ECON2015/4021/8013

Practice Questions Set 2

April 7, 2015

Comments

This second set of solved exercises is to help you prepare for the midterm (and final) exam(s). Apologies for getting it up on GitHub a bit later than expected. The topic is probability. The tutors and I are happy to discuss these problems during consultation hours if the solutions are not clear.

Questions

Question 1. Let Ω be a sample space, let $\mathbb{P}$ be a probability on Ω, and let A and B be events. Show that $\mathbb{P}(A) = \mathbb{P}(B) = 0$ implies $\mathbb{P}(A \cup B) = 0$.

Question 2. Let Ω be a sample space, let $\mathbb{P}$ be a probability on Ω, and let A and B be events satisfying $\mathbb{P}(A) = 1/2$ and $\mathbb{P}(B) = 2/3$. Show that

1. $1/6 \leq \mathbb{P}(A \cap B)$
2. $\mathbb{P}(A \cap B) \leq 1/2$

Question 3. Let Ω be a sample space, let $\mathbb{P}$ be a probability on Ω, and let A and B be events. Show that if A and B are independent, then so are A^c and B^c.

Question 4. Let Ω be a sample space, let A be any event, let A^c be the complement and let $\mathbb{1}_A$ and $\mathbb{1}_{A^c}$ be the respective indicator functions. Try to express $\mathbb{1}_{A^c}$ as a function of $\mathbb{1}_A$.

Question 5. Suppose we have two coins, one of which is fair (with probability of heads $= 1/2$) and one of which is rigged, with probability of heads $= 1/4$. We don't know which is which, and there is no obvious visual difference between the coins. One of the coins is flipped and lands on heads. What is the probability that this coin is the fair coin?

Question 6. Let X be any random variable and let $Y := \exp(X)$. Show carefully that

1. $y \leq 0$ implies $\mathbb{P}\{Y \leq y\} = 0$
2. $\mathbb{P}\{Y > 0\} = 1$

Question 7. Let $X \sim F$ and let $Y := \exp(X)$. Let G be the distribution of Y. Show that

$$G(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ F(\ln(y)) & \text{if } y > 0 \end{cases}$$

In the lectures it was shown that if X is a random variable with density p then the distribution function F of X is differentiable and satisfies $F'(x) = p(x)$ at every x such that p is continuous. In the next question you are asked to show a partial converse. Feel free to appeal to the Fundamental Theorem of Calculus, which is regarded as part of your prerequisite knowledge for this course. Also, it might help you to know that the integral of a function f over all of $\mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx \tag{1}$$

Question 8. Let X be a random variable with distribution function F. Show that if F is differentiable on $\mathbb{R}$ then X has a density.

Question 9. Let $X \sim F$ where F is the Cauchy cdf, and let $Y := 2X$. Show that the density of Y is

$$g(y) = \frac{1}{(2\pi)(1 + (y/2)^2)}$$

Question 10. Find the expected value of the random variable X with density

$$p(x) := \mathbb{1}\{-2 \leq x \leq 4\}\frac{|x|}{10}$$

Question 11. Let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} \exp(x+y) & \text{if } x \leq 0 \text{ and } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let (X,Y) be a random vector with joint density p. Show that X and Y are independent.