HYPERBOLIC DISCOUNTING NOTE

The aim is to produce a recursive expression that allows us to apply ADP methods to hyperbolic discounting. At this stage we just seek a recursive expression for lifetime value.

Lifetime value is given by

$$v(x) := \mathbb{E}_x \left[r(X_0) + \alpha r(X_1) + \sum_{t \geqslant 2} \beta^t r(X_t) \right]$$

Pointwise this is

$$v = r + \alpha P r + \sum_{t \geqslant 2} (\beta P)^t r$$

We generalize to allow state-dependent discounting, taking J and K to be positive linear operators and writing

$$v = r + Jr + \sum_{t \geqslant 2} K^t r \tag{1}$$

Rearranging gives

$$v = r + Jr + K^2 r + K \sum_{t \geqslant 2} K^t r$$

Using (1) again gives

$$\begin{aligned} v &= r + Jr + K^2 r + K(v - r - Jr) \\ &= r - Kr + Jr - KJr + K^2 r + Kv, \end{aligned}$$

or

$$v = (I - K)(r + Jr) + K^2 r + Kv$$

If $r(K) < 1$, then lifetime value is uniquely defined by this recursive expression.