SEMIGROUP NOTES

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1. Definitions

Let E be any set and let $(S_t) \coloneqq (S_t)_{t \geqslant 0}$ be a family of self-maps on E with index $t \in \mathbb{R}_+$. The pair $(E, (S_t))$ is called a semidynamical system if S_0 is the idendity and (S_t) has the semigroup property

$$S_{s+t} = S_t \circ S_s \quad \text{for all } s, t \in \mathbb{R}_+.$$

If E is a vector space and each S_t is linear, then $(E, (S_t))$ is called an algebraic operator (AO) semigroup. If, in addition, E is an Banach space and $t \mapsto S_t u$ is continuous for all $u \in E$, then $(E, (S_t))$ is called a C_0-semigroup. When E is understood, we say that (S_t) is a C_0-semigroup.

2. Continuity results

In what follows, E is a Banach space and $\mathcal{L}(E)$ is the set of bounded linear operators from E to itself. The symbol $\| \cdot \|$ denotes either the norm on E or the operator norm on $\mathcal{L}(E)$, depending on context.

Let K be a compact subset of $\mathbb{R}$ and let $\{S_t\}_{t \in K}$ be a subset of $\mathcal{L}(E)$. The following result is from Engel and Nagel (2006).

Lemma 2.1. *The following statements are equivalent:*

(i) *The map $t \mapsto S_t u$ is continuous on K for all $u \in E$.*

(ii) *$\|S_t\|$ is bounded over $t \in K$ and there exists a dense subset D of E such that $t \mapsto S_t u$ is continuous on K for all $u \in D$.*

(iii) *For any compact $C \subset E$, the map $(t, u) \mapsto S_t u$ is uniformly continuous on $K \times C$.*

Proof. ((i) $\implies$ (ii)) By (i), for any $u \in E$, the map $t \mapsto S_t u$ is continuous on a compact set and, therefore, its image is bounded in E. Hence, by the uniform boundedness principle, $\|S_t\|$ is bounded over $t \in K$. The statement in (ii) regarding continuity is obvious.

Date: February 2, 2024.

((ii) $\implies$ (iii)). Fix compact $C \subset E$ and $\varepsilon > 0$. We metrize $K \times C$ by setting $d((s,u),(t,v)) = \|u - v\| \vee |s - t|$. Choose $M \in \mathbb{N}$ such that $\|S_t\| \leqslant M$ for all $t \in K$. Let D be the dense set in (ii) and observe that the set of open balls $B(u, \varepsilon/M)$ over $u \in D$ provides an open cover of C. As such, we can choose a finite set $D_F \subset D$ such that C is contained in $\cup_{u \in D_F} B(u, \varepsilon/M)$. Since, for each $u \in D_F$, the map $t \mapsto S_t u$ is continuous on a compact set, it is also uniformly continuous. As a result, given $u \in D_F$, we can select a $\delta_u > 0$ such that

$$|s - t| < \delta_u \implies \|S_s u - S_t u\| < \varepsilon.$$

Let δ be the minimum of $\{\delta_u\}_{u \in D_F}$ and ε/M. If we take $u, v \in C$ and $s, t \in K$ with $d((s,u),(t,v)) < \delta$, then, choosing $w \in D_F$ with $\|u - w\| < \varepsilon/M$, we have

$$\begin{aligned} \|S_s u - S_t v\| &\leqslant \|S_s u - S_s w\| + \|S_s w - S_t w\| + \|S_t w - S_t v\| \\ &= \|S_s(u - w)\| + \|S_s w - S_t w\| + \|S_t(w - v)\| \\ &< M(\varepsilon/M) + \varepsilon + M(2\varepsilon/M) = 4\varepsilon. \end{aligned}$$

Hence $(t, u) \mapsto S_t u$ is uniformly continuous on $K \times C$, as claimed.

((iii) $\implies$ (i)) This implication is trivial (take C to be a singleton). $\square$

Lemma 2.2. *If $(S_t)_{t \geqslant 0}$ is a C_0-semigroup on E, then $\sup_{t \leqslant \delta} \|S_t\| < \infty$ for all $\delta > 0$.*

Proof. We first claim there exists an $\varepsilon > 0$ such that $\sup_{t \leqslant \varepsilon} \|S_t\| < \infty$. Indeed, if no such ε exists, then there exists a sequence $t_n \to 0$ such that $\|S_{t_n}\|$ is unbounded. But then, by the principle of uniform boundedness, there exists a $u \in E$ such that $\|S_{t_n} u\|$ is unbounded. This contradicts the continuity property of C_0-semigroups.

Now let ε be as above and choose $M \in \mathbb{N}$ with $\|S_t\| \leqslant M$ whenever $t \leqslant \varepsilon$. Fix $k \in \mathbb{N}$ and $t \leqslant k\varepsilon$. Since S_t is k compositions of $S_{t/k}$, and since $t/k < \varepsilon$, the semigroup property yields $\|S_t\| \leqslant kM$. Hence $t \mapsto S_t$ is bounded on $[0, k\varepsilon]$. Since k was an arbitrary element of $\mathbb{N}$, this proves the claim in Lemma 2.2. $\square$

Lemma 2.3. *An AO semigroup $(S_t)_{t \geqslant 0}$ on E is a C_0-semigroup on E if and only if $\lim_{t \downarrow 0} S_t u = u$ for all $u \in E$.*

Proof. Sufficiency is obvious. Regarding necessity, fix $u \in E$ and $t > 0$. We need to show that $\|S_{t+h} u - S_t u\| \to 0$ as $h \to 0$. Suppose first that $h \downarrow 0$. Then

$$\|S_{t+h} u - S_t u\| = \|S_t S_h u - S_t u\| \leqslant \|S_t\| \|S_h u - u\| \to 0.$$

If, on the other hand $h \uparrow 0$, then

$$\|S_{t+h}u - S_t u\| = \|S_{t+h}u - S_{t+h}S_{-h}u\| \leqslant \|S_{t+h}\| \|u - S_{-h}u\| \to 0.$$

In the last step we used the fact that $\|S_{t+h}\|$ is bounded over h by Lemma 2.2. ◻

Lemma 2.4. *Let $(S_t)_{t\geqslant 0}$ be an AO semigroup on E. If there exists a dense subset D of E such that $\lim_{t\downarrow 0} S_t u = u$ for all $u \in D$ and, in addition, $\sup_{t\leqslant\delta} \|S_t\| < \infty$ for some $\delta > 0$, then $(S_t)_{t\geqslant 0}$ is a C_0-semigroup.*

Proof. Fix $u \in E$. By Lemma 2.3 it suffices to show that, for a given sequence $t_n \downarrow 0$, we have $S_{t_n}u \to u$ as $n \to 0$. To see that this holds, fix $t_n \downarrow 0$ and choose a compact subset K of $\mathbb{R}_+$ such that $\{t_n\} \subset K$. Since K is compact, $K \ni t \mapsto S_t w$ is continuous when $w \in D$, and $\|S_t\|$ is bounded over $t \in K$, Lemma 2.1 implies that $K \ni t \mapsto S_t u$ is continuous. In particular, $S_{t_n}u \to u$ as $n \to 0$. ◻

3. Examples

3.1. **Left-shift semigroups.** Let $C_0(\mathbb{R}_+)$ be the set of all continuous real-valued functions f on $\mathbb{R}_+$ with $f(x) \to 0$ as $x \to \infty$. The set $C_0(\mathbb{R}_+)$ is paired with the supremum norm. Consider the left translation semigroup given by $(S_t^\ell f)(x) = f(x+t)$.

Lemma 3.1. *(S_t^ℓ) is a C_0-semigroup on $C_0(\mathbb{R}_+)$.*

Proof. Evidently $S_0^\ell f = f$. The semigroup property holds because, for $s, t \geqslant 0$, we have

$$(S_{s+t}^\ell f)(x) = f(x + s + t) = (S_t^\ell (S_s^\ell f))(x).$$

Regarding continuity, fix $f \in C_0(\mathbb{R}_+)$ and let (t_n) be a real sequence with $t_n \downarrow 0$. Fix $\varepsilon > 0$. Since f is uniformly continuous, we can select a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Let $N \in \mathbb{N}$ be such that $t_n < \delta$ when $n \geqslant N$. Then, for $n \geqslant N$,

$$\|S_{t_n}^\ell f - f\| = \sup_x |f(x + t_n) - f(x)| < \varepsilon.$$

Hence $S_t^\ell f \downarrow f$ and (S_t^ℓ) is a C_0-semigroup. ◻

Let $C_0^1(\mathbb{R}_+)$ be the set of all continuously differentiable $f \in C_0(\mathbb{R}+)$ with $f' \in C_0(\mathbb{R}+)$. The set $C_0^1(\mathbb{R}_+)$ is paired with the norm $\|f\| = \sup_x |f(x)| + \sup_x |f'(x)|$.

Lemma 3.2. *(S_t^ℓ) is a C_0-semigroup on $C_0^1(\mathbb{R}_+)$.*

Proof. In view of Lemma 3.1, we only need to check continuity. Fixing $f \in C_0^1(\mathbb{R}_+)$, we have

$$\|S_t^\ell f - f\| = \sup_x |f(x+t) - f(x)| + \sup_x |f'(x+t) - f'(x)|$$

Since f and f' are both in $C_0(\mathbb{R}_+)$, the proof of Lemma 3.1 implies that both terms on the right hand side converge to zero as $t \downarrow 0$. Hence continuity holds. □

3.2. **Right-shift semigroups.** Here we discuss right-shift semigroups. We will embed them in a space of integrable functions. Below λ denotes Lebesgue measure.

Let $C_c(\mathbb{R})$ be the set of all continuous real-valued functions f on $\mathbb{R}$ that vanish off a compact set. Let $L_1(\mathbb{R})$ be the set of Borel measurable real-valued functions on $\mathbb{R}$ with $\|f\| := \int |f| \,\mathrm{d}\lambda < \infty$. Let S_t be the linear operator on $L_1(\mathbb{R})$ defined by

$$(S_t f)(x) = f(x+t) \qquad (x \in \mathbb{R},\ t \geqslant 0).$$

Lemma 3.3. *(S_t^ℓ) is a C_0-semigroup on $L_1(\mathbb{R})$.*

Proof. It is simple to confirm that (S_t) is an AO semigroup on $L_1(\mathbb{R})$. Regarding continuity, note that $\|S_t f\| = \int |f(x-t)| \,\mathrm{d}x = \|f\|$, so $\|S_t\|$ is bounded in t. Since $C_c(\mathbb{R})$ is dense in $L_1(\mathbb{R})$ under this norm, Lemma 2.4 implies that, to show (S_t) is a C_0-semigroup on $L_1(\mathbb{R})$, it suffices to show that $\|S_t f - f\| \to 0$ for any $f \in C_c(\mathbb{R})$.

To this end, fix $f \in C_c(\mathbb{R})$ and let K be a compact set such that f vanishes off K. Fix $\varepsilon > 0$. By uniform continuity, we can take a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon / \lambda(K)$. If $t < \delta$, then

$$\|S_t f - f\| = \int |f(x-t) - f(x)| \,\mathrm{d}x \leqslant \lambda(K) \frac{\varepsilon}{\lambda(K)} = \varepsilon.$$

This completes the proof of C_0-continuity of (S_t) on $L_1(\mathbb{R})$. □

3.3. **Multiplication semigroups.** Let $(\mathsf{X}, \mathcal{B}, \mu)$ be a σ-finite measure space and let φ be a measurable map from X to $\mathbb{R}_+$. Define

$$S_t f = \exp(-t\varphi(x)) f(x) \qquad (x \in \mathsf{X},\ t \geqslant 0).$$

The family (S_t) is called a multiplication semigroup.

Lemma 3.4. *(S_t) is a C_0-semigroup on $L_1(\mathsf{X}, \mathcal{B}, \mu)$.*

Proof. It is simple to confirm that (S_t) is an AO semigroup on $L_1(\mathbb{R})$. Regarding continuity, fix $f \in L_1(\mathsf{X}, \mathcal{B}, \mu)$ and observe that

$$\|S_t f - f\| = \int |f(x)| |\exp(-t\varphi(x)) - 1| \mu(\mathrm{d}x).$$

It follows from the dominated convergence theorem that this integral converges to zero as $t \downarrow 0$. Hence (S_t) is a C_0-semigroup on $L_1(\mathsf{X}, \mathcal{B}, \mu)$. $\square$

3.4. **Uniformly continuous semigroups.** Let E be a Banach space and let $\mathcal{L}(E)$ be the bounded linear operators on E. Recall that the exponential of $A \in \mathcal{L}(E)$ is given by

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Fixing $A \in \mathcal{L}(E)$, consider the family of linear operators on E given by

$$S_t u = \exp(tA)u \qquad (u \in E,\ t \geqslant 0)$$

We recall that the exponential function $\varphi(t) := \exp(tA)$

(i) obeys $\varphi(0) = I$ and $\varphi(s+t) = \varphi(t)\varphi(s)$ for all $s, t \in \mathbb{R}$; and

(ii) is continuous as a map from $\mathbb{R}$ to $\mathcal{L}(E)$.

From (i) we can easily confirm that (S_t) is an algebraic operator semigroup on E. Regarding continuity, (i) and (ii) imply that

$$\lim_{t \downarrow 0} \|S_t - I\| = 0. \tag{1}$$

It follows from (1) that (S_t) is a C_0-semigroup on E.

Any operator semigroup (S_t) on E obeying (1) is called a uniformly continuous semigroup. In fact no other examples exist:

Theorem 3.5. *If (S_t) is a uniformly continuous semigroup on E, then there exists an $A \in \mathcal{L}(E)$ such that $S_t u = \exp(tA)u$ for all $u \in E$ and $t \geqslant 0$.*

The proof of Theorem 3.5 can be found 2(b) of Engel and Nagel (2006).

References

Engel, K.-J. and Nagel, R. (2006). *A Short Course on Operator Semigroups*. Springer Science & Business Media.