Automatic Differentiation Methods for Analyzing Wealth-Consumption Ratios and Stochastic Discount Factors

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ABSTRACT. To be written

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1. INTRODUCTION

To be added. Background on Newton's algorithm in $\mathbb{R}^N$.

Let f be a smooth map from $\mathbb{R}^N$ to itself. We want to find the $x \in \mathbb{R}^N$ that solves $f(x) = x$. Ordinary successive approximation uses

$$x_{k+1} = f(x_k) \tag{1}$$

Newton's method first sets $g(x) = f(x) - x$, so that we are seeking a root x satisfying $g(x) = 0$, and then iterates on

$$x_{k+1} = g(x_k) + J(x_k)^{-1} g(x_k) \tag{2}$$

where $J(x)$ is the Jacobian of g at x.

2. A LONG RUN RISK MODEL

Roadmap to be added.

To be added

Consumption growth and the growth rate of the preference shock are given by the generic formulas

$$g_{c,t+1} = g_c(X_t, X_{t+1}, \xi_{t+1}) \quad \text{and} \quad g_{\lambda,t+1} = g_\lambda(X_t, X_{t+1}, \xi_{t+1}), \tag{3}$$

where $\{X_t\}_{t \geqslant 0}$ is a discrete time Markov process on $\mathsf{X} \subset \mathbb{R}^d$, $\{\xi_t\}_{t \geqslant 1}$ is an IID process supported on $\mathbb{Y} \subset \mathbb{R}^k$, and $g_i \colon \mathsf{X} \times \mathsf{X} \times \mathbb{Y} \to \mathbb{R}$ is continuous for each $i \in \{c, \lambda\}$. The processes $\{X_t\}$ and $\{\xi_t\}$ are assumed to be independent.

2.1. **The SSY Model.** In the long run risk model of Schorfheide et al. (2018), the state process takes the form

$$X_t := (h_{\lambda,t}, h_{c,t}, h_{z,t}, z_t)$$

where, for $i \in \{z, c, \lambda\}$,

$$\begin{aligned} h_{i,t+1} &= \rho_i \, h_{i,t} + s_i \, \eta_{i,t+1} \\ \sigma_{i,t} &= \varphi_i \, \bar{\sigma} \, \exp(h_{i,t}), \\ z_{t+1} &= \rho \, z_t + (1 - \rho^2)^{1/2} \, \sigma_{z,t} \, \varepsilon_{t+1} \end{aligned}$$

Consumption growth is given by

$$g_{c,t+1} = \ln \frac{C_{t+1}}{C_t} = \mu_c + z_t + \sigma_{c,t} \, \xi_{c,t+1}. \tag{4}$$

The preference shock λ_t grows as

$$g_{\lambda,t+1} = \ln \frac{\lambda_{t+1}}{\lambda_t} = h_{\lambda,t+1}.$$

The innovations

$$\xi_{c,t}, \quad \varepsilon_t, \text{ and } (\eta_{i,t})_{i \in \{z,c,\lambda\}}$$

are all independent and standard normal.

Let $\mathbb{H}$ be the linear operator defined by

$$(\mathbb{H}g)(x) = \mathbb{E}_x \, g(X_{t+1}) \, \exp\left[\theta g_{\lambda,t+1} + (1 - \gamma) g_{c,t+1}\right] \tag{5}$$

at each $x \in \mathsf{X}$, where $\mathbb{E}_x$ conditions on $X_t = x$.

From Schorfheide et al. (2018), the stochatic discount factor process (M_t) takes the form

$$M_{t+1} = \beta^\theta \left(\frac{\lambda_{t+1}}{\lambda_t} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^{\theta - 1}, \tag{6}$$

where $w(X_t)$ is the aggregate wealth-consumption ratio $w(X_t) := W_t/C_t$.

In the model, the this ratio obeys

$$\beta^\theta \mathbb{E}_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^\theta \right] = 1$$

Rearranging the previous expression gives

$$(w(X_t) - 1)^\theta = \beta^\theta \mathbb{E}_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} w(X_{t+1})^\theta \right].$$

Conditioning on $X_t = x$, writing pointwise on X and using the definition of $\mathbb{H}$ yields $w = 1 + \beta \left(\mathbb{H} w^\theta\right)^{1/\theta}$. A function w solves this equation if and only if w is a fixed point of the operator $\mathbb{T}$ defined by

$$(\mathbb{T} w) = 1 + \beta \, (\mathbb{H} w^\theta)^{1/\theta}. \tag{7}$$

For computation, the state process is distcretized onto a state space $S = \{x_1, \ldots, x_N\}$ of size $N \in \mathbb{N}$ and the operator $\mathbb{H}$ is represented by a matrix $\mathbf{H}$, with

$$\mathbf{H}(n, n') = \sum_{n'=1}^{N} \exp\left\{\theta g_\lambda(x_n, x_{n'}, \xi) + (1-\gamma) g_c(x_n, x_{n'}, \xi)\right\} \mathbf{P}(n, n'), \tag{8}$$

where $\mathbf{P}$ is an $N \times N$ matrix with $\mathbf{P}(n, n')$ representing the probability that the discretized state process transitions from state x_n to state $x_{n'}$ in one unit of time.

The discretization of $\mathbb{T}$ is written as $\mathbf{T}$ and the problem is to find the fixed point of

$$(\mathbf{T} w) = 1 + \beta \, (\mathbf{H} w^\theta)^{1/\theta} \tag{9}$$

in the set of strictly positive vectors in $\mathbb{R}^N$.

REFERENCES

SCHORFHEIDE, F., D. SONG, AND A. YARON (2018): "Identifying long-run risks: A Bayesian mixed-frequency approach," *Econometrica*, 86, 617–654.