An Introduction to Computational Macroeconomics

Dynamic Programming: Chapter 7

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June – July 2022

Topics

1. MDPs with state-dependent discounting
2. Valuation and optimality theory
3. Application: Inventories revisited
4. Structural estimation
5. Optimal growth revisited
6. Refactoring the Bellman equation

MDPs with State-Dependent Discounting

Let A be a finite set, called the **action space**

The state variable is $X_t = (Y_t, Z_t)$, where

- $(Y_t)_{t \geqslant 0}$ is endogenous
- $(Z_t)_{t \geqslant 0}$ is purely exogenous and Q-Markov on Z
    - drives the discount factor process

The **state space** is $\mathsf{X} := \mathsf{Y} \times \mathsf{Z}$

An **MDP with state-dependent discounting** consists of elements (Γ, β, r, Q, R) where

1. Γ is a nonempty correspondence from $\mathsf{Y} \to \mathsf{A}$
2. β is a function from Z to $\mathbb{R}_+$
3. r is a function from $\mathsf{G} := \{(y, a) \in \mathsf{Y} \times \mathsf{A} : a \in \Gamma(y)\}$ to $\mathbb{R}$
4. Q is a stochastic matrix on Z
5. R is a stochastic kernel from G to Y

A summary of dynamics:

- (Z_t) is Q-Markov
- The **discount factor process** $(\beta_t)_{t \geqslant 0}$ obeys $\beta_t := \beta(Z_t)$
- Given $Y_t = y$ and current action a, current reward is $r(y, a)$
- Y_{t+1} is drawn from distribution $R(y, a, \cdot)$
- Y_{t+1} and Z_{t+1} are updated independently given (y, z, a)

The Bellman equation becomes

$$v(y,z) = \max_{a \in \Gamma(y)} \left\{ r(y,a) + \beta(z) \sum_{z',\, y'} v(y',z') Q(z,z') R(y,a,y') \right\}$$

for all $(y,z) \in \mathsf{X}$

Given $\sigma \in \Sigma$, the **policy operator** is

$$(T_\sigma\, v)(y,z) = r(y, \sigma(y,z)) + \\ \beta(z) \sum_{z',\, y'} v(y',z') Q(z,z') R(y, \sigma(y,z), y')$$

Notation

Let

$$\beta(x) = \beta(z, y) := \beta(z)$$

Given $\sigma \in \Sigma$, let

$$r_\sigma(x) := r_\sigma(y, z) := r(y, \sigma(y, z))$$

and

$$P_\sigma(x, x') := P_\sigma((y, z), (y', z')) := Q(z, z') R(y, \sigma(y, z), y')$$

- P_σ drives the state process $(X_t)_{t \geqslant 0}$ under σ

Fix $\sigma \in \Sigma$

Let (X_t) be P_σ-Markov with initial condition x

Proposition. Let L be defined by

$$L(z, z') := \beta(z) Q(z, z')$$

If $r(L) < 1$, then T_σ has a unique fixed point $\mathbb{R}^{\mathsf{X}}$, denoted by v_σ

Moreover,

$$v_\sigma(x) = \mathbb{E} \left\{ \sum_{t=0}^{\infty} \left[\prod_{i=0}^{t-1} \beta(X_i) \right] r_\sigma(X_t) \right\} \qquad (x \in \mathsf{X})$$

Assuming $r(L) < 1$, we can introduce the **value function** v^* via

$$v^*(x) = \max_{\sigma \in \Sigma} v_\sigma(x)$$

Given $v \in \mathbb{R}^{\mathsf{X}}$, a policy $\sigma \in \Sigma$ is called **v-greedy** if

$$\sigma(y, z) \in \operatorname*{argmax}_{a \in \Gamma(y)} \left\{ r(y, a) + \beta(z) \sum_{z', y'} v(y', z') Q(z, z') R(y, a, y') \right\}$$

for all $(y, z) \in \mathsf{X}$

A policy $\sigma \in \Sigma$ is called **optimal** if $v_\sigma = v^*$

Proposition If $r(L) < 1$, then

1. the Bellman operator T is globally stable on $\mathbb{R}^{\mathsf{X}}$ with unique fixed point v^* and
2. for each $\sigma \in \Sigma$, T_σ is globally stable on $\mathbb{R}^{\mathsf{X}}$ with unique fixed point v_σ
3. A feasible policy is optimal if and only it is v^*-greedy

Thus, the optimality results obtained for ordinary MDPs continue to hold whenever $r(L) < 1$

How tight is the condition $r(L) < 1$?

Ex. Show that $r(L) < 1$ when

$$\exists\, b < 1 \text{ such that } \beta(z) \leqslant b \text{ for all } z \in \mathsf{Z}$$

But this condition is too strict

Remember $\beta < 1$ and $\beta = 1/(1+r)$ implies $r > 0$

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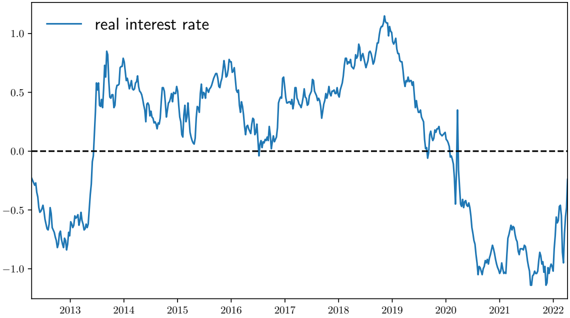

Figure: Real US interest rates are sometimes negative

Also, household preferences are sometimes assumed to have occasionally negative discount rates

- implies β_t sometimes > 1

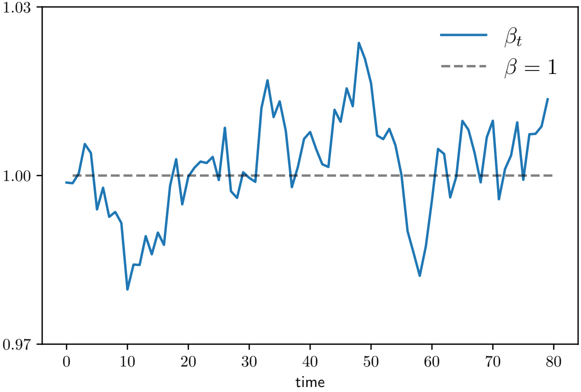

Figure: $(\beta)_{t \geqslant 0}$ process in Hills, Nakata and Schmidt (2019)

The process in Hills, Nakata and Schmidt (2019) is AR(1)

Following them, we discretize via Tauchen approximation

```
mc = tauchen(n, ρ, σ, 1 - ρ, m)
```

Parameters are as in Hills et al. (2019)

We find $r(L) = 0.9996$, so optimality results apply

In summary,

- $r(L) < 1$ allows the discount factor to exceed one at times
- Reasonable for economic applications
- Yet strong enough for optimality

Application: Inventory Management

Recall the inventory management model with Bellman equation

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{d \geqslant 0} v(m(x - d) + a)\varphi(d) \right\}$$

- $x \in \mathsf{X} := \{0, \dots, K\}$ is the current inventory level
- a is the current inventory order
- $r(x, a)$ is current profits
- $m(y) = y \vee 0$
- d is an IID demand shock with distribution φ

We now replace β with $\beta_t = \beta(Z_t)$

- $(Z_t)_{t \geqslant 0}$ is Q-Markov on Z

This is an MDP with state-dependent discounting

The Bellman equation becomes

$$v(x, z) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta(z) \sum_{d, z'} v(m(x - d) + a, z') \varphi(d) Q(z, z') \right\}$$

All optimality results hold when

- $L(z, z') := \beta(z) Q(z, z')$ and
- $r(L) < 1$

```julia
using LinearAlgebra, Distributions, OffsetArrays, QuantEcon

function create_sdd_inventory_model(;
        ρ=0.98, ν=0.002, n_z=20, b=0.97,  # Z state parameters
        K=40, c=0.2, κ=0.8, p=0.6)        # firm and demand parameters
    ϕ(d) = (1 - p)^d * p                  # demand pdf
    mc = tauchen(n_z, ρ, ν)
    z_vals, Q = mc.state_values .+ b, mc.p
    rL = maximum(abs.(eigvals(z_vals .* Q)))
    @assert  rL < 1 "Error: r(L) ≥ 1."    # check r(L) < 1
    return (; K, c, κ, p, ϕ, z_vals, Q)
end
```

Here

- $\beta_t := Z_t$
- the interest rate r_t is inferred from $\beta_t = 1/(1 + r_t)$

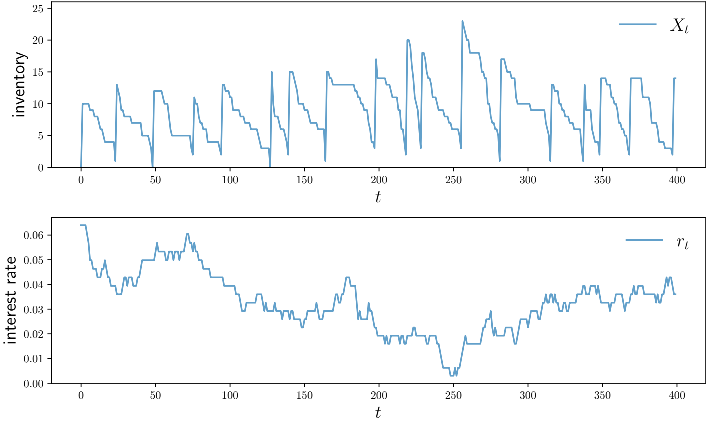

Figure: Inventory dynamics with time-varying interest rates