ECON2015/4021/8013

Practice Questions Set 1

March 14, 2015

Solutions

Solution 1: The point $x = 0$ is indeed a maximizer, since $f(x) = -|x| \leq 0 = f(0)$ for any $x \in [-1, 1]$. It is also a unique maximizer, since no other point is a maximizer (because $-|x| < 0$ for any other x). It is an interior maximizer since 0 is not an end point of $[-1, 1]$. It is not stationary because f is not differentiable at this point (sketch the graph if you like) and hence cannot satisfy $f'(x) = 0$. □

Solution 2: The set S of stationary points of f are the points $x \in \mathbb{R}$ such that $f'(x) = \cos(x) = 0$. By the definition of the cosine function this is the set

$$S := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \in \mathbb{Z}\}$$

Every point in the domain $\mathbb{R}$ is interior (i.e, not an end point) and the function f is differentiable, so the set of maximizers will be contained in the set of stationary points. The same is true of the set of minimizers. From the definition of the sine function, we have

$$f(\pi/2 + k\pi) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Hence the set of maximizers is

$$M^* := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \text{ an even integer}\}$$

The set of minimizers is

$$M_* := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \text{ an odd integer}\}$$

□

Solution 3: First, the cost function is strictly increasing in k and ℓ, so we would never produce more output than we need to. In particular, any pair (k, ℓ) with $f(k, \ell) > q$ is not a minimizer. Hence we need only consider the case $\min\{\alpha k, \beta\ell\} = q$. Second, no point (k, ℓ) with $\alpha k \neq \beta\ell$ is a minimizer. For example, if $\alpha k > \beta\ell$ then we can slightly reduce k without changing output $\min\{\alpha k, \beta\ell\}$. This will strictly reduce cost. A similar argument applies to the case $\alpha k < \beta\ell$.

These observations are enough to solve our problem. We know that any minimizer satisfies $\alpha k = \beta\ell$, and also that

$$\min\{\alpha k, \beta\ell\} = \alpha k = \beta\ell = q$$

The only possibility is therefore (k^*, ℓ^*) where $k^* = q/\alpha$ and $\ell^* = q/\beta$. The minimum value is

$$rk^* + w\ell^* = \frac{rq}{\alpha} + \frac{wq}{\beta}$$

□

Solution 4: This is not always true. For example, if $f(x) = x^2$ and $g(x) = 4x$, then $g \circ f$ and $f \circ g$ differ. Indeed, if we set $x = 1$, then

$$(g \circ f)(1) = g(f(1)) = 4(1^2) = 4,$$

while

$$(f \circ g)(1) = f(g(1)) = (4 \times 1)^2 = 16.$$

Hence $g \circ f \neq f \circ g$ as claimed. □

Solution 5: No, these two vectors do not form a basis of $\mathbb{R}^3$. If it did then $\mathbb{R}^3$ would be spanned by just two vectors. This is impossible. For example, it would imply by the exchange lemma that any three vectors in $\mathbb{R}^3$ are linearly dependent. We know this is false. □

Solution 6: Let A, B and C be any three sets, as in the question. Let

$$E := A \cap (B \cup C) \quad \text{and} \quad F := (A \cap B) \cup (A \cap C)$$

We need to show that $E = F$, or, equivalently, that $E \subset F$ and $F \subset E$.

To see that $E \subset F$, pick any $x \in E$. We claim that $x \in F$ also holds. To see this, observe that since $x \in E$, it must be true that x is in A as well as being in at least one of B and C. In the first case x is in both A and B. In the second case x is in both A and C. In either case we have $x \in F$ by the definition of F.

To see that $F \subset E$, pick any $x \in F$. We claim that $x \in E$ also holds. Indeed, since $x \in F$ we know that either x is in both A and B or x is in both A and C. In other words, x is in A and also at least one of B and C. Hence $x \in E$. □

Solution 7: This is a bit of a trick question, but to solve it you just need to look carefully at the definitions (as always). A linear subspace of $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ with certain properties. $\mathbb{R}^3$ is a collection of 3-tuples (x_1, x_2, x_3) where each x_i is a real number. Elements of $\mathbb{R}^2$ are 2-tuples (pairs), and hence not elements of $\mathbb{R}^3$. Therefore $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$, and in particular not a linear subspace of $\mathbb{R}^3$. □

Solution 8: Let T be as in the question. We need to show that $T\mathbf{0} = \mathbf{0}$. Here's one proof. We know from the definition of scalar multiplication that $0\mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$. Hence, letting $\mathbf{x}$ and $\mathbf{y}$ be any vectors in $\mathbb{R}^K$ and applying the definition of linearity,

$$T\mathbf{0} = T(0\mathbf{x} + 0\mathbf{y}) = 0T\mathbf{x} + 0T\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

□

Solution 9: Yes, S must be a linear subspace of $\mathbb{R}^N$. To see this, pick any $\mathbf{x}$ and $\mathbf{y}$ in S and any scalars α, β. To establish our claim we need to show that $\mathbf{z} := \alpha\mathbf{x} + \beta\mathbf{y}$ is in S. To see that this is so observe that by ($\star\star$) we have $\mathbf{u} := \alpha\mathbf{x} \in S$ and $\mathbf{v} := \beta\mathbf{y} \in S$. By ($\star$) we then have $\mathbf{u} + \mathbf{v} \in S$. In other words, $\mathbf{z} \in S$ as claimed. □

Solution 10: Let $\mathbf{x}_i \in S$ and $\alpha_i \in \mathbb{R}$ for $i = 1, 2, 3$. We claim that

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 \in S \tag{1}$$

To see this let $\mathbf{y} := \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$. By the definition of linear subspaces we know that $\mathbf{y} \in S$. Using the definition of linear subspaces again we have $\mathbf{y} + \alpha_3 \mathbf{x}_3 \in S$. Hence (1) is confirmed. □

Solution 11: The answer is yes. Here's one proof: Suppose to the contrary that $\{\gamma \mathbf{x}_1, \gamma \mathbf{x}_2\}$ is linearly dependent. Then one element can be written as a linear combination of the others. In our setting with only two vectors, this translates to $\gamma \mathbf{x}_1 = \alpha \gamma \mathbf{x}_2$ for some α. Since $\gamma \neq 0$ we can multiply each side by $1/\gamma$ to get $\mathbf{x}_1 = \alpha \mathbf{x}_2$. But now each $\mathbf{x}_i$ is a multiple of the other. This contradicts linear independence of $\{\mathbf{x}_1, \mathbf{x}_2\}$.

Here's another proof: Take any $\alpha_1, \alpha_2 \in \mathbb{R}$ with

$$\alpha_1 \gamma \mathbf{x}_1 + \alpha_2 \gamma \mathbf{x}_2 = \mathbf{0} \tag{2}$$

We need to show that $\alpha_1 = \alpha_2 = 0$. To see this, observe that

$$\alpha_1 \gamma \mathbf{x}_1 + \alpha_2 \gamma \mathbf{x}_2 = \gamma(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2)$$

Hence $\gamma(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) = \mathbf{0}$. Since $\gamma \neq 0$, the only way this could occur is that $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 = \mathbf{0}$. But $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent, so this implies that $\alpha_1 = \alpha_2 = 0$. The proof is done. □

Solution 12: There is an easy way to do this: We know that any linearly independent set of 3 vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$. Since $\mathbf{z} \in \mathbb{R}^3$, this will include $\mathbf{z}$. So all we need to do is show that X is linearly independent. To this end, take any scalars $\alpha_1, \alpha_2, \alpha_3$ with

$$\alpha_1 \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Written as 3 equations, this says that

$$\begin{aligned} -4\alpha_1 &= 0 \\ 2\alpha_2 &= 0 \\ -1\alpha_3 &= 0 \end{aligned}$$

Hence $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and therefore the set is linearly independent. □

Solution 13: By definition, $\text{rank}(\mathbf{I})$ is equal to the dimension of the span of its columns. Its columns are the N canonical basis vectors in $\mathbb{R}^N$, which we know span all of $\mathbb{R}^N$. Hence

$$\text{rank}(\mathbf{I}) = \dim(\mathbb{R}^N) = N$$

□

Solution 14: Let X and S be as in the statement of the question. If $\text{span}(X) \neq S$, then either $\text{span}(X)$ contains a vector not in S, or S contains a vector not in $\text{span}(X)$. The first case is impossible, because $X \subset S$ and S is a linear subspace, and hence any linear combination of elements of X also lies in S. Therefore the second case must hold. This verifies the existence claim in the question. □

Solution 15: Let $\{\mathbf{x}\}$ where $\mathbf{x}$ is a nonzero vector in S. This set is linearly independent. Indeed, given that $\mathbf{x} \neq \mathbf{0}$, if $\alpha\mathbf{x} = \mathbf{0}$ then we can be certain that $\alpha = 0$. Hence $\{\mathbf{x}\}$ satisfies the definition of linear independence. □

Solution 16: Let S, X and M be as stated in the question. Suppose to the contrary that $\text{span}(X) \neq S$. Then, as we saw in an earlier question, there exists some $\mathbf{x} \in S$ that does not lie in $\text{span}(X)$. It follows from the facts in lecture 7 that $X \cup \{\mathbf{x}\}$ is linearly independent. This set is a linearly independent subset of S with $M + 1$ elements. Existence of this set contradicts the definition of M. Hence $\text{span}(X) = S$ as claimed (and X is a basis of S). □

Solution 17: Let $T\colon \mathbb{R}^N \to \mathbb{R}^N$ be nonsingular and let T^{-1} be its inverse. To see that T^{-1} is linear we need to show that for any pair $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^N$ (which is the domain of T^{-1}) and any scalars α and β, the following equality holds:

$$T^{-1}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T^{-1}\mathbf{x} + \beta T^{-1}\mathbf{y}. \tag{3}$$

In the proof we will exploit the fact that T is by assumption a linear bijection.

So pick any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and any two scalars α, β. Since T is a bijection, we know that $\mathbf{x}$ and $\mathbf{y}$ have unique preimages under T. In particular, there exist unique vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$T\mathbf{u} = \mathbf{x} \quad \text{and} \quad T\mathbf{v} = \mathbf{y}$$

Using these definitions, linearity of T and the fact that T^{-1} is the inverse of T, we have

$$\begin{aligned} T^{-1}(\alpha\mathbf{x} + \beta\mathbf{y}) &= T^{-1}(\alpha T\mathbf{u} + \beta T\mathbf{v}) \\ &= T^{-1}(T(\alpha\mathbf{u} + \beta\mathbf{v})) \\ &= \alpha\mathbf{u} + \beta\mathbf{v} \\ &= \alpha T^{-1}\mathbf{x} + \beta T^{-1}\mathbf{y}. \end{aligned}$$

This chain of equalities confirms (3). □