ECON2015/4021/8013

Practice Questions Set 1

March 14, 2015

Comments

Some comments from your lecturer: First, these questions are not directly assessable and solutions are provided in a separate PDF file. The aim is to help you better understand the material we have covered so far and start to prepare for the mid-term exam.

Second, if you are having no particular problems with this course then please carry on to the questions below. If, on the other hand, you are having difficulty with the material, then please read the following before proceeding to the questions.

This is an upper level course on mathematics for economists that pushes you beyond the boundaries of the kind of things we do in high school or first year university (which is usually just a refresher course). Many people find this material hard at first. I certainly did. However, in my experience of teaching this kind of course, *anyone* who works diligently and consistently can and will do well.

Here are few tips on getting through and doing well:

1. It's hard to "wing" this course, even if you did well at maths in high school. It's also very hard to follow everything just from the lectures. It takes practice to do well, just like playing guitar or learning a language. The first resource is the lecture slides, and the more often you read them the more the definitions will stick and the material will gel in your head.

2. Each time you read about a new concept in the lecture slides try googling it. Have a look what Wikipedia says, or some of the other online resources. They might phrase or explain the concept in a way that fits better with your brain.

3. Work *consistently* throughout the semester. Concepts become clearer and more familiar the more times that you go over them—with at least one sleep in between to allow your brain to organize neurons and synapses to store and categorize this new information.

4. Make use of your tutors. They are very willing to put in time to help anyone who is genuinely trying (although much less inclined to help those who aren't).

5. Send me feedback if you think it's something I can help with (e.g., more practice questions on a certain topic) or drop in during my office hours to discuss.

Above all, remember that the course material is nontrivial for a reason. Doing straightforward calculations applying well known rules or memorizing "cookbooks" of facts are not particularly useful, mainly because computers are far, far better than humans at these kinds of activities. What is still very useful—probably more than ever—is understanding concepts and how they relate to each other, and building up your ability to digest technical material and think in a logical way. If you complete this course successfully you will have significantly upgraded your mathematical skills.

Finally, here are some tips on doing proofs:

1. In many instances there will be an easy way to do things, if you can spot it. A question that seems to require a long calculation will likely have an easy answer if you know the relevant fact.

2. If you feel stuck, remember that the hardest step is getting started, and for proofs the best place to start is *always* the relevant definitions. If you are asked to show that the range of a given function T is

a linear subspace, start by writing down those two definitions. They will tell you more specifically what you need to show. If you're still stuck, review any facts from the lecture slides related to those definitions. Is there a different way to describe the range of this function? Is there some fact related to linear subspaces that might be helpful?

3. If you're still stuck, try flipping the problem around. In the previous example, suppose that the range of the function is *not* a linear subspace. What would that imply? Can you show that such an outcome is impossible?

4. Be patient and don't rush. You'll get quicker naturally, with practice.

Questions

Question 1. Let $f\colon [-1,1] \to \mathbb{R}$ be defined by $f(x) = -|x|$, where $|x|$ is the absolute value of x. Is the point $x = 0$ a maximizer of f on $[-1,1]$? Is it a unique maximizer? Is it an interior maximizer? Is it stationary?

Question 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x)$. Write down the set of stationary points of this function. Which of these, if any, are maximizers, and which are minimizers?[1]

Question 3. Consider the production function on $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$ defined by $f(k,\ell) = \min\{\alpha k,\, \beta\ell\} :=$ smallest value out of αk and $\beta\ell$. Let costs be given by $c(k,\ell) = rk + w\ell$. Find the minimizer in the cost minimization problem

$$\min_{(k,\ell)\in\mathbb{R}^2_+} c(k,\ell) \;\text{ s.t. }\; f(k,\ell) \geq q$$

as well as the minimum (the value of the objective function at the minimizer). In doing so you can assume that the parameters α, β, r, w and q are all strictly positive.[2]

Question 4. Let f and g be any two functions from $\mathbb{R}$ to $\mathbb{R}$. Is it true that $g \circ f = f \circ g$ always holds?[3]

[1] Remarks: When we discussed these kinds of problems it was for functions of the form $f\colon [a,b] \to \mathbb{R}$. Now the domain is all of $\mathbb{R}$. However you can apply the same definitions and use similar reasoning. Also, feel free to look up and use any helpful facts on trigonometric functions.

[2] Hint, if you want it: The objective function is not differentiable here. Try to use direct reasoning. You might gain some intuition by sketching the contour lines and constraint in the same way we did when we looked at cost minimization with tangency conditions.

[3] Some remarks: There are two things implicit in this question. First, there is an implicit final sentence here, which is: If yes, prove it. If no, give a counterexample. Second, an equality sign between two functions means that they are the same function. Hence to show equality you need to show that they agree everywhere on the domain. To show inequality, you need to give just one point in the domain where the function values differ.

Question 5. Consider the matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0.5 & -12 \\ -2 & 7 \end{pmatrix}$$

Do the columns of this matrix form a basis of $\mathbb{R}^3$? Why or why not?

Question 6. Let A, B and C be any three sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.[4]

Question 7. Is $\mathbb{R}^2$ a linear subspace of $\mathbb{R}^3$? Why or why not?

Question 8. Show that if $T \colon \mathbb{R}^K \to \mathbb{R}^N$ is a linear function then $\mathbf{0} \in \ker(T)$.

Question 9. Let S be any nonempty subset of $\mathbb{R}^N$ with the following two properties:

$(\star)$ $\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S$

$(\star\star)$ $c \in \mathbb{R}$ and $\mathbf{x} \in S \implies c\mathbf{x} \in S$

Is S a linear subspace of $\mathbb{R}^N$?

Question 10. If S is a linear subspace of $\mathbb{R}^N$ then any linear combination of K elements of S is also in S. Show this for the case $K = 3$.

Question 11. Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a linearly independent set in $\mathbb{R}^2$ and let γ be a nonzero scalar. Is it true that $\{\gamma\mathbf{x}_1, \gamma\mathbf{x}_2\}$ is also linearly independent?

Question 12. Is

$$\mathbf{z} = \begin{pmatrix} -3.98 \\ 12.42 \\ -6.8 \end{pmatrix}$$

[4] Hint, if you need it: One way to show that $E = F$ is show that a arbitrary element of E must also be in F and vice versa.

in the span of $X := \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Why or why not?

Question 13. What is the rank of the $N \times N$ identity matrix $\mathbf{I}$?

Question 14. Let X be a finite subset of linear space $S \subset \mathbb{R}^N$. Show that if $\operatorname{span}(X) \neq S$, then there exists an $\mathbf{x} \in S$ with $\mathbf{x} \notin \operatorname{span}(X)$.

Question 15. Let S be a linear subspace of $\mathbb{R}^N$ such that $S \neq \{\mathbf{0}\}$. Show that S has at least one linearly independent subset.

One claim in the lectures was that every linear subspace of $\mathbb{R}^N$ distinct from $\{\mathbf{0}\}$ has a basis. The next question steps you through the proof.

Question 16. Let S be a nonzero subspace of $\mathbb{R}^N$. Since S is not $\{\mathbf{0}\}$, it contains at least one linearly independent subset. Let M be the size of the largest linearly independent set in S. Thus, by the definition of M, there exist a set $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_M\} \subset S$ that is linearly independent, and any subset of S with more vectors is linearly dependent. If $\operatorname{span}(X) = S$, then X is by definition a basis of S. Hence it remains only to show that this equality holds. Show that it does.

Question 17. Show that if $T\colon \mathbb{R}^N \to \mathbb{R}^N$ is nonsingular (i.e., a linear bijection) then T^{-1} is also linear.