

EQUILIBRIUM STORAGE WITH MULTIPLE COMMODITIES

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ABSTRACT. This paper introduces a multisector model of commodity markets with storage, where equilibrium is defined by profit maximization, arbitrage and market clearing conditions. We then solve for the decentralized equilibrium via a corresponding dynamic program. We also describe the dynamics of the model, establishing geometric ergodicity, a Law of Large Numbers and a Central Limit Theorem.

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1. INTRODUCTION

Given their volatility and importance to politically sensitive coalitions, markets for primary commodities have often attracted intervention by governments aimed at stabilizing producer incomes. Ideally, manipulation of endogenously determined variables requires careful analysis of the equilibrium process. For this and other reasons economists have been motivated to construct models which replicate the key features of primary commodity markets.

Perhaps the most successful attempt to frame such a model is found in the seminal work of Samuelson (1971) and Schectman and Scheinkman (1983). These analyses combine profit maximization, arbitrage

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conditions and market clearing to derive a complete system of equilibrium prices and quantities over time for a single market which features stochastic supply and speculative investment.¹

Given their simple arbitrage-based restrictions and endogenous determination of consumption, speculative investment and price, these commodity pricing models provide an attractive framework for empirical analysis. They have formed the basis of numerous quantitative studies, including Deaton and Laroque (1992, 1996), Chambers and Bailey (1996) and Ng and Ruge-Murcia (2000).

This paper introduces a multisector model of commodity prices in the spirit of Samuelson (1971) and Schectman and Scheinkman (1983). Our motivation is to provide the theoretical foundations for models which replicate commodity markets more closely by taking into account the joint determination of prices and quantities across related commodities. A multisector approach can accommodate the impact of demand and supply conditions for one commodity on the price and quantity observed in the market for another.

In general, prices and quantities of related commodities are jointly determined because of contemporaneously correlated shocks on the supply side and substitutability or complementarity on the demand side. For example, the markets for grains such as corn, sorghum, oats, wheat and barley have traditionally been closely integrated, as suggested by the scatter plot matrix for prices over the period from January 1994 to July 2007 shown in Figure 1.² Altered supply or demand conditions for one grain impact on prices for other grains, in this case due mainly to substitutability of feed grains (as a function of energy content).³

¹Numerous other authors have contributed to the theory of the one-sector commodity pricing model. See, for example, Wright and Williams (1991) or Routledge, Seppi and Spatt (2000).

²The data are monthly average prices in the US over the stated period. Source: Agriculture Statistics Board.

³It has also been argued that the prices of seemingly unrelated commodities are correlated even after controlling for relevant macroeconomic variables (Pindyck

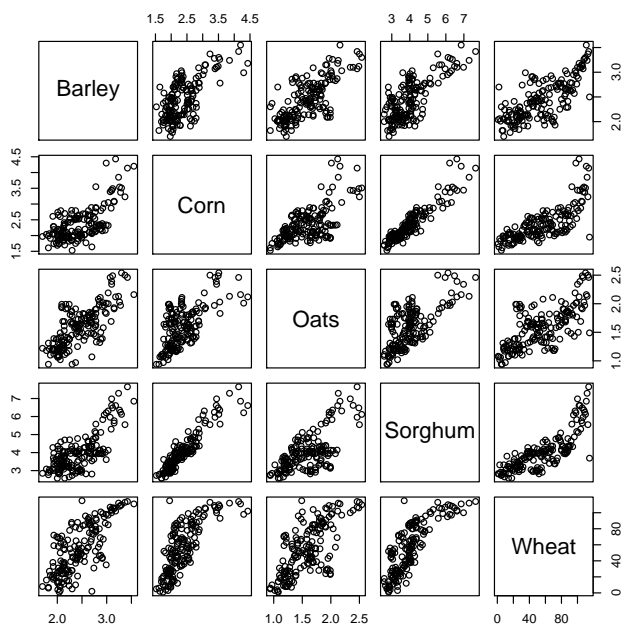


FIGURE 1. Price Correlations, Grains

The fact that a multisector equilibrium commodity pricing model has not been developed to date appears largely due to technical difficulties. While the one-sector model has proven to be highly tractable, the multisector model is considerably less so. For example, in the one-sector model interiority and first order conditions are straightforward, investment is monotone in quantity, and the state variable (supply) follows a renewal process which guarantees stationarity and ergodicity. In the multisector case none of these features remain, and analysis is correspondingly more difficult.

and Rotemberg, 1990). Our model is not suitable for addressing this issue as we envisage relationships between the commodities on either the demand or supply side. Without such relationships our M -sector model reduces to M decoupled one-sector models.

In this paper we show that despite these difficulties a multisector model can be successfully developed and analyzed. We frame a market with M commodities and define equilibrium prices and quantities via profit maximization, arbitrage and market clearing conditions. We then introduce a planner's problem, the first order conditions for which yield the arbitrage conditions of the decentralized market. The decentralized equilibrium can then be computed via dynamic programming techniques.

In the dynamic program, we have treated an unbounded reward problem via weighted supremum norms.⁴ Thus we require neither bounded rewards—which excludes common parametric formulations—nor bounded shocks. Bounded shocks are a standard technique used to compactify the state space and thereby bound rewards, but have the disadvantage of excluding many shock distributions used routinely in econometrics. We avoid placing additional assumptions on rewards and shocks by careful choice of the weighting function. (For example, we do not require the homogeneity conditions used in Boyd (1990) and Alvarez and Stokey (1998).)

Our other results concern dynamics of the state process. Establishing global stability (ergodicity) is considerably more complicated than in the one-sector model, where policies are monotone and the state follows a renewal process.⁵ Using an alternative approach, we are able to provide simple conditions under which the equilibrium process for the stock is asymptotically stationary and geometrically ergodic.

⁴See Hernández-Lerma and Lasserre (1999, Chapter 8) for an excellent discussion of the theory of dynamic programming with weighted sup-norms. For an alternative to the weighted sup-norm approach, see Le Van and Morhaim (2002) or Rincon-Zapatero and Rodriguez-Palmero (2003).

⁵The dynamics of the single sector commodity pricing model were investigated in detail by Scheinkman and Schectman (1983). Confirming a conjecture of Samuelson (1971), they show that the process for the state (the stock of the commodity) converges asymptotically to a unique stationary distribution. Bobenrieth, Bobenrieth and Wright (2002) established geometric ergodicity and investigated other properties of the stationary distribution.

Geometric ergodicity in turn leads to characterization of the sample paths, which are shown to satisfy a Strong Law of Large Numbers and a Central Limit Theorem. These properties open up a range of consistent estimation techniques (cf., e.g., the simulation-based methods of Duffie and Singleton, 1993).

1.1. Outline. The next section introduces the multisector model and defines competitive equilibrium. Section 3 sets up a corresponding dynamic program, and establishes the connection between this programming problem and the decentralized equilibrium. Section 4 considers dynamics under the equilibrium. Section 5 concludes, and any remaining proofs are given in the appendix.

2. SPECULATIVE PRICES

In this section we construct a multisector version of Samuelson's commodity pricing model. In subsequent sections we show that equilibrium storage by speculators in the decentralized market is equal to the optimal investment policy of a planner maximizing an discounted revenue stream for the final producers.

In what follows, $\mathbb{R}_+^M := [0, \infty)^M$ and $\mathbb{R}_{++}^M := (0, \infty)^M$. For x and y in \mathbb{R}^M the relation $x \leq y$ means that $y - x \in \mathbb{R}_+^M$, while $x \ll y$ means that $y - x \in \mathbb{R}_{++}^M$. The notation $[x, y]$ denotes an order interval: $[x, y]$ is all $z \in \mathbb{R}^M$ such that $x \leq z \leq y$; (x, y) is all z with $x \ll z \ll y$, and so on. Let $\partial\mathbb{R}_+^M$ be the boundary $\mathbb{R}_+^M \setminus \mathbb{R}_{++}^M$.

The inner product of x and y is denoted $\langle x, y \rangle$, and $\|x\| := \langle x, x \rangle^{1/2}$ is the Euclidean norm. For $g: \mathbb{R}^M \rightarrow \mathbb{R}$, the symbol ∇g denotes the vector of partial derivatives when it exists, and $D_h g(x)$ is the directional derivative of g at x in the direction h :

$$D_h g(x) := \lim_{\theta \downarrow 0} \frac{g(x + \theta h) - g(x)}{\theta}$$

We use λ to denote Lebesgue measure on \mathbb{R}_+^M , while $\mathcal{B}(\mathbb{R}_+^M)$ is the Borel sets and $L_1(\mathbb{R}_+^M)$ is the Lebesgue integrable functions. A *distribution* is a Borel probability measure on \mathbb{R}_+^M .

The market has M commodities, the vector of spot prices for which is given at time t by $p_t = (p_t^m)_{m=1}^M \in \mathbb{R}_+^M$. Demand for the commodities comes from firms (final producers), who use the commodities as inputs to their production process, and from speculators, who purchase the commodities for future sale. Let the risk-free interest rate r be constant, and set $\rho := (1 + r)^{-1}$.

The firms demand a vector C_t of the commodities according to the profit maximization problem

$$\max_{C_t} \Pi_t(C_t), \quad \Pi_t(x) := F(x) - \langle p_t, x \rangle$$

where $F(x)$ is the output of the final good given input vector x , and the price of the final good has been normalized to one.

Assumption 2.1. The function $F: \mathbb{R}_+^M \rightarrow \mathbb{R}_+$ is strictly concave, strictly increasing, continuous and differentiable on \mathbb{R}_{++}^M .⁶

In addition, for our analysis to succeed we require that equilibrium demand is strictly positive in each period. In the multisector case obtaining conditions on the primitives of the model under which this property holds is a nontrivial problem. We show that the following condition is sufficient.

Assumption 2.2. For any x on the boundary of \mathbb{R}_+^M and any vector h which points to the interior, the directional derivative of F at x in the direction h is infinite. That is, $D_h F(x) = \infty$ for all $x \in \partial \mathbb{R}_+^M$ and $h \in \mathbb{R}_+^M$ such that $x + h \in \mathbb{R}_{++}^M$.

Example 2.1. The Cobb-Douglas technology $F(x) = \prod_{m=1}^M (x^m)^{a_m}$ satisfies all of the conditions of Assumptions 2.1 and 2.2 when $a := \sum_{m=1}^M a_m < 1$. The only element of this claim which requires proof is the interiority condition in Assumption 2.2. For the proof, pick any $x \in \partial \mathbb{R}_+^M$ and any $h \in \mathbb{R}^M$ with $y := x + h \in \mathbb{R}_{++}^M$. We have

$$\theta^a F(y) = F(\theta y) \leq F((1 - \theta)x + \theta y) = F(x + \theta h)$$

⁶Strict concavity is assumed because other inputs such as labor and capital are held fixed, in which case constant returns to scale in all inputs requires decreasing returns to scale in the commodity vector alone.

$$\therefore \theta^{a-1}F(y) \leq \frac{F(x + \theta h) - F(x)}{\theta} \quad (\because F(x) = 0)$$

Since $y \gg 0$ we have $F(y) > 0$, and as $a < 1$ the left hand side converges to infinity when $\theta \downarrow 0$.

Aside from firms, there exists a unit mass of identical speculators who are able to store the commodities between periods. Purchasing I^m units of good m yields $\alpha^m I^m$ units next period, where $\alpha^m \in (0, 1)$ parameterizes storage cost, or depreciation. Hence $I_t = (I_t^m)_{m=1}^M \in \mathbb{R}_+^M$ carried over from time t yields ΛI_t at $t + 1$, where

$$\Lambda := \text{diag}(\alpha^1, \dots, \alpha^M) = \begin{pmatrix} \alpha^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha^M \end{pmatrix} \quad \text{and} \quad \alpha^m \in (0, 1)$$

Aggregate supply X_t in the market at time t is the sum of ΛI_{t-1} and a “harvest” W_t :

$$X_t = \Lambda I_{t-1} + W_t$$

Regarding the harvest process we make the following assumption:

Assumption 2.3. The shocks $(W_t)_{t \geq 1}$ are independent, identically distributed \mathbb{R}_+^M -valued random vectors with common distribution ϕ . In addition, $\phi(\partial \mathbb{R}_+^M) = 0$ and

$$(1) \quad \mu := \mathbb{E} \|W_t\| = \int \|z\| \phi(dz) < \infty$$

The assumption that $\phi(\partial \mathbb{R}_+^M) = 0$ implies that X_t lies in \mathbb{R}_{++}^M for all t with probability one, in which case we need not concern ourselves with infinite prices on the boundary $\partial \mathbb{R}_+^M$. The sequence of shocks $(W_t)_{t \geq 1}$ is defined on an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$,⁷ and \mathbb{E} denotes expectation with respect to \mathbb{P} .

To construct a competitive equilibrium for the commodity market we consider input demand, investment and pricing functions c, i and

⁷Thus, $\mathbb{P}\{W_t \in \cdot\} = \mathbb{P} \circ W_t^{-1} = \phi$ on $\mathcal{B}(\mathbb{R}_+^M)$.

p from $\mathbb{R}_{++}^M \rightarrow \mathbb{R}_+^M$. Given c, i and p , the process for quantities and prices evolves according to

$$(2) \quad I_t = i(X_t), \quad C_t = c(X_t), \quad p_t = p(X_t), \quad X_{t+1} = \Lambda I_t + W_{t+1},$$

For this system of functions to be an equilibrium requires market clearing and profit maximization, or

$$(3) \quad c(x) + i(x) = x \quad \text{and} \quad \nabla F(c(x)) = p(x)$$

for $x \in \mathbb{R}_{++}^M$. In addition, we introduce the following arbitrage condition. Given $x \in \mathbb{R}_+^M$ and investment policy i , we say that $h \in \mathbb{R}^M$ is a *feasible variation* at x if $i(x) + h \in [0, x]$. The set of all feasible variations at x is denoted $F_v^i(x)$. Assuming speculators are risk neutral, nonexistence of arbitrage requires that i and p satisfy

$$(4) \quad \rho \int \langle p(\Lambda i(x) + z), \Lambda h \rangle \phi(dz) \leq \langle p(x), h \rangle, \quad \forall h \in F_v^i(x)$$

for $x \in \mathbb{R}_{++}^M$. If the condition (4) fails for some x then a deviation from $i(x)$ in the feasible direction h yields strictly positive expected profits for speculators, and hence is not an equilibrium.

3. THE PLANNER'S PROBLEM

In order to construct a set of prices and quantities which satisfy (3) and (4) we introduce a planning problem, the optimal policy for which generates the decentralized equilibrium. The planning problem can be stated as

$$(5) \quad \max_{i \in \mathcal{I}} \mathbb{E} \sum_{t \geq 0} \rho^t F(X_t - i(X_t))$$

$$(6) \quad \text{subject to} \quad X_{t+1} = \Lambda i(X_t) + W_{t+1}, \quad X_0 \sim \psi_0$$

where $F, \rho, (W_t)_{t \geq 1}$ and Λ are as defined in Section 2, and \mathcal{I} denotes the set of investment policies $i: \mathbb{R}_+^M \rightarrow \mathbb{R}_+^M$ that are Borel measurable and satisfy the feasibility constraint $i(x) \leq x$.

3.1. Maximization of the Surplus. The control problem (5)–(6) maximizes the firm's discounted revenue stream.⁸ Before demonstrating

⁸Recall that the price of the final commodity is normalized to one.

that the solution to the planner's problem yields the decentralized equilibrium, we provide some intuition as to why maximizing the firm's discounted revenue stream yields the decentralized equilibrium. In particular, we show that (5)–(6) is a natural objective for the planner, as it is equivalent to maximizing total surplus in the market.

Total surplus is the sum of discounted firm profits, payments to the sellers of the commodities and returns to speculative investors:

$$\mathbb{E} \left\{ \sum_{t \geq 0} \rho^t \Pi_t(C_t) + \sum_{t \geq 0} \rho^t \langle p_t, W_t \rangle + \sum_{t \geq 1} \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle] \right\}$$

To see that maximization of this surplus subject to the constraints

$$X_{t+1} = \Lambda I_t + W_{t+1}, \quad I_t + C_t \leq X_t, \quad W_0 := X_0$$

is equivalent to (5)–(6), note that optimal paths satisfy $I_t + C_t = X_t$ for all $t \geq 0$, from which we obtain $I_t + C_t = \Lambda I_{t-1} + W_t$, and hence

$$\langle p_t, I_t \rangle + \langle p_t, C_t \rangle = \langle p_t, \Lambda I_{t-1} \rangle + \langle p_t, W_t \rangle$$

For each $T \in \mathbb{N}$ the sum $\sum_{t=1}^T \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle]$ becomes

$$\begin{aligned} & \sum_{t=1}^T \rho^t [\langle p_t, I_t \rangle + \langle p_t, C_t \rangle - \langle p_t, W_t \rangle - \langle p_{t-1}, I_{t-1} \rangle] \\ &= \sum_{t=1}^T \rho^t [\langle p_t, C_t \rangle - \langle p_t, W_t \rangle] + \rho^T \langle p_T, I_T \rangle - \langle p_0, I_0 \rangle \end{aligned}$$

Returning to our total surplus, then, for each $T \in \mathbb{N}$ the sum is

$$\begin{aligned} & \sum_{t=0}^T \rho^t \Pi_t(C_t) + \sum_{t=0}^T \rho^t \langle p_t, W_t \rangle + \sum_{t=1}^T \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle] \\ &= \sum_{t=0}^T \rho^t F(C_t) - \langle p_0, C_0 \rangle + \langle p_0, W_0 \rangle + \rho^T \langle p_T, I_T \rangle - \langle p_0, I_0 \rangle \end{aligned}$$

Since $X_0 = W_0$ we have $C_0 + I_0 = W_0$, and hence

$$\begin{aligned} & \sum_{t=0}^T \rho^t \Pi_t(C_t) + \sum_{t=0}^T \rho^t \langle p_t, W_t \rangle + \sum_{t=1}^T \rho^t [\langle p_t, \Lambda I_{t-1} \rangle - \langle p_{t-1}, I_{t-1} \rangle] \\ &= \sum_{t=0}^T \rho^t F(C_t) + \rho^T \langle p_T, I_T \rangle \end{aligned}$$

Taking the limit with respect to T , this becomes $\sum_{t \geq 0} \rho^t F(C_t)$. Hence the surplus maximization problem reduces to (5).

3.2. Solving the Planning Problem. Let v be the value function associated with (5)–(6). That is,

$$(7) \quad v(x) := \sup_{i \in \mathcal{I}} v_i(x), \quad \text{where } v_i(x) := \mathbb{E} \sum_{t \geq 0} \rho^t F(X_t - i(X_t))$$

Here $(X_t)_{t \geq 0}$ obeys the recursion in (6), but with $X_0 = x$. We call $i \in \mathcal{I}$ optimal if it attains the supremum in (7) for every $x \in \mathbb{R}_+^M$.

Under our assumptions the value function v is potentially unbounded on \mathbb{R}_+^M , and classical dynamic programming arguments do not apply. We use a weighted norm approach to characterize the optimal policy. Given fixed $\delta \in (\rho, 1)$ and the \mathbb{R}_+^M -valued process

$$(8) \quad Y_{t+1} = \Lambda Y_t + W_{t+1}, \quad Y_0 = x$$

we define the weighting function $\kappa: \mathbb{R}_+^M \rightarrow \mathbb{R}$ by the infinite sum

$$\kappa(x) := 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E} F(Y_t) \quad x \in \mathbb{R}_+^M$$

Lemma 3.1. *The function κ is finite, increasing and continuous everywhere on \mathbb{R}_+^M . For any $i \in \mathcal{I}$ we have $v_i \leq \kappa$ on \mathbb{R}_+^M .*

Hence v is finite and $v/\kappa \leq 1$. Define $w: \mathbb{R}_+^M \rightarrow \mathbb{R}$ to be κ -bounded if w/κ is bounded; that is, if

$$\|w\|_{\kappa} := \|w/\kappa\|_{\infty} := \sup_{x \in \mathbb{R}_+^M} |w(x)/\kappa(x)| < \infty$$

The function $w \mapsto \|w\|_{\kappa}$ is a norm on the set of all κ -bounded functions on \mathbb{R}_+^M . Define $b_{\kappa} \mathcal{B} \mathbb{R}_+^M$ to be the set of κ -bounded Borel measurable function on \mathbb{R}_+^M , and $b_{\kappa} c \mathbb{R}_+^M$ to be those functions which are in addition continuous. The collection of functions $b_{\kappa} \mathcal{B} \mathbb{R}_+^M$ endowed with the norm $\|\cdot\|_{\kappa}$ is a Banach space. Using continuity of κ , it can be shown that $b_{\kappa} c \mathbb{R}_+^M$ is a $\|\cdot\|_{\kappa}$ -closed subset of $b_{\kappa} \mathcal{B} \mathbb{R}_+^M$.

The Bellman operator $T: b_\kappa \mathcal{B}\mathbb{R}_+^M \rightarrow b_\kappa \mathcal{B}\mathbb{R}_+^M$ is defined by

$$(9) \quad Tw(x) = \sup_{0 \leq \xi \leq x} \left\{ F(x - \xi) + \rho \int w(\Lambda \xi + z) \phi(dz) \right\} \quad x \in \mathbb{R}_+^M$$

The operator T is a contraction of modulus $\gamma := \rho/\delta < 1$ on $b_\kappa \mathcal{B}\mathbb{R}_+^M$ with respect to the $\|\cdot\|_\kappa$ -norm:

Proposition 3.1. *T is a well-defined map from $b_\kappa \mathcal{B}\mathbb{R}_+^M$ to itself, and*

$$\|Tw - Tu\|_\kappa \leq \gamma \|w - u\|_\kappa \quad w, u \in b_\kappa \mathcal{B}\mathbb{R}_+^M$$

If w is continuous then so is Tw , and hence T sends $b_\kappa c\mathbb{R}_+^M$ into itself.

We can now characterize the value function and the unique optimal policy in the following way.

Theorem 3.1. *The value function v is the unique fixed point of T in $b_\kappa \mathcal{B}\mathbb{R}_+^M$, and for each $w \in b_\kappa \mathcal{B}\mathbb{R}_+^M$ we have $\|T^n w - v\|_\kappa \rightarrow 0$ as $n \rightarrow \infty$. In addition, v is continuous, strictly increasing and strictly concave on \mathbb{R}_+^M . A unique optimal policy $I \in \mathcal{I}$ exists. It is continuous, and satisfies*

$$(10) \quad I(x) = \operatorname{argmax}_{0 \leq \xi \leq x} \left\{ F(x - \xi) + \rho \int v(\Lambda \xi + z) \phi(dz) \right\} \quad x \in \mathbb{R}_+^M$$

Figure 2 shows (an approximation to) the value function for the two commodity case, where $\alpha^1 = \alpha^2 = \rho = 0.9$, $F(x, y) = x^{0.4}y^{0.4}$ and $W = (e^\xi, e^\eta)$ with (ξ, η) independent standard normal. The approximation was carried out by iterating the Bellman operator, starting at $F \in b_\kappa c\mathbb{R}_+^M$. The sequence converges to v at rate $O(\gamma^n)$.⁹ Figure 3 gives the corresponding investment functions $I^1(x^1, x^2)$ and $I^2(x^1, x^2)$ (at top and bottom respectively), obtained by solving (10) with v the approximate value function given in Figure 2.

We also have the following envelope condition:

⁹Each iterate was approximated using a continuous piecewise affine function constructed as the infimum of 324 supporting hyperplanes. This technique is related to the method proposed by Santos and Vigo (1998), who suggest approximating value functions by continuous piecewise affine interpolation. (Our algorithm leads to a convex approximation from above, rather than below.)

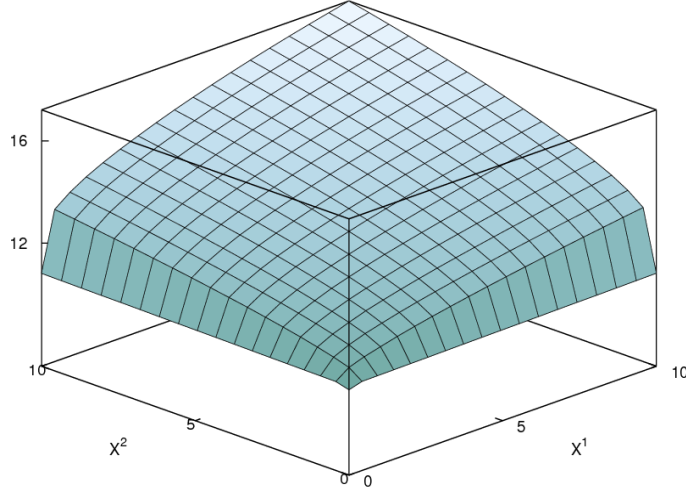


FIGURE 2. Value Function

Proposition 3.2. *If $x \gg 0$ then $x - I(x) \gg 0$. In addition, the value function v is differentiable on \mathbb{R}_{++}^M and*

$$(11) \quad \nabla v(x) = \nabla F(x - I(x)) \quad x \in \mathbb{R}_{++}^M$$

3.3. The Decentralized Equilibrium. We are now ready to connect the equilibrium for the decentralized commodity market defined in Section 2 with the planner's problem. To do so, notice that if one chooses $i: \mathbb{R}_{++}^M \rightarrow \mathbb{R}_+^M$ with $i(x) \in [0, x)$ for all x , then for competitive equilibrium to obtain the consumption and price functions c and p are completely determined on \mathbb{R}_{++}^M by i via the two equations in (3). As a result, to determine competitive equilibria we can focus on the investment function alone. This leads to the following definition.

Definition 3.1. A function $i: \mathbb{R}_{++}^M \rightarrow \mathbb{R}_+^M$ with $i(x) \in [0, x)$ for all x in \mathbb{R}_{++}^M is said to be a competitive equilibrium investment function for the commodity market if i and the induced pricing function $p(x) := \nabla F(x - i(x))$ satisfy (4).

Theorem 3.2. *The optimal investment policy I defined in Theorem 3.1 is a competitive equilibrium investment function for the commodity market.*

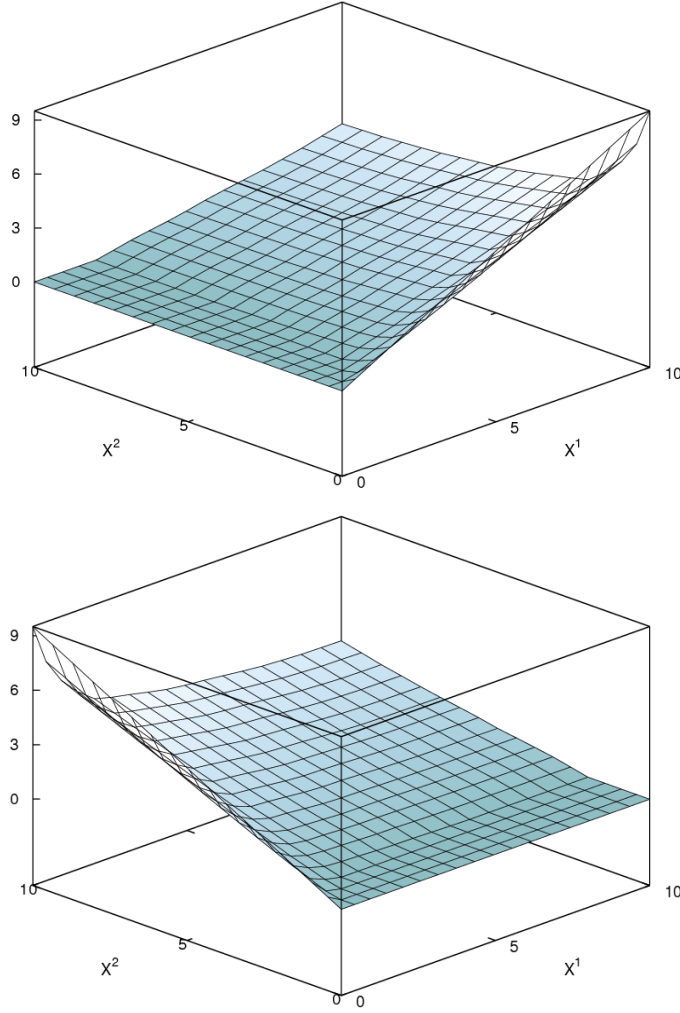


FIGURE 3. Investment in Commodities 1 (top) and 2 (bottom)

Moreover, it is unique in the sense that if i is a competitive equilibrium investment function then $i = I$ on \mathbb{R}_{++}^M .

Proof. We begin with a definition: Let f_x be the concave function

$$f_x(\xi) := F(x - \xi) + \rho \int v(\Lambda \xi + z) \phi(dz), \quad \xi \in [0, x]$$

It is not difficult to show that ξ^* maximizes f_x over $[0, x]$ iff

$$(12) \quad D_h f_x(\xi^*) \leq 0 \text{ for all } h \text{ with } \xi^* + h \in [0, x]$$

Now we establish that I defined in Theorem 3.1 is a competitive equilibrium investment function in the sense that I and the induced pricing function $P(x) := \nabla F(x - I(x))$ satisfy

$$(13) \quad \rho \int \langle P(\Lambda I(x) + z), \Lambda h \rangle \phi(dz) \leq \langle P(x), h \rangle \quad \forall h \in F_v^I(x)$$

for all $x \in \mathbb{R}_{++}^M$. To see this, fix $x \in \mathbb{R}_{++}^M$ and $h \in F_v^I(x)$, so that $I(x) + h \in [0, x]$. Since v is differentiable and $x - I(x)$ is interior the directional derivative of f_x at $I(x)$ is

$$D_h f_x(I(x)) = -\langle \nabla F(x - I(x)), h \rangle + \rho \int \langle \nabla v(\Lambda I(x) + z), \Lambda h \rangle \phi(dz)$$

Since $I(x)$ maximizes f_x over $[0, x]$ and $I(x) + h \in [0, x]$ the inequality in (12) gives

$$\rho \int \langle \nabla v(\Lambda I(x) + z), \Lambda h \rangle \phi(dz) \leq \langle \nabla F(x - I(x)), h \rangle$$

Using $P(x) = \nabla v(x) = \nabla F(x - I(x))$ we obtain (13). Thus I is a competitive equilibrium investment function.

Next we consider uniqueness. Since f_x is strictly concave its maximizer $I(x)$ is unique for all x . Since the first order condition (12) is necessary and sufficient, it follows that I is the unique function that satisfies (12) for all x . Writing this directional derivative in terms of I we see that I is the only function which satisfies

$$\begin{aligned} \rho \int \langle \nabla F[\Lambda I(x) + z - I(\Lambda I(x) + z)], \Lambda h \rangle \phi(dz) \\ \leq \langle \nabla F(x - I(x)), h \rangle \quad \forall h \in F_v^I(x) \end{aligned}$$

Now suppose that i is a competitive equilibrium investment function, so in particular $p(x) = \nabla F(x - i(x))$ satisfies (4). Then

$$\begin{aligned} \rho \int \langle \nabla F[\Lambda i(x) + z - i(\Lambda i(x) + z)], \Lambda h \rangle \phi(dz) \\ \leq \langle \nabla F(x - i(x)), h \rangle \quad \forall h \in F_v^i(x) \end{aligned}$$

for all $x \in \mathbb{R}_{++}^M$. But we have just established that I is the only function which has this property. Hence $i = I$. \square

4. DYNAMICS

Next we turn to equilibrium dynamics of $(X_t)_{t \geq 0}$ given the optimal investment policy I defined in (10). The process is Markovian and obeys the stochastic recursive sequence

$$(14) \quad X_{t+1} = \Lambda I(X_t) + W_{t+1}, \quad (W_t)_{t \geq 0} \stackrel{\text{iid}}{\sim} \phi$$

The sequence $(X_t)_{t \geq 0}$ can also be seen as the equilibrium time path given in (2), as discussed in Section 2.

Throughout this section we maintain Assumptions 2.1 and 2.3. Assumption 2.2 is not required. Instead, additional restrictions on the nature of the shock distribution are necessary:

Assumption 4.1. The distribution ϕ of the shock W can be represented by a density, which we again denote by ϕ . The density ϕ is continuous everywhere on \mathbb{R}_+^M and positive on its interior.

Many standard distributions satisfy all of our assumptions, a useful example being the multivariate lognormal density. Heavy tailed densities are also possible, provided that μ in (1) remains finite. It would appear that the latter assumption is difficult to weaken substantially while maintaining our stability results.¹⁰

The dynamics in (14) can be encapsulated in the Markov density kernel

$$(15) \quad q(x, y) := \phi(y - \Lambda I(x)) \quad x, y \in \mathbb{R}_+^M$$

Intuitively, $q(x, y)$ is the conditional density of X_{t+1} when $X_t = x$.¹¹ If $y \not\geq \Lambda I(x)$ then $y - \Lambda I(x) \notin \mathbb{R}_+^M$ and $\phi(y - \Lambda I(x))$ is not defined.

¹⁰The assumption that ϕ is a density can perhaps be relaxed without losing the stability results given below. However, the density assumption is suitable for empirical applications and allows slightly more direct proofs, as well as a more explicit construction of the Markov process generated by the optimal policy.

¹¹To see this, observe that for any $B \in \mathcal{B}(\mathbb{R}_+^M)$ the change of variable $z = y - \Lambda I(x)$ yields $\int \mathbb{1}_B(y) \phi(y - \Lambda I(x)) dy = \int \mathbb{1}_B(\Lambda I(x) + z) \phi(z) dz = \mathbb{P}\{\Lambda I(x) + W \in B\} = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$, where $\mathbb{1}_B$ denotes the indicator function of B .

For such values of x and y we take $q(x, y) = 0$. Alternatively, one can regard ϕ as defined on all of \mathbb{R}^M and equal to zero on $\mathbb{R}^M \setminus \mathbb{R}_+^M$.

Using standard arguments¹² we can deduce that if X_t has any distribution ψ_t (not necessarily a density), then X_{t+1} has a distribution represented by density ψ_{t+1} , where

$$(16) \quad \psi_{t+1}(y) = \int q(x, y) \psi_t(dx) \quad y \in \mathbb{R}_+^M$$

Let \mathbf{M} be a map from the set of distributions on \mathbb{R}_+^M into the set of densities on \mathbb{R}_+^M defined by $\psi \mapsto \psi \mathbf{M}$,

$$(17) \quad \psi \mathbf{M}(y) = \int q(x, y) \psi(dx) \quad y \in \mathbb{R}_+^M$$

This map is called the Markov operator corresponding to q .¹³ In light of (16), the marginal distributions (ψ_t) of (X_t) satisfy $\psi_{t+1} = \psi_t \mathbf{M}$. Iterating backwards we obtain $\psi_t = \psi_0 \mathbf{M}^t$, where \mathbf{M}^t is the t -th composition of \mathbf{M} with itself, and, as above, ψ_0 is the distribution of X_0 .

A distribution ψ^* on \mathbb{R}_+^M is called stationary for the optimal process (14) if ψ^* is a fixed point of \mathbf{M} . The interpretation is that if $X_t \sim \psi^*$, then $X_{t+1} \sim \psi^* \mathbf{M} = \psi^*$, and hence probabilities are unchanged. Since \mathbf{M} maps distributions into densities, any fixed point ψ^* of \mathbf{M} must be a density (because ψ^* is the image of itself under \mathbf{M}). Hence in what follows we need concern ourselves only with stationary densities, rather than stationary distributions. For such a stationary density, the defining condition $\psi^* \mathbf{M} = \psi^*$ translates to

$$(18) \quad \int q(x, y) \psi^*(x) dx = \psi^*(y) \quad y \in S$$

We measure the distance between densities ψ and ψ' according to their deviation with respect to the norm on $L_1(\mathbb{R}_+^M)$:

$$d_1(\psi, \psi') := \int |\psi(x) - \psi'(x)| dx$$

¹²See, for example, Lasota and Mackey (1994) or Stachurski (2002).

¹³As is traditional, \mathbf{M} acts on distributions to the left.

By Scheffé's Identity, we also have

$$(19) \quad d_1(\psi, \psi') = \sup_{|h| \leq 1} \left| \int h(x) \psi(x) dx - \int h(x) \psi'(x) dx \right|$$

Here the supremum is with respect to all Borel measurable bounded functions with supremum norm less than 1.¹⁴

To state our results, we introduce two classes \mathcal{H}_s and \mathcal{G}_s

$$\mathcal{H}_s := \{h \in m\mathcal{B}(\mathbb{R}_+^M) : |h(x)| \leq \|x\| + s, \forall x \in \mathbb{R}_+^M\} \quad (s \geq 1)$$

$$\mathcal{G}_s := \{h \in m\mathcal{B}(\mathbb{R}_+^M) : h(x)^2 \leq \|x\| + s, \forall x \in \mathbb{R}_+^M\} \quad (s \geq 1)$$

where $m\mathcal{B}(\mathbb{R}_+^M)$ is the Borel measurable, real-valued functions on \mathbb{R}_+^M . We can now state the main stability result of the paper.

Theorem 4.1. *The following statements are true:*

- (1) *The optimal process (14) has a unique stationary density ψ^* .*
- (2) *The stationary density ψ^* satisfies $\int \|x\| \psi^*(dx) < \infty$.*
- (3) *If ψ_0 is any distribution with $\int \|x\| \psi_0(dx) < \infty$, then for all $s \geq 1$ there is an $M < \infty$ and a $\beta < 1$ such that, $\forall t \in \mathbb{N}$,*

$$\sup_{h \in \mathcal{H}_s} \left| \int h(x) \psi_0 \mathbf{M}^t(x) dx - \int h(x) \psi^*(x) dx \right| \leq \beta^t M$$

We present several corollaries to the theorem:

Corollary 4.1. *Let $(X_t)_{t \geq 0}$ be the optimal process starting at $x_0 \in \mathbb{R}_+^M$. For any such x_0 , the density ψ_t of X_t converges in $L_1(\mathbb{R}_+^M)$ to ψ^* at a geometric rate.*

For the next corollary some additional notation is useful. Let $(X_t^*)_{t \geq 0}$ be a stationary version of the process. That is, $X_{t+1}^* = \Lambda I(X_t^*) + W_{t+1}$ and $X_0^* \sim \psi^*$. Given $h: \mathbb{R}_+^M \rightarrow \mathbb{R}$, we set

$$m_h^* := \int h(x) \psi^*(x) dx = \mathbb{E}h(X_0^*)$$

$$v_h^* := \mathbb{E}[h(X_0^*) - m_h^*]^2 + 2 \sum_{t \geq 1} \mathbb{E}[h(X_0^*) - m_h^*][h(X_t^*) - m_h^*]$$

¹⁴From (19) it is easy to see that L_1 convergence of densities implies uniform (and hence weak) convergence of distribution functions.

Corollary 4.2. *Let ψ_0 be an arbitrary initial condition and let $(X_t)_{t \geq 0}$ be the process starting at $X_0 \sim \psi_0$. If $h \in \mathcal{H}_s$ for some $s \geq 1$, then m_h^* is finite, and*

$$(LLN) \quad \frac{1}{n} \sum_{t=1}^n h(X_t) \rightarrow m_h^* \quad \mathbb{P}\text{-a.s. as } n \rightarrow \infty$$

If, in addition, $h \in \mathcal{G}_s$ for some $s \geq 1$, then v_h^ is finite, and*

$$(CLT) \quad \frac{1}{n} \sum_{t=1}^n h(X_t) \rightarrow N(m_h^*, v_h^*) \quad \text{in distribution as } n \rightarrow \infty$$

The two most important consequences of Corollary 4.2 are as follows. First, for any $B \in \mathcal{B}(\mathbb{R}_+^M)$ we have $\frac{1}{n} \sum_{t=1}^n \mathbb{1}_B(X_t) \rightarrow \psi^*(B)$, and hence the steady state probability $\psi^*(B)$ is approximately equal to the fraction of time that the equilibrium quantity spends in B as the time horizon tends to infinity. This is the standard concept of ergodicity. Second, expectations and probabilities vis-a-vis the stationary distribution can be computed by simulation, appealing (LLN). For such calculations, (CLT) provides (asymptotic) error bounds.

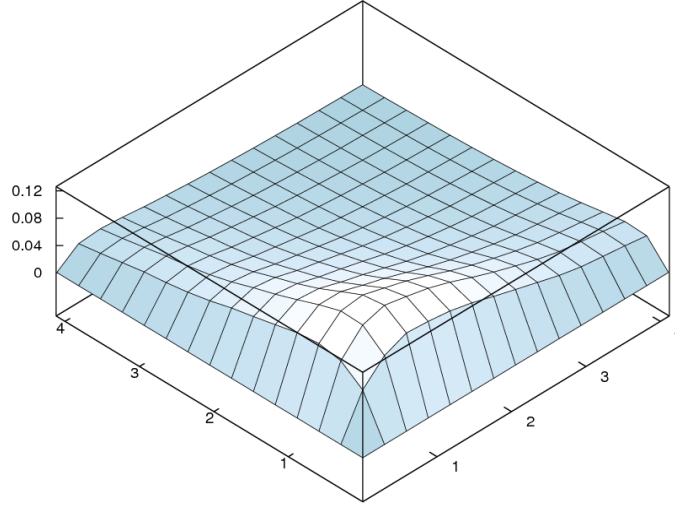
As an application of the second point, we compute an estimate ψ_n^* of the stationary density ψ^* via conditional Monte Carlo (Glynn and Henderson, 2001). Fix $y \in \mathbb{R}_+^M$. Since $x \mapsto q(x, y)$ is bounded it is an element of \mathcal{H}_1 . By Corollary 4.2, then, we have

$$\psi_n^*(y) := \frac{1}{n} \sum_{t=1}^n q(X_t, y) \rightarrow \int q(x, y) \psi^*(x) dx \quad \mathbb{P}\text{-a.s. as } n \rightarrow \infty$$

By (18), the right hand side is precisely $\psi^*(y)$, so $\psi_n^*(y) \rightarrow \psi^*(y)$ almost surely for all y . Figure 4 displays an instance of ψ_n^* for the same parameters as in Figure 2, where $n = 2000$.

5. CONCLUSION

In this paper we introduced a multisector commodity pricing model and described the competitive equilibrium. We indicated how one can solve for equilibrium prices and quantities via a corresponding

FIGURE 4. Approximation ψ_n^* of ψ^*

dynamic program. We also showed that the equilibrium state process is stationary and geometrically ergodic. This sets out a cohesive framework for future empirical analysis.

A number of extensions to our model can be considered. One is to include stochastic demand on the part of firms driven by shocks to production and output prices. Another is correlated shocks for the harvest processes. In terms of the dynamics, the impact of correlated shocks is unclear. While establishing the existence of a stationary distribution should be possible via continuity arguments, stability is highly problematic given the lack of monotonicity and the obvious problems in establishing irreducibility. Such topics are left for future research.

6. APPENDIX

This section collects all remaining proofs. Throughout the proofs we adopt the new notation $\bar{\alpha} := \max_{1 \leq m \leq M} \alpha^m$. Evidently $\|\Lambda x\| \leq \bar{\alpha} \|x\|$, $\forall x \in \mathbb{R}^M$.

6.1. Optimality. Our first task is to prove Lemma 3.1. Recall our definition of the auxiliary process $(Y_t)_{t \geq 0}$ by $Y_{t+1} = \Lambda Y_t + W_{t+1}$ with $Y_0 = x$. Alternatively, $Y_t = \Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j$. Since F is concave there exist positive

constants b_0 and b_1 such that $F(x) \leq b_0 + b_1\|x\|$ for all $x \in \mathbb{R}_+^M$. Moreover, $\|\Lambda^k z\| \leq \bar{\alpha}^k \|z\|$ for any $z \in \mathbb{R}^M$. Consequently,

$$\begin{aligned} \mathbb{E}F\left(\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right) \\ \leq b_0 + b_1 \mathbb{E}\left\|\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right\| \leq b_0 + b_1\|x\| + \frac{b_1 \mu}{1 - \bar{\alpha}} =: M(x) \end{aligned}$$

Given this bound the finiteness of $\kappa(x) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E}F(Y_t)$ is immediate:

$$\kappa(x) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E}F\left(\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right) \leq 1 + \frac{M(x)}{1 - \delta}$$

That κ is increasing follows from monotonicity of F . Continuity of κ follows from continuity of F and the Dominated Convergence Theorem.

The remainder of the proof of Lemma 3.1 is straightforward. Since F is increasing and $X_t - i(X_t) \leq X_t \leq Y_t$ pointwise on Ω for any $i \in \mathcal{J}$ we have $v_i \leq \kappa$. Since $v(x)$ is defined as $\sup_{i \in \mathcal{J}} v_i(x)$ and since $v_i(x) \leq \kappa(x)$ for every $i \in \mathcal{J}$ the function v exists and is dominated by κ . This completes the proof of Lemma 3.1.

Next we turn to the proof of Proposition 3.1. For this proof some extra notation is useful. In particular, for any appropriately integrable function h on \mathbb{R}_+^M we define the new function $\mathbf{N}h$ by

$$\mathbf{N}h(x) := \int h(\Lambda x + z) \phi(dz)$$

so that $\mathbf{N}h(x)$ is the expectation of $h(Y_t)$ given $Y_{t-1} = x$.¹⁵ Evidently $h \leq h'$ implies $\mathbf{N}h \leq \mathbf{N}h'$, and $\mathbf{N}\mathbb{1} = \mathbb{1}$. We let \mathbf{N}^t be the t -th iterate, in which case $\mathbf{N}^t h(x)$ is the expectation of $h(Y_t)$ given $Y_0 = x$. In particular, $\mathbf{N}^t F(x) = \mathbb{E}F(Y_t)$ for all t , and we can express κ as

$$\kappa(x) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E}F(Y_t) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t F(x)$$

To prove that T is well-defined and contracting on $b_\kappa \mathcal{B}\mathbb{R}_+^M$ we need

Lemma 6.1. *For any $x \in \mathbb{R}_+^M$ the weight function κ satisfies*

$$\sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z) \phi(dz) \leq \frac{\kappa(x)}{\delta}$$

¹⁵The operator \mathbf{N} corresponds to T in Stokey, Lucas and Prescott (1989, §8.1).

Proof. Pick any $x \in \mathbb{R}_+^M$. Since κ is increasing,

$$\sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z) \phi(dz) \leq \int \kappa(\Lambda x + z) \phi(dz) = \mathbf{N}\kappa(x)$$

But from the definitions and the Dominated Convergence Theorem,

$$\begin{aligned} \mathbf{N}\kappa(x) &= 1 + \mathbf{N} \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t F(x) \\ &= 1 + \sum_{t=0}^{\infty} \delta^t \mathbf{N}^{t+1} F(x) \\ &= 1 + (1/\delta) \sum_{t=0}^{\infty} \delta^{t+1} \mathbf{N}^{t+1} F(x) \\ &\leq 1 + (1/\delta) \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t F(x) \leq (1/\delta) + (1/\delta) \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t F(x) \end{aligned}$$

This last expression is just $(1/\delta)\kappa(x)$, and the proof is done. \square

Using Lemma 6.1 we can show that the Bellman operator T does send $b_\kappa \mathcal{B}\mathbb{R}_+^M$ into itself—in particular, Tw is κ -bounded whenever w is. Indeed, for any $w \in b_\kappa \mathcal{B}\mathbb{R}_+^M$ we have

$$\begin{aligned} |Tw(x)| &\leq \sup_{0 \leq \xi \leq x} \left| F(x - \xi) + \rho \int w(\Lambda \xi + z) \phi(dz) \right| \\ &\leq F(x) + \rho \sup_{0 \leq \xi \leq x} \int |w(\Lambda \xi + z)| \phi(dz) \\ &\leq F(x) + \rho \|w\|_\kappa \sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z) \phi(dz) \\ &\leq F(x) + \frac{\rho \|w\|_\kappa \kappa(x)}{\delta} \end{aligned}$$

Since $F(x) \leq \kappa(x)$ it follows that for any $x \in \mathbb{R}_+^M$ we have

$$\frac{|Tw(x)|}{\kappa(x)} \leq 1 + \frac{\rho \|w\|_\kappa}{\delta}$$

Thus Tw is κ -bounded, as was to be shown.

In order to prove that T is a contraction of modulus $\gamma = \rho/\delta$ we use the following extension of Blackwell's sufficient condition, which is proved in Hernández-Lerma and Lasserre (1999, Proposition 7.2.9).

Lemma 6.2. *If T is monotone and, for any $c \in \mathbb{R}_+$ and $w \in b_\kappa \mathcal{B}\mathbb{R}_+^M$,*

$$(20) \quad T(w + c\kappa) \leq Tw + \gamma c\kappa$$

then T is a $\|\cdot\|_\kappa$ -contraction of modulus γ on $b_\kappa \mathcal{B}\mathbb{R}_+^M$.

By monotonicity is meant that for any pair $w, w' \in b_\kappa \mathcal{B}\mathbb{R}_+^M$ with $w \leq w'$ we have $Tw \leq Tw'$. This property is easily shown and the proof is omitted. To verify (20), observe that

$$\begin{aligned} T(w + c\kappa)(x) &= \\ \sup_{0 \leq \xi \leq x} \left\{ F(x - \xi) + \rho \int w(\Lambda\xi + z)\phi(dz) + c\rho \int \kappa(\Lambda\xi + z)\phi(dz) \right\} \\ &\leq Tw(x) + c\rho \sup_{0 \leq \xi \leq x} \int \kappa(\Lambda\xi + z)\phi(dz) \end{aligned}$$

In light of Lemma 6.1 we have

$$\begin{aligned} \sup_{0 \leq \xi \leq x} \int \kappa(\Lambda\xi + z)\phi(dz) &= \int \kappa(\Lambda x + z)\phi(dz) \leq \frac{\kappa(x)}{\delta} \\ \therefore T(w + c\kappa)(x) &\leq Tw(x) + \frac{\rho}{\delta} c\kappa(x) \end{aligned}$$

Since $\gamma = \rho/\delta$ the proof is complete.

The only claim in Proposition 3.1 that remains to be verified is that T maps the set of continuous κ -bounded functions $b_\kappa c\mathbb{R}_+^M$ into itself. In particular, we need to check that if w is continuous κ -bounded then Tw is continuous. To see this, pick any such w . In view of Berge's Theorem of the Maximum, the function

$$Tw(x) = \sup_{0 \leq \xi \leq x} \left\{ F(x - \xi) + \rho \int w(\Lambda\xi + z)\phi(dz) \right\}$$

will be continuous provided that

$$(x, \xi) \mapsto F(x - \xi) + \rho \int w(\Lambda\xi + z)\phi(dz)$$

is jointly continuous on $\{(x, \xi) : x \in \mathbb{R}_+^M, 0 \leq \xi \leq x\}$. The only nontrivial assertion is that if $(\xi_n) \subset \mathbb{R}_+^M$ with $\xi_n \rightarrow \xi$, then

$$\int w(\Lambda\xi_n + z)\phi(dz) \rightarrow \int w(\Lambda\xi + z)\phi(dz)$$

In view of continuity of w and the Dominated Convergence Theorem, it is sufficient to show that $|w(\Lambda\xi_n + z)|$ is dominated pointwise by some integrable function for all n . But if x is any vector with $\xi_n \leq x$ for all n , then for any $n \in \mathbb{N}$ and any $z \in \mathbb{R}_+^M$ we have

$$|w(\Lambda\xi_n + z)| \leq \|w\|_\kappa \kappa(\Lambda\xi_n + z) \leq \kappa(\Lambda x + z)$$

The integral of the right hand side is finite by Lemma 6.1. This completes the proof of Proposition 3.1.

Next we prove Theorem 3.1. Since T is a $\|\cdot\|_\kappa$ -contraction on the Banach space $b_\kappa \mathcal{B}\mathbb{R}_+^M$ it follows that T has a unique fixed point $\bar{w} \in b_\kappa \mathcal{B}\mathbb{R}_+^M$ and $\|T^n w - \bar{w}\|_\kappa \rightarrow 0$ as $n \rightarrow \infty$ for any $w \in b_\kappa \mathcal{B}\mathbb{R}_+^M$. Moreover, $\bar{w} \in b_\kappa c\mathbb{R}_+^M$ and is therefore continuous, as $b_\kappa c\mathbb{R}_+^M$ is a closed subset of $b_\kappa \mathcal{B}\mathbb{R}_+^M$ on which T is invariant. The proof that \bar{w} is in fact equal to the value function v is almost identical to the standard argument (i.e., the argument for bounded rewards) and is omitted.

Existence of a maximizer $I(x)$ for each x follows from continuity of the objective and compactness of the constraint. Continuity of I follows from Berge's Theorem of the Maximum. That v is strictly increasing and strictly concave can be proved by a small modification of the usual technique.¹⁶

Let us now consider the proof of Proposition 3.2. First we show that if $x \gg 0$ then $c(x) := x - I(x) \gg 0$. To this end, pick any $x \gg 0$ and let $c := c(x)$. Suppose instead that $c \in \partial\mathbb{R}_+^M$. Let $d := x - c$. By Assumption 2.2 we have

$$\lim_{\theta \downarrow 0} \frac{F(c + \theta d) - F(c)}{\theta} = \infty$$

For v the value function, define the new function g by

$$g(s) = \rho \int v(\Lambda(x - s) + z) \phi(dz)$$

There is no difficulty in checking that g is well-defined and concave on $(-\infty, x]$. It follows that

$$h(\theta) := \frac{g(c + \theta d) - g(c)}{\theta}$$

is well-defined and (by concavity) decreasing on $(-\infty, 0) \cup (0, 1]$. As a result, the limit $\lim_{\theta \downarrow 0} h(\theta)$ exists and is finite.¹⁷ Now since c is optimal, and

¹⁶It is easy to show that T maps the set \mathcal{C} of increasing concave functions in $b_\kappa c\mathbb{R}_+^M$ into itself. Moreover, a simple argument shows that $\|\cdot\|_\kappa$ -convergence implies pointwise convergence, which in turn preserves monotonicity and concavity. Hence \mathcal{C} is $\|\cdot\|_\kappa$ -closed. As $T: \mathcal{C} \rightarrow \mathcal{C}$ and \mathcal{C} is $\|\cdot\|_\kappa$ -closed we have $v \in \mathcal{C}$. Finally, T maps elements of \mathcal{C} into strictly increasing, strictly concave functions in \mathcal{C} , so v is strictly increasing and strictly concave (because $Tv = v$).

¹⁷The value $h(\theta)$ increases monotonically as $\theta \downarrow 0$ and is bounded by $h(-1)$.

since the alternative $c + \theta d$ is less than x and therefore feasible at x for $\theta \in (0, 1]$, we must have

$$\begin{aligned} F(c) + \rho \int v(\Lambda(x - c) + z) \phi(dz) \\ \geq F(c + \theta d) + \rho \int v(\Lambda(x - (c + \theta d)) + z) \phi(dz) \end{aligned}$$

Using the function g this can be rewritten as

$$F(c) + g(c) \geq F(c + \theta d) + g(c + \theta d)$$

Rearranging and dividing through by θ gives

$$\frac{F(c + \theta d) - F(c)}{\theta} \leq -\frac{g(c + \theta d) - g(c)}{\theta} = -h(\theta)$$

The left hand side diverges to infinity as $\theta \downarrow 0$, while the right hand side converges to a finite constant. This contradicts our assumption that $c \in \partial \mathbb{R}_+^M$, and we conclude that $c = x - I(x) \gg 0$.

To complete the proof of Proposition 3.2 we show that v is differentiable on \mathbb{R}_{++}^M and satisfies the envelope condition $\nabla v(x) = \nabla F(x - I(x))$. We use the well-known techniques developed by Mirman and Zilcha (1975) and Benveniste and Scheinkman (1979, Lemma 1). In particular, if $x \in \mathbb{R}_{++}^M$ and w is any concave differentiable function defined on a neighborhood N of x and satisfying $w(x) = v(x)$ and $w(y) \leq v(y)$ for all $y \in N$, then v is differentiable at x and $\nabla v(x) = \nabla w(x)$.

Although investment is not interior, the interiority of demand is sufficient for this technique to work. To see this, pick any $x_0 \gg 0$, and let $i_0 := I(x_0)$. Since $c(x_0) = x_0 - i_0 \gg 0$ we have $x_0 \gg i_0$, and there exists an open neighborhood N of x_0 with $N \subset \mathbb{R}_{++}^M$ and $i_0 \leq x$ for all $x \in N$. On the set N define

$$w(x) = F(x - i_0) + \rho \int v(\Lambda i_0 + z) \phi(dz)$$

Note that w is well-defined on N , as $i_0 \leq x$ for all $x \in N$. In addition, for each $x \in N$ investment i_0 is feasible, so

$$w(x) \leq v(x) = F(x - I(x)) + \rho \int v(\Lambda I(x) + z) \phi(dz)$$

Evidently w is concave and $v(x_0) = w(x_0)$. Finally, w is differentiable at x_0 with $\nabla w(x_0) = \nabla F(x_0 - I(x_0))$. Hence $\nabla v(x_0) = \nabla F(x_0 - I(x_0))$.

6.2. Dynamics. Now we turn to dynamics with a view to proving Theorem 4.1. Recall the definition of q in (15). With respect to this q we define q -small sets, aperiodicity and irreducibility.¹⁸

Definition 6.1. A set $C \in \mathcal{B}(\mathbb{R}_+^M)$ is called q -small if

$$(21) \quad \exists g \in L_1(\mathbb{R}_+^M) \text{ such that } g \geq 0, g \neq 0 \text{ and } \forall x \in C, q(x, \cdot) \geq g$$

If such set C exists and $\int_C g > 0$, then $(X_t)_{t \geq 0}$ is called aperiodic.¹⁹

Definition 6.2. The optimal process $(X_t)_{t \geq 0}$ is called irreducible if, $\forall x_0 \in \mathbb{R}_+^M$ and $\forall B \in \mathcal{B}(\mathbb{R}_+^M)$ with $\lambda(B) > 0$, the process $(X_t)_{t \geq 0}$ started at $X_0 \equiv x_0$ satisfies $\mathbb{P} \cup_{t \geq 1} \{X_t \in B\} > 0$.

Let $s \geq 1$, and let $V(x) := \|x\| + s$. By Meyn and Tweedie (1993), Theorem 16.1.2, if $(X_t)_{t \geq 0}$ is aperiodic, irreducible and possesses a q -small set C such that

$$(22) \quad \int V(y)q(x, y)dy \leq \gamma V(x) + b\mathbb{1}_C(x) \quad x \in \mathbb{R}_+^M$$

for some $\gamma < 1$ and $b < \infty$, then $(X_t)_{t \geq 0}$ is V -uniformly ergodic. In particular, there exists a unique stationary distribution (in this case a density) ψ^* ; the density ψ^* satisfies $\int V(x)\psi^*(x)dx < \infty$; and, moreover, there is a $\beta < 1$ and $N < \infty$ such that

$$(23) \quad \sup_{|h| \leq V} \left| \int h(y)\delta_x \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \leq \beta^t N V(x)$$

for all $t \in \mathbb{N}$ and all $x \in \mathbb{R}_+^M$. Here δ_x is the distribution concentrated at x , so that $\delta_x \mathbf{M}^t$ is the density of X_t when $X_0 \equiv x$.

Below we establish that $(X_t)_{t \geq 0}$ is V -uniformly ergodic for any choice of the constant $s \in [1, \infty)$. The conclusions of Theorem 4.1 then follow in a straightforward way. Parts (1) and (2) are immediate. To see that (3) is true, pick any $s \geq 1$ and any distribution ψ_0 such that $\int \|x\|\psi_0(dx)$ is finite. Now take any $h \in \mathcal{H}_s$. Consider the term

$$\begin{aligned} & \left| \int h(y)\psi_0 \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \\ &= \left| \int h(y) \left[\int \delta_x \mathbf{M}^t(y)\psi_0(dx) \right] dy - \int h(y)\psi^*(y)dy \right| \end{aligned}$$

¹⁸See Meyn and Tweedie (1993) for more details on these concepts.

¹⁹Our definitions of small sets and aperiodicity are slightly stronger than the standard definitions. See Meyn and Tweedie (1993, Chapter 5).

Since $|h| \leq V = \|\cdot\| + s$, equation (23) implies that the right hand side is dominated by

$$\begin{aligned} & \int \left| \int h(y) \delta_x \mathbf{M}^t(y) dy - \int h(y) \psi^*(y) dy \right| \psi_0(dx) \leq \beta^t N \int V(x) \psi_0(dx) \\ \therefore & \left| \int h(y) \psi_0 \mathbf{M}^t(y) dy - \int h(y) \psi^*(y) dy \right| \leq \beta^t N \left(\int \|x\| \psi_0(dx) + s \right) \end{aligned}$$

As h is an arbitrary element of \mathcal{H}_s the conclusion of Theorem 4.1 follows.

In summary, to prove Theorem 4.1, it is sufficient to establish that for any $s \geq 1$, the optimal process $(X_t)_{t \geq 0}$ is irreducible, aperiodic and possesses a small set C such that (22) holds for $V(x) = \|x\| + s$. We begin with irreducibility:

Proposition 6.1. *The optimal process $(X_t)_{t \geq 0}$ is irreducible.*

Proof. Fix $x_0 \in \mathbb{R}_+^M$ and $B \in \mathcal{B}(\mathbb{R}_+^M)$ with $\lambda(B) > 0$. Let $\mathbf{1} \in \mathbb{R}_+^M$ be the vector of ones. Evidently one can select a strictly positive scalar a with the property $\lambda([a\mathbf{1}, \infty) \cap B) > 0$. To prove Proposition 6.1 we need the following two lemmas concerning a .

Lemma 6.3. *If $x \in (0, a\mathbf{1}]$, then*

$$\int_B q(x, y) dy = \int_B \phi(y - \Lambda I(x)) dy > 0$$

Proof. For $y \in [a\mathbf{1}, \infty)$, the interiority of x implies that

$$\begin{aligned} y & \geq a\mathbf{1} \geq x \gg \Lambda x \geq \Lambda I(x) \\ \therefore \quad \phi(y - \Lambda I(x)) & > 0 \end{aligned}$$

Since $[a\mathbf{1}, \infty) \cap B$ has positive Lebesgue measure, it follows that

$$\begin{aligned} & \int_{B \cap [a\mathbf{1}, \infty)} \phi(y - \Lambda I(x)) dy > 0 \\ \therefore \quad \int_B \phi(y - \Lambda I(x)) dy & \geq \int_{B \cap [a\mathbf{1}, \infty)} \phi(y - \Lambda I(x)) dy > 0 \end{aligned}$$

□

Hence $X_t \in (0, a\mathbf{1}]$ implies $X_{t+1} \in B$ with positive probability.

Lemma 6.4. *For each $x_0 \in \mathbb{R}_+^M$ there is an $n \geq 0$ such that $\mathbb{P}\{X_n \in (0, a\mathbf{1}]\} > 0$, where $(X_t)_{t \geq 0}$ is the process starting at x_0 .*

Proof. Let $\|x\|_\infty := \max_{1 \leq m \leq M} x^m$ for any $x = (x^m)_{m=1}^M \in \mathbb{R}_+^M$. Note that $\|\Lambda x\|_\infty \leq \bar{\alpha} \|x\|_\infty$ holds for any x . Note also that $\|\cdot\|_\infty$ is consistent with the ordering on \mathbb{R}_+^M , in the sense that $x \leq y$ implies $\|x\|_\infty \leq \|y\|_\infty$.

Since $a > 0$ and $\bar{\alpha} < 1$, clearly we can choose an $n \geq 0$ and a $z_0 \gg 0$ such that

$$\bar{\alpha}^n \|x_0\|_\infty + \|z_0\|_\infty \frac{1}{1 - \bar{\alpha}} \leq a$$

Let E be the event $\cap_{t \leq n} \{W_t \leq z_0\}$. Evidently E has positive probability, so it suffices to prove that $X_n \leq a\mathbf{1}$ on E . To this end, observe that on E we have $X_t \leq \Lambda X_{t-1} + z_0$ for $t = 1, \dots, n$.

$$\begin{aligned} \therefore X_n &\leq \Lambda^n x_0 + \Lambda^0 z_0 + \Lambda^1 z_0 + \dots + \Lambda^{n-1} z_0 \\ \therefore \|X_n\|_\infty &\leq \bar{\alpha}^n \|x_0\|_\infty + \|z_0\|_\infty \frac{1}{1 - \bar{\alpha}} \leq a \\ \therefore X_n &\leq a\mathbf{1} \end{aligned}$$

The proof of Lemma 6.4 is now complete. \square

It remains to complete the proof of Proposition 6.1. Regarding irreducibility, fix $x_0 \in \mathbb{R}_+^M$ and let $(X_t)_{t \geq 0}$ be the process starting at x_0 . By Lemma 6.4 there is an $n \geq 0$ such that $\mathbb{P}\{X_n \in (0, a\mathbf{1}]\} > 0$. We show that $\mathbb{P}\{X_{n+1} \in B\} > 0$, which implies irreducibility. Note that

$$\begin{aligned} \mathbb{P}\{X_{n+1} \in B\} &\geq \mathbb{P}[\{X_n \in (0, a]\} \cap \{X_{n+1} \in B\}] \\ &= \mathbb{P}[\mathbb{P}[\{X_n \in (0, a]\} \cap \{X_{n+1} \in B\} \mid \mathcal{F}_n]] \end{aligned}$$

and

$$\mathbb{P}[\{X_n \in (0, a]\} \mathbb{P}[\{X_{n+1} \in B\} \mid \mathcal{F}_n]] = \mathbb{P}\left[\{X_n \in (0, a]\} \int_B q(X_n, y) dy\right]$$

where $\mathcal{F}_t := \sigma(W_1, \dots, W_t)$. This last term is strictly positive, because $\mathbb{P}\{X_n \in (0, a]\} > 0$ by the definition of n , and on $\{X_n \in (0, a]\}$ the integral is strictly positive (Lemma 6.3). The proof is done. \square

Next we address the existence of small sets.

Lemma 6.5. *All bounded Borel measurable subsets of \mathbb{R}_+^M are q -small.*

Proof. Since measurable subsets of small sets are small, it suffices to prove that all sets of the form $C = [0, c]$, $c \gg 0$, are q -small. Pick any $c \gg 0$ and set $C := [0, c]$. Let α' be any number satisfying $\bar{\alpha} < \alpha' < 1$, and set

$$K := \{(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^M : x \in C, \alpha' c \leq y \leq c\}$$

For $(x, y) \in K$ we have

$$y - \Lambda I(x) \geq y - \Lambda x \geq y - \Lambda c \geq \alpha' c - \Lambda c \gg 0$$

Since $\phi(z) > 0$ whenever $z \gg 0$, it follows that $\phi(y - \Lambda I(x)) > 0$. Combining this observation with the compactness of K and the continuity of ϕ it follows that $\epsilon := \min\{\phi(y - \Lambda I(x)) : (x, y) \in K\}$ is strictly positive.

Let $g := \epsilon \mathbb{1}_{[\alpha' c, c]}$. Since $\epsilon > 0$, $c \gg 0$ and $\alpha' < 1$, the function g is nontrivial. We claim that g satisfies (21). To see this, pick any $x \in C$. If $y \notin [\alpha' c, c]$ then $g(y) = 0$, and (21) must hold. On the other hand, if $y \in [\alpha' c, c]$, then $(x, y) \in K$, and, by the definition of ϵ , $\phi(y - \Lambda I(x)) \geq \epsilon = g(y)$. Either way we have $q(x, y) = \phi(y - \Lambda I(x)) \geq g(y)$ as claimed. \square

For this C and g we have $\int_C g(y) dy = \epsilon \lambda([\alpha' c, c]) > 0$, implying $(X_t)_{t \geq 0}$ is aperiodic. To prove Theorem 4.1, then, it remains only to show that there exists a q -small set C such that

$$\int V(y) q(x, y) dy \leq \gamma V(x) + b \mathbb{1}_C(x) \quad x \in \mathbb{R}_+^M$$

for some $\gamma < 1$ and $b < \infty$, where $V(x) = \|x\| + s$. Using the change of variable $z = y - \Lambda I(x)$,

$$\int V(y) q(x, y) dy = \int V(y) \phi(y - \Lambda I(x)) dy = \int V(\Lambda I(x) + z) \phi(z) dz$$

From $V(x) = \|x\| + s$ this gives

$$\begin{aligned} \int V(y) q(x, y) dy &= \int \|\Lambda I(x) + z\| \phi(z) dz + s \\ &\leq \bar{\alpha} \|x\| + \int \|z\| \phi(z) dz + s \\ &\leq \bar{\alpha} V(x) + b, \quad b := \int \|z\| \phi(z) dz + s \end{aligned}$$

The constant b is finite by Assumption 2.3.

Let γ be any number in $(\bar{\alpha}, 1)$. Choose a vector $c \gg 0$ such that

$$x \not\leq c \implies \bar{\alpha} + \frac{b}{V(x)} \leq \gamma$$

It follows that if $x \not\leq c$, then

$$\int V(y) q(x, y) dy \leq \bar{\alpha} V(x) + b \leq \gamma V(x)$$

Defining $C := [0, c]$ now gives

$$\int V(y) q(x, y) dy \leq \gamma V(x) + b \mathbb{1}_C(x) \quad x \in \mathbb{R}_+^M$$

as required. As C is q -small (Lemma 6.5) the proof is done.

Finally, let us turn to the proofs of Corollary 4.1 and Corollary 4.2.

Proof of Corollary 4.1. The proof is almost trivial: Let $x_0 \in \mathbb{R}_0^M$, and let $\psi_0 = \delta_{x_0}$. Evidently the conditions of Theorem 4.1 part (3) hold, and

$$\sup_{h \in \mathcal{H}_1} \left| \int h(x) \psi_0 \mathbf{M}^t(x) dx - \int h(x) \psi^*(x) dx \right| = O(\beta^t)$$

for some $\beta \in (0, 1)$. Since \mathcal{H}_1 contains all Borel measurable real-valued functions h with $|h| \leq 1$, it follows from (19) that $d_1(\psi_t, \psi^*) = O(\beta^t)$. \square

Proof of Corollary 4.2. Since $(X_t)_{t \geq 0}$ has been shown to be V -uniformly ergodic, both the LLN and the CLT results are immediate from Meyn and Tweedie (1993, Theorem 17.0.1). \square

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