# SELECTED SOLUTIONS FOR WEEK 4 TUTORIAL

## Question 2

PART 4.

*Proof.* In this question , based on the condition that $f$ being a bijection, we are required to show that

$$f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2)$$

**Step 1.** Verify that $f^{-1}(C_1 \cap C_2) \subset f^{-1}(C_1) \cap f^{-1}(C_2)$
$\forall x \in f^{-1}(C_1 \cap C_2)$, we have: $\exists y \in C_1 \cap C_2$ s.t. $x = f^{-1}(y)$
So, $y \in C_1$ and $y \in C_2$ s.t. $x = f^{-1}(y)$
So, $y \in C_1$ s.t. $x = f^{-1}(y)$ and $y \in C_2$ s.t. $x = f^{-1}(y)$
So, $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$
So, $x \in f^{-1}(C_1) \cap f^{-1}(C_2)$
In this way, we showed that $f^{-1}(C_1 \cap C_2) \subset f^{-1}(C_1) \cap f^{-1}(C_2)$
**Step 2**. Verify that $f^{-1}(C_1) \cap f^{-1}(C_2) \subset f^{-1}(C_1 \cap C_2)$
$\forall x \in f^{-1}(C_1) \cap f^{-1}(C_2)$, we have: $x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2)$
So, $\exists y_1 \in C_1$ s.t. $x = f^{-1}(y_1)$ and $\exists y_2 \in C_2$ s.t. $x = f^{-1}(y_2)$
Since $f^{-1}$ is a bijection, which means $f^{-1}$ is one-to-one. Since $f^{-1}(y_1) = x = f^{-1}(y_2)$, by the definition of one-to-one function, we must have: $y_1 = y_2$.
$y := y_1 = y_2$, so we have: $y \in C_1 \cap C_2$ and $x = f^{-1}(y)$
So, $\exists y \in C_1 \cap C_2$, s.t. $x = f^{-1}(y)$, i.e., $x \in f^{-1}(C_1 \cap C_2)$
In this way, we showed that $f^{-1}(C_1) \cap f^{-1}(C_2) \subset f^{-1}(C_1 \cap C_2)$ □

## Question 4

PART 1.

*Proof.* $\forall x, y \in S_1 \cap S_2 \cap ... \cap S_n$, $\forall \alpha, \beta \in \mathbb{R}$, we need to verify that $\alpha x + \beta y \in S_1 \cap S_2 \cap ... \cap S_n$

Since $S_1, ..., S_n$ are all linear subspaces, we have: $\alpha x + \beta y \in S_1$,$\alpha x + \beta y \in S_2$, ..., $\alpha x + \beta y \in S_n$, which means $\alpha x + \beta y \in S_1 \cap S_2 \cap ... \cap S_n$ □

PART 2

*Proof.* Consider the counter example:
$S_1 := span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathbb{R}\right\}$
$S_1 := span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)) = \left\{\begin{bmatrix} y \\ 0 \end{bmatrix} : y \in \mathbb{R}\right\}$

It is obvious that $S_1$ and $S_2$ are all linear subspaces of $\mathbb{R}^2$(Prove it by the definition of linear subspace).

In this case, $S_1 \cup S_2 = \left\{\begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} y \\ 0 \end{bmatrix} : y \in \mathbb{R}\right\}$. In particular, a typical element of $S_1 \cup S_2$ is an element in $\mathbb{R}^2$ that must have at least one coordinate being 0.

However, $\forall \alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \neq 0$, we have $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \notin S_1 \cup S_2$ since both coordinates of the vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are not 0.

This tells us that $S_1 \cup S_2$ is not a linear subspace of $\mathbb{R}^2$ since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S_1 \subset S_1 \cup S_2$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S_2 \subset S_1 \cup S_2$ □