Stability of Stochastic Optimal Growth Models: A New Approach

Kazuo Nishimura and John Stachurski

March 24, 2005

Structure of Talk

1. Introduction to the Problem
2. Discrete Time Markov Chains
3. The Model
4. Results
5. Discussion of Proofs

Introduction to the Problem

Recall *deterministic* optimal growth: save to max sum of discounted utilities.

Introduction to the Problem

Recall *deterministic* optimal growth: save to max sum of discounted utilities.

Under Inada conditions $f'(\infty) = 0$, $f'(0) > 1/\varrho$, optimal savings implies

1. $\exists$ steady state for income $y^* > 0$.

Introduction to the Problem

Recall *deterministic* optimal growth: save to max sum of discounted utilities.

Under Inada conditions $f'(\infty) = 0$, $f'(0) > 1/\varrho$, optimal savings implies

1. $\exists$ steady state for income $y^* > 0$.

2. Stability: $y_t \to y^*$ independent of y_0.

Introduction to the Problem

Recall *deterministic* optimal growth: save to max sum of discounted utilities.

Under Inada conditions $f'(\infty) = 0$, $f'(0) > 1/\varrho$, optimal savings implies

1. $\exists$ steady state for income $y^* > 0$.

2. Stability: $y_t \to y^*$ independent of y_0.

In the stochastic OGM, steady state is a distribution ψ^*.

Introduction to the Problem

Recall *deterministic* optimal growth: save to max sum of discounted utilities.

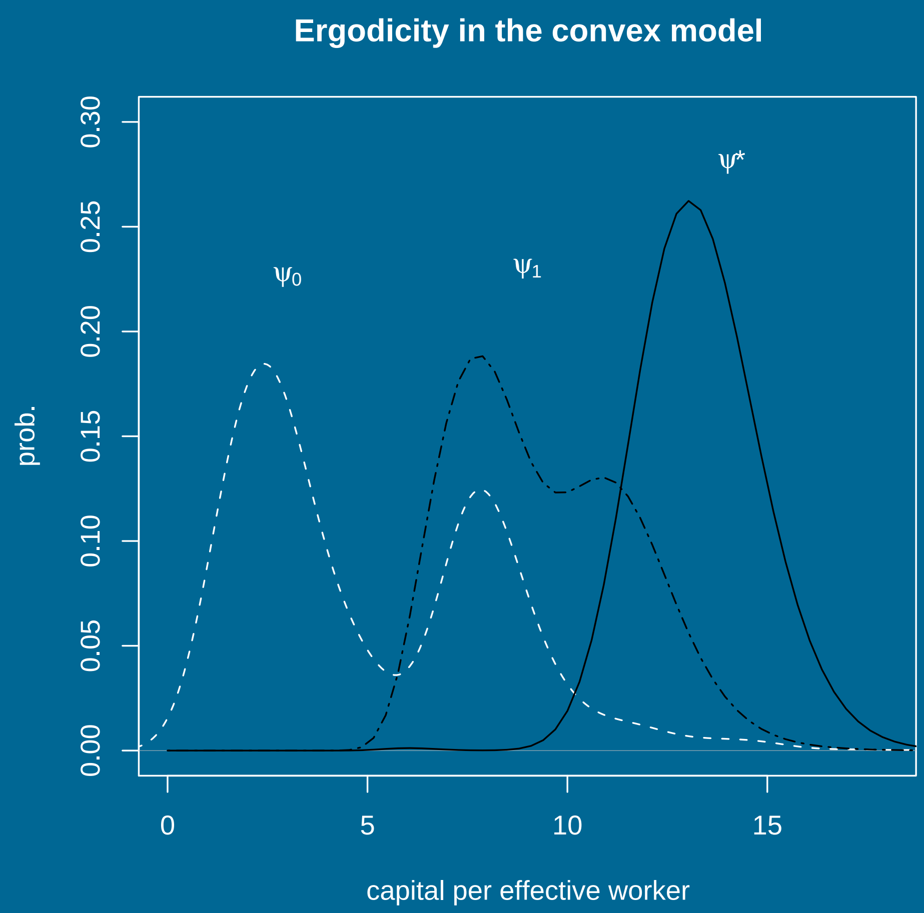
Under Inada conditions $f'(\infty) = 0$, $f'(0) > 1/\varrho$, optimal savings implies

1. $\exists$ steady state for income $y^* > 0$.
2. Stability: $y_t \to y^*$ independent of y_0.

In the stochastic OGM, steady state is a distribution ψ^*.

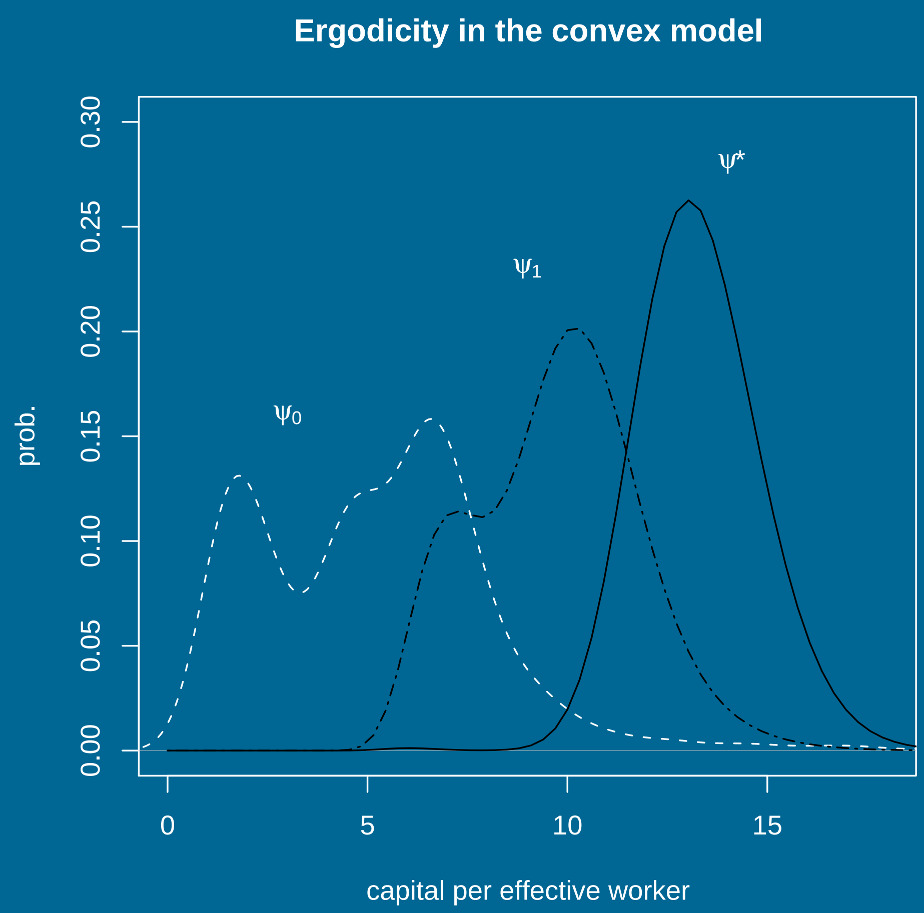
Stability means that the distribution of y_t converges to ψ^*.

Back to top



Ergodicity in the convex model
prob.
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
5
10
15
capital per effective worker
ψ0
ψ1
ψ*

Back to top



Ergodicity in the convex model
prob.
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
5
10
15
capital per effective worker
ψ_0
ψ_1
ψ^*

Back to top

Brock-Mirman 1972: The first to prove global stability in stochastic case.

Brock-Mirman 1972: The first to prove global stability in stochastic case.

But with some problematic assumptions:

Brock-Mirman 1972: The first to prove global stability in stochastic case.

But with some problematic assumptions:

⋆ Productivity shock $\varepsilon_t \in [a, b] \subset (0, \infty)$.

Brock-Mirman 1972: The first to prove global stability in stochastic case.

But with some problematic assumptions:

- $\star$ Productivity shock $\varepsilon_t \in [a, b] \subset (0, \infty)$.

- $\star$ Production function f is concave.

Brock-Mirman 1972: The first to prove global stability in stochastic case.

But with some problematic assumptions:

- ⋆ Productivity shock $\varepsilon_t \in [a, b] \subset (0, \infty)$.
- ⋆ Production function f is concave.
- ⋆ Inada condition $f'(0) = \infty$.

Brock-Mirman 1972: The first to prove global stability in stochastic case.

But with some problematic assumptions:

- ⋆ Productivity shock $\varepsilon_t \in [a, b] \subset (0, \infty)$.
- ⋆ Production function f is concave.
- ⋆ Inada condition $f'(0) = \infty$.

SLP 89, HP 92, Amir 97 etc: not much progress weakening these assumptions. . .

Other interesting questions:

Other interesting questions:

If distribution of y_t converges to ψ^*, then maybe

$$\frac{1}{T}\sum_{t=0}^{T} y_t \to \mathbb{E}_{\psi^*}[y_t] := \int y\psi^*(dy). \tag{1}$$

Other interesting questions:

If distribution of y_t converges to ψ^*, then maybe

$$\frac{1}{T}\sum_{t=0}^{T} y_t \to \mathbb{E}_{\psi^*}[y_t] := \int y\psi^*(dy). \tag{1}$$

And maybe

$$\frac{1}{\sqrt{T}}\sum_{t=0}^{T}(y_t - \mathbb{E}_{\psi^*}[y_t]) \to N(0,\sigma^2) \text{ in distribution.} \tag{2}$$

Other interesting questions:

If distribution of y_t converges to ψ^*, then maybe

$$\frac{1}{T}\sum_{t=0}^{T} y_t \to \mathbb{E}_{\psi^*}[y_t] := \int y\psi^*(dy). \tag{1}$$

And maybe

$$\frac{1}{\sqrt{T}}\sum_{t=0}^{T}(y_t - \mathbb{E}_{\psi^*}[y_t]) \to N(0,\sigma^2) \text{ in distribution.} \tag{2}$$

Previously no studies connected these to Inada conditions and stability. . .

Back to top

Our approach to the problem is as follows.

Our approach to the problem is as follows.

Use a “Lyapunov function” approach with drift condition.

Our approach to the problem is as follows.

Use a "Lyapunov function" approach with drift condition.

Our trick is to use marginal utility of consumption as the Lyap function.

Our approach to the problem is as follows.

Use a "Lyapunov function" approach with drift condition.

Our trick is to use marginal utility of consumption as the Lyap function.

Then the Euler equation gives us the drift in a very direct way.

Our approach to the problem is as follows.

Use a “Lyapunov function” approach with drift condition.

Our trick is to use marginal utility of consumption as the Lyap function.

Then the Euler equation gives us the drift in a very direct way.

Leads to tighter conditions.

Our approach to the problem is as follows.

Use a “Lyapunov function” approach with drift condition.

Our trick is to use marginal utility of consumption as the Lyap function.

Then the Euler equation gives us the drift in a very direct way.

Leads to tighter conditions.

Moreover, get stability in a strong sense, which then leads to LLN and CLT.

Back to top

Discrete Time Markov Chains

Back to top

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Construction of a MC usually begins with a *Markov kernel* P:

$$\text{for all } x \in S, \; P(x, dy) \in \mathcal{P}(S).$$

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Construction of a MC usually begins with a *Markov kernel* P:

$$\text{for all } x \in S, \; P(x, dy) \in \mathcal{P}(S).$$

Heuristically, $P(x, dy)$ is distribution of X_{t+1} given $X_t = x$.

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Construction of a MC usually begins with a *Markov kernel* P:

$$\text{for all } x \in S,\ P(x, dy) \in \mathcal{P}(S).$$

Heuristically, $P(x, dy)$ is distribution of X_{t+1} given $X_t = x$.

Example

$$S = \mathbb{R}, \quad X_{t+1} = \alpha X_t + W_{t+1}, \quad W_t \sim \text{IID } N(0, \sigma^2). \tag{3}$$

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Construction of a MC usually begins with a *Markov kernel* P:

$$\text{for all } x \in S,\ P(x, dy) \in \mathcal{P}(S).$$

Heuristically, $P(x, dy)$ is distribution of X_{t+1} given $X_t = x$.

Example

$$S = \mathbb{R}, \quad X_{t+1} = \alpha X_t + W_{t+1}, \quad W_t \sim \text{IID } N(0, \sigma^2). \tag{3}$$

Then set

$$P(x, dy) =$$

Space of distributions denoted by $\mathcal{P}(S)$, $S =$ state space.

Construction of a MC usually begins with a *Markov kernel* P:

$$\text{for all } x \in S, \; P(x, dy) \in \mathcal{P}(S).$$

Heuristically, $P(x, dy)$ is distribution of X_{t+1} given $X_t = x$.

Example

$$S = \mathbb{R}, \quad X_{t+1} = \alpha X_t + W_{t+1}, \quad W_t \sim \text{IID } N(0, \sigma^2). \tag{3}$$

Then set

$$P(x, dy) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y - \alpha x)^2}{2\sigma^2}\right) dy.$$

Back to top

Construct the chain from P as follows:

$$X_0 \text{ given}, \quad X_1 \sim P(X_0, dy), \quad X_2 \sim P(X_1, dy), \text{ etc., etc.} \tag{4}$$

Construct the chain from P as follows:

$$X_0 \text{ given}, \quad X_1 \sim P(X_0, dy), \quad X_2 \sim P(X_1, dy), \text{ etc., etc.} \tag{4}$$

Let $P^j(x, dy) :=$ distribution of X_{t+j} given $X_t = x$.

Construct the chain from P as follows:

$$X_0 \text{ given}, \quad X_1 \sim P(X_0, dy), \quad X_2 \sim P(X_1, dy), \text{ etc., etc.} \tag{4}$$

Let $P^j(x, dy) :=$ distribution of X_{t+j} given $X_t = x$. Note that $P^t(x, \cdot)$ is the distrib of X_t when $X_0 \equiv x$.

Construct the chain from P as follows:

$$X_0 \text{ given}, \quad X_1 \sim P(X_0, dy), \quad X_2 \sim P(X_1, dy), \text{ etc., etc.} \tag{4}$$

Let $P^j(x, dy) :=$ distribution of X_{t+j} given $X_t = x$. Note that $P^t(x, \cdot)$ is the distrib of X_t when $X_0 \equiv x$.

$$\psi^* \in \mathcal{P}(S) \text{ stationary } \iff \psi^*(dy) = \int P(x, dy) \psi^*(dx).$$

Construct the chain from P as follows:

$$X_0 \text{ given}, \quad X_1 \sim P(X_0, dy), \quad X_2 \sim P(X_1, dy), \text{ etc., etc.} \tag{4}$$

Let $P^j(x, dy) :=$ distribution of X_{t+j} given $X_t = x$. Note that $P^t(x, \cdot)$ is the distrib of X_t when $X_0 \equiv x$.

$$\psi^* \in \mathcal{P}(S) \text{ stationary } \iff \psi^*(dy) = \int P(x, dy)\psi^*(dx).$$

The process $(X_t)_{t=0}^{\infty}$ generated by P is called ergodic if

$$\exists \text{ unique } P\text{-stationary } \psi^* \in \mathcal{P}(S) \text{ and } \lim_{t \to \infty} P^t(x, \cdot) = \psi^*, \ \forall x \in S. \tag{5}$$

Back to top

What does $\lim_{t\to\infty} P^t(x,\cdot) = \psi^*$ mean?

What does $\lim_{t\to\infty} P^t(x,\cdot) = \psi^*$ mean?

Here we use the TV distance $\|P^t(x,\cdot) - \psi^*\| = \sup_B |P^t(x,B) - \psi^*(B)|$.

What does $\lim_{t \to \infty} P^t(x, \cdot) = \psi^*$ mean?

Here we use the TV distance $\|P^t(x, \cdot) - \psi^*\| = \sup_B |P^t(x, B) - \psi^*(B)|$.

Stronger than the "weak star" notion used in SLP, BM, HP, etc.

What does $\lim_{t\to\infty} P^t(x, \cdot) = \psi^*$ mean?

Here we use the TV distance $\|P^t(x, \cdot) - \psi^*\| = \sup_B |P^t(x, B) - \psi^*(B)|$.

Stronger than the "weak star" notion used in SLP, BM, HP, etc.

Advantages:

1. Useful quantitative interpretation.

What does $\lim_{t\to\infty} P^t(x, \cdot) = \psi^*$ mean?

Here we use the TV distance $\|P^t(x, \cdot) - \psi^*\| = \sup_B |P^t(x, B) - \psi^*(B)|$.

Stronger than the "weak star" notion used in SLP, BM, HP, etc.

Advantages:

1. Useful quantitative interpretation.

2. Connections to α-mixing and hence LLN, CLT.

Back to top

Probabilistic MC Techniques

Huge progress in last 30 years.

Probabilistic MC Techniques

Huge progress in last 30 years.

Start two P-MCs $(X_t)_{t=0}^{\infty}$ and $(\hat{X}_t)_{t=0}^{\infty}$ with $X_0 \equiv x$, $\hat{X}_0 \sim \psi^*$.

Probabilistic MC Techniques

Huge progress in last 30 years.

Start two P-MCs $(X_t)_{t=0}^{\infty}$ and $(\hat{X}_t)_{t=0}^{\infty}$ with $X_0 \equiv x$, $\hat{X}_0 \sim \psi^*$.

Define $\tau = \inf\{t \,:\, X_t = \hat{X}_t\}$.

Probabilistic MC Techniques

Huge progress in last 30 years.

Start two P-MCs $(X_t)_{t=0}^{\infty}$ and $(\hat{X}_t)_{t=0}^{\infty}$ with $X_0 \equiv x$, $\hat{X}_0 \sim \psi^*$.

Define $\tau = \inf\{t : X_t = \hat{X}_t\}$.

The the *coupling inequality* says that $\|P^t(x, \cdot) - \psi^*\| \leq \mathbb{P}\{\tau > t\}$.

Probabilistic MC Techniques

Huge progress in last 30 years.

Start two P-MCs $(X_t)_{t=0}^{\infty}$ and $(\hat{X}_t)_{t=0}^{\infty}$ with $X_0 \equiv x$, $\hat{X}_0 \sim \psi^*$.

Define $\tau = \inf\{t : X_t = \hat{X}_t\}$.

The the *coupling inequality* says that $\|P^t(x, \cdot) - \psi^*\| \leq \mathbb{P}\{\tau > t\}$.

Big progress in learning how to bound $\mathbb{P}\{\tau > t\}$ for general state chains.

Probabilistic MC Techniques

Huge progress in last 30 years.

Start two P-MCs $(X_t)_{t=0}^{\infty}$ and $(\hat{X}_t)_{t=0}^{\infty}$ with $X_0 \equiv x$, $\hat{X}_0 \sim \psi^*$.

Define $\tau = \inf\{t \,:\, X_t = \hat{X}_t\}$.

The the *coupling inequality* says that $\|P^t(x, \cdot) - \psi^*\| \leq \mathbb{P}\{\tau > t\}$.

Big progress in learning how to bound $\mathbb{P}\{\tau > t\}$ for general state chains.

Technique used here: show that compact sets have lots of mixing (C-sets), and that $\exists$ norm-like $w \colon S \to [0, \infty)$ and $\lambda < 1$, $b < \infty$ with

$$\mathbb{E}[w(X_{t+1}) \,|\, X_t] \leq \lambda w(X_t) + b.$$

Back to top

Gives ergodicity: $\|P^t(x, \cdot) - \psi^*\| \to 0$ as $t \to \infty$.

Gives ergodicity: $\|P^t(x,\cdot) - \psi^*\| \to 0$ as $t \to \infty$.

In fact *geometric* ergodicity: $\|P^t(x,\cdot) - \psi^*\| = O(\alpha^t)$ for $\alpha < 1$.

Gives ergodicity: $\|P^t(x,\cdot) - \psi^*\| \to 0$ as $t \to \infty$.

In fact *geometric* ergodicity: $\|P^t(x,\cdot) - \psi^*\| = O(\alpha^t)$ for $\alpha < 1$.

And α-mixing: $\mathbb{P}\{X_t \in A, X_{t+j} \in B\} \to \mathbb{P}\{X_t \in A\}\mathbb{P}\{X_{t+j} \in B\}$.

Gives ergodicity: $\|P^t(x, \cdot) - \psi^*\| \to 0$ as $t \to \infty$.

In fact *geometric* ergodicity: $\|P^t(x, \cdot) - \psi^*\| = O(\alpha^t)$ for $\alpha < 1$.

And α-mixing: $\mathbb{P}\{X_t \in A, X_{t+j} \in B\} \to \mathbb{P}\{X_t \in A\}\mathbb{P}\{X_{t+j} \in B\}$.

Which in turn implies LLN and CLT.

Gives ergodicity: $\|P^t(x, \cdot) - \psi^*\| \to 0$ as $t \to \infty$.

In fact *geometric* ergodicity: $\|P^t(x, \cdot) - \psi^*\| = O(\alpha^t)$ for $\alpha < 1$.

And α-mixing: $\mathbb{P}\{X_t \in A, X_{t+j} \in B\} \to \mathbb{P}\{X_t \in A\}\mathbb{P}\{X_{t+j} \in B\}$.

Which in turn implies LLN and CLT.

Note: Don't need compactness of S.

Gives ergodicity: $\|P^t(x, \cdot) - \psi^*\| \to 0$ as $t \to \infty$.

In fact *geometric* ergodicity: $\|P^t(x, \cdot) - \psi^*\| = O(\alpha^t)$ for $\alpha < 1$.

And α-mixing: $\mathbb{P}\{X_t \in A, X_{t+j} \in B\} \to \mathbb{P}\{X_t \in A\}\mathbb{P}\{X_{t+j} \in B\}$.

Which in turn implies LLN and CLT.

Note: Don't need compactness of S.

Not critical to have continuity, monotonicity.

Back to top

The Model

The economy produces a single good, representative agent.

The Model

The economy produces a single good, representative agent.

At the start of time t, the agent observes income y_t, which is then divided between savings and consumption.

The Model

The economy produces a single good, representative agent.

At the start of time t, the agent observes income y_t, which is then divided between savings and consumption.

Savings is added one-for-one to existing capital stock and depreciation total: current savings $= k_t$.

The Model

The economy produces a single good, representative agent.

At the start of time t, the agent observes income y_t, which is then divided between savings and consumption.

Savings is added one-for-one to existing capital stock and depreciation total: current savings $= k_t$.

Production then takes place, yielding at the start of next period output

$$y_{t+1} = f(k_t)\,\varepsilon_t, \tag{6}$$

Back to top

Assumption 1. The function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuously differentiable and satisfies

$$\text{Inada condition 1:} \qquad f'(\infty) = 0.$$

Assumption 1. The function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuously differentiable and satisfies

$$\text{Inada condition 1:} \qquad f'(\infty) = 0.$$

Classic example:

$\star$ Brock–Mirman: f concave, $f'(0) = \infty$

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

Assumption 2. The shock ε is distributed according to φ, a density on $\mathbb{R}_+$.

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

Assumption 2. The shock ε is distributed according to φ, a density on $\mathbb{R}_+$. Density φ is continuous, strictly positive.

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

Assumption 2. The shock ε is distributed according to φ, a density on $\mathbb{R}_+$. Density φ is continuous, strictly positive. Also, the moments $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ both finite for some $p \in \mathbb{N}$.

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

Assumption 2. The shock ε is distributed according to φ, a density on $\mathbb{R}_+$. Density φ is continuous, strictly positive. Also, the moments $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ both finite for some $p \in \mathbb{N}$.

For example, the class of lognormal distributions satisfies Assumption 2 for every $p \in \mathbb{N}$.

In $y_{t+1} = f(k_t)\,\varepsilon_t$ the sequence $(\varepsilon_t)_{t=0}^{\infty}$ is i.i.d.

Assumption 2. The shock ε is distributed according to φ, a density on $\mathbb{R}_+$. Density φ is continuous, strictly positive. Also, the moments $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ both finite for some $p \in \mathbb{N}$.

For example, the class of lognormal distributions satisfies Assumption 2 for every $p \in \mathbb{N}$.

Restrictions $\mathbb{E}(\varepsilon^p) < \infty$ and $\mathbb{E}(1/\varepsilon) < \infty$ to be thought of as bounds on right and left hand tails of φ respectively.

Preferences: $\varrho \in (0,1)$ is the discount factor, and u is the period utility function.

Preferences: $\varrho \in (0, 1)$ is the discount factor, and u is the period utility function.

Assumption 3 The function $u \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies "usual conditions."

Preferences: $\varrho \in (0,1)$ is the discount factor, and u is the period utility function.

Assumption 3 The function $u \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies "usual conditions."

A **feasible savings policy** is a (Borel) function π from $\mathbb{R}_+$ to itself such that $0 \le \pi(y) \le y$ for all y.

Preferences: $\varrho \in (0,1)$ is the discount factor, and u is the period utility function.

Assumption 3 The function $u \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies "usual conditions."

A **feasible savings policy** is a (Borel) function π from $\mathbb{R}_+$ to itself such that $0 \leq \pi(y) \leq y$ for all y.

Each feasible π defines a Markov process for income via the recursion

$$y_{t+1} = f(\pi(y_t))\,\varepsilon_t. \tag{7}$$

Back to top

Agent solves

$$\max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c^\pi(y_t))\right], \quad c^\pi(y) := y - \pi(y), \quad y_{t+1} = f(\pi(y_t))\varepsilon_t. \quad (8)$$

Agent solves

$$\max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c^\pi(y_t))\right], \quad c^\pi(y) := y - \pi(y), \quad y_{t+1} = f(\pi(y_t))\varepsilon_t. \tag{8}$$

A policy π is called **optimal** if it is feasible and attains maximum in (8).

Agent solves

$$\max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c^\pi(y_t))\right], \quad c^\pi(y) := y - \pi(y), \quad y_{t+1} = f(\pi(y_t))\varepsilon_t. \quad (8)$$

A policy π is called **optimal** if it is feasible and attains maximum in (8).

Theorem 1. Under Assumptions 1–3 there is at least one optimal policy for (8).

Ideally every optimal policy satisfies an Euler equation...

Ideally every optimal policy satisfies an Euler equation...

Assumption 4. Some more technical stuff to get interiority.

Ideally every optimal policy satisfies an Euler equation...

Assumption 4. Some more technical stuff to get interiority.

If in addition Assumption 4 holds, then every optimal policy π interior, satisfies for each $y > 0$ the Euler equation

$$u' \circ c^\pi(y) = \varrho \int u' \circ c^\pi[f(\pi(y))z] f'(\pi(y)) z \varphi(z) dz.$$

Back to top

In all of what follows let π be a fixed *optimal* policy.

In all of what follows let π be a fixed *optimal* policy.

We study the dynamics of the MC generated by

$$y_{t+1} = f(\pi(y_t))\, \varepsilon_t. \tag{9}$$

In all of what follows let π be a fixed *optimal* policy.

We study the dynamics of the MC generated by

$$y_{t+1} = f(\pi(y_t))\,\varepsilon_t. \tag{9}$$

Implies a Markov kernel $P(y, B) = \int \mathbf{1}\{f(\pi(y))z \in B\}\varphi(z)dz$.

In all of what follows let π be a fixed *optimal* policy.

We study the dynamics of the MC generated by

$$y_{t+1} = f(\pi(y_t))\,\varepsilon_t. \tag{9}$$

Implies a Markov kernel $P(y, B) = \int \mathbf{1}\{f(\pi(y))z \in B\}\varphi(z)dz$.

Each y_t is a random variable taking values in $(0, \infty)$, has distribution $P^t(y_0, \cdot)$ on same.

Results

Theorem 2. If the inequality

$$f'(0) > \frac{\mathbb{E}(1/\varepsilon)}{\varrho} \tag{10}$$

holds, then $(y_t)_{t=0}^{\infty}$ has a unique stationary distribution ψ^*,

Results

Theorem 2. If the inequality

$$f'(0) > \frac{\mathbb{E}(1/\varepsilon)}{\varrho} \tag{10}$$

holds, then $(y_t)_{t=0}^{\infty}$ has a unique stationary distribution ψ^*, and, moreover, there is a constant $\alpha \in (0,1)$ and function $y \mapsto K(y) < \infty$ such that

$$\|P^t(y_0, \cdot) - \psi^*\| \leq \alpha^t K(y_0), \quad \forall y_0 > 0, \ \forall\, t \in \{0\} \cup \mathbb{N}. \tag{11}$$

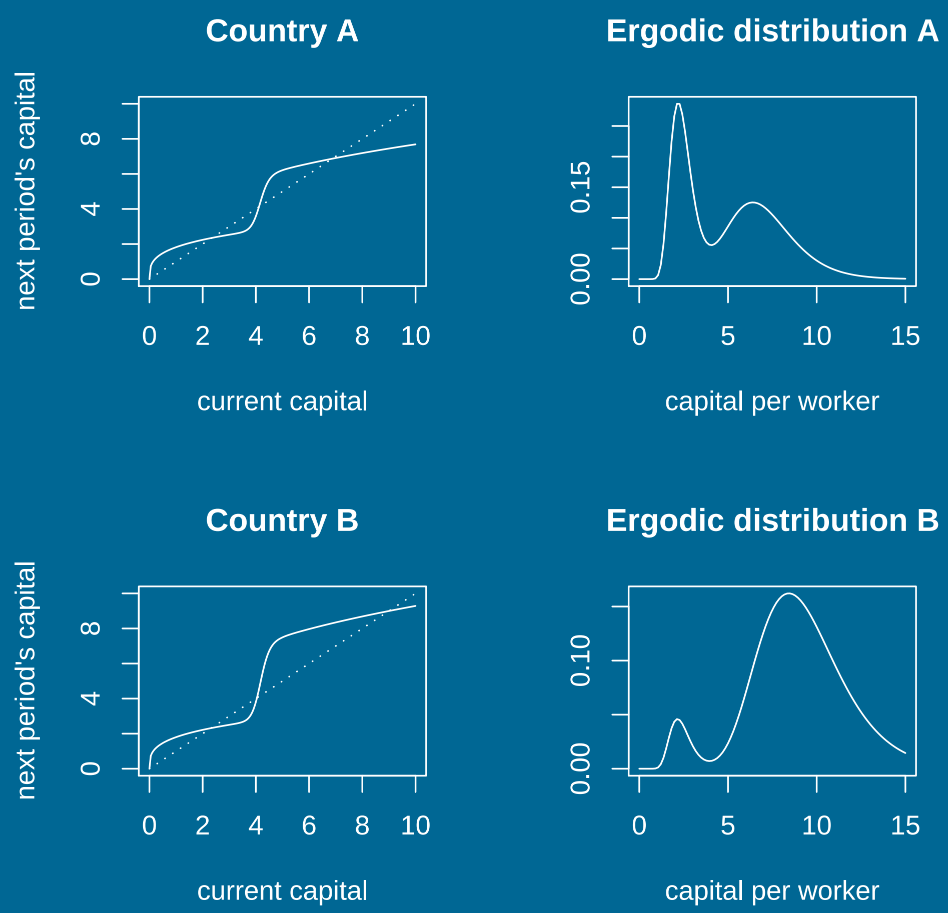
Back to top

What about models with nonconvexities?

What about models with nonconvexities?

Dechert-Nishimura shows that global stability can fail. . .

Back to top


Country A
next period's capital
0
4
8
0
2
4
6
8
10
current capital
Ergodic distribution A
0.00
0.15
0
5
10
15
capital per worker
Country B
next period's capital
0
4
8
0
2
4
6
8
10
current capital
Ergodic distribution B
0.00
0.10
0
5
10
15
capital per worker

Back to top

For h a real function on $(0,\infty)$, define the rv $S_n(h) := \sum_{t=1}^{n} h \circ y_t$.

For h a real function on $(0, \infty)$, define the rv $S_n(h) := \sum_{t=1}^n h \circ y_t$.

Theorem 3. Let (10) hold, and let ψ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^\infty$.

For h a real function on $(0, \infty)$, define the rv $S_n(h) := \sum_{t=1}^{n} h \circ y_t$.

Theorem 3. Let (10) hold, and let ψ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^{\infty}$. If $h\colon (0, \infty) \longrightarrow \mathbb{R}$ is any Borel function satisfying $|h| \leq V$, then the Law of Large Numbers holds for h.

For h a real function on $(0, \infty)$, define the rv $S_n(h) := \sum_{t=1}^{n} h \circ y_t$.

Theorem 3. Let (10) hold, and let ψ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^{\infty}$. If $h \colon (0, \infty) \to \mathbb{R}$ is any Borel function satisfying $|h| \leq V$, then the Law of Large Numbers holds for h. That is,

$$\mathbb{E}_{\psi^*}(h) := \int h \, d\psi^* < \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{S_n(h)}{n} = \mathbb{E}_{\psi^*}(h) \;\; \mathbb{P}\text{-a.s.} \qquad (12)$$

For h a real function on $(0, \infty)$, define the rv $S_n(h) := \sum_{t=1}^{n} h \circ y_t$.

Theorem 3. Let (10) hold, and let ψ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^{\infty}$. If $h \colon (0, \infty) \to \mathbb{R}$ is any Borel function satisfying $|h| \leq V$, then the Law of Large Numbers holds for h. That is,

$$\mathbb{E}_{\psi^*}(h) := \int h \, d\psi^* < \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{S_n(h)}{n} = \mathbb{E}_{\psi^*}(h) \quad \mathbb{P}\text{-a.s.} \qquad (12)$$

If in addition $h^2 \leq V$, then the Central Limit Theorem also holds for h.

For h a real function on $(0, \infty)$, define the rv $S_n(h) := \sum_{t=1}^n h \circ y_t$.

Theorem 3. Let (10) hold, and let ψ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^\infty$. If $h\colon (0, \infty) \to \mathbb{R}$ is any Borel function satisfying $|h| \leq V$, then the Law of Large Numbers holds for h. That is,

$$\mathbb{E}_{\psi^*}(h) := \int h \, d\psi^* < \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{S_n(h)}{n} = \mathbb{E}_{\psi^*}(h) \quad \mathbb{P}\text{-a.s.} \tag{12}$$

If in addition $h^2 \leq V$, then the Central Limit Theorem also holds for h. Precisely, there is a constant $\sigma^2 \in \mathbb{R}_+$ such that

$$\frac{S_n(h - \mathbb{E}_{\psi^*}(h))}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2). \tag{13}$$

Back to top

Discussion of Proofs

Discussion of Proofs

We use mixing on compacts and drift conditions for stability.

Discussion of Proofs

We use mixing on compacts and drift conditions for stability.

Need a norm-like function and a certain “expected contraction” condition.

Discussion of Proofs

We use mixing on compacts and drift conditions for stability.

Need a norm-like function and a certain "expected contraction" condition.

Marginal utility of consumption will be the norm-like function.

Discussion of Proofs

We use mixing on compacts and drift conditions for stability.

Need a norm-like function and a certain "expected contraction" condition.

Marginal utility of consumption will be the norm-like function.

Establish this expected contraction using Euler-equation.

Discussion of Proofs

We use mixing on compacts and drift conditions for stability.

Need a norm-like function and a certain "expected contraction" condition.

Marginal utility of consumption will be the norm-like function.

Establish this expected contraction using Euler-equation.

Equating marginal rate of substitution with marginal returns to investment implies a contraction condition on the norm-like function.

Recall the condition $\exists$ norm-like $w\colon S \to [0, \infty)$ and $\lambda < 1$, $b < \infty$ with

$$\mathbb{E}[w(X_{t+1}) \mid X_t] \leq \lambda w(X_t) + b.$$

Recall the condition $\exists$ norm-like $w \colon S \to [0, \infty)$ and $\lambda < 1$, $b < \infty$ with

$$\mathbb{E}[w(X_{t+1}) \,|\, X_t] \leq \lambda w(X_t) + b.$$

Present Case: There is a norm-like function w on $(0, \infty)$ and constants $\lambda < 1$ and a $b < \infty$ such that

$$\int w[f(\pi(y))z]\varphi(z)dz \leq \lambda w(y) + b, \quad \forall y \in (0, \infty). \tag{14}$$

Back to top

Our construction of norm-like function:

Find (i) a w_1 with $\lim_{x \downarrow 0} w_1(x) = \infty$ and a $\lambda_1 < 1$ such that

$$\int w_1[f(\pi(y))z]\varphi(z)dz \le \lambda_1 w_1(y), \quad \forall \text{ small } y, \tag{15}$$

Our construction of norm-like function:

Find (i) a w_1 with $\lim_{x\downarrow 0} w_1(x) = \infty$ and a $\lambda_1 < 1$ such that

$$\int w_1[f(\pi(y))z]\varphi(z)dz \leq \lambda_1 w_1(y), \quad \forall \text{ small } y, \tag{15}$$

and (ii) a w_2 with $\lim_{x\uparrow\infty} w_2(x) = \infty$ and a $\lambda_2 < 1$ such that

$$\int w_2[f(\pi(y))z]\varphi(z)dz \leq \lambda_2 w_2(y), \quad \forall \text{ large } y. \tag{16}$$

Our construction of norm-like function:

Find (i) a w_1 with $\lim_{x\downarrow 0} w_1(x) = \infty$ and a $\lambda_1 < 1$ such that

$$\int w_1[f(\pi(y))z]\varphi(z)dz \leq \lambda_1 w_1(y), \quad \forall \text{ small } y, \tag{15}$$

and (ii) a w_2 with $\lim_{x\uparrow\infty} w_2(x) = \infty$ and a $\lambda_2 < 1$ such that

$$\int w_2[f(\pi(y))z]\varphi(z)dz \leq \lambda_2 w_2(y), \quad \forall \text{ large } y. \tag{16}$$

Then $w := w_1 + w_2$ satisfies (14).

The hard part is to show there is a w_1 with $\lim_{x \downarrow 0} w_1(x) = \infty$ and a $\lambda_1 < 1$ such that

$$\int w_1[f(\pi(y))z]\varphi(z)dz \leq \lambda_1 w_1(y), \quad \forall \text{ small } y, \tag{17}$$

The hard part is to show there is a w_1 with $\lim_{x \downarrow 0} w_1(x) = \infty$ and a $\lambda_1 < 1$ such that

$$\int w_1[f(\pi(y))z]\varphi(z)dz \leq \lambda_1 w_1(y), \quad \forall \text{ small } y, \tag{17}$$

Proof will use the Euler equation

$$u' \circ c^\pi(y) = \varrho f'(\pi(y)) \int u' \circ c^\pi[f(\pi(y))z]z\varphi(dz). \tag{18}$$

Back to top

Some manipulation of Euler equation gives

$$\int \sqrt{u' \circ c^{\pi}}[f(\pi(y))z]\varphi(dz) \leq \left[\frac{\mathbb{E}(1/\varepsilon)}{\varrho f'(\pi(y))}\right]^{1/2} \sqrt{u' \circ c^{\pi}}(y). \qquad (19)$$

Some manipulation of Euler equation gives

$$\int \sqrt{u' \circ c^\pi}[f(\pi(y))z]\varphi(dz) \le \left[\frac{\mathbb{E}(1/\varepsilon)}{\varrho f'(\pi(y))}\right]^{1/2} \sqrt{u' \circ c^\pi}(y). \tag{19}$$

Provided $\varrho f'(0) > \mathbb{E}(1/\varepsilon)$, we get

$$\int \sqrt{u' \circ c^\pi}[f(\pi(y))z]\varphi(dz) \le \lambda_1 \sqrt{u' \circ c^\pi}(y), \quad \forall \text{ small } y. \tag{20}$$

Other Projects

- ⋆ Other stochastic growth problems
- ⋆ Density forecasting
- ⋆ Simulated moments estimator
- ⋆ Estimation with piecewise linear functions
- ⋆ Development and corruption
- ⋆ Numerical dynamic programming

Back to top