SEMIGROUP NOTES

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1. Definitions

Let E be any set and let $\mathbb{T} := (T_t)_{t \geqslant 0}$ be a family of self-maps on E. The pair $(E, \mathbb{T})$ is called a semidynamical system if T_0 is the idendity and $\mathbb{T}$ has the semigroup property

$$T_{s+t} = T_t \circ T_s \quad \text{for all } s, t \in \mathbb{R}_+.$$

If E is a vector space and each $T_t \in \mathbb{T}$ is linear, then $(E, \mathbb{T})$ is called an algebraic operator (AO) semigroup. If, in addition, E is an Banach space and $t \mapsto T_t u$ is continuous for all $u \in E$, then $(E, \mathbb{T})$ is called a C_0-semigroup. When E is understood, we say that $\mathbb{T}$ is a C_0-semigroup.

2. Continuity results

In what follows, E is a Banach space and $\mathcal{L}(E)$ is the set of bounded linear operators from E to itself. The symbol $\| \cdot \|$ denotes either the norm on E or the operator norm on $\mathcal{L}(E)$, depending on context.

Let K be a compact subset of $\mathbb{R}$ and let $\{T_t\}_{t \in K}$ be a subset of $\mathcal{L}(E)$. The following result is from Engel and Nagel (2006).

Lemma 2.1. *The following statements are equivalent:*

(i) *The map $t \mapsto T_t u$ is continuous on K for all $u \in E$.*

(ii) *$\|T_t\|$ is bounded over $t \in K$ and there exists a dense subset D of E such that $t \mapsto T_t u$ is continuous on K for all $u \in D$.*

(iii) *For any compact $C \subset E$, the map $(t, u) \mapsto T_t u$ is uniformly continuous on $K \times C$.*

Proof. ((i) $\implies$ (ii)) By (i), for any $u \in E$, the map $t \mapsto T_t u$ is continuous on a compact set and, therefore, its image is bounded in E. Hence, by the uniform boundedness principle, $\|T_t\|$ is bounded over $t \in K$. The statement in (ii) regarding continuity is obvious.

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((ii) $\implies$ (iii)). Fix compact $C \subset E$ and $\varepsilon > 0$. We metrize $K \times C$ by setting $d((s,u),(t,v)) = \|u - v\| \vee |s - t|$. Choose M such that $\|T_t\| \leqslant M$ for all $t \in K$. Let D be the dense set in (ii) and observe that the set of open balls $B(u, \varepsilon/M)$ over $u \in D$ provides an open cover of C. As such, we can choose a finite set $D_F \subset D$ such that C is contained in $\cup_{u \in D_F} B(u, \varepsilon/M)$. Since, for each $u \in D_F$, the map $t \mapsto T_t u$ is continuous on a compact set, it is also uniformly continous. As a result, we can select a $\delta_u > 0$ such that

$$|s - t| < \delta_u \implies \|T_s u - T_t u\| < \varepsilon.$$

Let δ be the minimum of $\{\delta_u\}_{u \in D_F}$ and ε/M. If we take $u, v \in C$ and $s, t \in K$ with $d((s,u),(t,v)) < \delta$, then, choosing $w \in D_F$ with $\|u - w\| < \varepsilon/M$, we have

$$\begin{aligned}\|T_s u - T_t v\| &\leqslant \|T_s u - T_s w\| + \|T_s w - T_t w\| + \|T_t w - T_t v\| \\ &= \|T_s(u - w)\| + \|T_s w - T_t w\| + \|T_t(w - v)\| < M(\varepsilon/M) + \varepsilon + M(2\varepsilon/M) = 4\varepsilon.\end{aligned}$$

Hence $(t,u) \mapsto T_t u$ is uniformly continuous on $K \times C$, as claimed.

((iii) $\implies$ (i)) This claim is also obvious (take C to be a singleton). □

Lemma 2.2. *If $(T_t)_{t \geqslant 0}$ is a C_0-semigroup on E, then $\sup_{t \leqslant \delta} \|T_t\| < \infty$ for all $\delta > 0$.*

Proof. We first claim there exists an $\varepsilon > 0$ such that $\sup_{t \leqslant \varepsilon} \|T_t\| < \infty$. Indeed, if no such ε exists, then there exists a sequence $t_n \to 0$ such that $\|T_{t_n}\|$ is unbounded. But then, by the principle of uniform boundedness, there exists a $u \in E$ such that $\|T_{t_n} u\|$ is unbounded. This contradicts the continuity property of C_0-semigroups.

Now let ε be as above and choose $M \in \mathbb{N}$ with $\|T_t\| \leqslant M$ whenever $t \leqslant \varepsilon$. Fix $k \in \mathbb{N}$ and $t \leqslant k\varepsilon$. Since T_t is k compositions of $T_{t/k}$, and since $t/k < \varepsilon$, the semigroup property yields $\|T_t\| \leqslant kM$. Hence $t \mapsto T_t$ is bounded on $[0, k\varepsilon]$. Since k was an arbitrary element of $\mathbb{N}$, this proves the claim in Lemma 2.2. □

Lemma 2.3. *An AO semigroup $(T_t)_{t \geqslant 0}$ on E is a C_0-semigroup on E if and only if $\lim_{t \downarrow 0} T_t u = u$ for all $u \in E$.*

Proof. Sufficiency is obvious. Regarding necessity, fix $u \in E$ and $t > 0$. We need to show that $\|T_{t+h} u - T_t u\| \to 0$ as $h \to 0$. Suppose first that $h \downarrow 0$. Then

$$\|T_{t+h} u - T_t u\| = \|T_t T_h u - T_t u\| \leqslant \|T_t\| \|T_h u - u\| \to 0.$$

If, on the other hand $h \uparrow 0$, then

$$\|T_{t+h} u - T_t u\| = \|T_{t+h} u - T_{t+h} T_{-h} u\| \leqslant \|T_{t+h}\| \|u - T_{-h} u\| \to 0.$$

In the last step we used the fact that $\|T_{t+h}\|$ is bounded over h by Lemma 2.2. □

Lemma 2.4. *Let* $(T_t)_{t \geqslant 0}$ *be an AO semigroup on* E. *If there exists a dense subset* D *of* E *such that* $\lim_{t \downarrow 0} T_t u = u$ *for all* $u \in D$ *and, in addition,* $\sup_{t \leqslant \delta} \|T_t\| < \infty$ *for some* $\delta > 0$, *then* $(T_t)_{t \geqslant 0}$ *is a* C_0*-semigroup on* E.

Proof. Fix $u \in E$. In view of Lemma 2.3 it suffices to show that, for a given sequence $t_n \downarrow 0$, we have $T_{t_n} u \to u$ as $n \to 0$.

To see that this holds, fix $t_n \downarrow 0$ and choose a compact set K such that $\{t_n\} \subset K$. Since K is compact, $K \ni t \mapsto T_t w$ is continuous when $w \in D$, and $\|T_t\|$ is bounded over $t \in K$, Lemma 2.1 implies that $K \ni t \mapsto T_t u$ is continuous. In particular, $T_{t_n} u \to u$ as $n \to 0$. □

References

Engel, K.-J. and Nagel, R. (2006). *A Short Course on Operator Semigroups*. Springer Science & Business Media.