Stability of Markov Semigroups

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Outline of the Talk

Notation.

Introduce the L_1-approach of Lasota.

Our contribution.

Applications to stochastic growth theory.

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Some notation:

Let U be a space and let $F: U \to U$. Pair (U, F) a “dynamical system.”

The set $\{F^t x\}_{t=0}^{\infty}$ the “trajectory” of x under F.

Point x^* is “fixed,” or “stationary” if $Fx^* = x^*$.

(U, F) is “globally stable” if exists unique f.p. x^* and

$$F^t x \to x^* \text{ for all } x \in U.$$

Let $S \subset \mathbb{R}^n$ be the state space.

Let $L_1(S)$ be all $f\colon S \to \mathbb{R}$ with $\|f\| := \int |f(x)|dx < \infty$.

Let $D(S) :=$ all $f \in L_1(S)$ with $f \geq 0$ and $\int f = \|f\| = 1$.

A "Markov operator" is a linear operator $P\colon L_1(S) \to L_1(S)$ such that

$$f \in D(S) \text{ implies } Pf \in D(S).$$

We are interested in dynamical systems $(D(S), P)$, where P a MO.

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Why?!

Example: Representative household maximizes

$$\max_{k_0,k_1,\ldots} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(y_t - k_t)\right], \quad \text{s.t. } y_{t+1} = f(k_t)\, z_{t+1}, \tag{1}$$

$u'(0) = \infty$, increasing, concave, f maybe nonconvex, $(z_t)_{t \geq 1}$ IID LN(0,1).

We study the optimal process $(y_t)_{t \geq 0}$ given by $y_{t+1} = f(\pi(y_t))\, z_{t+1}$.

$$y_{t+1} | y_t \text{ lognorm}, \quad p(y' \mid y_t) = \frac{1}{\sqrt{2\pi} y'} \exp\left(-\frac{(\ln y' - \ln f \circ \pi(y_t))^2}{2}\right).$$

Let $\psi_t :=$ distrib of y_t. Claim that $\psi_{t+1} = P\psi_t$ for some MO P.

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$$\mathbb{P}\{y_{t+1} \in B\} = \mathbb{E}[\,\mathbb{P}\{y_{t+1} \in B \;\mid y_t\}\,] = \mathbb{E}\left[\int_B p(y' \mid y_t) dy'\right]$$

$$\therefore \quad \mathbb{P}\{y_{t+1} \in B\} = \int_B \mathbb{E}[p(y' \mid y_t)] dy'.$$

$$\therefore \quad \psi_{t+1}(y') = \mathbb{E}[p(y' \mid y_t)].$$

$$\therefore \quad \psi_{t+1}(y') = \int p(y' \mid y) \psi_t(y) dy.$$

$$\therefore \quad \psi_{t+1} = P\psi_t \text{ when } Pf := \int p(\cdot \mid y) f(y) dy.$$

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So is $(D(S), P)$ globally stable? (The Brock-Mirman problem)

Easy to show that if $\psi, \psi' \in D(S)$, then $\|P\psi - P\psi'\| \leq \|\psi - \psi'\|$.

Much harder to show that exists $\alpha < 1$ with

$$\|P\psi - P\psi'\| \leq \alpha\|\psi - \psi'\|, \quad \forall \psi, \psi' \in D(S).$$

However, Lasota showed we can often get the following condition:

$$\text{T2 contraction:} \quad \forall \psi \neq \psi' \in D(S), \quad \|P\psi - P\psi'\| < \|\psi - \psi'\|. \tag{2}$$

Under what conditions do we get this?

P said to “overlap supports” if

$$\forall \psi, \psi' \in D(S), \quad \operatorname{supp} P\psi \cap \operatorname{supp} P\psi' \text{ non-negligable.}$$

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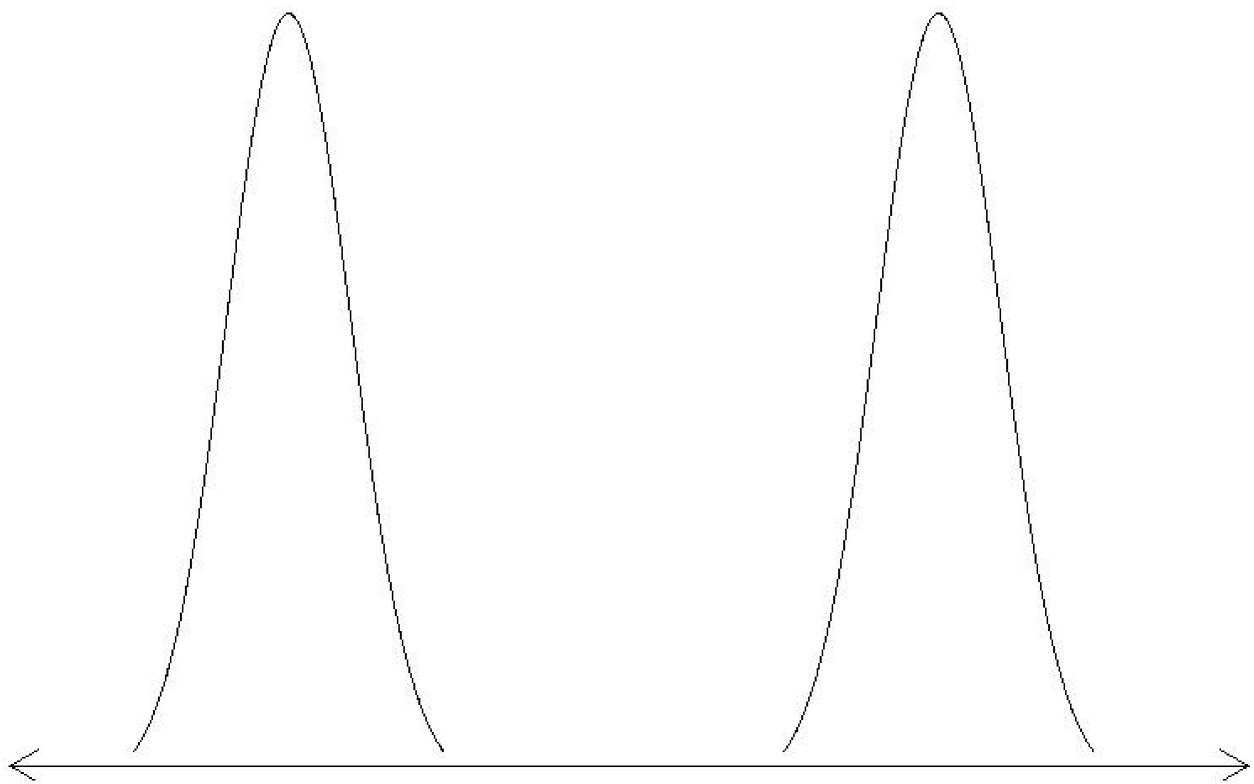

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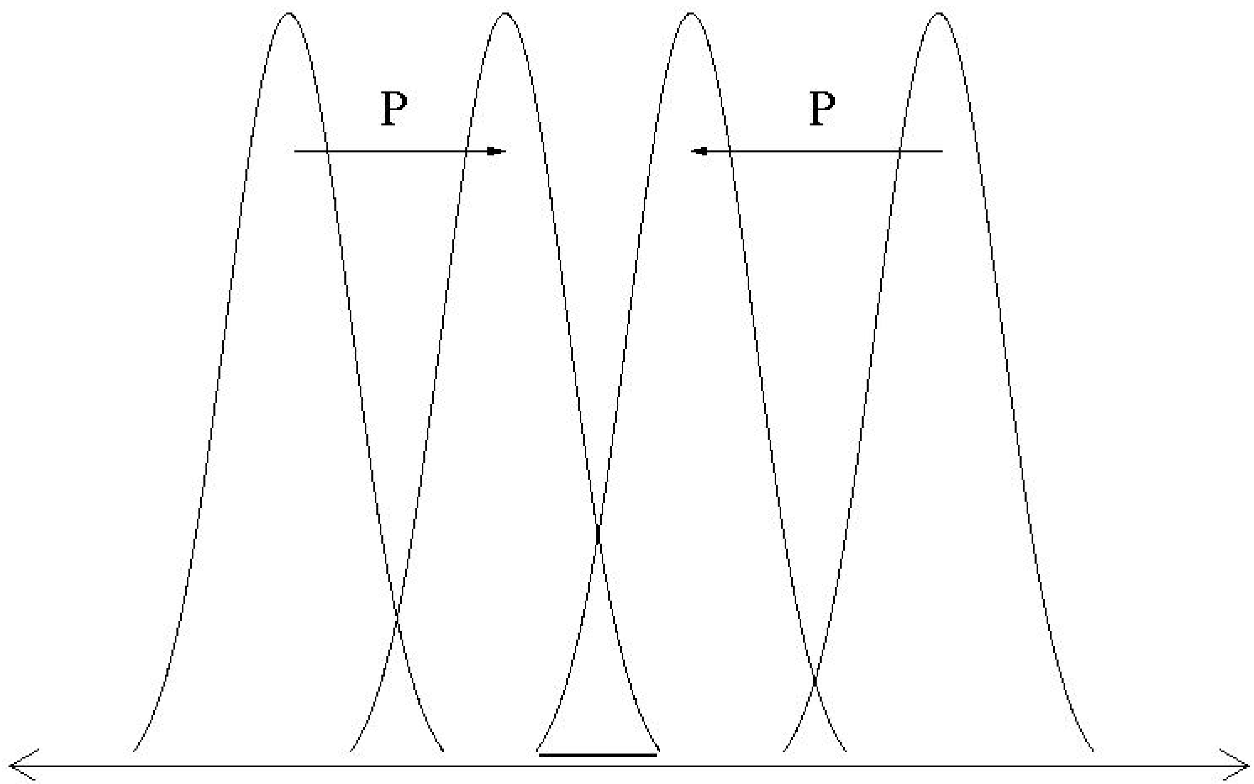

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The optimal growth P overlaps supports given lognormal assumption:

$$
\begin{aligned}
P\psi(y') &= \int p(y' \mid y)\psi(y)dy \\
&= \int \frac{1}{\sqrt{2\pi}y'} \exp\left(-\frac{(\ln y' - \ln f \circ \pi(y))^2}{2}\right) \psi(y)dy \\
&> 0
\end{aligned}
$$

$$
\therefore \quad \forall \psi \in D(S), \quad \operatorname{supp} P\psi = S. \tag{3}
$$

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Lasota: if P overlaps supports, then P a T2 contraction.

First implication of the T2 property: If $\psi \neq \psi'$ two f.p.s, then

$$\|P\psi - P\psi'\| < \|\psi - \psi'\| \text{ and } \|P\psi - P\psi'\| = \|\psi - \psi'\|. \tag{4}$$

Reason: this is a mixing condition.

Also, T2 contraction $P\colon$ compact $\to$ compact is globally stable.

Therefore, if $D(S)$ were compact, $(D(S), P)$ would be globally stable!

It's not, but enough to show that $\{P^t\psi\}$ is precompact for each $\psi \in D(S)$.

Reason: P maps $\{P^t\psi\}$ to itself.

A bit hard to check when $\{P^t\psi\}$ is precompact. . .

But! Lasota points out that if P is an integral operator:

$$P\psi = \int p(\cdot \mid y)\psi(y)dy, \tag{5}$$

then images of *weakly* precompact sets are strongly precompact.

Therefore sufficient to show that $\{P^t\psi\}$ is weakly precompact.

But how?

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Thm. Dunford & Pettis (1940): tight + UI = weakly precompact.

Let $D_0 \subset D(S)$. D_0 is called tight if

$$\forall \varepsilon > 0,\ \exists K \subset\subset S \text{ s.t. } \left\{ \int_{K^c} \psi(x)dx < \varepsilon,\ \ \forall \psi \in D_0 \right\}.$$

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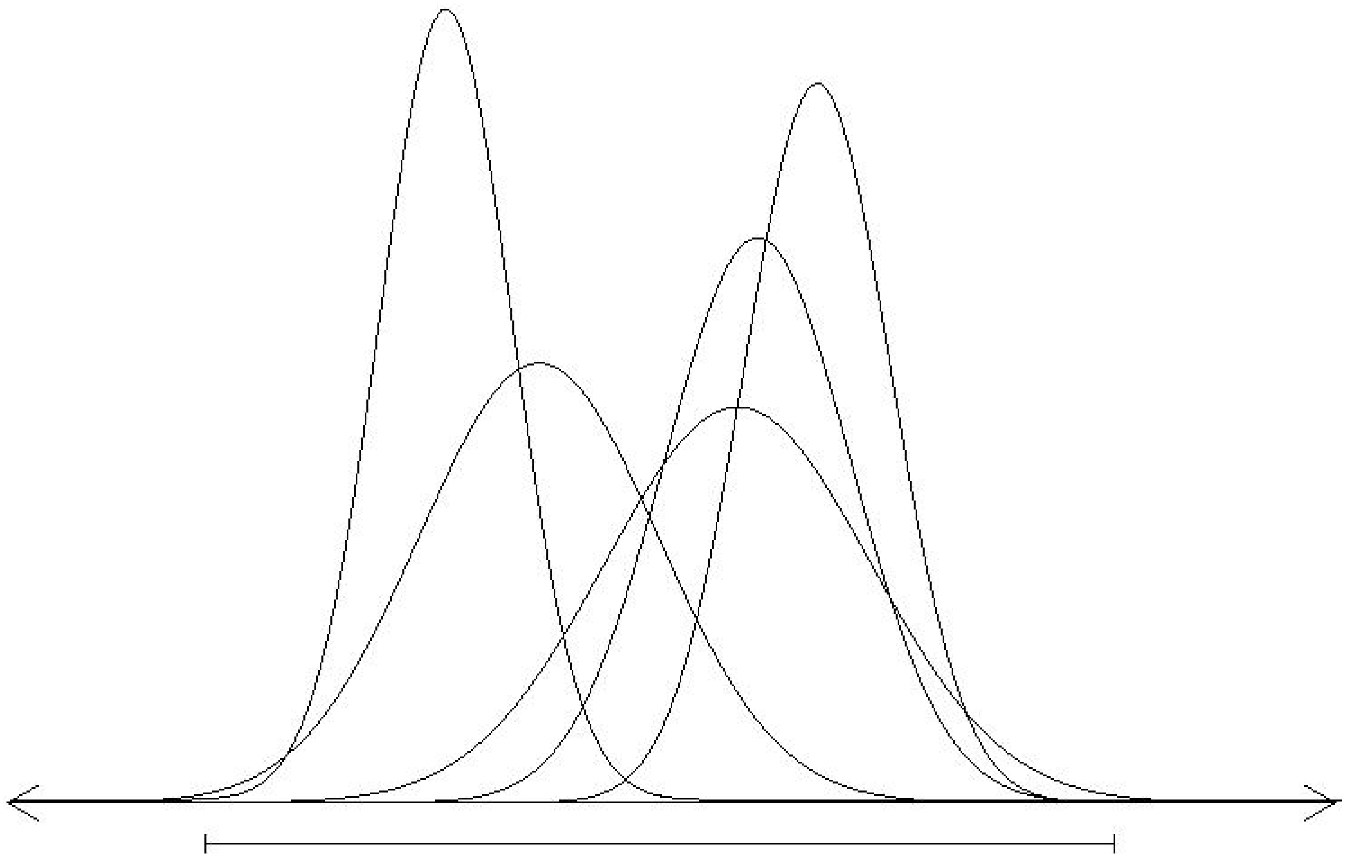

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The collection D_0 is called uniformly integrable if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \lambda(A) < \delta \implies \left\{ \int_A \psi(x)dx < \varepsilon,\ \ \forall \psi \in D_0 \right\}.$$

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Our main contribution:

If $\{P^t\psi\}$ is tight, and

$$\exists \text{ continuous } h\colon S \to \mathbb{R} \text{ with } \sup_y p(y' \mid y) \leq h(y') \tag{6}$$

then $\{P^t\psi\}$ is also UI, and hence weakly ($\Rightarrow$ strongly) precompact.

Example: optimal growth P with $S = (0, \infty)$:

$$p(y' \mid y) = \frac{1}{\sqrt{2\pi}y'} \exp\left(-\frac{(\ln y' - \ln f \circ \pi(y))^2}{2}\right) \leq \frac{1}{\sqrt{2\pi}y'} =: h(y').$$

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In summary: for our model, $\{P^t \psi\}$ tight implies $(D(S), P)$ is g.s.

In optimal growth case, $S = (0, \infty)$, $\{P^t \psi\}$ tight iff: $\forall \varepsilon$,

1. $\exists M < \infty$ s.t. $\mathrm{Prob}\{y_t \geq M\} \leq \varepsilon$, $\forall t \in \mathbb{N}$.

2. $\exists m > 0$ s.t. $\mathrm{Prob}\{y_t \leq m\} \leq \varepsilon$, $\forall t \in \mathbb{N}$.

Then $\mathrm{Prob}\{y_t \notin [m, M]\} \leq 2\varepsilon$ for all t.

For the first one:

$$\mathrm{Prob}\{y_t \geq M\} \leq \frac{\mathbb{E}[\, y_t \,]}{M}. \tag{7}$$

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Hard one is: $\forall \varepsilon$, $\exists m > 0$ s.t. $\sup_t \mathrm{Prob}\{y_t \le m\} \le \varepsilon$.

Recall that in deterministic case relevant condition is $f'(0) > 1/\beta$.

An example with nonconvex f:

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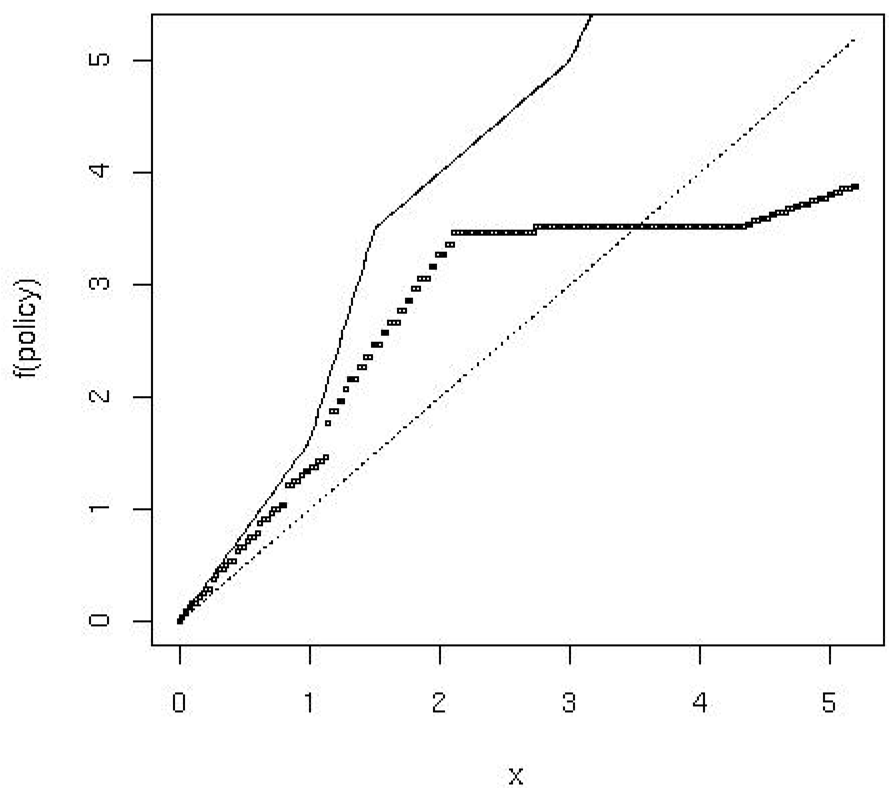

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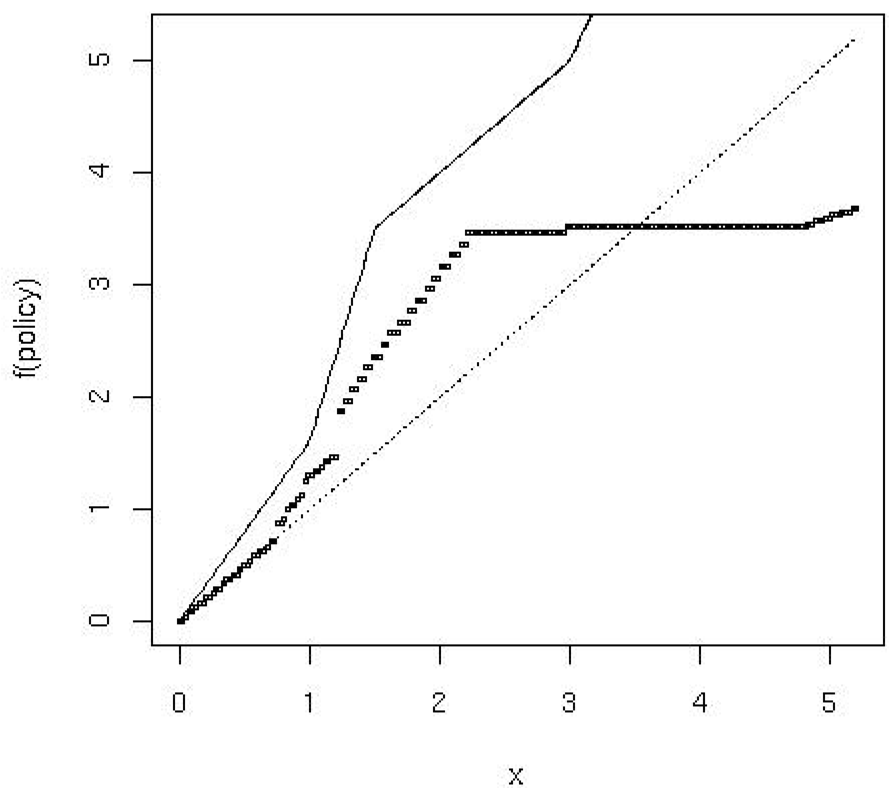

f(policy)
x
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1
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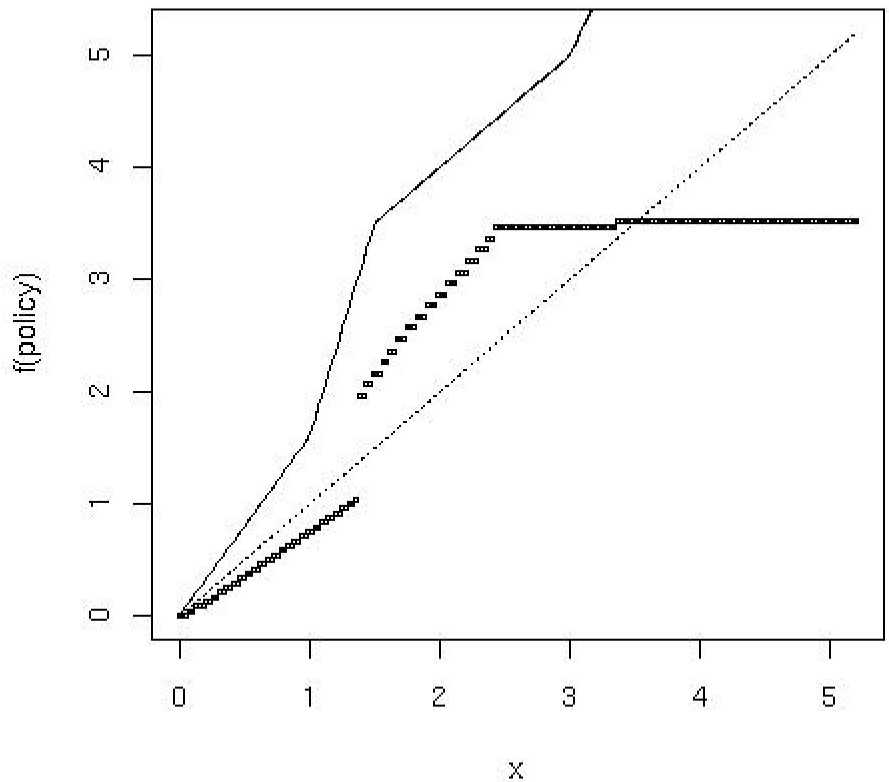

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What is the stochastic equivalent of $f'(0) > 1/\beta$?

We are looking for a similar condition, but accounting for distrib of shock.

Nishimura and Stachurski (2005):

$$f'(0) > \frac{1}{\beta} \cdot \mathbb{E}[\,1/z_t\,] \Rightarrow \forall \varepsilon,\ \exists m > 0 \text{ s.t. } \sup_t \mathrm{Prob}\{y_t \leq m\} \leq \varepsilon.$$

Therefore tight. Therefore tight + UI.

Therefore globally stable.

Simple proof, and we don't need any continuity conditions.

Open Questions

Question 1: What is the precise condition for tightness in the OGM?

In particular, how much better than

$$f'(0) > \frac{1}{\beta} \cdot \mathbb{E}(1/z)$$

can we do?

Some simulations:

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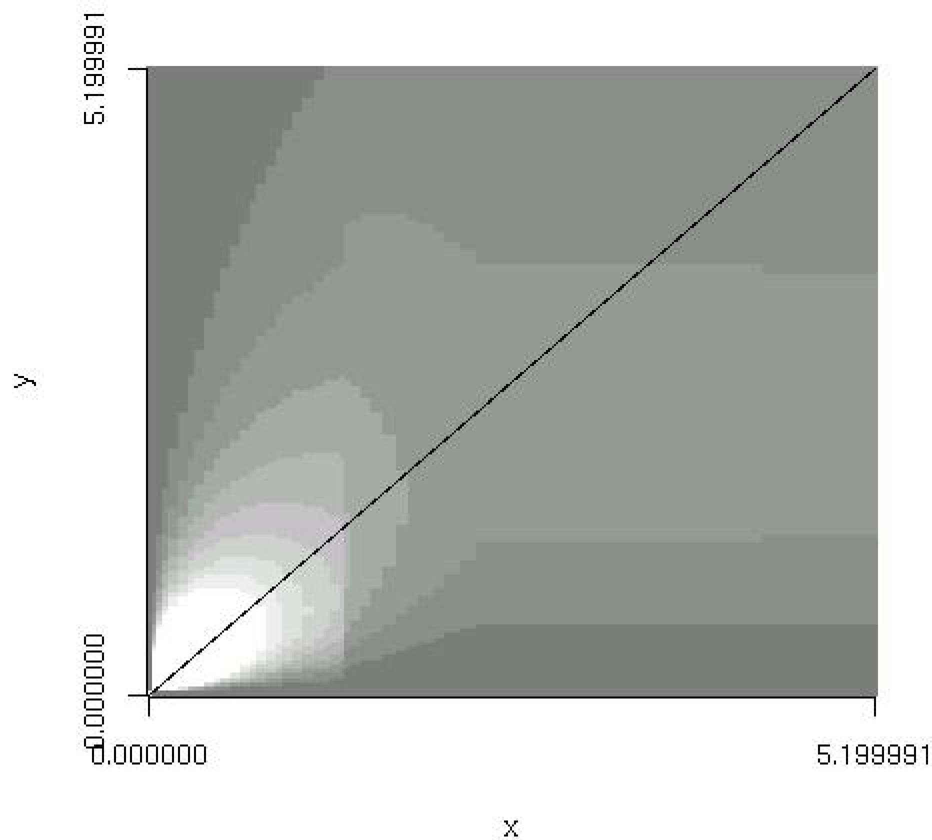
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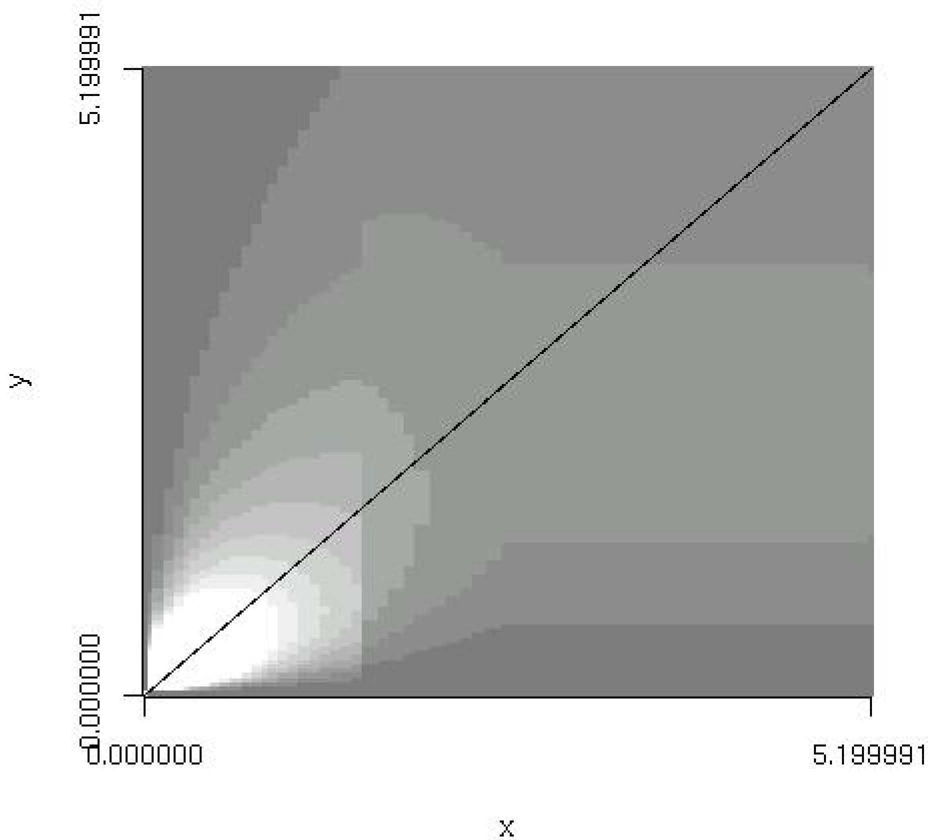

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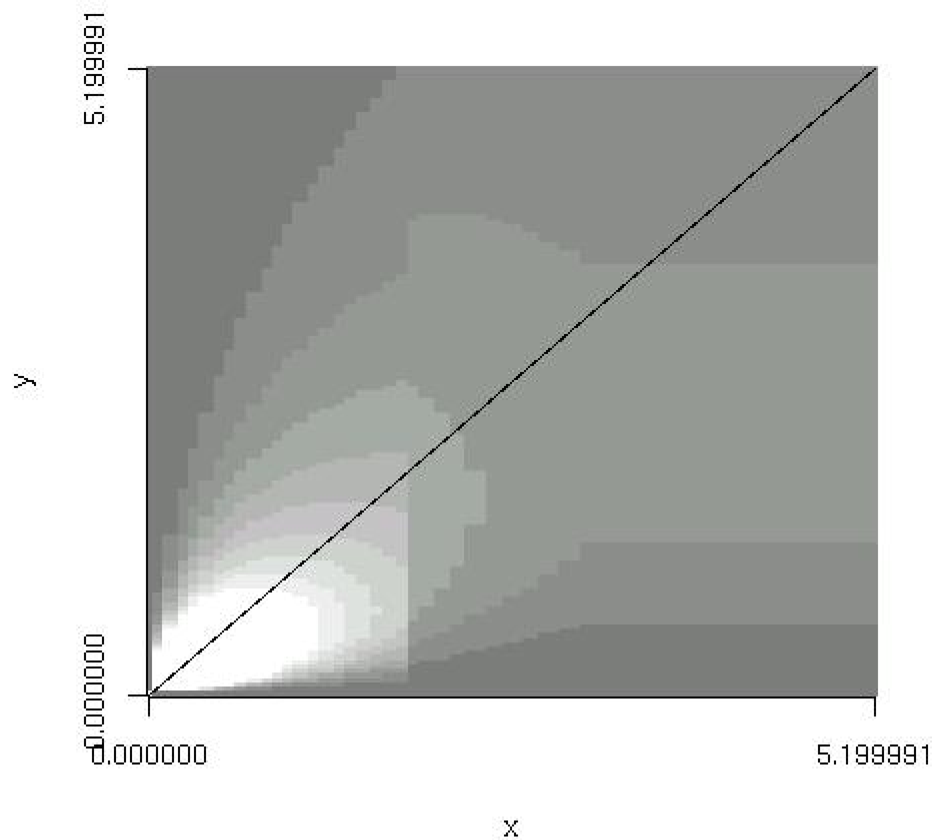
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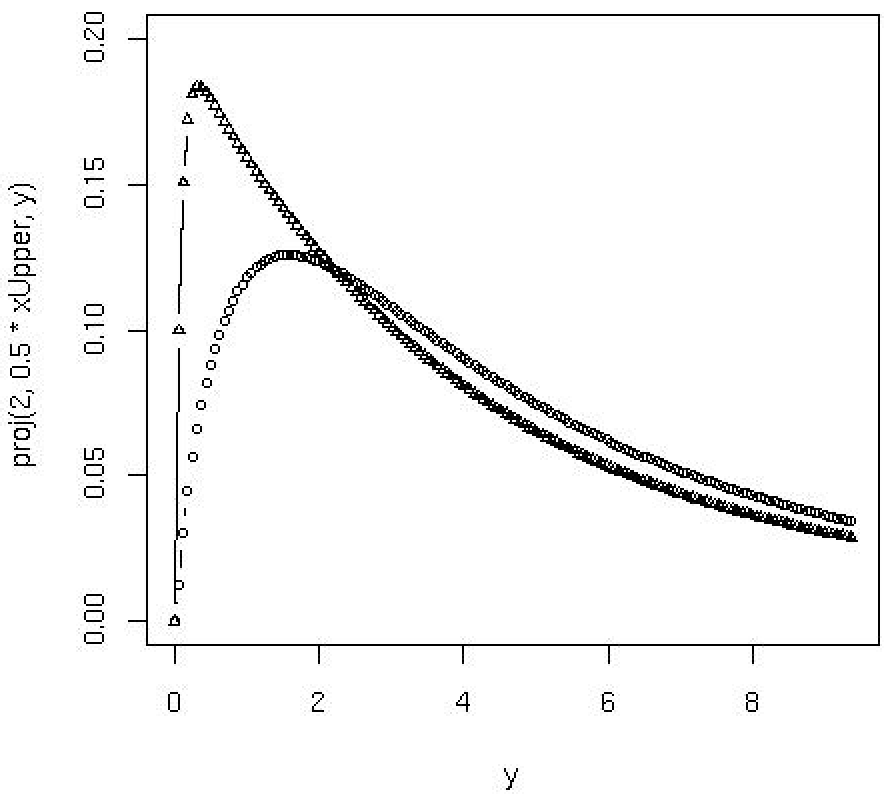

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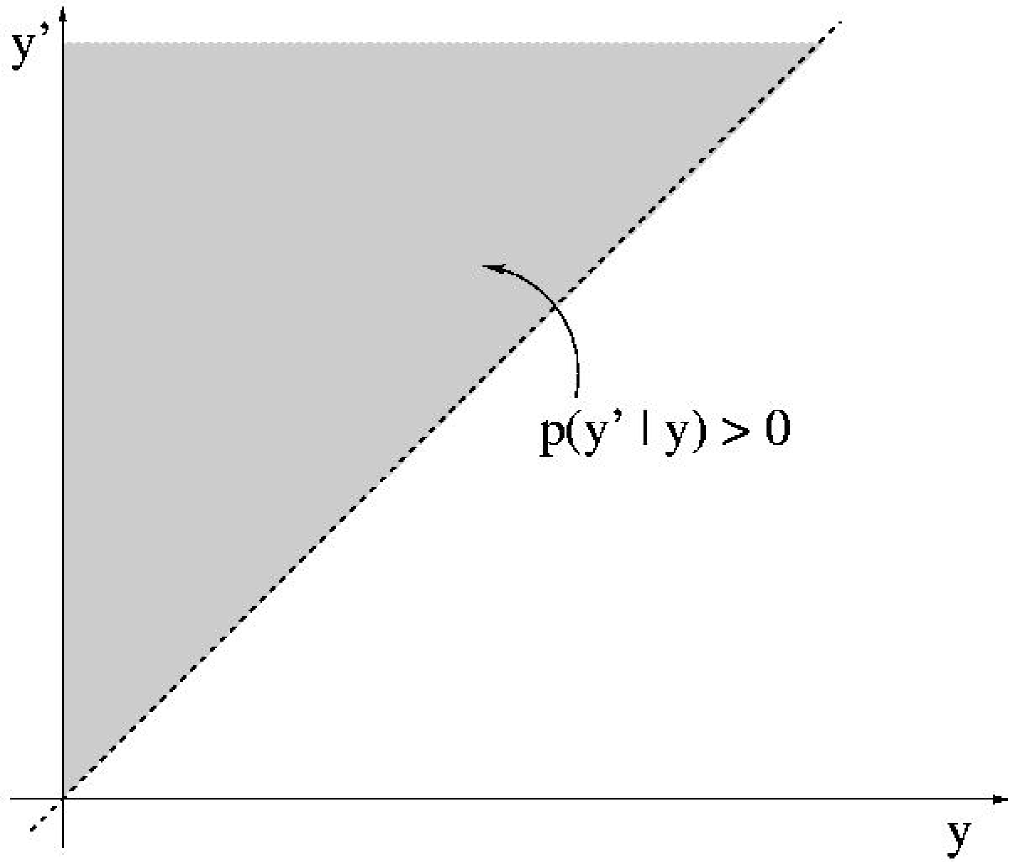
Some other open questions:

1. What is the relationship between overlappings supports and irreducibility?

$$\mathrm{Prob}\{X_t \in B \text{ for some } t \mid X_0 = x\} > 0, \quad \forall x \in S, \forall \text{ non-neg. } B. \tag{8}$$



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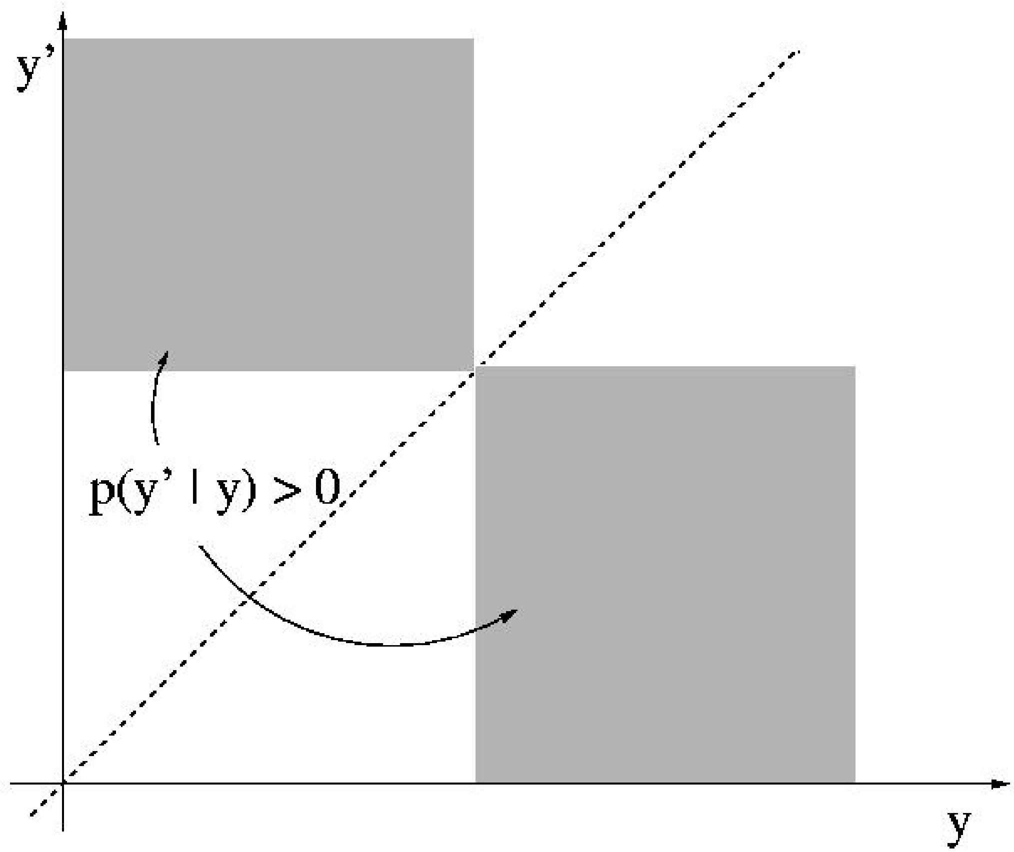

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Also:

When does T2 contraction imply Banach contraction?

What are minimal conditions for overlapping supports in OGM?

Can we extend this analysis to more interesting models?