SEMIGROUP NOTES

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In what follows,

- E denotes a real Banach space,
- $\mathcal{L}(E)$ is the set of bounded linear operators on E,
- E' represents the dual space (i.e., the set of bounded linear functionals on E),
- $\|\cdot\|$ denotes either the norm on E or the operator norm on $\mathcal{L}(E)$, depending on context, and
- $\mathscr{C}$ is the set of continuous functions from $\mathbb{R}_+$ to E.

1. Linear Differential Equations and Exponential Paths

This section contains a brief review of linear ordinary differential equations (ODEs) in Banach space when the linear operator that defines the ODE is a bounded linear operator.

To set the stage, recall that an (autonomous) linear ODE in $\mathbb{R}^n$ has the form

$$\dot{x}(t) = Ax(t)$$

where $x(t)$ is an n-vector for each t and A is $n \times n$. The matrix A is sometimes called the "vector field." A first step to extend these concepts to an abstract Banach space is to set $\dot{x}(t) = Ax(t)$, where $x(t)$ is understood as a point in the space and A is a *bounded* linear operator from the space to itself. (This includes the finite-dimensional case, including the one-dimensional case as taught in elementary lectures on ODEs, since all linear operators on finite-dimensional spaces are bounded. The derivative $\dot{x}$ is defined below.)

It turns out that the boundedness restriction leaves out many interesting and important systems. On the other, it illustrates the ideal case, and highlights some of the properties we hope to discover for the solutions of general ODEs (driven by possibly unbounded linear operators) in Banach space.

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1.1. **Preliminaries.** First we recall some facts concerning calculus for Banach space valued functions. To this end, let u be an element of $\mathscr{C}$. If, for some $t > 0$, the limit

$$\dot{u}(t) := \frac{\mathrm{d}}{\mathrm{d}t} u(t) := \lim_{h \to 0} \frac{\|u(t+h) - u(t)\|}{h}$$

exists, then we say that u is differentiable at t and call $\dot{u}(t)$ the derivative of u at t. In the case where $t = 0$ we use the same definitions after replacing $h \to 0$ with $h \downarrow 0$. We say that u is differentiable on $\mathbb{R}_+$ if u is differentiable at t for all $t \in \mathbb{R}_+$.

Next we define an integral for continuous Banach space valued functions. Fix $u \in \mathscr{C}$. the (Riemann) integral of u over $[a, b]$ is the unique point $I \in E$ such that

$$\int_a^b \langle u(s), u' \rangle \, \mathrm{d}s = \langle I, u' \rangle \qquad \text{for all } u' \in E' \tag{1}$$

In (1), the left hand side is an ordinary Riemann integral for real-valued functions. We write $I = \int_a^b u(s)\, \mathrm{d}s$, so that $\int_a^b \langle u(s), u' \rangle \, \mathrm{d}s = \left\langle \int_a^b u(s)\, \mathrm{d}s, u' \right\rangle$ for all $u' \in E$. A proof of existence and uniqueness, as well as the properties listed below, can be found in Section 5.1.2 of Bühler and Salamon (2018).

Lemma 1.1. *If u is continuous on $\mathbb{R}_+$ and $a, b \in \mathbb{R}_+$ with $a < b$, then*

(i) $\int_a^c u(s)\, \mathrm{d}s = \int_a^b u(s)\, \mathrm{d}s + \int_b^c u(s)\, \mathrm{d}s$ *whenever $a < b < c$,*
(ii) $\int_a^b u(s + c)\, \mathrm{d}s = \int_{a+c}^{b+c} u(s)\, \mathrm{d}s$ *whenever $c \geqslant 0$,*
(iii) $\| \int_a^b u(s)\, \mathrm{d}s \| \leqslant \int_a^b \|u(s)\|\, \mathrm{d}s$,
(iv) $u(t) = \lim_{h \to 0} (1/h) \int_t^{t+h} u(s)\, \mathrm{d}s$ *for all $t \in \mathbb{R}_+$, and*
(v) $L \in \mathcal{L}(E)$ *implies* $L \int_a^b u(s)\, \mathrm{d}s = \int_a^b Lu(s)\, \mathrm{d}s$.

If, in addition, u is continuously differentiable on $[a, b]$ with derivative $\dot{u}$, then

$$u(b) = u(a) + \int_a^b \dot{u}(s)\, \mathrm{d}s. \tag{2}$$

1.2. **Exponentials of Linear Operators.** The scalar exponential of $a \in \mathbb{R}$ can be defined by $\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!}$. Analogously, the exponential of $A \in \mathcal{L}(E)$ is given by

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \cdots \tag{3}$$

We list some properties of the exponential below.

Lemma 1.2. *Let A and B be elements of $\mathcal{L}(E)$. The following properties hold:*

(i) $\exp(A)$ *is a well-defined element of $\mathcal{L}(E)$ and* $\| \exp(A) \| \leqslant \exp(\|A\|)$.

(ii) *If* $AB = BA$, *then* $\exp(A + B) = \exp(A)\exp(B)$.

(iii) $\exp(0) = I$.

It is immediate from (ii) that if m is any positive integer, then $\exp(mA) = (\exp(A))^m$. The next lemma considers exponential flows in $\mathcal{L}(E)$.

Lemma 1.3. *Fix* $A \in \mathcal{L}(E)$ *and consider the function* $e(t) := \exp(tA)$ *from* $\mathbb{R}$ *to* $\mathcal{L}(E)$. *The following statements are true:*

(i) *The map* e *is differentiable in* t, *with*

$$\dot{e}(t) = Ae(t) = e(t)A. \tag{4}$$

(ii) *The fundamental theorem of calculus holds, in the sense that*

$$e(b) = e(a) + \int_a^b \dot{e}(s)\,\mathrm{d}s \quad \textit{for all } a \leqslant b. \tag{5}$$

For example, to prove (i) of Lemma 1.3, we fix $A \in \mathcal{L}(E)$ and consider $e(t) = \exp(tA)$. Applying (ii) of Lemma 1.2 yields

$$\frac{e(t+h) - e(t)}{h} = \frac{\exp(tA + hA) - \exp(tA)}{h} = \exp(tA)\frac{\exp(hA) - I}{h}$$

Using the definition (3) and taking the limit as $h \downarrow 0$ gives $\dot{e}(t) = e(t)A$. A small variation on the argument shows that $\dot{e}(t) = Ae(t)$ also holds.

1.3. **Initial Value Problems.** Now fix $A \in \mathcal{L}(E)$ and consider the initial value problem (IVP) $\dot{u}(t) = Au(t)$ for all $t \geqslant 0$ with given initial condition $u(0) = u_0 \in E$. A solution to this problem is $u \in \mathscr{C}$ that is differentiable and obeys the IVP conditions.

Proposition 1.4. *Given* $A \in \mathcal{L}(E)$, *the function* u *defined by* $u(t) = \exp(tA)u_0$ *is the unique solution to the IVP stated above.*

That u solves the IVP is immediate from the properties in Lemmas 1.2–1.3. We prove uniqueness in a more general setting below.

Proposition 1.4 gives a complete description of the solution of ODEs in Banach space when the vector field A is a bounded linear operator. As well as studying flows of the form $t \mapsto \exp(tA)u_0$ in E we can also directly analyze the dynamics of flows $t \mapsto \exp(tA)$ in $\mathcal{L}(E)$. We notice that these flows are continuous (in fact differentiable, by Lemma 1.3) and obey $\exp((s + t)A) = \exp(tA)\exp(sA)$ for all $s, t \geqslant 0$. The latter property is called the *semigroup* property of the flow.

In order to generalize these ideas and handle ODEs with unbounded vector fields, we now study a more general notion of continuous semigroups in $\mathcal{L}(E)$.

2. Introduction to Semigroups

Add roadmap.

2.1. **Definitions.** Let $(S_t) := (S_t)_{t \geqslant 0}$ be a family of linear operators in $\mathcal{L}(E)$ with index $t \in \mathbb{R}_+$. (S_t) is called an algebraic operator (AO) semigroup if S_0 is the idendity and (S_t) has the semigroup property

$$S_{s+t} = S_t \circ S_s \quad \text{for all } s, t \in \mathbb{R}_+.$$

If, in addition, $t \mapsto S_t u$ is continuous for all $u \in E$, then (S_t) is called a C_0-semigroup. When E is understood, we say that (S_t) is a C_0-semigroup. Given $u \in E$, the function $t \mapsto S_t u$ is called the orbit or trajectory of u under (S_t). The point u is called the initial condition.

2.2. **Continuity Results.** Let K be a compact subset of $\mathbb{R}$ and let $\{S_t\}_{t \in K}$ be a subset of $\mathcal{L}(E)$. The following result is from Engel and Nagel (2006).

Lemma 2.1. *The following statements are equivalent:*

(i) *The map $t \mapsto S_t u$ is continuous on K for all $u \in E$.*

(ii) *$\|S_t\|$ is bounded over $t \in K$ and there exists a dense subset D of E such that $t \mapsto S_t u$ is continuous on K for all $u \in D$.*

(iii) *For any compact $C \subset E$, the map $(t, u) \mapsto S_t u$ is uniformly continuous on $K \times C$.*

Proof. ((i) $\implies$ (ii)) By (i), for any $u \in E$, the map $t \mapsto S_t u$ is continuous on a compact set and, therefore, its image is bounded in E. Hence, by the uniform boundedness principle, $\|S_t\|$ is bounded over $t \in K$. The statement in (ii) regarding continuity is obvious.

((ii) $\implies$ (iii)). Fix compact $C \subset E$ and $\varepsilon > 0$. We metrize $K \times C$ by setting $d((s,u),(t,v)) = \|u - v\| \vee |s - t|$. Choose $M \in \mathbb{N}$ such that $\|S_t\| \leqslant M$ for all $t \in K$. Let D be the dense set in (ii) and observe that the set of open balls $B(u, \varepsilon/M)$ over $u \in D$ provides an open cover of C. As such, we can choose a finite set $D_F \subset D$ such that C is contained in $\cup_{u \in D_F} B(u, \varepsilon/M)$. Since, for each $u \in D_F$, the map $t \mapsto S_t u$ is continuous on a compact set, it is also uniformly continuous. As a result, given $u \in D_F$, we can select a $\delta_u > 0$ such that

$$|s - t| < \delta_u \implies \|S_s u - S_t u\| < \varepsilon.$$

Let δ be the minimum of $\{\delta_u\}_{u\in D_F}$ and ε/M. If we take $u,v \in C$ and $s,t \in K$ with $d((s,u),(t,v)) < \delta$, then, choosing $w \in D_F$ with $\|u-w\| < \varepsilon/M$, we have

$$\begin{aligned}\|S_s u - S_t v\| &\leqslant \|S_s u - S_s w\| + \|S_s w - S_t w\| + \|S_t w - S_t v\| \\ &= \|S_s(u-w)\| + \|S_s w - S_t w\| + \|S_t(w-v)\| \\ &< M(\varepsilon/M) + \varepsilon + M(2\varepsilon/M) = 4\varepsilon.\end{aligned}$$

Hence $(t,u) \mapsto S_t u$ is uniformly continuous on $K \times C$, as claimed.

((iii) $\implies$ (i)) This implication is trivial (take C to be a singleton). □

Lemma 2.2. *If $(S_t)_{t\geqslant 0}$ is a C_0-semigroup on E, then $\sup_{t\leqslant\delta}\|S_t\| < \infty$ for all $\delta > 0$.*

Proof. We first claim there exists an $\varepsilon > 0$ such that $\sup_{t\leqslant\varepsilon}\|S_t\| < \infty$. Indeed, if no such ε exists, then there exists a sequence $t_n \to 0$ such that $\|S_{t_n}\|$ is unbounded. But then, by the principle of uniform boundedness, there exists a $u \in E$ such that $\|S_{t_n}u\|$ is unbounded. This contradicts the continuity property of C_0-semigroups.

Now let ε be as above and choose $M \in \mathbb{N}$ with $\|S_t\| \leqslant M$ whenever $t \leqslant \varepsilon$. Fix $k \in \mathbb{N}$ and $t \leqslant k\varepsilon$. Since S_t is k compositions of $S_{t/k}$, and since $t/k < \varepsilon$, the semigroup property yields $\|S_t\| \leqslant kM$. Hence $t \mapsto S_t$ is bounded on $[0, k\varepsilon]$. Since k was an arbitrary element of $\mathbb{N}$, this proves the claim in Lemma 2.2. □

Lemma 2.3. *If (S_t) is an AO semigroup on E, then the following statements are equivalent:*

(i) *(S_t) is a C_0-semigroup on E.*
(ii) *$\lim_{t\downarrow 0} S_t u = u$ for all $u \in E$.*

Proof. That (i) implies (ii) is obvious. For the reverse implication, fix $u \in E$ and $t > 0$. We need to show that $\|S_{t+h}u - S_t u\| \to 0$ as $h \to 0$. Suppose first that $h \downarrow 0$. Then

$$\|S_{t+h}u - S_t u\| = \|S_t S_h u - S_t u\| \leqslant \|S_t\|\|S_h u - u\| \to 0.$$

If, on the other hand $h \uparrow 0$, then

$$\|S_{t+h}u - S_t u\| = \|S_{t+h}u - S_{t+h}S_{-h}u\| \leqslant \|S_{t+h}\|\|u - S_{-h}u\| \to 0.$$

In the last step we used the fact that $\|S_{t+h}\|$ is bounded over h by Lemma 2.2. □

Lemma 2.4. *Let $(S_t)_{t\geqslant 0}$ be an AO semigroup on E. If there exists a dense subset D of E such that $\lim_{t\downarrow 0} S_t u = u$ for all $u \in D$ and, in addition, $\sup_{t\leqslant\delta}\|S_t\| < \infty$ for some $\delta > 0$, then $(S_t)_{t\geqslant 0}$ is a C_0-semigroup.*

Proof. Fix $u \in E$. By Lemma 2.3 it suffices to show that, for a given sequence $t_n \downarrow 0$, we have $S_{t_n} u \to u$ as $n \to 0$. To see that this holds, fix $t_n \downarrow 0$ and choose a compact subset K of $\mathbb{R}_+$ such that $\{t_n\} \subset K$. Since K is compact, $K \ni t \mapsto S_t w$ is continuous when $w \in D$, and $\|S_t\|$ is bounded over $t \in K$, Lemma 2.1 implies that $K \ni t \mapsto S_t u$ is continuous. In particular, $S_{t_n} u \to u$ as $n \to 0$. □

2.3. **Examples.**

2.3.1. *Left-Shift Semigroups.* Let $C_0(\mathbb{R}_+)$ be the set of all continuous real-valued functions f on $\mathbb{R}_+$ with $f(x) \to 0$ as $x \to \infty$. The set $C_0(\mathbb{R}_+)$ is paired with the supremum norm. Consider the left translation semigroup given by $(S_t f)(x) = f(x+t)$.

Lemma 2.5. *(S_t) is a C_0-semigroup on $C_0(\mathbb{R}_+)$.*

Proof. Evidently $S_0 f = f$. The semigroup property holds because, for $s, t \geqslant 0$, we have

$$(S_{s+t} f)(x) = f(x+s+t) = (S_t(S_s f))(x).$$

Regarding continuity, fix $f \in C_0(\mathbb{R}_+)$ and let (t_n) be a real sequence with $t_n \downarrow 0$. Fix $\varepsilon > 0$. Since f is uniformly continuous, we can select a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Let $N \in \mathbb{N}$ be such that $t_n < \delta$ when $n \geqslant N$. Then, for $n \geqslant N$,

$$\|S_{t_n} f - f\| = \sup_x |f(x+t_n) - f(x)| < \varepsilon.$$

Hence $S_t f \downarrow f$ and (S_t) is a C_0-semigroup. □

Let $C_0^1(\mathbb{R}_+)$ be the set of all continuously differentiable $f \in C_0(\mathbb{R}+)$ with $f' \in C_0(\mathbb{R}+)$. The set $C_0^1(\mathbb{R}_+)$ is paired with the norm $\|f\| = \sup_x |f(x)| + \sup_x |f'(x)|$.

Lemma 2.6. *(S_t) is a C_0-semigroup on $C_0^1(\mathbb{R}_+)$.*

Proof. In view of Lemma 2.5, we only need to check continuity. Fixing $f \in C_0^1(\mathbb{R}_+)$, we have

$$\|S_t f - f\| = \sup_x |f(x+t) - f(x)| + \sup_x |f'(x+t) - f'(x)|$$

Since f and f' are both in $C_0(\mathbb{R}_+)$, the proof of Lemma 2.5 implies that both terms on the right hand side converge to zero as $t \downarrow 0$. Hence continuity holds. □

2.3.2. *Right-Shift Semigroups.* Here we discuss right-shift semigroups. We will embed them in a space of integrable functions. Below λ denotes Lebesgue measure.

Let $C_c(\mathbb{R})$ be the set of all continuous real-valued functions f on $\mathbb{R}$ that vanish off a compact set. Let $L_1(\mathbb{R})$ be the set of Borel measurable real-valued functions on $\mathbb{R}$ with $\|f\| := \int |f| \, \mathrm{d}\lambda < \infty$. Let S_t be the linear operator on $L_1(\mathbb{R})$ defined by

$$(S_t f)(x) = f(x+t) \qquad (x \in \mathbb{R}, \; t \geqslant 0).$$

Lemma 2.7. *(S_t) is a C_0-semigroup on $L_1(\mathbb{R})$.*

Proof. It is simple to confirm that (S_t) is an AO semigroup on $L_1(\mathbb{R})$. Regarding continuity, note that $\|S_t f\| = \int |f(x-t)| \, \mathrm{d}x = \|f\|$, so $\|S_t\|$ is bounded in t. Since $C_c(\mathbb{R})$ is dense in $L_1(\mathbb{R})$ under this norm, Lemma 2.4 implies that, to show (S_t) is a C_0-semigroup on $L_1(\mathbb{R})$, it suffices to show that $\|S_t f - f\| \to 0$ for any $f \in C_c(\mathbb{R})$.

To this end, fix $f \in C_c(\mathbb{R})$ and let K be a compact set such that f vanishes off K. Fix $\varepsilon > 0$. By uniform continuity, we can take a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon / \lambda(K)$. If $t < \delta$, then

$$\|S_t f - f\| = \int |f(x-t) - f(x)| \, \mathrm{d}x \leqslant \lambda(K) \frac{\varepsilon}{\lambda(K)} = \varepsilon.$$

This completes the proof of C_0-continuity of (S_t) on $L_1(\mathbb{R})$. □

2.3.3. *Multiplication Semigroups.* Let $(\mathsf{X}, \mathcal{B}, \mu)$ be a σ-finite measure space and let φ be a measurable map from X to $\mathbb{R}_+$. Define

$$S_t f = \exp(-t\varphi(x)) f(x) \qquad (x \in \mathsf{X}, \; t \geqslant 0).$$

The family (S_t) is called a multiplication semigroup.

Lemma 2.8. *(S_t) is a C_0-semigroup on $L_1(\mathsf{X}, \mathcal{B}, \mu)$.*

Proof. It is simple to confirm that (S_t) is an AO semigroup on $L_1(\mathbb{R})$. Regarding continuity, fix $f \in L_1(\mathsf{X}, \mathcal{B}, \mu)$ and observe that

$$\|S_t f - f\| = \int |f(x)| |\exp(-t\varphi(x)) - 1| \mu(\mathrm{d}x).$$

It follows from the dominated convergence theorem that this integral converges to zero as $t \downarrow 0$. Hence (S_t) is a C_0-semigroup on $L_1(\mathsf{X}, \mathcal{B}, \mu)$. □

2.3.4. *Uniformly Continuous Semigroups.* Fixing $A \in \mathcal{L}(E)$, consider the family of linear operators on E given by

$$S_t u = \exp(tA)u \qquad (u \in E,\ t \geqslant 0)$$

We recall that the exponential function $\varphi(t) := \exp(tA)$

(i) obeys $\varphi(0) = I$ and $\varphi(s+t) = \varphi(t)\varphi(s)$ for all $s, t \in \mathbb{R}$; and

(ii) is continuous as a map from $\mathbb{R}$ to $\mathcal{L}(E)$.

From (i) we can easily confirm that (S_t) is an algebraic operator semigroup on E. Regarding continuity, (i) and (ii) imply that

$$\lim_{t \downarrow 0} \|S_t - I\| = 0. \tag{6}$$

It follows from (6) that (S_t) is a C_0-semigroup on E.

Any operator semigroup (S_t) on E obeying (6) is called a uniformly continuous semigroup. In fact no other examples exist:

Theorem 2.9. *If (S_t) is a uniformly continuous semigroup on E, then there exists an $A \in \mathcal{L}(E)$ such that $S_t u = \exp(tA)u$ for all $u \in E$ and $t \geqslant 0$.*

The proof of Theorem 2.9 can be found 2(b) of Engel and Nagel (2006).

3. Generators and Resolvents

Add roadmap.

3.1. Infinitesmial Generators.

Let E be a Banach space and let (S_t) be an AO semigroup on E. Set

$$\mathcal{D}(A) = \left\{ u \in E \text{ s.t. } \lim_{t \downarrow 0} \frac{S_t u - u}{t} \text{ exists} \right\} \quad \text{and} \quad Au = \lim_{t \downarrow 0} \frac{S_t u - u}{t} \text{ on } \mathcal{D}(A).$$

The map A is called the infinitesimal generator of (S_t) and $\mathcal{D}(A)$ is its domain.

Lemma 3.1. *The set $\mathcal{D}(A)$ is a linear subspace of E and A is linear on $\mathcal{D}(A)$.*

Proof. Fix $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathcal{D}(A)$. Let $w = \alpha u + \beta v$. Since

$$\frac{S_t w - w}{t} = \alpha \frac{S_t u - u}{t} + \beta \frac{S_t v - v}{t},$$

we see that $w \in \mathcal{D}(A)$ and, moreover, $Aw = \alpha A u + \beta A v$. □

The next result shows that any trajectory of (S_t) starting from a point $\bar{u}$ in $\mathcal{D}(A)$ obeys a linear differential equation evolving in $\mathcal{D}(A)$, with A as the "vector field."

Lemma 3.2. *If (S_t) is a C_0-semigroup and $u \in E$, then, for all $t \geqslant 0$,*

$$\int_0^t S_s\, u\, \mathrm{d}s \in \mathcal{D}(A) \quad \textit{and} \quad S_t\, u = u + A \int_0^t S_s\, u\, \mathrm{d}s. \tag{7}$$

Proof. We begin by proving (7). Fix $u \in E$ and $t \geqslant 0$. For $h > 0$, using properties of the integral described in Lemma 1.1, we have

$$\begin{aligned} S_h \int_0^t S_s\, u\, \mathrm{d}s - \int_0^t S_s\, u\, \mathrm{d}s &= \int_0^t S_{h+s}\, u\, \mathrm{d}s - \int_0^t S_s\, u\, \mathrm{d}s \\ &= \int_h^{t+h} S_s\, u\, \mathrm{d}s - \int_0^t S_s\, u\, \mathrm{d}s = \int_t^{t+h} S_s\, u\, \mathrm{d}s - \int_0^h S_s\, u\, \mathrm{d}s \end{aligned}$$

Multiplying by $(1/h)$ and taking the limit as $h \downarrow 0$ gives $A \int_0^t S_s\, u\, \mathrm{d}s = S_t u - u$. Thus (7) is verified. □

Proposition 3.3. *If (S_t) is a C_0-semigroup, then $\mathcal{D}(A)$ is dense in E. Moreover, for any $u \in \mathcal{D}(A)$ and $t \geqslant 0$,*

(i) $S_t u \in \mathcal{D}(A)$,

(ii) *the map $s \mapsto S_s u$ is differentiable on $\mathbb{R}_+$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t u = A S_t u = S_t A u, \quad \textit{and} \tag{8}$$

(iii) $S_t u = u + \int_0^t S_s A u\, \mathrm{d}s$ *for all $u \in \mathcal{D}(A)$ and $t \geqslant 0$.*

Proof. To see that $\mathcal{D}(A)$ is dense, consider $v_h := (1/h) \int_0^h S_s\, u\, \mathrm{d}s$ with $h \geqslant 0$. By Lemma 1.1, $v_h \to u$ as $h \downarrow 0$. Moreover, (7) implies that v_h is in $\mathcal{D}(A)$ for all $h > 0$. Hence $\mathcal{D}(A)$ is dense in E.

Regarding (i), fix $t \geqslant 0$ and $u \in \mathcal{D}(A)$. We have

$$A S_t u = \lim_{h \downarrow 0} \frac{S_h S_t u - S_t u}{h} = \lim_{h \downarrow 0} \frac{S_t S_h u - S_t u}{h} = \lim_{h \downarrow 0} S_t \frac{S_h u - u}{h} = S_t A u,$$

where the last equality is by continuity of S_t (i.e., $S_t \in \mathcal{L}(E)$). This proves both (i) and the second equality in (8).

We now show that $s \mapsto S_s u$ is differentiable at $t \in \mathbb{R}_+$ with derivative $S_t A u$, which will complete the proof of (ii). We choose $t > 0$ because the case $t = 0$ is trivial. The right derivative of $S_t u$ is

$$\lim_{h \downarrow 0} \frac{S_{t+h} u - S_t u}{h} = \lim_{h \downarrow 0} \frac{S_t S_h u - S_t u}{h}$$

which has already been shown to equal $S_t A u$.

Regarding the left-hand derivative, we take $0 < h < t$ and use

$$\frac{S_{t-h}u - S_t u}{-h} - S_t Au = S_{t-h}\left(\frac{u - S_h u}{-h} - S_h Au\right)$$

and the boundedness of $\|S_t\|$ over bounded sets (Lemma 2.2) to obtain a finite M such that

$$\left\|\frac{S_{t-h}u - S_t u}{-h} - S_t Au\right\| \leqslant M \left\|\frac{u - S_h u}{-h} - Au\right\| + M \left\|Au - S_h Au\right\|.$$

Both terms on the right converge to zero in h, so we have proved that the left derivative is also to $S_t Au$. This completes the proof of (ii).

Regarding (iii), let $\varphi(s) = S_s u$ for all $s \geqslant 0$. We have shown in (ii) that φ is differentiable on $\mathbb{R}_+$ with $\dot{\varphi}(s) = S_s Au$. With this notation, the claim in (iii) can be expressed as $\varphi(t) = \varphi(0) + \int \dot{\varphi}(s)\, \mathrm{d}s$. Since $s \mapsto S_s Au$ is continuous in s, φ is also continuously differentiable. The claim in (iii) now follows from (2). □

The next result shows that semigroups are uniquely identified by their generators.

Proposition 3.4. *Let (S_t) and (T_t) be two C_0-semigroups on E with infinitesimal generators $(A, \mathcal{D}(A))$ and $(B, \mathcal{D}(B))$. If $(A, \mathcal{D}(A)) = (B, \mathcal{D}(B))$, then $(S_t) = (T_t)$.*

Proof. Let the statement hold, so that (S_t) and (T_t) are C_0-semigroups $(A, \mathcal{D}(A))$. Fix $t > 0$. We claim that $S_t = T_t$ on E. Since $\mathcal{D}(A)$ is dense in E, it suffices to show that S_t and T_t agree on $\mathcal{D}(A)$.

To this end, fix $u \in \mathcal{D}(A)$ and let $s \mapsto v(s)$ be a differentiable function on $\mathbb{R}_+$ with $\dot{v} = Av$ and $v(0) = u$. Define $w(s) := S_{t-s}v(s)$ for all $s \in (0, t)$. For h close to zero we have

$$\begin{aligned} w(s+h) - w(s) &= S_{t-s-h}v(s+h) - S_{t-s}v(s) \\ &= S_{t-s-h}(v(s+h) - v(s)) + S_{t-s-h}v(s) - S_{t-s}v(s) \end{aligned}$$

Dividing by h gives

$$\frac{w(s+h) - w(s)}{h} = S_{t-s-h}\frac{v(s+h) - v(s)}{h} - S_{t-s}\frac{S_{-h}v(s) - v(s)}{-h} \tag{9}$$

Consider the first term in (9). Letting

$$d(h) := \left\|S_{t-s-h}\frac{v(s+h) - v(s)}{h} - S_{t-s}Av(s)\right\|$$

and, choosing M suitably large, we have

$$\begin{aligned} d(h) &\leqslant \|S_{t-s-h}\| \left\| \frac{v(s+h)-v(s)}{h} - S_h A v(s) \right\| \\ &\leqslant M \left\| \frac{v(s+h)-v(s)}{h} - Av(s) \right\| + M\|Av(s) - S_h A v(s)\| \end{aligned}$$

This confirms that $d(h) \to 0$ as $h \to 0$, which means that the first term in (9) converges to $S_{t-s}Av(s)$.

The second term also converges to $S_{t-s}Av(s)$. Hence w is differentiable on $(0,t)$ with $\dot{w} = 0$. It follows from (2) that $w(s) = w(0) = S_t u$ for all $s < t$. Hence $S_{t-s}v(s) = S_t u$ for all $s < t$. Taking $s \to t$ and applying continuity gives $v(t) = S_t u$.

To complete the proof we consider the case $v(s) = T_s u$. According to Proposition 3.3, the function v is differentiable on $\mathbb{R}_+$ with $\dot{v} = Av$ and $v(0) = v$. Hence $T_t u = S_t u$. This confirms that S_t and T_t agree on $\mathcal{D}(A)$. □

3.2. **Implications for IVPs.** Let's now translate the results above in to findings for IVPs with potentially unbounded generators. We confirm that many of the results from the bounded case (see §1.3) either carry over or have direct analogs with the general case.

In the next theorem, $C_1(\mathbb{R}_+, E)$ is the set of continuously differentiable functions from $\mathbb{R}_+$ to E.

Theorem 3.5. *Let (S_t) be a C_0-semigroup with infinitesimal generator $(A, \mathcal{D}(A)$ and let u_0 be a point in $\mathcal{D}(A)$. The unique solution in $C(\mathbb{R}_+, E)$ of the initial value problem $\dot{u} = Au$ with $u(0) = u_0$ is the function defined by $u(t) := S_t u_0$. Moreover, $u(t)$ is continuously differentiable on $\mathbb{R}_+$ and*

$$\dot{u}(t) = AS_t u = S_t A u_0.$$

Proof. Most of the claims in Theorem 3.5 follow directly from Proposition 3.3. The only missing components is uniqueness. To prove it, fix $u_0 \in \mathcal{D}(A)$ and let v be a differentiable function from $\mathbb{R}_+$ to E with $\dot{v}(t) = Av(t)$ for all t and $v(0) = u_0$. We already showed in the proof of Proposition 3.4 that for any such function we have $v(t) = S_t u_0 = u(t)$. Hence uniqueness also holds. □

References

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