary Firm Size Distributon with Benford Distributed Entrants
$\log P\{\varphi > \varphi^s\} = \log C - \delta \log(\varphi^s)$
Log Counter-cdf $\log P\{\varphi > \varphi^s\}$ at state $\log\varphi^s$
Log firm size state $\log\varphi^s$
−5.8
−6.0
−6.2
−6.4
−6.6
−6.8
89.0
89.5
90.0
90.5
91.0
91.5

7. What's Next

- Work on my Conjecture and prove that
  - The stationary firm size can be Pareto, given whatever entrants' distributions.
- Work on other interesting parts of the firm dynamic model.
- Find data in the real world, employ empirical tools to analyze quantitative implications of the model and my conjecture.

The End. Thank You!