# ECON2015/4021/8013

# Practice Questions Set 2 — Solutions

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**Question 1.** Let $\Omega$ be a sample space, let $\mathbb{P}$ be a probability on $\Omega$, and let $A$ and $B$ be events. Show that $\mathbb{P}(A) = \mathbb{P}(B) = 0$ implies $\mathbb{P}(A \cup B) = 0$.

*Solution to question 1.* Let $\mathbb{P}$, $A$ and $B$ be as stated in the question, with $\mathbb{P}(A) = \mathbb{P}(B) = 0$. From the lecture slides we know that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for any $A, B$. Hence $\mathbb{P}(A \cup B) \leq 0$. Since $\mathbb{P}$ is a probability, the inequality $\mathbb{P}(A \cup B) \geq 0$ also holds. Hence $\mathbb{P}(A \cup B) = 0$, as was to be shown. □

**Question 2.** Let $\Omega$ be a sample space, let $\mathbb{P}$ be a probability on $\Omega$, and let $A$ and $B$ be events satisfying $\mathbb{P}(A) = 1/2$ and $\mathbb{P}(B) = 2/3$. Show that

1. $1/6 \leq \mathbb{P}(A \cap B)$
2. $\mathbb{P}(A \cap B) \leq 1/2$

*Solution to question 2.* Let $\mathbb{P}$, $A$ and $B$ be as stated in the question. From the lecture slides we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

or, rearranging,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \tag{1}$$

Regarding part 1 of the question, since $\mathbb{P}(A \cup B) \leq 1$, or, equivalently, $-\mathbb{P}(A \cup B) \geq -1$, it follows from (1) that

$$\mathbb{P}(A) + \mathbb{P}(B) - 1 \leq \mathbb{P}(A \cap B)$$

Inserting $\mathbb{P}(A) = 1/2$ and $\mathbb{P}(B) = 2/3$ gives $1/6 \leq \mathbb{P}(A \cap B)$ as required.

Regarding part 2, note that $B \subset A \cup B$, so $2/3 = \mathbb{P}(B) \leq \mathbb{P}(A \cup B)$. Hence $-\mathbb{P}(A \cup B) \leq -2/3$, and therefore, using (1),

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B) - 2/3 = 1/2 + 2/3 - 2/3 = 1/2$$

□

**Question 3.** Let $\Omega$ be a sample space, let $\mathbb{P}$ be a probability on $\Omega$, and let $A$ and $B$ be events. Show that if $A$ and $B$ are independent, then so are $A^c$ and $B^c$.

*Solution to question 3.* Using various facts from the lectures and independence of $A$ and $B$, we have

$$\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}(A^c) + \mathbb{P}(B^c) - \mathbb{P}(A^c \cup B^c) \\
&= \mathbb{P}(A^c) + \mathbb{P}(B^c) - \mathbb{P}((A \cap B)^c) \\
&= 1 - \mathbb{P}(A) + 1 - \mathbb{P}(B) - 1 + \mathbb{P}(A \cap B) \\
&= 1 - \mathbb{P}(A) + 1 - \mathbb{P}(B) - 1 + \mathbb{P}(A)\mathbb{P}(B) \\
&= (1 - \mathbb{P}(A))(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A^c)\mathbb{P}(B^c)
\end{aligned}$$

Hence $A^c$ and $B^c$ are independent as claimed. □

**Question 4.** Let $\Omega$ be a sample space, let $A$ be any event, let $A^c$ be the complement and let $\mathbb{1}_A$ and $\mathbb{1}_{A^c}$ be the respective indicator functions. Try to express $\mathbb{1}_{A^c}$ as a function of $\mathbb{1}_A$.

*Solution to question 4.* The solution is $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$. To show this we just need to confirm that the left hand side and right hand side agree pointwise, which means at every $\omega \in \Omega$. This is true because

$$\omega \in A^c \implies 1 - \mathbb{1}_A(\omega) = 1 - 0 = 1$$

while

$$\omega \in A \implies 1 - \mathbb{1}_A(\omega) = 1 - 1 = 0$$

These values are the same as for $\mathbb{1}_{A^c}(\omega)$. □

**Question 5.** Suppose we have two coins, one of which is fair (with probability of heads $= 1/2$) and one of which is rigged, with probability of heads $= 1/4$. We don't know which is which, and there is no obvious visual difference between the coins. One of the coins is flipped and lands on heads. What is the probability that this coin is the fair coin?

*Solution to question 5.* Let $H$ represent heads for this coin, $F$ represent fair coin and $R$ represent rigged coin. Using Bayes' theorem and the law of total probability, we have

$$\begin{aligned}\mathbb{P}(F \mid H) &= \frac{\mathbb{P}(H \mid F)\mathbb{P}(F)}{\mathbb{P}(H)} \\ &= \frac{\mathbb{P}(H \mid F)\mathbb{P}(F)}{\mathbb{P}(H \mid F)\mathbb{P}(F) + \mathbb{P}(H \mid R)\mathbb{P}(R)}\end{aligned}$$

Since the coin was selected at random, $\mathbb{P}(F) = \mathbb{P}(H) = 1/2$. Plugging in the rest of the numbers gives

$$\mathbb{P}(F \mid H) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}} = \frac{2}{3}$$

□

**Question 6.** Let $X$ be any random variable and let $Y := \exp(X)$. Show carefully that

1. $y \leq 0$ implies $\mathbb{P}\{Y \leq y\} = 0$
2. $\mathbb{P}\{Y > 0\} = 1$

*Solution to question 6.* Let $X$ and $Y$ be as described. Regarding part 1, if $y \leq 0$, then $\exp(x) > y$ for any $x \in \mathbb{R}$, so

$$\{Y \leq y\} = \{\exp(X) \leq y\} = \{\omega \in \Omega : \exp(X(\omega)) \leq y\} = \emptyset$$

Since $\mathbb{P}(\emptyset) = 0$ we have $\mathbb{P}\{Y \leq y\} = 0$.

Regarding part 2, if we set $y = 0$ in part 1 we get $\mathbb{P}\{Y \leq 0\} = 0$. Since

$$\{Y > 0\} = \{Y \leq 0\}^c$$

we have

$$\mathbb{P}\{Y > 0\} = 1 - \mathbb{P}\{Y \leq 0\} = 1 - 0 = 1$$

□

**Question 7.** Let $X \sim F$ and let $Y := \exp(X)$. Let $G$ be the distribution of $Y$. Show that

$$G(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ F(\ln(y)) & \text{if } y > 0 \end{cases}$$

*Solution to question 7.* Let $X$ and $Y$ be as described and let $G$ be the cdf of $Y$. If $y \leq 0$, then

$$G(y) = \mathbb{P}\{Y \leq y\} = 0$$

as shown above. On the other hand, if $y > 0$, then

$$G(y) = \mathbb{P}\{Y \leq y\} = \mathbb{P}\{\exp(X) \leq y\} = \mathbb{P}\{X \leq \ln(y)\}$$

This equals $F(\ln(y))$, as was to be shown. □

In the lectures it was shown that if $X$ is a random variable with density $p$ then the distribution function $F$ of $X$ is differentiable and satisfies $F'(x) = p(x)$ at every $x$ such that $p$ is continuous. In the next question you are asked to show a partial converse. Feel free to appeal to the Fundamental Theorem of Calculus, which is regarded as part of your prerequisite knowledge for this course. Also, it might help you to know that the integral of a function $f$ over all of $\mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n \to \infty} \int_{-n}^{n} f(x)dx \tag{2}$$

**Question 8.** Let $X$ be a random variable with distribution function $F$. Show that if $F$ is differentiable on $\mathbb{R}$ then $X$ has a density.

*Solution to question 8.* By the definition of the statement "$X$ has a density" from the lecture slides, we need to show the existence of a density $p$ such that

$$\mathbb{P}\{a < X \leq b\} = \int_a^b p(x)dx \tag{3}$$

for any $a < b$. As a candidate for $p$ take $p = F'$. This function exists since $F$ is differentiable. By the Fundamental Theorem of Calculus and the definition of $F$ we have, for any $a < b$,

$$\int_a^b p(x)dx = F(b) - F(a) \tag{4}$$

It now follows from the fact $F(b) - F(a) = \mathbb{P}\{a < X \leq b\}$ that (3) is valid.

The only thing we haven't verified is that $p$ is in fact a density. To see this observe that $p(x) \geq 0$ for all $x$, since $p = F'$ and $F$ is increasing. Moreover, in light of (4), (2) and the properties of cdfs,

$$\int_{-\infty}^{\infty} p(x)dx = \lim_{n\to\infty}(F(n) - F(-n)) = 1 - 0 = 1$$

□

**Question 9.** Let $X \sim F$ where $F$ is the Cauchy cdf, and let $Y := 2X$. Show that the density of $Y$ is

$$g(y) = \frac{1}{(2\pi)(1 + (y/2)^2)}$$

*Solution to question 9.* Let $X$, $F$ and $Y$ be as described and let $G$ be the cdf of $Y$. We have

$$G(y) = \mathbb{P}\{Y \leq y\} = \mathbb{P}\{2X \leq y\} = \mathbb{P}\{X \leq y/2\} = F(y/2)$$

Since $F$ is the Cauchy cdf this translates to

$$G(y) = \frac{\arctan(y/2)}{\pi} + \frac{1}{2}$$

Differentiating gives

$$g(y) = G'(y) = \frac{1}{(2\pi)(1 + (y/2)^2)}$$

as claimed. □

**Question 10.** Find the expected value of the random variable $X$ with density

$$p(x) := \mathbb{1}\{-2 \leq x \leq 4\} \frac{|x|}{10}$$

*Solution to question 10.* The expectation is

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x p(x) dx \\
&= \int_{-2}^{4} x \frac{|x|}{10} dx \\
&= \int_{-2}^{0} x \frac{-x}{10} dx + \int_{0}^{4} x \frac{x}{10} dx \\
&= \frac{-1}{10} \int_{-2}^{0} x^2 dx + \frac{1}{10} \int_{0}^{4} x^2 dx = \frac{28}{15}
\end{aligned}$$

□

**Question 11.** Let $p \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} \exp(x+y) & \text{if } x \leq 0 \text{ and } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $(X, Y)$ be a random vector with joint density $p$. Show that $X$ and $Y$ are independent.

*Solution to question 11.* Let $\phi_X$ be the marginal density of $X$ and $\phi_Y$ be the marginal density of $Y$. It suffices to show that

$$p(x,y) = \phi_X(x)\phi_Y(y) \quad \text{for all } (x,y) \in \mathbb{R}^2 \tag{5}$$

To see that this is the case, first we obtain the marginal distributions by integrating out the other variable. To start let $x \leq 0$ and observe that

$$\phi_X(x) = \int_{-\infty}^{0} p(x,y) dy = \int_{-\infty}^{0} e^x e^y dy = e^x \int_{-\infty}^{0} e^y dy = e^x \int_{0}^{\infty} e^{-y} dy = e^x$$

On the other hand, if $x > 0$, then $p(x,y) = 0$ for any $y$, and hence

$$\phi_X(x) = \int_{-\infty}^{0} p(x,y) dy = 0$$

In summary,

$$\phi_X(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

A similar argument gives

$$\phi_Y(y) = \begin{cases} e^y & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

It then follows easily that (5) is valid. □