ECON2125/4021/8013*
Week 9 Tutorial Questions (1/5/2015)

Semester 1 2015

***Question* 1**

X and **Y** are random vectors, and **a** is a nonrandom vector of same size. A and B are conformable matrices. Try to verify the following:

1. $\mathbb{E}[A\mathbf{X}] = A\mathbb{E}[\mathbf{X}]$
2. $\mathbb{E}[\mathbf{X}A] = \mathbb{E}[\mathbf{X}]A$
3. $\mathbb{E}[A\mathbf{X} + B\mathbf{Y}] = A\mathbb{E}[\mathbf{X}] + B\mathbb{E}[\mathbf{Y}]$
4. $\text{var}[\mathbf{a} + B\mathbf{X}] = B\text{var}[\mathbf{X}]B'$
5. $\text{var}[\mathbf{X}]$ is always symmetric and nonnegative definite.

***Question* 2**

Let X and Y be two random variables. The **law of iterated expectation** tells us that $\mathbb{E}(Y) = \mathbb{E}_X [E(Y|X)]$, where $\mathbb{E}(Y|X)$ is defined to be the conditional expectation of Y given X, and $\mathbb{E}_X$ denotes the expectation with respect to the random variable X.

1. Prove the law of iterated expectation for the density case. Use the following notation: $p(x, y)$ denotes the joint density of X and Y. $p(x)$ and $p(y)$ represent the marginal density of X and Y, respectively. $p(y|x)$ denotes the conditional density of Y given $X = x$. In this case, $\mathbb{E}(Y|X) := \int_{-\infty}^{\infty} yp(y|x)dy$.

2. Let $g(\cdot)$ be a real-valued function. Use the law of iterated expectation to show that

$$\mathbb{E}[g(X, Y)] = \mathbb{E}_X \left[\mathbb{E}_{Y|X} g(X, Y)\right] := \mathbb{E}_X \{\mathbb{E}[g(X, Y)|X]\}$$

where $\mathbb{E}_{Y|X} g(X, Y) := \mathbb{E}[g(X, Y)|X]$, i.e., the conditional expectation of $g(X, Y)$ given X. In the density case, $\mathbb{E}_{Y|X} g(X, Y)$ is defined to be the random variable

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$\int_{-\infty}^{\infty} g(X, y)p(y|X)dy$, which is a function of the random variable X.

3. Based on the first two parts of the question, show that

$$\mathrm{var}\left[g(X,Y)\right] = \mathbb{E}_X\left\{\mathrm{var}_{Y|X}\left[g(X,Y)\right]\right\} + \mathrm{var}_X\left\{\mathbb{E}_{Y|X}\left[g(X,Y)\right]\right\}$$

where var_X denotes the variance with respect to the random variable X, and $\mathrm{var}_{Y|X}\left[g(X,Y)\right] := \mathrm{var}_X\left[g(X,Y)|X\right] := \mathbb{E}_{Y|X}\left\{\left[g(X,Y) - \mathbb{E}_{Y|X}g(X,Y)\right]^2\right\}$ denotes the variance of the random variable Y given X. In the density case, $\mathrm{var}_{Y|X}\left[g(X,Y)\right]$ is defined to be the random variable $\int_{-\infty}^{\infty}\left[g(X,y) - \mathbb{E}_{Y|X}g(X,Y)\right]^2 p(y|X)dy$, which is a function of the random variable X.

***Question* 3**

As mentioned in the lecture, the Law of Large Numbers (LLN) still works if correlations die out sufficiently quickly. This question guides you to prove that case, i.e., the LLN when there is correlation. Before working on this question, have a careful review of the LLN you learned in the lecture and the proof of it.

1. Let $\{X_n\}_{n\geq 1}$ be a sequnce of random variables, and $(\beta_k)_{k\geq 0}$ be a positive sequence in $\mathbb{R}$. Prove that if the sequence$\{X_n\}$ satisfies $\mathrm{cov}\left(X_i, X_{i+k}\right) \leq \beta_k$ for all $i \geq 1$, and the sequence (β_k) satisfies $\sum_{k\geq 0}\beta_k < \infty$, then $var\left(\bar{X}_n\right) \to 0$ as $n \to \infty$.

2. Prove the LLN when there is correlation: If (1) $\mathrm{cov}\left(X_i, X_{i+k}\right) \leq \beta_k$ for all $i \geq 1$ where (β_k) is positive and satisfies $\sum_{k\geq 0}\beta_k < \infty$, and (2) $\mathbb{E}[X_n] \to m \in \mathbb{R}$ as $n \to \infty$, then $\bar{X}_n \to m$ in probability as $n \to \infty$.