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Practice Questions Set 3

May 26, 2015

Comments

This third and last set of solved exercises is to help you prepare for the final exam. There is a mix of topics. It is possible to solve these questions using only basic algebra, a bit of logic and the facts and definitions from the slides. The solutions to the exercises are in this same file, at the end.

Some of the questions are harder than what you can expect for exam questions. This reflects the fact that you won't solve them under exam conditions. Don't be afraid to spend a while on each question. Write down the definition of each property you need to prove. Review the lecture slides for any facts connected to these definitions. If you still can't make progress, think what would happen if the property in question failed. Would that lead to some kind of contradiction?

Similarly, if you end up having to look up the answer to some of the questions don't be too concerned. (The last couple of questions in particular are very challenging.) But do review questions you couldn't get out later on. For example, try to reproduce the argument the next day without looking.

For results connected to analysis, sketching can be helpful. If you're asked to show that a certain set is open, try to sketch it. Then pick any point in that set and try to think why it should be interior. I also recommend that you first read over some of the simple proofs in lecture 16, say, in order to become more comfortable with the style of argument.

Note that properties you are asked to show will sometimes have a "there exists" qualifier. For example, to show that a point $\mathbf{x}$ is interior to a set A, you need to show that “there exists" an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset A$. When attempting to show this property, you need to construct a specific ϵ using the information you have at hand.

Finally, hints are in footnotes, which makes for an excess of superscripts with a variety of meanings (powers, products, footnotes, etc.). I couldn’t figure out a better way to arrange the information so please read carefully.

Questions

Question 1. Show that the function $f(x) = -|x|$ from $\mathbb{R}$ to $\mathbb{R}$ is concave.

Question 2. Let $\mathbf{A}$ be the 1×1 matrix (a). Give a necessary and sufficient condition on a (that is, an “if and only if" condition on a) under which $\mathbf{A}$ is nonsingular.

Question 3. Consider the function f from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) = (cx)^2 + z$$

Give a necessary and sufficient (if and only if) condition on c under which f has a unique minimizer.

Question 4. Let $\mathbf{C}$ be an $N \times K$ matrix, let $z \in \mathbb{R}$ and consider the function f from $\mathbb{R}^K$ to $\mathbb{R}$ defined by

$$f(\mathbf{x}) = \mathbf{x}'\mathbf{C}'\mathbf{C}\mathbf{x} + z$$

Show that f has a unique minimizer on $\mathbb{R}^K$ if and only if $\mathbf{C}$ has linearly independent columns.[1]

[1] Hint: Obviously, you should draw intuition from the preceding question. Also, what does linear independence of the columns of $\mathbf{C}$ say about the vector $\mathbf{C}\mathbf{x}$ for different choices of $\mathbf{x}$?

Question 5. Show that the Cobb-Douglas production function $f(k, \ell) = k^\alpha \ell^\beta$ from $A := [0, \infty) \times [0, \infty)$ to $\mathbb{R}$ is continuous everywhere on A.[2]

Question 6. Let $\beta \in (0, 1)$. Show that the utility function $U(c_1, c_2) = \sqrt{c_1} + \beta\sqrt{c_2}$ from $A := [0, \infty) \times [0, \infty)$ to $\mathbb{R}$ is continuous everywhere on A.

Question 7. Let B be the set of all consumption pairs (c_1, c_2) such that $c_1, c_2 \geq 0$ and $p_1 c_1 + p_2 c_2 \leq m$. Here p_1, p_2 and m are positive constants. Show that B is a closed subset of $\mathbb{R}^2$.[3]

Question 8. Consider the maximization problem

$$\max_{c_1, c_2} (\sqrt{c_1} + \beta\sqrt{c_2})$$

subject to $c_1, c_2 \geq 0$ and $p_1 c_1 + p_2 c_2 \leq m$. Here p_1, p_2 and m are nonnegative constants, and $\beta \in (0, 1)$. Show that this problem has a solution if and only if p_1 and p_2 are both strictly positive.

Question 9. Let $\mathbf{A}$ be any square matrix. Show that $\mathbf{A}$ and $\mathbf{A}'$ share the same eigenvalues.

Question 10. Show that, for conformable and suitably invertible matrices $\mathbf{A}$, $\mathbf{U}$ and $\mathbf{V}$, we have

$$(\mathbf{A} + \mathbf{U}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{I} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\mathbf{A}^{-1}$$

You don't need to prove that matricies are invertible, just that the expression for the inverse is valid.[4]

[2]Hint: You can use the fact that, for any $a \in \mathbb{R}$ the function $g(x) = x^a$ is continuous at any $x \in [0, \infty)$. This was mentioned in passing in lecture 17. Also, remember that norm convergence implies element by element convergence.

[3]Hint: Weak inequalities are preserved under limits!

[4]Hint: Look at the definition of the inverse and any subsequent facts. What is the minimum you need to prove to show that $\mathbf{B}$ is the inverse of $\mathbf{A}$?

Question 11. Show that for any vector $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ the inequality

$$\left| \sum_{n=1}^{N} x_n \right| \leq \|\mathbf{x}\| \sqrt{N}$$

always holds.[5]

Question 12. Let $\mathbf{D}$ and $\mathbf{A}$ be square matrices. Let $\mathbf{Q} := \mathbf{A}'\mathbf{D}\mathbf{A}$. Show that if $\mathbf{D}$ is positive definite and $\mathbf{A}$ is nonsingular, then $\mathbf{Q}$ is positive definite.

Question 13. Show directly, using the "ϵ, N" definition of convergence, that if $\{x_n\}$ is a sequence in $\mathbb{R}$ with $x_n \to x$ for some $x \in \mathbb{R}$, and if r is any constant in $\mathbb{R}$, then $rx_n \to rx$.[6]

Question 14. Let A be a nonempty bounded set and let $B := \{b \in \mathbb{R} : b = 2a \text{ for some } a \in A\}$. Obtain $\sup B$ in terms of $\sup A$. Justify your answer.

Question 15. Find the infimum of the following sets, justifying your answer

1. $\{1/n : n \in \mathbb{N}\}$
2. $\mathbb{Q}$, the rational numbers

Question 16. Prove that $\{x_n\}$ defined by $x_n = (n+1)/n$ is a Cauchy sequence. (Use any fact from the slides that will make your job easier.)

Question 17. Let $T \colon \mathbb{R}^K \to \mathbb{R}^N$ be a linear function. Show that the range of T is a linear subspace of $\mathbb{R}^N$.

Question 18. Is it true that for each square matrix $\mathbf{A}$ we have $\det(\mathbf{A}^2) = \det(\mathbf{A})^2$?

[5] Hint: Look carefully at the properties of norms that we learned about when we first introduced norms in lecture 6. Four major properties were stated on one of the slides. One of the four properties will be very helpful.

[6] Hint: If you have trouble, try treating the cases $r = 0$ and $r \neq 0$ separately.

Question 19. Let X and W be independent random variables. Let ρ be a number satisfying $|\rho| < 1$. Show that if $X \sim N(0,1)$, $W \sim N(0,1)$ and

$$Y = \rho X + \sqrt{1-\rho^2} W$$

then X, W and Y are identically distributed.

Question 20. The version of the Intermediate Value Theorem in the lecture slides stated that if $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then f has a zero in $[a,b]$. Using this fact, show that if $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(b) < 0 < f(a)$, then f has a zero in $[a,b]$.[7]

Question 21. Let A be a subset of $\mathbb{R}$. Show that if $\max A$ exists then $\sup A = \max A$.

Question 22. It is a fact that a subset A of $\mathbb{R}^K$ is closed and bounded $\iff$ every sequence in A has a subsequence which converges to a point of A Prove the $\implies$ part of this claim when $A \subset \mathbb{R}$.[8]

Remark: Sets with the property that every sequence in the set has a subsequence which converges to a point of the set are called compact.

Question 23. Let $\mathbb{D}$ be all $\mathbf{x} \in \mathbb{R}^N$ such that are nonnegative and sum to one. That is, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x}'\mathbf{1} = 1$. (This set is called the "unit simplex" in $\mathbb{R}^N$.) Let $T\colon \mathbb{D} \to \mathbb{D}$ be a continuous function. Show that T has at least one fixed point in $\mathbb{D}$.[9]

For the next question we note that a function f from a subset of $\mathbb{R}$ into $\mathbb{R}$ is called strictly increasing if $x < y \implies f(x) < f(y)$.

[7] Hint: A good approach is to come up with a transformation of f that (i) satisfies the conditions of the Intermediate Value Theorem, and (ii) has the same zeros as f. Can you think of one? Try sketching the problem.

[8] Hint: Use the Bolzano-Weierstrass theorem.

[9] Hint: You are only asked to show existence of a fixed point, not uniqueness, etc. So look in the lecture slides for a fixed point result that gives you only existence. Then try to check the conditions one by one.

Question 24. Let $A \subset \mathbb{R}^K$, let

- $f\colon A \to B \subset \mathbb{R}$
- $h\colon B \to \mathbb{R}$ and $g := h \circ f$

In the lecture we stated the fact that if h is strictly increasing, then

$$\operatorname*{argmax}_{\mathbf{x} \in A} f(\mathbf{x}) = \operatorname*{argmax}_{\mathbf{x} \in A} g(\mathbf{x})$$

We also proved the $\subset$ part of this claim. (Remember that these are sets, so equality is the same as both $\subset$ and $\supset$.) Now prove the other direction. That is, show that, under the stated assumptions,

$$\mathbf{a}^* \in \operatorname*{argmax}_{\mathbf{x} \in A} g(\mathbf{x}) \implies \mathbf{a}^* \in \operatorname*{argmax}_{\mathbf{x} \in A} f(\mathbf{x})$$

Question 25. Show that $(0, \infty)$ is an open subset of $\mathbb{R}$.

Question 26. Show that $S_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ is an open subset of $\mathbb{R}^2$.

Question 27. Although it wasn't stated in the slides, it is a well known fact that the intersection of any two open sets is open. Use this fact to prove that

$$P := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}$$

is an open set.

Question 28. Let $f\colon [a, b] \to \mathbb{R}$. Recall the statement of the Intermediate Value Theorem from the lecture slides: If $f(a) < 0 < f(b)$ and f is continuous, then f has a zero in $[a, b]$. Let's walk through the argument, breaking it down into exercises. For starters, define

- $Q := \{x \in [a, b] : f(x) < 0\}$
- $\bar{x} := \sup Q$

Show the following, which together prove that $f(\bar{x}) = 0$:

1. $\bar{x}$ exists (and is not ∞)
2. $f(\bar{x}) \leq 0$
3. $f(\bar{x}) \geq 0$

Hints:

- For 2, recall from the lecture slides that if s is the supremum of a set S, then there exists a sequence in that set converging up to s. If $S = Q$ and $s = \bar{x}$, what properties would such a sequence have? Also, you need to use the fact that f is continuous.
- For 3, suppose instead that $f(\bar{x}) < 0$. Intuitively, since f is continuous and therefore contains no jumps, we could then take a point y slightly larger than $\bar{x}$ but still realizing $f(y) < 0$. This y would contradict the definition of $\bar{x}$ as the supremum of Q. Can you formalize this argument? Perhaps by considering sequences?

Question 29. The reverse triangle inequality tells us that, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^K$, we have

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$$

Try to prove this inequality using your knowledge of norms.[10]

Question 30. Show that the function $f(\mathbf{x}) = \|\mathbf{x}\|$ is continuous on $\mathbb{R}^K$.[11]

Question 31. One of the claims in the lecture slides was that a continuous function on a closed bounded set has a maximizer and a minimizer (the

[10] Hint: Use the triangle inequality and the "add and subtract strategy" (e.g., $\mathbf{x} = \mathbf{x} - \mathbf{y} + \mathbf{y}$ and so on).

[11] Hint: Use the reverse triangle inequality.

Weierstrass extreme value theorem). This result turned on the fact that if $f\colon A \to \mathbb{R}$ is continuous and A is closed and bounded, then so is

$$f(A) := \{f(\mathbf{x}) : \mathbf{x} \in A\}$$

Try to prove this. Use the fact from question 22 that $A \subset \mathbb{R}^K$ is closed and bounded if and only if every sequence in A has a subsequence converging to a point in A.

Question 32. Suppose that $\{x_n\}$ is a bounded sequence in $\mathbb{R}$ that does not converge. Show that $\{x_n\}$ has at least two subsequences that converge to different limits.

Turn over for solutions

Solutions

Solution to question 1. Proof 1: We already showed that $f(\mathbf{x}) = \|\mathbf{x}\|$ is convex on $\mathbb{R}^K$, and that f convex implies $-f$ concave. Setting $K = 1$ gives the desired result.

Proof 2 (direct proof): Pick any $x, y \in \mathbb{R}$ and any $\lambda \in [0,1]$. By the triangle inequality, we have

$$|\lambda x + (1-\lambda)y| \leq |\lambda x| + |(1-\lambda)y|$$

and hence

$$-|\lambda x + (1-\lambda)y| \geq -|\lambda x| - |(1-\lambda)y| = -\lambda|x| - (1-\lambda)|y|$$

That is, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$. Hence f is concave as claimed. $\square$

Solution to question 2. The condition for nonsingularity of (a) is $a \neq 0$. There are many ways we could show this. One is that $\mathbf{A}$ is nonsingular when $\mathbf{A}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$. Here this translates to $ax = 0 \implies x = 0$. The question then becomes, for what a is this implication true? It is true exactly when $a \neq 0$, for if $a \neq 0$ and $ax = 0$, the only possibility is that $x = 0$. $\square$

Solution to question 3. The function f has a unique minimizer at $x^* = 0$ if and only if $c \neq 0$. Here's one proof: If $c \neq 0$ then the function is strictly convex. Moreover, it is stationary at $x^* = 0$. Hence, by our facts on minimization under convexity, x^* is the unique minimizer. The condition is necessary and sufficient because if $c = 0$, then f is a constant function, which clearly does not have a unique minimizer.

Here's a second (more direct) proof that the correct condition is $c \neq 0$. Suppose first that $c \neq 0$ and pick any $x \in \mathbb{R}$. We have

$$f(x) = (cx)^2 + z \geq z = f(0)$$

This tells us that $x^* = 0$ is a minimizer. Moreover,

$$f(x) = (cx)^2 + z > z = f(0) \quad \text{whenever} \quad x \neq x^*$$

Hence $x^* = 0$ is the unique minimizer.

Suppose next that $x^* = 0$ is the unique minimizer. Then it must be that $c \neq 0$, for if $c = 0$ then $f(x) = f(x^*)$ for every $x \in \mathbb{R}$. □

Solution to question 4. Suppose first that $\mathbf{C}$ has linearly independent columns. We claim that $\mathbf{x} = \mathbf{0}$ is the unique minimizer of f on $\mathbb{R}^K$. To see this observe that if $\mathbf{x} = \mathbf{0}$ then $f(\mathbf{x}) = z$. On the other hand, if $\mathbf{x} \neq \mathbf{0}$, then, by linear independence, $\mathbf{Cx}$ is not the origin, and hence $\|\mathbf{Cx}\| > 0$. Therefore

$$f(\mathbf{x}) = \mathbf{x}'\mathbf{C}'\mathbf{Cx} + z = (\mathbf{Cx})'\mathbf{Cx} + z = \|\mathbf{Cx}\|^2 + z > z$$

Thus $\mathbf{x} = \mathbf{0}$ is the unique minimizer of f on $\mathbb{R}^K$ as claimed.

Since this is an "if and only if" proof we also need to show that when f has a unique minimizer on $\mathbb{R}^K$, it must be that $\mathbf{C}$ has linearly independent columns. Suppose to the contrary that the columns of $\mathbf{C}$ are not linearly independent. We will show that multiple minimizers exist.

Since $f(\mathbf{x}) = \|\mathbf{Cx}\|^2 + z$ it is clear that $f(\mathbf{x}) \geq z$, and hence $\mathbf{x} = \mathbf{0}$ is one minimizer. (At this point, f evaluates to z.) Since the columns of $\mathbf{C}$ are not linearly independent, there exists a nonzero vector $\mathbf{y}$ such that $\mathbf{Cy} = \mathbf{0}$. At this vector we clearly have $f(\mathbf{y}) = z$. Hence $\mathbf{y}$ is another minimizer. □

Solution to question 5. Let (k, ℓ) be any point in A, and let $\{(k_n, \ell_n)\}$ be any sequence converging to (k, ℓ) in the sense of convergence in $\mathbb{R}^2$. We wish to show that

$$f(k_n, \ell_n) \to f(k, \ell)$$

Since $(k_n, \ell_n) \to (k, \ell)$ in $\mathbb{R}^2$, we know from the facts on convergence in norm that the individual components converge in $\mathbb{R}$. That is,

$$k_n \to k \quad \text{and} \quad \ell_n \to \ell \tag{1}$$

We also know from the facts that, for any a, the function $g(x) = x^a$ is continuous at x. It follows from the definition of continuity and (1) that $k_n^\alpha \to k^\alpha$ and $\ell_n^\beta \to \ell^\beta$. Moreover, we know that, for any sequences $\{y_n\}$ and $\{z_n\}$, if $y_n \to y$ and $z_n \to z$, then $y_n z_n \to yz$. Hence

$$k_n^\alpha \ell_n^\beta \to k^\alpha \ell^\beta$$

That is, $f(k_n, \ell_n) \to f(k, \ell)$. Hence f satisfies the definition of continity. □

Solution to question 6. The proof is very similar to the proof of continuity of the Cobb-Douglas production function given above, and hence is omitted. □

Solution to question 7. To show that B is closed, we need to show that the limit of any sequence contained in B is also in B. To this end, let $\{\mathbf{x}_n\}$ be an arbitrary sequence in B coverging to a point $\mathbf{x} \in \mathbb{R}^2$. Since $\mathbf{x}_n \in B$ for all n we have $\mathbf{x}_n \geq \mathbf{0}$ in the sense of the vector inequality (lecture 17) and $\mathbf{x}_n'\mathbf{p} \leq m$, where $\mathbf{p} = (p_1, p_2)$. We need to show that the same is true for $\mathbf{x}$.

Since $\mathbf{x}_n \to \mathbf{x}$, we have $\mathbf{x}_n'\mathbf{p} \to \mathbf{x}'\mathbf{p}$. Since limits preserve weak inequalities and $\mathbf{x}_n'\mathbf{p} \leq m$ for all n, we have $\mathbf{x}'\mathbf{p} \leq m$. Hence it remains only to show that $\mathbf{x} \geq 0$. Again using the fact that weak inequalities are preserved under limits, combined with $\mathbf{x}_n \geq \mathbf{0}$ for all n, gives $\mathbf{x} \geq \mathbf{0}$ as required. □

Solution to question 8. As we have seen, $U(c_1, c_2) = \sqrt{c}_1 + \beta\sqrt{c_2}$ is continuous and

$$B := \{(c_1, c_2) : c_i \geq 0 \text{ and } p_1 c_1 + p_2 c_2 \leq m\}$$

is closed. Hence, by the Weierstrass extreme value theorem, a maximizer will exist whenever B is bounded. If p_1 and p_2 are strictly positive then B is bounded. This is intuitive but we can also show it formally by observing that $(c_1, c_2) \in B$ implies $c_i \leq m/p_i$ for $i = 1, 2$. Hence

$$\mathbf{c} := (c_1, c_2) \in B \implies \|\mathbf{c}\| \leq M := \sqrt{\left(\frac{m}{p_1}\right)^2 + \left(\frac{m}{p_2}\right)^2}$$

We also need to show that if one price is zero then no maximizer exists. Suppose to the contrary that $p_1 = 0$. Intuitively, no maximizer exists because we can always consumer more of good one, thereby increasing our utility. To formalize this we can suppose that a maximizer exists and derive a contradiction. To this end, suppose that $\mathbf{c}^* = (c_1^*, c_2^*)$ is a maximizer of U over B. Since $p_1 = 0$, the fact that $(c_1^*, c_2^*) \in B$ implies $\mathbf{c}^{**} := (c_1^* + 1, c_2^*) \in B$. Since U is strictly increasing in its first argument, we also have $U(\mathbf{c}^{**}) > U(\mathbf{c}^*)$. This contradicts the statement that $\mathbf{c}^*$ is a maximizer of U over B. □

Solution to question 9. The eigenvalues of $\mathbf{A}$ are exactly the values of λ that solve

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Let λ be a solution to this equation. Recalling that the determinant is unaffected by the transpose operation, this implies that

$$\det((\mathbf{A} - \lambda \mathbf{I})') = 0$$

Using symmetry of $\mathbf{I}$ and other basic rules for transposes, we then have

$$\det(\mathbf{A}' - \lambda \mathbf{I}') = \det(\mathbf{A}' - \lambda \mathbf{I}) = 0$$

Hence λ is an eigenvalue of $\mathbf{A}'$.

We have shown that any eigenvalue of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}'$. The same argument works in the other direction. This proves the claim. □

Solution to question 10. It suffices to show that the product of the right and left hand sides is the identity. We have

$$\begin{aligned}
&(\mathbf{A} + \mathbf{U}\mathbf{V}) \left[\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U} \left(\mathbf{I} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U} \right)^{-1} \mathbf{V}\mathbf{A}^{-1} \right] \\
&\quad = \mathbf{I} + \mathbf{U}\mathbf{V}\mathbf{A}^{-1} - (\mathbf{U} + \mathbf{U}\mathbf{V}\mathbf{A}^{-1}\mathbf{U})(\mathbf{I} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1} \\
&\quad = \mathbf{I} + \mathbf{U}\mathbf{V}\mathbf{A}^{-1} - \mathbf{U}(\mathbf{I} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})(\mathbf{I} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1} \\
&\quad = \mathbf{I} + \mathbf{U}\mathbf{V}\mathbf{A}^{-1} - \mathbf{U}\mathbf{V}\mathbf{A}^{-1} = \mathbf{I}
\end{aligned}$$

□

Solution to question 11. The result follows immediately from the Cauchy-Schwarz inequality $|\mathbf{x}'\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ after setting $\mathbf{y} = \mathbf{1}$. □

Solution to question 12. Pick any nonzero $\mathbf{x}$. We aim to show that $\mathbf{x}'\mathbf{Q}\mathbf{x} > 0$ under the stated assumptions.

Since $\mathbf{x}$ is not the zero vector, neither is $\mathbf{y} := \mathbf{A}\mathbf{x}$ (otherwise independence of the columns of $\mathbf{A}$—and hence nonsingularity—would be contradicted). As a result, using positive definiteness of $\mathbf{D}$,

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = \mathbf{x}'\mathbf{A}'\mathbf{D}\mathbf{A}\mathbf{x} = \mathbf{y}'\mathbf{D}\mathbf{y} > 0$$

The proof is done. □

Solution to question 13. Let $\{x_n\}$, x and r be as in the statement of the question. Let $y_n = rx_n$ and $y = rx$. If $r = 0$, then the claim is that $0 \to 0$, which is trivial. So suppose instead that $r \neq 0$. Fix any $\epsilon > 0$. We aim to show existence of an $N \in \mathbb{N}$ such that

$$n \geq N \implies |y_n - y| < \epsilon \tag{2}$$

To this end, observe that since $x_n \to x$, we can select an $N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n - x| < \frac{\epsilon}{|r|}$$

(The right hand side is finite because $r \neq 0$.) From this it follows that

$$n \geq N \implies |r||x_n - x| < \epsilon$$

It is not hard to see that N satisfies (2). □

Solution to question 14. Let A and B be as stated in the question. We claim that $\sup B = \bar{b}$ where $\bar{b} := 2 \sup A$. According to the definition of the supremum, to prove this we need to show that

1. $b \leq \bar{b}$ for all $b \in B$

2. $\bar{b} \leq u$ for all $u \in U(B)$

Regarding part 1, pick any $b \in B$. By definition, $b = 2a$ for some $a \in A$. We know that $a \leq \sup A$ and hence $2a \leq 2 \sup A$. Therefore $b = 2a \leq \bar{b}$, as was to be shown.

Regarding part 2, take any $u \in U(B)$. For any $a \in A$ we have $2a \in B$ and hence $2a \leq u$. Therefore $a \leq u/2$ for all $a \in A$, and hence $u/2$ is an upper bound of A. Therefore $\sup A \leq u/2$. Rearranging gives $\bar{b} \leq u$, as claimed. □

Solution to question 15. Regarding part 1, the infimum of $A := \{1/n : n \in \mathbb{N}\}$ is 0. Clearly 0 is a lower bound. Moreover, if ℓ is any lower bound of A, then $\ell \leq 1/n$ for all n. Since the weak inequality is preserved under limits, this gives $\ell \leq 0$. In other words, 0 is the greatest lower bound.

Regarding part 2, recall that between any two real numbers we can find a rational number. It follows that the set of lower bounds of $\mathbb{Q}$ is the empty set, since any $\ell \in \mathbb{R}$ has a rational number q with $q < \ell$. Hence, by definition, $\inf \mathbb{Q} = -\infty$. □

Solution to question 16. As discussed in the slides, every convergent sequence is Cauchy, so it suffices to show that $\{x_n\}$ is convergent, and in particular that $x_n \to 1$. To see this, observe that

$$x_n = \frac{n+1}{n} = \frac{1 + 1/n}{1}.$$

Since $1/n \to 0$, it follows that $x_n \to 1$. □

Solution to question 17. Let $S := \operatorname{rng}(T)$. By definition,

$$S = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = T\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^K\}$$

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two vectors in S and let α and β be any two scalars. We need to show that $\alpha\mathbf{y}_1 + \beta\mathbf{y}_2 \in S$, or, equivalently, that

$$\exists\, \mathbf{z} \in \mathbb{R}^K \text{ s.t. } T\mathbf{z} = \alpha\mathbf{y}_1 + \beta\mathbf{y}_2$$

To see that this is so, observe that, since each $\mathbf{y}_i$ is in S, we must have vectors $\mathbf{x}_i \in \mathbb{R}^K$ such that $T\mathbf{x}_i = \mathbf{y}_i$ for $i = 1, 2$. Let $\mathbf{z} := \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$. Then, by linearity of T,

$$T\mathbf{z} = T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha T\mathbf{x}_1 + \beta T\mathbf{x}_2 = \alpha\mathbf{y}_1 + \beta\mathbf{y}_2$$

as was to be shown. □

Solution to question 18. Yes, this is true. We know that for any square and conformable $\mathbf{A}$ and $\mathbf{B}$ we have $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. Setting $\mathbf{B} = \mathbf{A}$ proves the claim. □

Solution to question 19. We need to show that $Y \sim N(0,1)$. Since Y is a linear combination of normal random variables it must itself be normal. Hence it remains only to show that $\mathbb{E}[Y] = 0$ and $\operatorname{var}[Y] = 1$. Regarding the first equality, linearity of $\mathbb{E}$ gives

$$\mathbb{E}[Y] = \rho\mathbb{E}[X] + \sqrt{1-\rho^2}\mathbb{E}[W] = 0$$

Regarding the second, using our rules for variance of linear combinations and the fact that X and Y are independent, we have

$$\operatorname{var}[Y] = \rho^2 \operatorname{var}[X] + (1-\rho^2)\operatorname{var}[W] = 1$$

□

Solution to question 20. Let $f : [a,b] \to \mathbb{R}$ be continuous with $f(b) < 0 < f(a)$. If we define $g = -f$ then $g(a) < 0 < g(b)$. Moreover, g is continuous, since multiplication of continuous functions by scalars (in this case -1) preserves continuity. Hence, by the Intermediate Value Theorem, there exists a $\bar{x} \in [a,b]$ with $g(\bar{x}) = 0$. Clearly $f(\bar{x}) = 0$ also holds, so f has a zero in $[a,b]$. □

Solution to question 21. Let $a^* := \max A$. We claim that $\sup A = a^*$. To show this we need to show that

1. a^* is an upper bound of A

2. $a^* \leq u$ for any other upper bound u

Since $a^* = \max A$ we have $a \leq a^*$ for all $a \in A$. Hence statement 1 holds. Since $a^* = \max A$ we also have $a^* \in A$. If u is any upper bound of A then, by definition, $a \leq u$ for all $a \in A$, and, in particular, $a^* \leq u$. Hence statement 2 holds as well. □

Solution to question 22. Let $A \subset \mathbb{R}$ be closed and bounded. Take any sequence $\{x_n\}$ contained in A. Since A is bounded, $\{x_n\}$ is bounded, and hence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence. Since the subsequence is also contained in A and A is closed, it must be that the limit lies in A. □

Solution to question 23. By the Brouwer fixed point theorem, since T is assumed to be continuous, it suffices to show that $\mathbb{D}$ is closed, bounded and convex. The proof that $\mathbb{D}$ is closed was given for the $N = 2$ case in the lectures and the general argument is no different. To see that $\mathbb{D}$ is convex, it is enough to show that $\mathbb{D}$ is the intersection of two convex sets. We can write $\mathbb{D}$ as $\mathbb{D} = C \cap P$ where $C := \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}'\mathbf{1} = 1\}$ and P is the "positive cone" $\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} \geq \mathbf{0}\}$. We showed in the lectures that P is convex. To see that C is convex, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two elements of C. Fixing $\lambda \in [0, 1]$, we claim that $\mathbf{y} := \lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in C$. This is clear since

$$\begin{aligned} \mathbf{y}'\mathbf{1} &= (\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)'\mathbf{1} \\ &= \lambda\mathbf{x}_1'\mathbf{1} + (1-\lambda)\mathbf{x}_2'\mathbf{1} = \lambda + (1-\lambda) = 1 \end{aligned}$$

It follows that C and hence $\mathbb{D}$ is convex.

It only remains to show that $\mathbb{D}$ is bounded. To see this, observe that each element of $\mathbf{x} \in \mathbb{D}$ is necessarily weakly less than one, and hence, for such an $\mathbf{x}$,

$$\|\mathbf{x}\| \leq \sqrt{\sum_{n=1}^{N} 1^2} = \sqrt{N}$$

Hence $\mathbb{D}$ is also bounded. $\square$

Solution to question 24. Let A, f, g and h be as in the statement of the problem. Let $\mathbf{a}^* \in \operatorname{argmax}_{\mathbf{x}\in A} g(\mathbf{x})$. Pick any $\mathbf{x} \in A$. By definition we have

$$h(f(\mathbf{x})) \leq h(f(\mathbf{a}^*)) \tag{3}$$

Since h is strictly increasing, it follows that $f(\mathbf{x}) \leq f(\mathbf{a}^*)$, because otherwise the inequality in (3) could not be valid.

We have now shown that $f(\mathbf{x}) \leq f(\mathbf{a}^*)$ for any $\mathbf{x} \in A$. Hence $\mathbf{a}^* \in \operatorname{argmax}_{\mathbf{x}\in A} f(\mathbf{x})$ as claimed $\square$

Solution to question 25. We did something similar in the slides but let's give a direct proof in any case. We need to show that any arbitrary point x in this set $(0, \infty)$ is interior. That is, given any strictly positive number x, there is an $\epsilon > 0$ such that every element of $B_\epsilon(x)$ is strictly positive. Consider the value $\epsilon := x/2$. If $y \in B_\epsilon(x)$, then $y > x - \epsilon = x/2 > 0$. This proves the claim. $\square$

Solution to question 26. We need to show that every point in S_1 is interior to S_1. To this end, pick any $\mathbf{x} = (x_1, x_2) \in S_1$. We need to exhibit an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset S_1$. Consider $\epsilon := x_1/2$. Since $(x_1, x_2) \in S_1$ we have $x_1 > 0$ and hence ϵ is indeed a positive number. Now take any $\mathbf{y} = (y_1, y_2) \in B_\epsilon(\mathbf{x})$. We claim that $\mathbf{y} \in S_1$ also holds.

To see this, observe that

$$\begin{aligned}
(y_1, y_2) \in B_\epsilon(\mathbf{x}) &\implies \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < \epsilon \\
&\implies (y_1 - x_1)^2 + (y_2 - x_2)^2 < \epsilon^2 \\
&\implies (y_1 - x_1)^2 < \epsilon^2 \\
&\implies |y_1 - x_1| < \epsilon \quad \text{(take the square root)} \\
&\implies x_1 - \epsilon < y_1 < x_1 + \epsilon \\
&\implies x_1/2 < y_1
\end{aligned}$$

Hence $y_1 > 0$, and $(y_1, y_2) \in S_1$ as claimed. $\square$

Solution to question 27. P can be written as $P = S_1 \cap S_2$ where $S_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ and $S_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. We showed above that S_1 is open. The proof that S_2 is open is essentially identical. Hence P is the intersection of two open sets, and therefore open. □

Solution to question 28. For part 1 of the question, note that Q is bounded, since it lies in the bounded set $[a, b]$. Every bounded set has a finite supremum.

For part 2 we need to show that $f(\bar{x}) \leq 0$. Recall from the lecture slides that if s is the supremum of a set, then there exists a sequence in that set converging up to s. In particular, since $\bar{x} = \sup Q$, there exists $\{x_n\} \subset Q$ with $x_n \to \bar{x}$. By the assumed properties of x_n and f we have

1. $f(x_n) \leq 0$ for all n, and
2. $f(x_n) \to f(\bar{x})$.

(Actually we have the stronger property $f(x_n) < 0$ for all n but we don't need it for this part.) Weak inequalities are preserved under limits, so $f(\bar{x}) \leq 0$.

For part 3 we need to show that $f(\bar{x}) \geq 0$. Suppose to the contrary that $f(\bar{x}) < 0$. Then $\bar{x}$ could not be the supremum of Q. Indeed, the sequence $x_n := \bar{x} + 1/n$ converges down to $\bar{x}$, and hence $f(x_n) \to f(\bar{x})$. Thus $f(x_n)$ is eventually in every ϵ-ball containing $f(\bar{x})$. It follows (why?) that $f(x_n) < 0$ for some n. Thus, $x_n = \bar{x} + 1/n$ is in Q. But now we have a point in Q strictly larger than $\bar{x}$. This contradicts the assumption that $\bar{x} = \sup Q$. □

Solution to question 29. From the triangle inequality we have

$$\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$$

It follows that

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

A similar argument reversing the roles of $\mathbf{x}$ and $\mathbf{y}$ gives

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

Combining these inequalities gives

$$-\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

In other words,

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$$

□

Solution to question 30. Pick any $\mathbf{x} \in \mathbb{R}^K$ and take any $\mathbf{x}_n \to \mathbf{x}$. Since $\mathbf{x}_n \to \mathbf{x}$ we have $\|\mathbf{x}_n - \mathbf{x}\| \to 0$. Moreover,

$$|f(\mathbf{x}_n) - f(\mathbf{x})| = |\|\mathbf{x}_n\| - \|\mathbf{x}\|| \leq \|\mathbf{x}_n - \mathbf{x}\|$$

It follows that $f(\mathbf{x}_n) \to f(\mathbf{x})$. Hence f is continuous at $\mathbf{x}$. Since $\mathbf{x}$ was arbitrary, we conclude that f is continuous everywhere. □

Solution to question 31. Let A and f be as described in the question. As discussed in the question statement, subsets of $\mathbb{R}^K$ are closed and bounded if and only if each sequence in the set has a subsequence converging to a point in the set. Hence it suffices to show that every sequence in $f(A)$ has a subsequence converging to a point in $f(A)$.

To see this, Let $\{\mathbf{y}_n\}$ be a sequence in $f(A)$. By definition, we can take $\{\mathbf{x}_n\} \subset A$ with $f(\mathbf{x}_n) = \mathbf{y}_n$ for each n. Since A itself is closed and bounded, there exists subsequence $\{\mathbf{x}_{n_k}\}$ with $\mathbf{x}_{n_k} \to \mathbf{x} \in A$. By continuity of f we have $f(\mathbf{x}_{n_k}) \to f(\mathbf{x})$. Since $\mathbf{x} \in A$ we have $f(\mathbf{x}) \in f(A)$. In summary, the subsequence $\mathbf{y}_{n_k} = f(\mathbf{x}_{n_k})$ converges to a point in $f(A)$. We conclude that $f(A)$ is closed and bounded. □

Solution to question 32. Since $\{x_n\}$ is bounded, we know from the Bolzano-Weierstrass theorem that this sequence possesses at least one convergent subsequence. Take any such subsequence $\{x_{n_k}\}$, converging to some point x. Because the whole sequence does not converge, it does not converge to x, and hence it must be the case that for some ϵ-ball $B_\epsilon(x)$ around x, the sequence leaves this ϵ-ball infinitely often. It follows that we can find a second subsequence of $\{x_n\}$ that lies entirely outside $B_\epsilon(x)$. Moreover, this second subsequence is clearly bounded, and hence itself contains a convergent subsequence. The convergent subsequence converges to a point other than x because it lies outside $B_\epsilon(x)$. We have now found a second convergent subsequence of $\{x_n\}$, converging to a different point. Hence the claim is verified. $\square$