Parametric Continuity of Stationary Distributions

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Structure of the Seminar

Outline of the Problem

We consider a Markov process on S:

$$X_{t+1} = T_\alpha(X_t, \xi_t), \quad \xi_t \sim \nu. \tag{1}$$

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Here $\alpha \in W$ a "vector" of parameters.

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Convergence in distribution is convergence in $w(\mathscr{P}(S), C_b(S))$.

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We extend these results to unbounded S under various assumptions.

Noncompact is also needed (e.g., $X_{t+1} = \alpha X_t + \xi_t$, Gaussian shock).

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Need to show that

$$s_n \to s \implies \int u \, d\varphi_{s_n} \to \int u \, d\varphi_s. \tag{4}$$

Parametric Continuity Results

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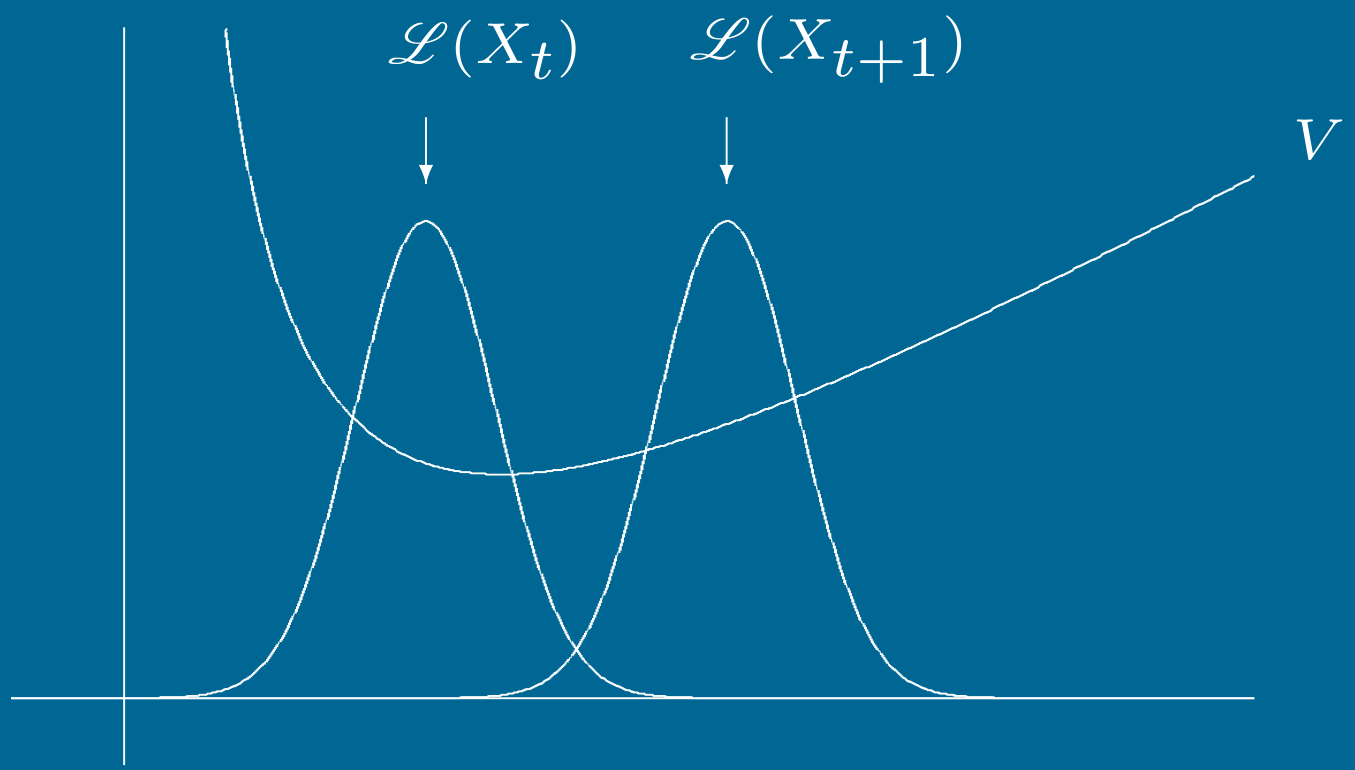
Definition. A real function $V : S \to [0, \infty)$ called norm-like if all sublevel sets precompact.

Assumption 2. $\forall\, \alpha \in W$, exists norm-like V and $\lambda, b \in [0, \infty)$ with $\lambda < 1$ and

$$\mathbb{E}_{\xi}\, V[T_{\alpha}(x, \xi)] \leq \lambda V(x) + b, \quad \forall x \in S. \tag{5}$$



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In other words, $\exists\, \psi \in \mathscr{P}(S)$, possibly depending on the parameter α, such that $\psi(B) > 0$ implies

$$\mathrm{Prob}_x\{X_t \in B \text{ for some } t \in \mathbb{N}\} > 0, \quad \forall x \in S,$$

where Prob_x is the distribution of $(X_t)_{t=0}^\infty$ when $X_0 \equiv x$.

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Theorem 2. Let Assumptions 1–5 hold, and take $\{\alpha_n\}_{n=1}^{\infty} \cup \{\alpha\} \subset W$.

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- $\star$ If for each $K \subset\subset S$ there exists an $M < \infty$ which can be chosen independent of α_n to satisfy

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μ_{α_n}, μ_α exist, unique, and, moreover, $\alpha_n \to \alpha$ in W implies $\mu_{\alpha_n} \xrightarrow{d} \mu_\alpha$.

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Now set $b := \sup_n b_n < \infty$.

We also need $\forall\, K \subset\subset S$, $\exists\, M < \infty$ independent of s_n to satisfy

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$$\begin{aligned}\int |s_n f(k) z - s_n f(k') z| \nu(dz) &\leq \int z\nu(dz) |f(k) - f(k')| \\ &\leq \int z\nu(dz) A|k - k'|.\end{aligned}$$