An Introduction to Computational Macroeconomics

Dynamic Programming: Review Slides

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Review Slides

Let's review some of our major topics

- Markov dynamics
- Order
- Valuation
- Fixed point theory

Neumann Series Lemma

Suppose b is a column vector in $\mathbb{R}^n$ and A is $n \times n$

Let I be the $n \times n$ identity matrix

Theorem. If $r(A) < 1$, then

1. $I - A$ is nonsingular,
2. the sum $\sum_{k \geqslant 0} A^k$ converges,
3. $(I - A)^{-1} = \sum_{k \geqslant 0} A^k$, and
4. the vector equation $x = Ax + b$ has the unique solution

$$x^* := (I - A)^{-1} b = \sum_{k \geqslant 0} A^k b$$

Fixed Points

Recall that, if S is any set then

- T is a **self-map** on S if T maps S into itself
- $x^* \in S$ is called a **fixed point** of T in S if $Tx^* = x^*$

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$r(A) < 1 \implies x^* := (I - A)^{-1} b \text{ is the unique f.p. of } T \text{ in } S$$

Example. If $S \subset \mathbb{R}$, $Tx = x \iff T$ meets the 45 degree line

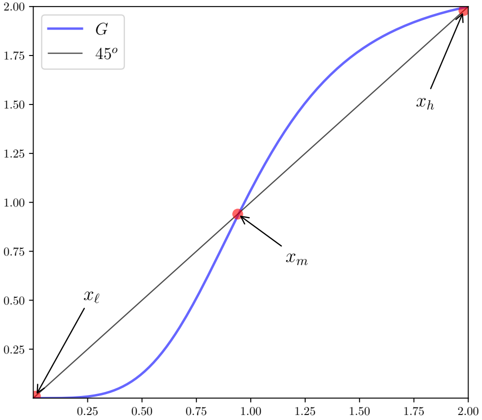


Figure: Graph and fixed points of $G \colon x \mapsto 2.125/(1 + x^{-4})$

Self-map T is called **globally stable** on S if

1. ?
2. ?

Let T be a self-map on $S \subset \mathbb{R}^n$

We call $C \subset S$ **invariant** for T if ?

Lemma. If T is globally stable on $S \subset \mathbb{R}^n$ with fixed point u^* and C is nonempty, closed and invariant for T, then $u^* \in C$

Successive Approximation

A natural algorithm for approximating the fixed point in S:

---

fix x_0 and $k = 0$
while *some stopping condition fails* **do**
  $x_{k+1} \leftarrow T x_k$
  $k \leftarrow k + 1$
end
return x_k

If T is globally stable on S, then $(x_k) = (T^k x_0)$ converges to x^*

hence output $\approx x^*$

The algorithm just described is called **successive approximation**

Norms in Vector Space

A function $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$ is called a **norm** on $\mathbb{R}^n$ if, for any $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$,

(a) ?

(b) ?

(c) ?

(d) ?

Example. The **Euclidean norm** $\|u\| := \sqrt{\langle u, u \rangle}$

Example. The **supremum norm**, defined by

$$\|u\|_\infty := \max_{i=1}^{n} |u_i|$$

Contractions

Let

- U be a nonempty subset of $\mathbb{R}^n$,
- $\| \cdot \|$ be a norm on $\mathbb{R}^n$, and
- T be a self-map on U

T is called a **contraction** on U with respect to $\| \cdot \|$ if

$$\exists\, \lambda < 1 \text{ such that } \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{for all} \quad u, v \in U$$

Ex. Prove: If T is a contraction on U, then T has at most one fixed point in U

Banach's Contraction Mapping Theorem

Theorem If

1. U is closed in $\mathbb{R}^n$ and
2. T is a contraction of modulus λ on U with respect to some norm $\|\cdot\|$ on $\mathbb{R}^n$,

then T has a unique fixed point u^* in U and

$$\|T^n u - u^*\| \leqslant \lambda^n \|u - u^*\| \quad \text{for all } n \in \mathbb{N} \text{ and } u \in U$$

In particular, T is globally stable on U

Proof: See the course notes

Nonnegative Matrices

Matrix A is called

- **nonnegative**, and we write $A \geqslant 0$, if all elements of A are nonnegative
- **positive**, and we write $A \gg 0$, if every element of A is strictly positive

Ex. When is a matrix **irreducible**?

Note: positive $\Longrightarrow$ irreducible $\Longrightarrow$ nonnegative

Theorem. (Perron–Frobenius) If $A \geqslant 0$, then $r(A)$ is an eigenvalue of A with nonnegative, real-valued right and left eigenvectors

In particular, there exists

- a nonnegative, nonzero column vector e s.t. $Ae = r(A)e$
- a nonnegative, nonzero row vector ε s.t. $\varepsilon A = r(A)\varepsilon$

If A **is irreducible**, then these eigenvectors are everywhere positive and have multiplicity of one

If A **is positive**, then with e and ε such that $\langle \varepsilon, e \rangle = 1$, we have

$$r(A)^{-t} A^t \to e\,\varepsilon \qquad (t \to \infty)$$

Stochastic Matrices

Let P be a square matrix

P is called **stochastic** if $P \geqslant 0$ and $P\mathbb{1} = \mathbb{1}$

Ex. Show that P is stochastic $\implies r(P) = 1$

A row vector ψ is called a **stationary distribution** of P if

??

Let P be a stochastic matrix

Ex. Does P has at least one stationary distribution?

In other words, does there always

$\exists$ a nonzero, nonnegative row vector φ satisfying $\varphi P = \varphi$?

Under what condition is this stationary distribution unique?

Markov Chains

Let

- $\mathsf{X} = \{x_1, \ldots, x_n\}$ = arbitrary finite set
- P be an $n \times n$ stochastic matrix

A Markov chain is generated by some stochastic matrix P

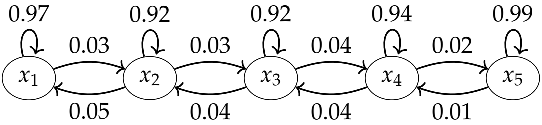
Interpretation:

$$P_{ij} = \text{ probability of moving from } x_i \text{ to } x_j \text{ in one step}$$

Example.

$$P = \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.05 & 0.92 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix}$$

Transition probabilities:



Notation: We use the identification $P_{ij} =: P(x_i, x_j)$

In this notation, P is a stochastic matrix iff

$$P \geqslant 0 \quad \text{and} \quad \sum_{x' \in \mathsf{X}} P(x, x') = 1 \text{ for all } x \in \mathsf{X}$$

Equivalent:

$$P \geqslant 0 \quad \text{and} \quad P\mathbb{1} = \mathbb{1}$$

Equivalent:

$$P(x, \cdot) \in \mathcal{D}(\mathsf{X}) \quad \text{for all } x \in \mathsf{X}$$

We call P **a stochastic matrix on** X

Let

- $(X_t)_{t \geqslant 0}$ be a sequence of X-valued random variables
- P be a stochastic matrix on X

Def. We call $(X_t)_{t \geqslant 0}$ P-**Markov** if

$$\mathbb{P}\{X_{t+1} = x' \mid X_0, X_1, \ldots, X_t\} = P(X_t, x') \quad \text{for all} \quad t \geqslant 0, \ x' \in \mathsf{X}$$

Standard terminology

- $(X_t)_{t \geqslant 0}$ is a **Markov chain**
- P is the **transition matrix** of $(X_t)_{t \geqslant 0}$
- We call either X_0 or its distribution ψ_0 the **initial condition**

Let

1. P be a stochastic matrix on X
2. ψ_0 be an element of $\mathcal{D}(\mathsf{X})$

This algorithm yields a P-Markov chain with initial condition ψ_0

$t \leftarrow 0$
$X_t \leftarrow$ a draw from ψ_0
while $t < \infty$ **do**
    $X_{t+1} \leftarrow$ a draw from the distribution $P(X_t, \cdot)$
    $t \leftarrow t + 1$
end

Example. Assume $0 < \alpha, \beta < 1$ and let

$$\mathsf{X} = \{0, 1\} \quad \text{and} \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Ex. Show that

$$\psi^* := \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \end{pmatrix} = \text{ unique stationary distribution}$$

Multistep transitions

Fix a state space X and transition matrix P on X

Recall

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\} \quad \text{for any } P\text{-chain } (X_t)_{t \geqslant 0}$$

Lemma. The following statements are equivalent:

1. P is ????
2. For any P-chain (X_t) and any $x, x' \in \mathsf{X}$, there exists a $k \geqslant 0$ such that

$$\mathbb{P}\{X_k = x' \mid X_0 = x\} > 0$$

Application: S-s Dynamics

Inventory $(X_t)_{t \geqslant 0}$ obeys

$$X_{t+1} = \max\{X_t - D_{t+1}, 0\} + S\mathbb{1}\{X_t \leqslant s\}, \tag{1}$$

where

- $(D_t)_{t \geqslant 1}$ is demand, IID with $D_t \stackrel{d}{=} \varphi \in \mathcal{D}(\mathbb{Z}_+)$
- S = amount of stock ordered when inventory $\leqslant s$

We assume φ obeys the geometric distribution:

$$\varphi(d) = \mathbb{P}\{D_t = d\} = p(1-p)^d \text{ for } d \in \mathbb{Z}_+$$

We take $\mathsf{X} := \{0, \ldots, S + s\}$ to be the state space

If

$$h(x, d) = \max\{x - d, 0\} + S\mathbb{1}\{x \leqslant s\}$$

then

$$X_{t+1} = h(X_t, D_{t+1}) \quad \text{for all } t \geqslant 0$$

The transition matrix can be expressed as

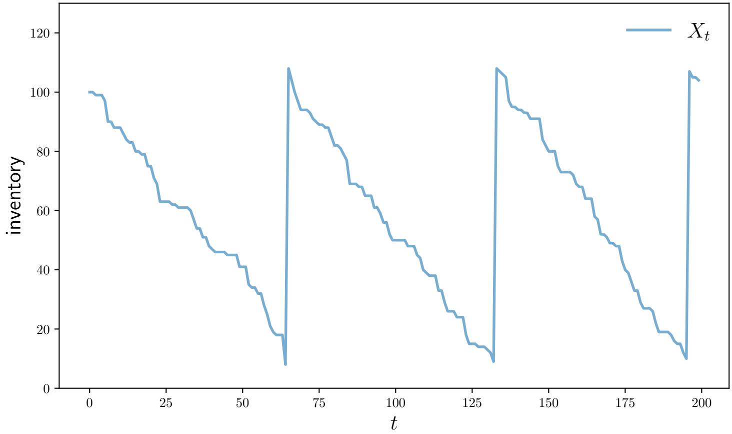
$$\begin{aligned} P(x, x') &= \mathbb{P}\{h(x, D_{t+1}) = x'\} \\ &= \sum_{d \geqslant 0} \mathbb{1}\{h(x, d) = x'\}\varphi(d) \end{aligned}$$

(In calculations we truncate the sum)

```julia
using Distributions, IterTools, QuantEcon

function create_inventory_model(; S=100,  # Order size
                                  s=10,   # Order threshold
                                  p=0.4)  # Demand parameter
    ϕ = Geometric(p)
    h(x, d) = max(x - d, 0) + S*(x <= s)
    return (; S, s, p, ϕ, h)
end

"Simulate the inventory process."
function sim_inventories(model; ts_length=200)
    (; S, s, p, ϕ, h) = model
    X = Vector{Int32}(undef, ts_length)
    X[1] = S  # Initial condition
    for t in 1:(ts_length-1)
        X[t+1] = h(X[t], rand(ϕ))
    end
    return X
end
```



X_t
inventory
t
0
20
40
60
80
100
120
0
25
50
75
100
125
150
175
200

Ex. Recreate this plot

- see `notebooks/sim_ss_ex` in the repo

Dynamics of Marginals

Fix a stochastic matrix P on X and let (X_t) be a P-chain

Let $\psi_t \stackrel{d}{:=} X_t$ for all t

Then

$$\psi_{t+1} = \psi_t P \quad \text{for all } t$$

Recall ψ^* is called **stationary** for P if

$$\psi^* = \psi^* P$$

Meaning?

Example. Recall the model

$$\mathsf{X} = \{0, 1\}, \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

and

$$\psi^* := \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \end{pmatrix}$$

If P is ???, ergodicity holds:

$$\mathbb{P}\left\{ \lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\} = \psi^*(x) \right\} = 1$$

Ex. If $\alpha = 1$ and $\beta = 0.5$, does ??? hold?

Conditional Expectations

Let P be any stochastic matrix on X

For each $h \in \mathbb{R}^{\mathsf{X}}$, $k \in \mathbb{N}$ and $x \in \mathsf{X}$, we define

$$(P^k h)(x) := \sum_{x' \in \mathsf{X}} h(x') P^k(x, x')$$

Interpretation:

$$(P^k h)(x) = \mathbb{E}[h(X_{t+k}) \mid X_t = x]$$

When updating distributions we use row vectors:

$$(\psi P)(x') = \sum_{x \in \mathsf{X}} P(x, x')\psi(x)$$

- sum is down column x'

When taking conditional expectations we use column vectors:

$$(Ph)(x) := \sum_{x' \in \mathsf{X}} h(x')P(x, x')$$

- sum is along row x

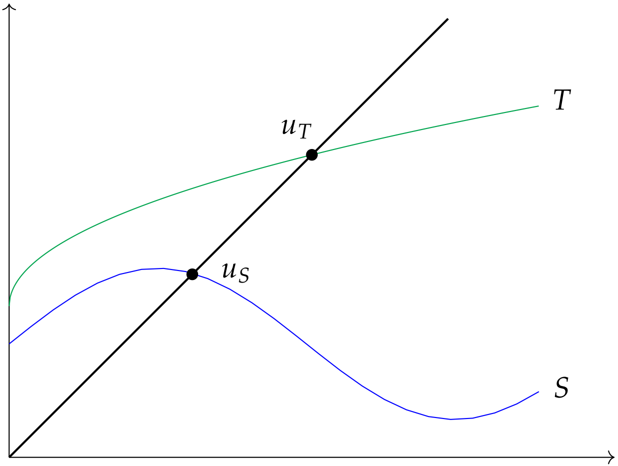
Let

- S and T be self-maps on $M \subset \mathbb{R}^n$
- $\leqslant$ be the pointwise partial order on M

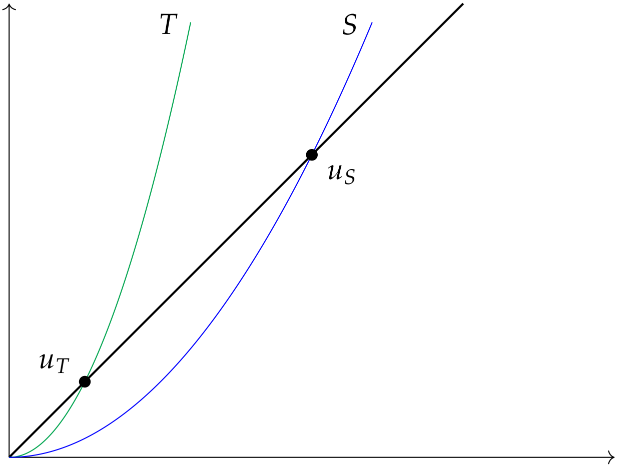
Questions:

1. What does $S \leqslant T$ mean?
2. If $S \leqslant T$, then are the fixed points of T larger?

Sometimes true:



And sometimes false:



Let

- S and T be self-maps on $M \subset \mathbb{R}^n$
- $\leqslant$ be the pointwise partial order on M

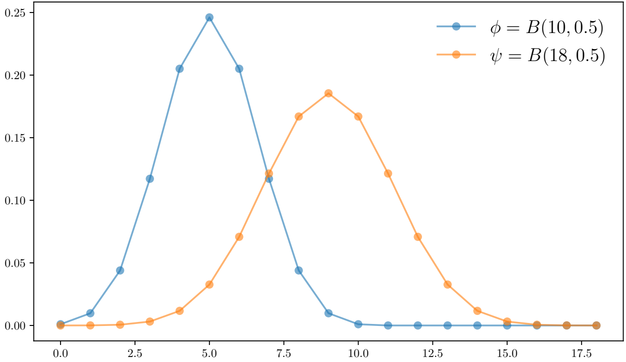
Proposition. If

1. T dominates S on M and
2. T is order-preserving and globally stable on M,

then the unique fixed point of T dominates any fixed point of S

Stochastic Dominance

Distribution ψ seems "larger than" φ



Let X be a finite set partially ordered by $\preceq$

Fix $\varphi, \psi \in \mathcal{D}(\mathsf{X})$

Write $\langle u, \varphi \rangle$ for $\sum_x u(x)\varphi(x)$, etc.

We say that ψ **stochastically dominates** φ and write $\varphi \preceq_{\mathrm{F}} \psi$ if

$$u \in i\mathbb{R}^{\mathsf{X}} \quad \Longrightarrow \quad \langle u, \varphi \rangle \leqslant \langle u, \psi \rangle$$

Monotone Markov Chains

A stochastic matrix P on $\mathsf{X} \times \mathsf{X}$ is called **monotone increasing** if

$$x, y \in \mathsf{X} \text{ and } x \preceq y \quad \Longrightarrow \quad P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$$

True or false: P is monotone increasing iff P is invariant on $i\mathbb{R}^{\mathsf{X}}$?

Example. Consider the AR(1) model $X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$

Question: After Tauchen discretization, is P always monotone increasing?

Valuation

Given $\beta \in \mathbb{R}_+$ and $h \in \mathbb{R}^{\mathsf{X}}$, let

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) \quad \text{where} \quad (X_t) \text{ is } P\text{-Markov on } \mathsf{X}$$

Lemma. If $\beta \in (0, 1)$, then v is finite, $I - \beta P$ is invertible and

$$v = \sum_{t \geqslant 0} (\beta P)^t h = (I - \beta P)^{-1} h \tag{2}$$

Proof: ??

Generalized Geometric Sums

Suppose

- $h \in \mathbb{R}^{\mathsf{X}}$ and $b \in \mathbb{R}^{\mathsf{X} \times \mathsf{X}}$
- $(X_t)_{t \geqslant 0}$ is P-Markov, $H_t = h(X_t)$ and $B_t = b(X_{t-1}, X_t)$
- K is the matrix on X defined by $K(x, x') := b(x, x')P(x, x')$

Theorem. If $r(K) < 1$, then

$$v(x) = \mathbb{E}_x \left\{ \sum_{t=0}^{\infty} \left[\prod_{i=1}^{t} B_i \right] H_t \right\} \qquad \text{with } \prod_{i=1}^{0} B_i := 1$$

is finite-valued

Moreover, $I - K$ is nonsingular and $v = (I - K)^{-1} h$

Example. Valuation of a firm when

$$v(x) = \mathbb{E}_x \left\{ \sum_{t=0}^{\infty} \left[\prod_{i=0}^{t-1} \beta_i \right] \pi_t \right\}$$

Suppose

- $r_t = r(X_t)$ for $r \in \mathbb{R}^{\mathsf{X}}$
- Set $\beta(x) := 1/(1 + r(x))$

Let

$$K(x, x') := \beta(x) P(x, x') \qquad ((x, x') \in \mathsf{X} \times \mathsf{X})$$

Proposition. If $r(K) < 1$, then the firm valuation is finite and satisfies

$$v = (I - K)^{-1} \pi$$

Proof: Apply the last theorem with

$$b(X_{t-1}, X_t) = \beta(X_{t-1}) = \frac{1}{1 + r(X_{t-1})}$$

and $h = \pi$

Example. Pricing dividend streams

The price of a claim on dividend stream $(D_t)_{t \geqslant 0}$ obeys

$$\Pi_t = \mathbb{E}_t \, M_{t+1}(\Pi_{t+1} + D_{t+1})$$

Let

- $D_t = d(X_t)$ where $(X_t)_{t \geqslant 0}$ is P-Markov
- $\pi(x) =$ current price given $X_t = x$

We get

$$\pi(x) = \sum_{x'} m(x, x')(\pi(x') + d(x'))P(x, x') \qquad (x \in \mathsf{X})$$

Rewrite the last expression as

$$\pi = A\pi + Ad$$

where

$$A(x, x') := m(x, x')P(x, x')$$

Questions:

1. When is there a unique solution π^*?
2. How can we compute it?