Stationarity and Ergodicity of Equilibrium Price Processes

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Outline

Iterated Random Maps

The Commodity Pricing Model

Exact Sampling

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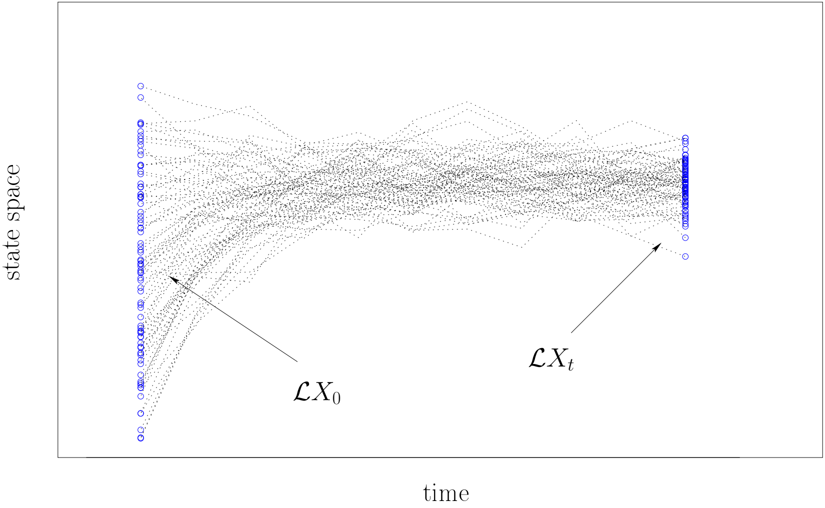
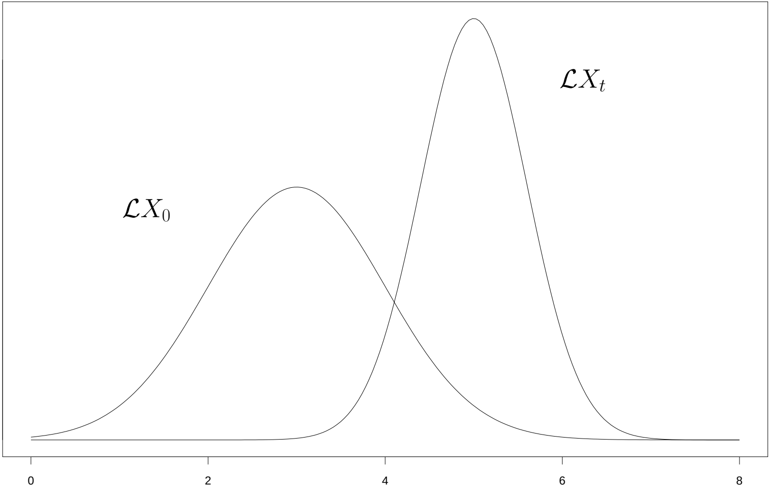
Let $\mathcal{L}X_t$ be the distribution of X_t.


state space
$\mathcal{L}X_0$
$\mathcal{L}X_t$
time



$\mathcal{L}X_t$
$\mathcal{L}X_0$
0
2
4
6
8

A distribution ψ^* is called stationary if

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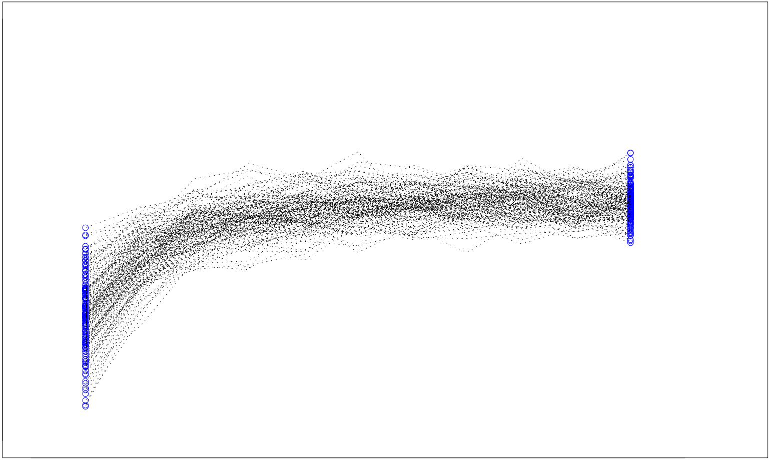
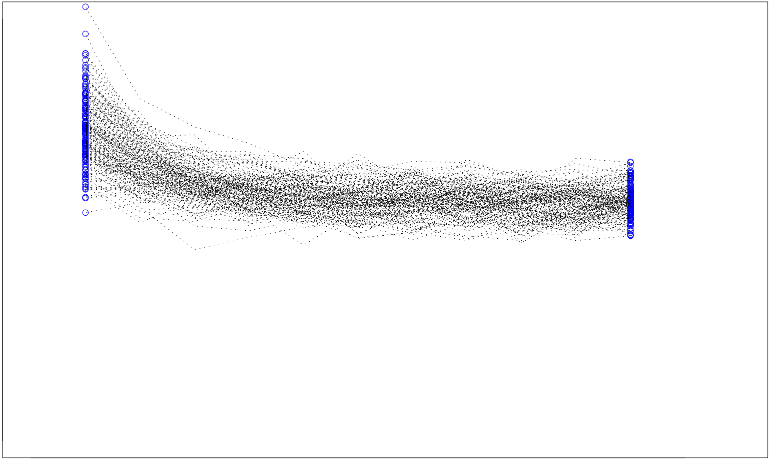
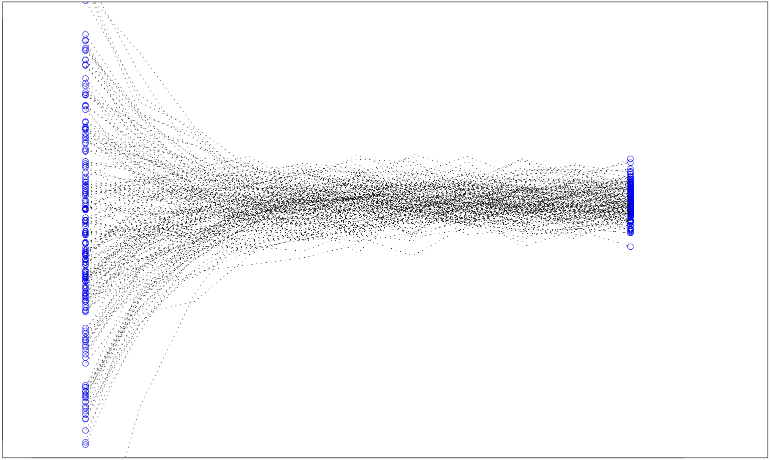
Call ψ^* globally stable if $\mathcal{L}X_t \to \psi^*$ independent of X_0.







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- To study the equilibrium.
- Simulation-based econometrics.

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- returns exact draw from ψ^*.

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- Exposit Samuelson's model, and
- show how to sample from its stationary distribution.

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Distribution of ξ_t is φ.

Spot price is p_t.

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Easy to show that there is a maximal state $\bar{s}$.

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Profit maximization (PM) requires

$$I_t = 0 \text{ whenever } \alpha \mathbf{E}_t p_{t+1} < p_t$$

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Theorem. The following results hold.

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Equilibrium consumption: $C(x) = D(p^E(x))$

Equilibrium investment: $I(x) = x - D(p^E(x))$

Relationship to growth theory:

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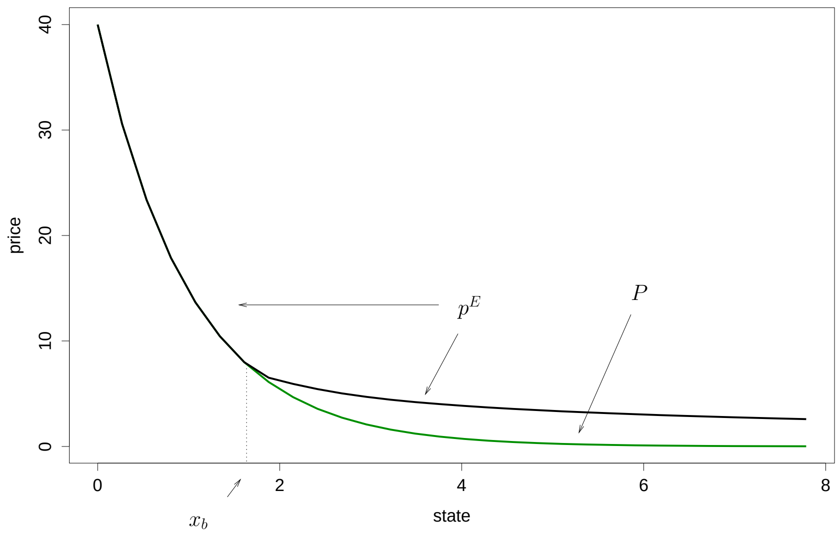
Equilibrium investment function I is the solution to

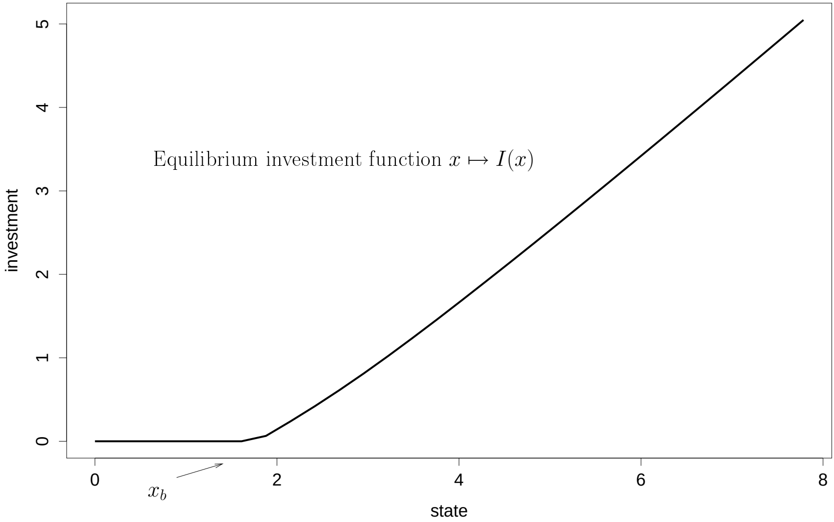
$$\max_i \mathbf{E}\left[\sum_{t=0}^{\infty} U(X_t - i(X_t))\right]$$

subject to

$$X_{t+1} = \alpha\, i(X_t) + \xi_{t+1}$$

when U is chosen to satisfy $U' = P = D^{-1}$.


price
state
p^E
P
x_b
0
10
20
30
40
0
2
4
6
8



Equilibrium investment function $x \mapsto I(x)$
investment
state
x_b
0
1
2
3
4
5
0
2
4
6
8

Dynamics

Equilibrium dynamics:

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Observe time zero value of sequence from infinite past.

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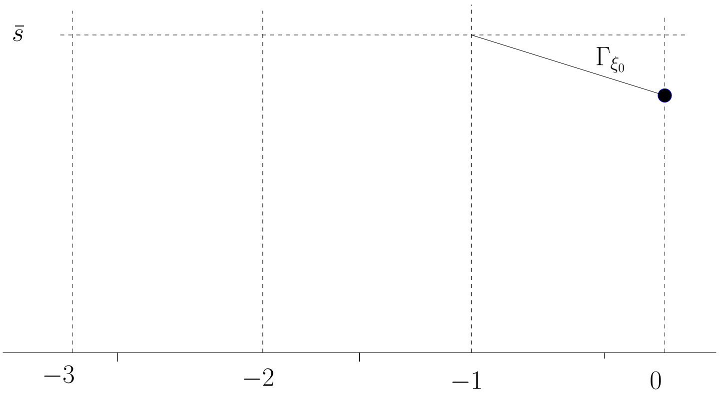
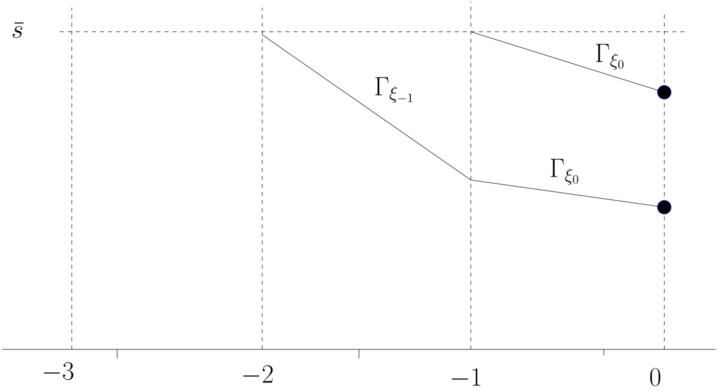
Remarkably, we can do this in finite time.

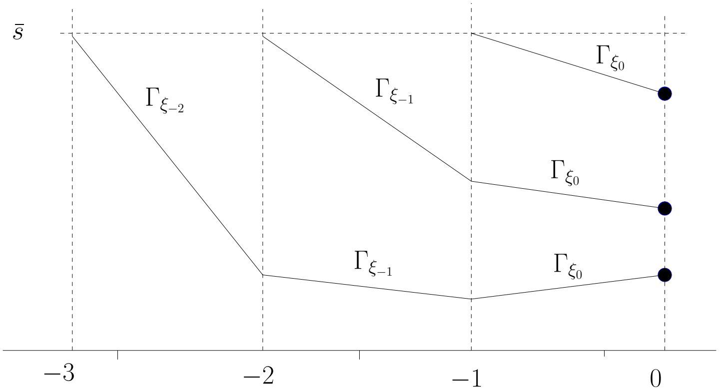
Harvests: (ξ_0)



Harvests: (ξ_{-1}, ξ_0)



Harvests: $(\xi_{-2}, \xi_{-1}, \xi_0)$



Two important properties of paths:

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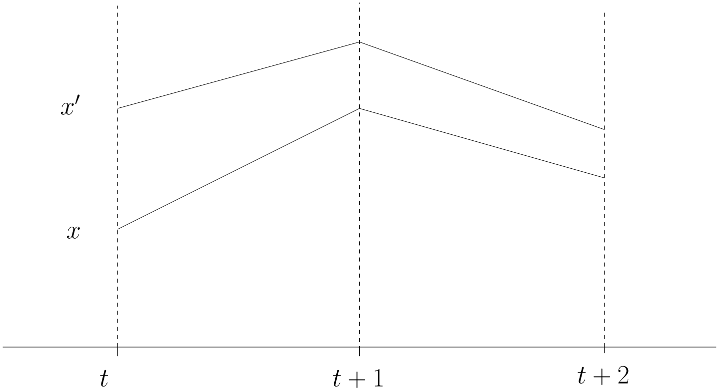
- Monotonicity: $x \leq x'$ implies

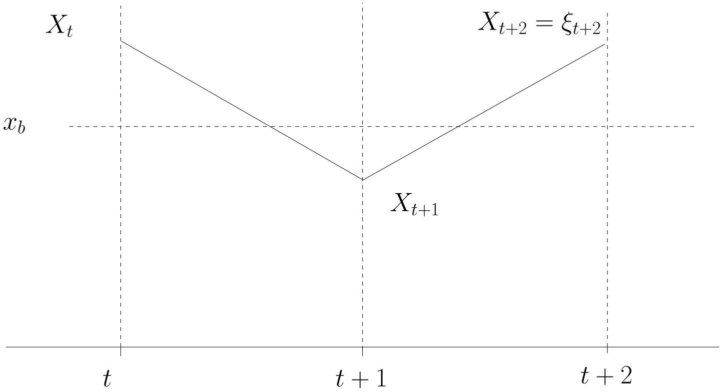
$$\alpha I(x) + z \leq \alpha I(x') + z$$



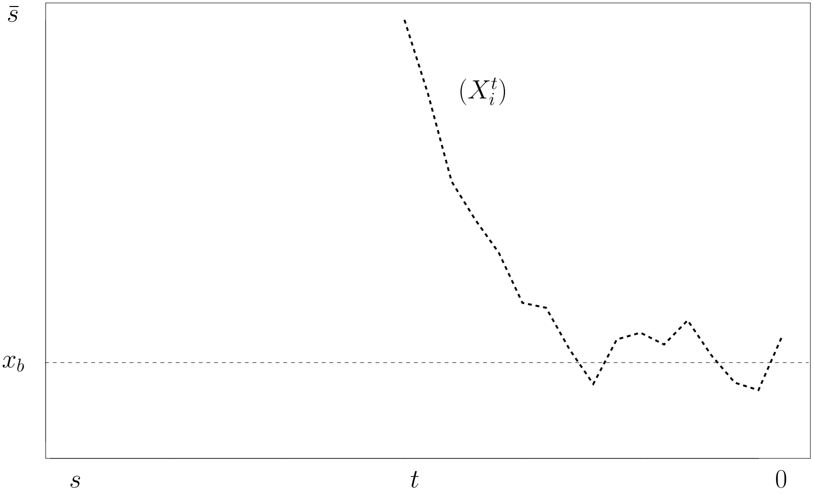
x'
x
t
$t+1$
$t+2$

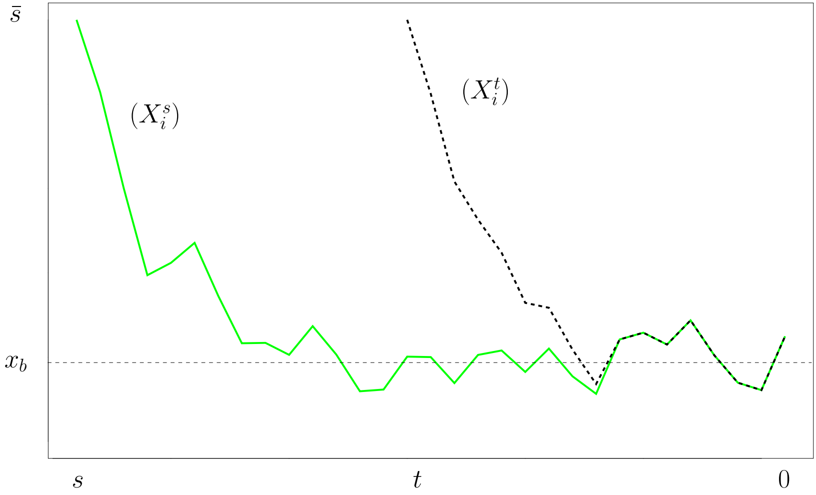
- Non-interiority: $x \leq x_b$ implies

$$\alpha I(x) + z = z$$



X_t
$X_{t+2} = \xi_{t+2}$
x_b
X_{t+1}
t
$t+1$
$t+2$


$\bar{s}$
(X_i^t)
x_b
s
t
0



$\bar{s}$
(X_i^s)
(X_i^t)
x_b
s
t
0

set $t = C = 0$; set $X = \bar{s}$; set $Z = (\,)$;

repeat

  draw $\xi_t \sim \varphi$ and append to list Z ;

  for *i* in $t, \ldots, 0$ **do** `// iterate until time zero`

    set $X = \Gamma_{\xi_i}(X)$;

    if $X \leq x_b$ *and* $i < 0$ **then** set $C = 1$;

  end

  if $C = 1$ **then** **break** ; `// coupling successful`

  else

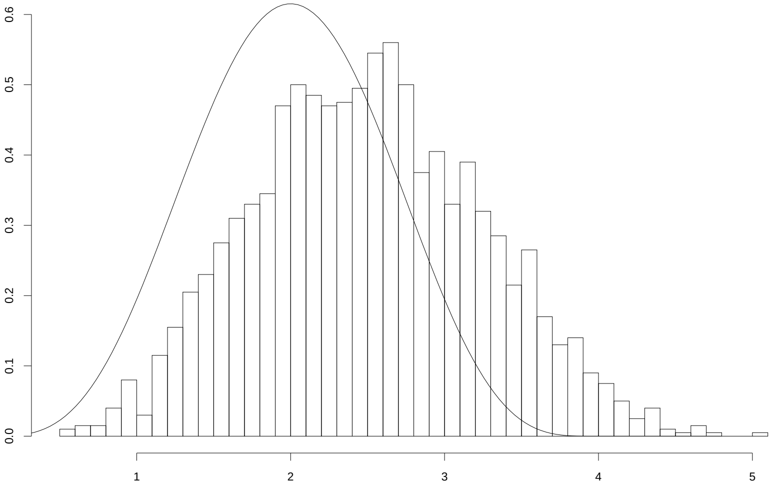
    set $t = t - 1$;

    set $X = \bar{s}$;

  end

end

return X


0.0
0.1
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1
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5