An Introduction to Computational Macroeconomics

Dynamic Programming: Chapter 5

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Optimal Stopping

Many decision making problems involve **choosing when to act**

When to

- accept a job
- exit or enter a market
- bring a new product to market
- default on a loan
- exercise a real or financial option

We now discuss optimal stopping problems via dynamic programming

We begin with the standard theory of optimal stopping

Then we consider alternative approaches

- continuation values
- threshold policies
- etc.

One key objective: provide a rigorous discussion of optimality

This clarifies our intuitive analysis in the context of job search

Theory

Let X be a finite set

An **optimal stopping problem** with state space X consists of

- a stochastic matrix P on X
- a discount factor $\beta \in (0, 1)$
- a **continuation reward function** $c \in \mathbb{R}^{\mathsf{X}}$ and
- an **exit reward function** $e \in \mathbb{R}^{\mathsf{X}}$

Given a P-Markov chain $(X_t)_{t \geqslant 0}$, the problem evolves as follows:

An agent observes X_t each period, continues or stops

If she chooses to stop, she receives $e(X_t)$ and the process ends

If she continues, she receives $c(X_t)$ and the process repeats

Lifetime rewards are given by

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t R_t,$$

where R_t equals

- $c(X_t)$ while the agent continues
- $e(X_t)$ when the agent stops, and zero thereafter

Example. In the infinite-horizon job search problem

- wage offer process (W_t) is $\overset{\text{IID}}{\sim}$ φ on W
- stopping $=$ accepting and continuing $=$ receiving unemployment compensation and repeating

This is an optimal stopping problem with

- state space $\mathsf{X} = \mathsf{W}$
- stochastic matrix P with all rows equal to φ
- exit reward function $e(x) = x/(1-\beta)$ and
- the continuation reward function $\equiv$ unemployment compensation

Example. Infinite-horizon American call option

Provides the right to buy a given asset at strike price K at every future date

The market price of the asset is given by $S_t = s(X_t)$

- (X_t) is P-Markov on finite set X
- the interest rate is $r > 0$

When to exercise is an optimal stopping problem, with

- discount factor and $\beta = 1/(1+r)$
- exit reward function $e(x) = s(x) - K$ and
- continuation reward zero

A **policy function** is a map σ from X to $\{0,1\}$

Interpretation: observe state x, respond with action $\sigma(x)$, where

- 0 means "continue"
- 1 means "stop"

State must contain enough information to decide!

Let $\Sigma =$ all $\sigma \colon \mathsf{X} \to \{0,1\}$

Let $v_\sigma(x) =$ expected lifetime value of following policy σ now and forever, given current state $x \in \mathsf{X}$

We call v_σ the **σ-value function**

Our aim:

- choose a policy that maximizes lifetime value!

In particular, $\sigma^* \in \Sigma$ is called **optimal** if

$$v_{\sigma^*}(x) = \max_{\sigma \in \Sigma} v_\sigma(x) \quad \text{for all } x \in \mathsf{X}$$

Before we can compute optimal policies, we need to be able to evaluate v_σ for each $\sigma \in \Sigma$

- How can we do this?

Some thought will convince you that v_σ must satisfy

$$v_\sigma(x) = \sigma(x)e(x) + (1 - \sigma(x)) \left[c(x) + \beta \sum_{x' \in \mathsf{X}} v_\sigma(x') P(x, x') \right]$$

<u>Case 1</u>: $\sigma(x) = 1$

Then the above states $v_\sigma(x) = e(x)$, which is the right value

<u>Case 2</u>: $\sigma(x) = 0$

Then

$$v_\sigma(x) = c(x) + \beta \sum_{x' \in \mathsf{X}} v_\sigma(x') P(x, x')$$

A natural recursion

To repeat, we need to solve

$$v_\sigma(x) = \sigma(x)e(x) + (1 - \sigma(x)) \left[c(x) + \beta \sum_{x' \in \mathsf{X}} v_\sigma(x') P(x, x') \right]$$

Let

- $r_\sigma(x) := \sigma(x)e(x) + (1 - \sigma(x))c(x)$
- $P_\sigma(x, x') := (1 - \sigma(x))P(x, x')$

Then the equation becomes $v_\sigma = r_\sigma + \beta P_\sigma v_\sigma$

Ex. Show that $r(\beta P_\sigma) < 1$ and hence

$$v_\sigma = (I - \beta P_\sigma)^{-1} r_\sigma$$

We can also view v_σ as the fixed point of the **policy operator**

$$(T_\sigma v)(x) = \sigma(x)e(x) + (1 - \sigma(x)) \left[c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right]$$

Pointwise this is

$$T_\sigma v = r_\sigma + \beta P_\sigma v$$

We know that v_σ is the unique solution to $v = r_\sigma + \beta P_\sigma v$

Hence v_σ is the unique fixed point of point of T_σ in $\mathbb{R}^{\mathsf{X}}$

Ex. Prove that T_σ is order-preserving on $\mathbb{R}^{\mathsf{X}}$

Ex. Show that T_σ is a contraction map on $\mathbb{R}^{\mathsf{X}}$

Given $f, g \in \mathbb{R}^{\mathsf{X}}$ and $x \in \mathsf{X}$, we have

$$|(T_\sigma f)(x) - (T_\sigma g)(x)| = \left| (1 - \sigma(x))\beta \sum_{x' \in \mathsf{X}} (g(x') - f(x'))P(x, x') \right|$$

$$\leqslant \beta \left| \sum_{x'} [f(x') - g(x')] P(x, x') \right|$$

Applying the triangle inequality and $\sum_{x' \in \mathsf{X}} P(x, x') = 1$ yields

$$|(T_\sigma f)(x) - (T_\sigma g)(x)| \leqslant \beta \| f - g \|_\infty$$

Hence

$$\| T_\sigma f - T_\sigma g \|_\infty \leqslant \beta \| f - g \|_\infty$$

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Hence

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We define the **value function** of the optimal stopping problem as

$$v^*(x) := \max_{\sigma \in \Sigma} v_\sigma(x) \qquad (x \in \mathsf{X})$$

Thus, $v^*(x) =$ max lifetime value given current state x

Next steps

1. introduce the **Bellman equation**

$$v(x) = \max \left\{ e(x), c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\}$$

2. prove that the Bellman equation has a unique solution in $\mathbb{R}^{\mathsf{X}}$
3. show that this solution equals v^*

The **Bellman operator** for the optimal stopping is defined by

$$(Tv)(x) = \max \left\{ e(x),\, c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\}$$

Ex. Prove that T is an order preserving self-map on $\mathbb{R}^{\mathsf{X}}$

Proof: Fix $f, g \in \mathbb{R}^{\mathsf{X}}$ with $f \leqslant g$

Since $P \geqslant 0$, we have $Pf \leqslant Pg$

Hence $c + \beta Pf \leqslant c + \beta Pg$

$$\therefore \quad Tf = e \vee (c + \beta Pf) \leqslant e \vee (c + \beta Pg) = Tg$$

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Ex. Prove that, for all $\sigma \in \Sigma$, T dominates T_σ on $\mathbb{R}^{\mathsf{X}}$

Proof: For all $a, b \in \mathbb{R}$ and $\lambda \in [0,1]$, we have

$$\lambda a + (1-\lambda) b \leqslant a \vee b$$

Now fix $\sigma \in \Sigma$ and $v \in \mathbb{R}^{\mathsf{X}}$

We have

$$(T_\sigma v)(x) = \sigma(x) e(x) + (1 - \sigma(x)) \left[c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right]$$

$$\leqslant \max \left\{ e(x),\, c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\} = (Tv)(x)$$

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Value function vs Bellman equation

Prop. Under the stated conditions

1. T is a contraction map of modulus β on $\mathbb{R}^{\mathsf{X}}$ and
2. the unique fixed point of T on $\mathbb{R}^{\mathsf{X}}$ is the value function v^*

In particular, v^* is the unique solution to the Bellman equation

The first step of the proof is the next exercise

Ex. Given f, g in $\mathbb{R}^{\mathsf{X}}$, show that

$$\|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

Proof: Recall the bound $|z \vee a - z \vee b| \leqslant |a - b|$

From this we have

$$|Tf - Tg| = |e \vee (c + \beta Pf) - [e \vee (c + \beta Pg)]|$$

$$\leqslant |c + \beta Pf - (c + \beta Pg)| = \beta \left| \sum_{x'} [f(x') - g(x')] P(x, x') \right|$$

$$\therefore \quad |(Tf)(x) - (Tg)(x)| \leqslant \beta \sum_{x'} |f(x') - g(x')| P(x, x')$$

$$\therefore \quad \|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

Now we know T has a unique fixed point $\bar{v}$ in $\mathbb{R}^{\mathsf{X}}$

Next we claim that

$$\bar{v} = v^*$$

We show

- $\bar{v} \leqslant v^*$ and
- $\bar{v} \geqslant v^*$

To prove $\bar{v} \leqslant v^*$, define $\sigma \in \Sigma$ by

$$\sigma(x) := \mathbb{1}\left\{e(x) \geqslant c(x) + \beta \sum_{x' \in \mathsf{X}} \bar{v}(x')P(x, x')\right\}$$

For this choice of σ, for any $x \in \mathsf{X}$,

$$(T_\sigma \bar{v})(x) = \sigma(x)e(x) + (1 - \sigma(x))\left[c(x) + \beta \sum_{x' \in \mathsf{X}} \bar{v}(x')P(x, x')\right]$$

$$= \max\left\{e(x),\, c(x) + \beta \sum_{x' \in \mathsf{X}} \bar{v}(x')P(x, x')\right\}$$

$$= (T\bar{v})(x) = \bar{v}(x)$$

In particular,

$$T_\sigma \, \bar{v} = \bar{v}$$

But the only fixed point of T_σ in $\mathbb{R}^\mathsf{X}$ is the σ-value function v_σ!

Hence $\bar{v} = v_\sigma$

But then $\bar{v} \leqslant v^*$, by the definition of v^*

- why?

It only remains to prove $\bar{v} \geqslant v^*$

Fix $\sigma \in \Sigma$

Since

1. T dominates T_σ on $\mathbb{R}^{\mathsf{X}}$ and
2. T is order-preserving and globally stable

we have $v_\sigma \leqslant \bar{v}$

- why?

Taking the max over $\sigma \in \Sigma$ implies $v^* \leqslant \bar{v}$

We have now proved that v^* is the unique solution to the Bellman equation!

Finding optimal policies

For $v \in \mathbb{R}^{\mathsf{X}}$, we call $\sigma \in \Sigma$ **v-greedy** if

$$\forall\, x \in \mathsf{X}, \quad \sigma(x) = \mathbb{1}\left\{ e(x) \geqslant c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\}$$

- treats v as the value function and optimizes

Ex. Show that $\sigma \in \Sigma$ is v^*-greedy if and only if $T_\sigma v^* = v^*$

Proof: We have

$$\sigma \in \Sigma \text{ is } v^*\text{-greedy} \iff \sigma e + (1 - \sigma)(c + \beta P v^*) = e \vee (c + \beta P v^*)$$

$$\iff T_\sigma v^* = v^*$$

Finding optimal policies

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$$\forall\, x \in \mathsf{X}, \quad \sigma(x) = \mathbb{1}\left\{ e(x) \geqslant c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\}$$

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Prop. $\sigma \in \Sigma$ is optimal $\iff$ σ is v^*-greedy

Proof: For $\sigma \in \Sigma$, the following are all equivalent

1. σ is v^*-greedy
2. $T_\sigma v^* = v^*$
3. $v^* = v_\sigma$

- Why are 2 and 3 equivalent?

This result is a version of **Bellman's principle of optimality**

Prop. $\sigma \in \Sigma$ is optimal $\iff$ σ is v^*-greedy

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1. σ is v^*-greedy
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- Why are 2 and 3 equivalent?

This result is a version of **Bellman's principle of optimality**

Corollary. The optimal stopping problem has exactly one optimal policy

Proof: For each $v \in \mathbb{R}^{\mathsf{X}}$, the greedy policy

$$\sigma^*(x) := \mathbb{1}\left\{ e(x) \geqslant c(x) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, x') \right\} \qquad (x \in \mathsf{X})$$

is uniquely defined

By the last Proposition, a policy is optimal iff it is v^*-greedy

Hence exactly one optimal policy exists

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Firm Valuation with Exit

Previously we discussed firm valuation under various scenarios

Value was defined as expected present value of profit stream

- a standard and popular methodology
- easy to apply

But it ignores an important fact

- firms have the option to exit and sell remaining assets

Now we consider firm valuation in the presence of this exit option

We consider a firm where

- $\pi_t = \pi(Z_t)$ for some fixed $\pi \in \mathbb{R}^{\mathsf{Z}}$
- $(Z_t)_{t \geqslant 0}$ is Q-Markov on finite set $\mathsf{Z} \subset \mathbb{R}$

At the start of each period, the firm decides whether to

- remain in operation, receiving current profit π_t, or
- exit, receiving $s > 0$ for sale of all assets

Discounting is at fixed rate r and $\beta := 1/(1+r)$

We assume that $r > 0$

Let Σ be all $\sigma \colon \mathsf{Z} \to \{0,1\}$

For given $\sigma \in \Sigma$ and $v \in \mathbb{R}^{\mathsf{Z}}$, the policy operator is

$$(T_\sigma v)(z) = \sigma(z)s + (1 - \sigma(z)) \left[\pi(z) + \beta \sum_{z'} v(z')Q(z,z') \right]$$

Recall that

- T_σ has a unique fixed point v_σ
- $v_\sigma(z) :=$ the value of following policy σ forever, given $Z_0 = z$

The Bellman operator is the order-preserving self-map T on $\mathbb{R}^{\mathsf{Z}}$ defined by

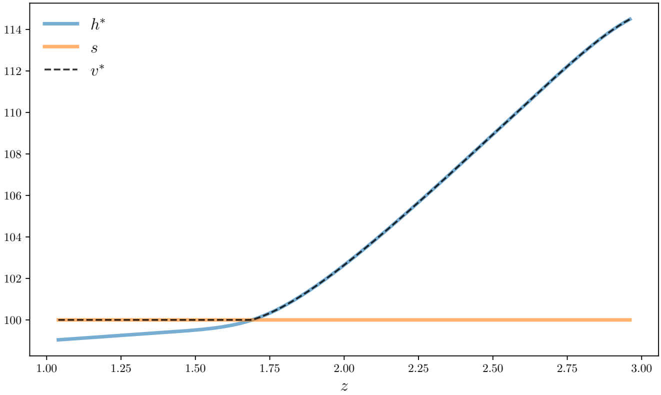
$$(Tv)(z) = \max \left\{ s, \pi(z) + \beta \sum_{z'} v(z') Q(z, z') \right\}$$

Pointwise, $Tv = s \vee (\pi + \beta Q v)$

Let v^* be the value function for this problem

- v^* is the unique fixed point of T in $\mathbb{R}^{\mathsf{Z}}$
- successive approximation from any $v \in \mathbb{R}^{\mathsf{Z}}$ converges to v^*
- a policy is optimal if and only if it is v^*-greedy

```julia
"Creates an instance of the firm exit model."
function create_exit_model(;
        n=200,                     # productivity grid size
        ρ=0.95, μ=0.1, ν=0.1,      # persistence, mean and volatility
        β=0.98, s=100.0            # discount factor and scrap value
    )
    mc = tauchen(n, ρ, ν, μ)
    z_vals, Q = mc.state_values, mc.p
    return (; n, z_vals, Q, β, s)
end
```


h^*
s
v^*
114
112
110
108
106
104
102
100
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
z

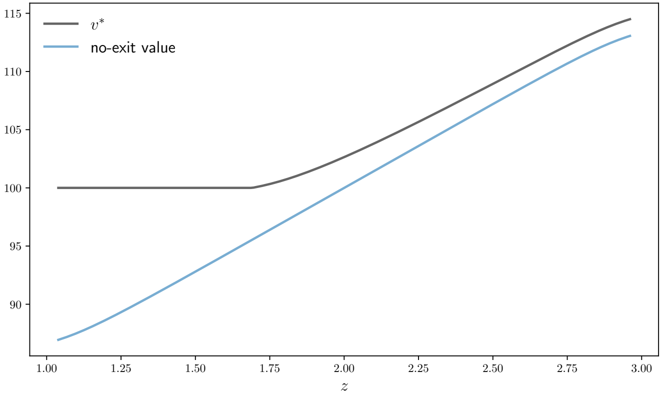
Exit vs No-Exit

Define w by

$$w(z) = \mathbb{E}_z \sum_{t \geqslant 0} \beta^t \pi_t$$

for all $z \in \mathsf{Z}$

- the value of the firm given $Z_0 = z$ when the firm never exits
- w evaluates the firm according to expected present value of the profit stream


v^*
no-exit value
115
110
105
100
95
90
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
z

Intuitively, $\underline{w \leqslant v^* \text{ due to the option value of exit}}$ — let's prove it

Since $\beta < 1$, we have $w = (I - \beta Q)^{-1} \pi$

Rearranging gives $w = \pi + \beta Q w$

If $\sigma \equiv 0$, then $T_\sigma v = \pi + \beta Q v$

Hence the unique fixed point of T_σ is w

Therefore $w = v_\sigma$ for $\sigma \equiv 0$

But then $w \leqslant v^*$

- why?

Suppose now productivity is constant but prices are stochastic

The price process (P_t) for the final good is Q-Markov

Let

- ℓ be labor input
- w be the wage rate (constant)

One-period profit for a given price p is $\max_{\ell \geqslant 0} \pi(\ell, p)$

Suppose that $\pi(\ell, p) = p\ell^{1/2} - w\ell$

Ex. Write down the Bellman equation for this model

For the model described, the Bellman equation takes the form

$$v(p) = \max\left\{ s,\, \max_{\ell \geqslant 0} \pi(\ell, p) + \beta \sum_{p'} v(p') Q(p, p') \right\}$$

Calculus shows that maximized one-period profits are

$$\pi(p) = \frac{p^2}{4w}$$

Hence the final expression is

$$v(p) = \max\left\{ s,\, \frac{p^2}{4w} + \beta \sum_{p'} v(p') Q(p, p') \right\}$$

Monotonicity

Now we consider monotonicity in values and actions

We return to the general optimal stopping problem, where

- X is the state space
- $(X_t)_{t \geqslant 0}$ is P-Markov on X
- β is the discount factor
- e is the exit reward function
- c is the continuation reward function

Let v^* be the value function

The **continuation value function** h^* is defined by

$$h^*(x) := c(x) + \beta \sum_{x' \in \mathsf{X}} v^*(x') P(x, x')$$

(continuation value function $\neq$ continuation reward function)

Pointwise this is $h^* = c + \beta P v^*$

Let

- X be partially ordered and
- $i\mathbb{R}^{\mathsf{X}}$ be the increasing functions in $\mathbb{R}^{\mathsf{X}}$

Lemma. If

1. $e, c \in i\mathbb{R}^{\mathsf{X}}$ and
2. P is monotone increasing,

then h^* and v^* are both increasing

Proof: Let the stated conditions hold

Since P is monotone increasing, P is invariant on $i\mathbb{R}^{\mathsf{X}}$

Since $Tv = e \vee (c + \beta P v)$, this implies T is invariant on $i\mathbb{R}^{\mathsf{X}}$

Since $i\mathbb{R}^{\mathsf{X}}$ is closed and nonemply, v^* is in $i\mathbb{R}^{\mathsf{X}}$

Since $h^* = c + \beta P v^*$, the same is true for h^*

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then h^* and v^* are both increasing

Proof: Let the stated conditions hold

Since P is monotone increasing, P is invariant on $i\mathbb{R}^{\mathsf{X}}$

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Since $h^* = c + \beta P v^*$, the same is true for h^*

Example. Consider the firm problem with exit

The Bellman operator is

$$Tv = s \vee (\pi + \beta Q v)$$

Since s is constant, it follows directly that v^* and h^* are both increasing functions when

- $\pi \in i\mathbb{R}^{\mathsf{Z}}$ and
- Q is monotone increasing

Monotone Actions

Suppose $\mathsf{X} \subset \mathbb{R}$

Lemma. σ^* is decreasing on X whenever e is decreasing on X and h^* is increasing on X

Proof: Fix $x, x' \in \mathsf{X}$ with $x \leqslant x'$

Suffices to show that $\sigma^*(x) = 0$ implies $\sigma^*(x') = 0$

So suppose that $\sigma^*(x) = 0$

Then $e(x) < h^*(x)$

As $x \leqslant x'$, we have $e(x') < h^*(x')$

Hence $\sigma^*(x') = 0$

Example. In the firm problem with exit, h^* is increasing when

1. $\pi \in i\mathbb{R}^{\mathsf{Z}}$ and
2. Q is monotone increasing

Since the exit reward constant, the optimal policy is decreasing

- exit is optimal when the state is small and
- continuing is optimal when z is large

Intuition?

Ex. Prove that σ^* is increasing on X whenever $e \in i\mathbb{R}^{\mathsf{X}}$ and h^* is decreasing on X

Example. In the IID job search problem

1. $e(w) = w/(1-\beta)$ is increasing and
2. h^* is constant

Therefore $\sigma^*(w) = \mathbb{1}\{w \geqslant (1-\beta)h^*\}$ is increasing

- The agent accepts all sufficiently large wage offers

Continuation Values

We used a "continuation value" approach to solving the job search problem with IID draws

1. compute the continuation value h^* directly
2. set the optimal policy to $\sigma^*(w) = \mathbb{1}\{w/(1-\beta) \geqslant h^*\}$

More efficient: one-dimensional rather than $|\mathsf{W}|$-dimensional

But the same trick does not work for the job search problem with Markov state

So when are continuation value methods better?

Let's start by thinking about how to compute h^* directly

Note that $v^* = e \vee h^*$

$$\therefore \quad c + \beta P v^* = c + \beta P(e \vee h^*)$$

$$\therefore \quad h^* = c + \beta P(e \vee h^*)$$

Motivates us to introduce the **continuation value operator**

$$(Ch)(x) = c(x) + \beta \sum_{x' \in \mathsf{X}} \max\left\{e(x'), h(x')\right\} P(x, x')$$

- An order-preserving self-map on $\mathbb{R}^{\mathsf{X}}$

Lemma. The following statements are true:

1. C is a contraction of modulus β on $\mathbb{R}^{\mathsf{X}}$
2. the unique fixed point of C in $\mathbb{R}^{\mathsf{X}}$ is h^*

Provides an alternative method to compute σ^*

1. Use successive approximation with C to compute h^*

   (up to an approximation)

2. Set $\sigma^*(x) = \mathbb{1}\{e(x) \geqslant h^*(x)\}$

Ex. Prove the contraction property using norm $\| \cdot \|_\infty$

Proof that $Ch = h$ implies $h = h^*$

Suppose $Ch = h$ and set $v := e \vee h$

Then

$$v = e \vee (Ch) = e \vee [c + \beta P(e \vee h)] = e \vee [c + \beta P v]$$

Hence $v = Tv$, implying $v = v^*$

- why?

Now observe that

$$h = Ch = c + \beta P(e \vee h) = c + \beta P v = c + \beta P v^* = h^*$$

Dimensionality Reduction

Let W and Z be finite

Suppose that

- the continuation reward depends only on z
- $\varphi \in \mathcal{D}(\mathsf{W})$ and Q is a stochastic matrix on Z
- (W_t) is IID with distribution φ
- (Z_t) is Q-Markov on Z, independent of (W_t)

Then $X_t := (W_t, Z_t)$ is P-Markov on X, where

$$P(x, x') = P((w, z), (w', z')) = \varphi(w') Q(z, z')$$

Bellman operator:

$$(Tv)(w,z) = \max\left\{e(w,z),\, c(z) + \beta \sum_{w'} \sum_{z'} v(w',z')\varphi(w')Q(z,z')\right\}$$

Continuation value operator:

$$(Ch)(z) = c(z) + \beta \sum_{w'} \sum_{z'} \max\left\{e(w',z'), h(z')\right\} \varphi(w')Q(z,z')$$

Note lower dimensionality!

Example. Consider the firm valuation problem with exit

Suppose now that scrap value is $(S_t) \stackrel{\text{IID}}{\sim} \varphi$ on $\mathbb{R}_+$

The Bellman operator is

$$(Tv)(z,s) = \max \left\{ s, \pi(z) + \beta \sum_{z'} \int v(z', s') \varphi(s') \, \mathrm{d}s' Q(z, z') \right\}$$

The continuation value operator is defined at $h \in \mathbb{R}^{\mathsf{Z}}$ by

$$(Ch)(z) = \pi(z) + \beta \sum_{z'} \int \max\{s', h(z')\} \varphi(s') \, \mathrm{d}s' Q(z, z')$$

Remaining function depends only on $z \in \mathsf{Z}$

Finite-Horizon American Call Options

Give the right to buy a given asset at any time up to T

- Strike prices is K
- Market prices is S_t
- Option can be exercised at $t \in \mathbb{N}$ with $t \leqslant T$

Let $\mathsf{T} := \{1, \ldots, T+1\}$ and $m(t) := (t+1) \wedge (T+1)$

Time is updated via $t' = m(t)$

(Bounding time at $T+1$ keeps the state space finite)

The share price is affected by a persistent and purely transient component:

$$S_t = Z_t + W_t$$

where

- $(W_t)_{t \geqslant 0} \overset{\text{IID}}{\sim} \varphi \in \mathcal{D}(\mathrm{W})$
- $(Z_t)_{t \geqslant 0}$ is Q-Markov on finite set Z

We choose parameters such that $(Z_t)_{t \geqslant 0}$ is close to a random walk

- price changes are difficult to predict

Can be viewed as an optimal stopping problem

- State space is $\mathsf{X} := \mathsf{T} \times \mathsf{W} \times \mathsf{Z}$ and
- $P((t, w, z), (t', w', z')) := \mathbb{1}\{t' = m(t)\} \varphi(w') Q(z, z')$
- Continuation reward is identically zero
- Discount rate is $\beta := 1/(1 + r)$ where $r > 0$
- Exit reward is $\mathbb{1}\{t \leqslant T\}(S_t - K)$

Hence all optimality results from optimal stopping theory apply

Bellman equation:

$$v(t, w, z) = \max\left\{e(t, w, z),\ \beta \sum_{w'} \sum_{z'} v(t', w', z')\varphi(w')Q(z, z')\right\}$$

- $v(t, w, z) =$ the state-contingent risk-neutral value of the option

The maximum of

- current exercise value and
- the discounted expected value of of carrying the option over to the next period

We can reduce dimensionality by switching to the continuation value operator

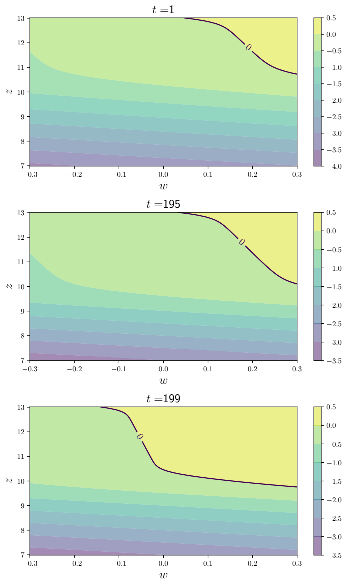
$$(Ch)(t,z) = \beta \sum_{z'} \sum_{w'} \max \left\{ e(t',w',z'),\, h(t',z') \right\} \varphi(w') Q(z,z')$$

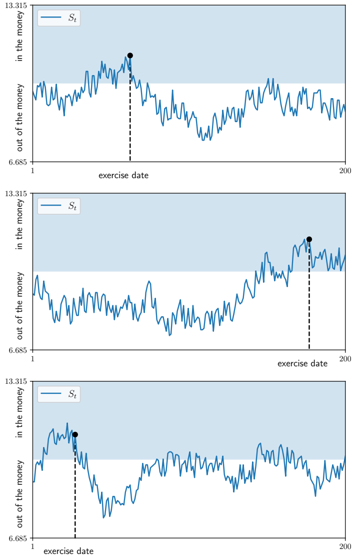
We know that

- the unique fixed point of C is h^*
- and $C^k h \to h^*$ as $k \to \infty$ for all $h \in \mathbb{R}^{\mathsf{X}}$
- the optimal policy is $\sigma^*(t,w,z) = \mathbb{1}\left\{ e(t,w,z) \geqslant h^*(t,z) \right\}$

```julia
using QuantEcon, LinearAlgebra, IterTools

"Creates an instance of the option model with log S_t = Z_t + W_t."
function create_american_option_model(;
        n=100, μ=10.0,  # Markov state grid size and mean value
        ρ=0.98, ν=0.2,  # persistence and volatility for Markov state
        s=0.3,          # volatility parameter for W_t
        r=0.01,         # interest rate
        K=10.0, T=200)  # strike price and expiration date
    t_vals = collect(1:T+1)
    mc = tauchen(n, ρ, ν)
    z_vals, Q = mc.state_values .+ μ, mc.p
    w_vals, φ, β = [-s, s], [0.5, 0.5], 1 / (1 + r)
    e(t, i_w, i_z) = (t ≤ T) * (z_vals[i_z] + w_vals[i_w] - K)
    return (; t_vals, z_vals, w_vals, Q, φ, T, β, K, e)
end
```


t =1
t =195
t =199
w
z
0


13.315
6.685
in the money
out of the money
S_t
1
200
exercise date
13.315
6.685
in the money
out of the money
S_t
1
200
exercise date
13.315
6.685
in the money
out of the money
S_t
1
200
exercise date

Research and Development

Consider a firm that engages in costly R&D

Choose when to

1. continue investing in the project or
2. stop and bring the product to market

Bringing to market yields single payoff $\pi_t = \pi(X_t)$

- $(X_t)_{t \geqslant 0}$ is P-Markov on finite set X

The flow cost of investing in R&D is C_t per period

Discount factor is $\beta := 1/(1+r)$ with $r > 0$

Suppose that $C_t \equiv c \in \mathbb{R}_+$ for all t

An optimal stopping problem with

- exit reward $e = \pi$ and
- constant continuation reward $-c$

Bellman equation:

$$v(x) = \max \left\{ \pi(x), -c + \beta \sum_{x'} v(x') P(x, x') \right\}$$

Ex. Prove that $h^* \in i\mathbb{R}^{\mathsf{X}}$ whenever $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone increasing

Now suppose $(C_t)_{t \geqslant 0} \overset{\text{IID}}{\sim} \varphi \in \mathcal{D}(\mathsf{W})$

The Bellman equation becomes

$$v(c, x) = \max \left\{ \pi(x), -c + \beta \sum_{x'} \sum_{c'} v(c', x') \varphi(c') P(x, x') \right\}$$

The continuation value function is

$$h(c, x) := -c + \beta \sum_{x'} \sum_{c'} v(c', x') \varphi(c') P(x, x')$$

Still depends on c and z !

- no dimensionality reduction

However, suppose we define

$$g(x) := \beta \sum_{x'} \sum_{c'} v(c', x') \varphi(c') P(x, x')$$

The Bell. eq. becomes $v(c', x') = \max \{\pi(x'), -c' + g(x')\}$

$$\therefore \quad \sum_{x'} \sum_{c'} v(c', x') \varphi(c') P(x, x')$$

$$= \sum_{x'} \sum_{c'} \max \{\pi(x'), -c' + g(x')\} \varphi(c') P(x, x')$$

Using the definition of g again gives

$$g(x) = \beta \sum_{x'} \sum_{c'} \max \{\pi(x'), -c' + g(x')\} \varphi(c') P(x, x')$$

To solve this functional equation we introduce

$$(Rg)(x) = \beta \sum_{x'} \sum_{c'} \max \left\{ \pi(x'), -c' + g(x') \right\} \varphi(c') P(x, x')$$

Ex. Prove that R is a contraction of mod β on $\mathbb{R}^{\mathsf{X}}$

The unique solution g^* can be computed by successive approx

With g^* in hand, we can compute the optimal policy via

$$\sigma^*(c, x) = \mathbb{1} \left\{ \pi(x), -c + g^*(x) \right\}$$

- there are multiple "versions" of the Bellman operator
- some are much more efficient than others