An Introduction to Computational Macroeconomics

Dynamic Programming: Chapter 3

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Introduction

Summary of this lecture:

- Introduction to partial orders
- Pointwise orders
- Order-preserving maps
- Fixed points and order
- Monotone Markov chains

Order

The next few slides give a quick introduction to **order theory**

One of the foundational subjects of maths, on par with

- algebra
- geometry
- topology
- number theory
- set theory

But not commonly taught in foundational math courses

Why?

Rarely used in

- physics
- chemistry
- biology, etc.

Math courses are biased toward these subjects!

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Math courses are biased toward these subjects!

But **very important for econ and related fields**

Examples.

- Does consumer X <u>prefer</u> good A or good B?
- Is welfare <u>greater</u> under policy A or policy B?
- Does R & D <u>increase</u> profits?
- How can firm Y <u>minimize</u> costs?

For these lectures, we need order for

- studying optimality
- fixed point results

Partial orders

Let P be a nonempty set

A **partial order** on a P is a binary relation $\preceq$ on $P \times P$ satisfying, for any p, q, r in P,

$p \preceq p$,

$p \preceq q$ and $q \preceq p$ implies $p = q$ and

$p \preceq q$ and $q \preceq r$ implies $p \preceq r$

(Reflexivity, antisymmetry, transitivity)

We call $(P, \preceq)$ (or just P) a **partially ordered set**

Ex.

1. Show that the usual order $\leqslant$ on $\mathbb{R}$ is a partial order on $\mathbb{R}$
2. Given set M, show that $\subset$ is a partial order on $\wp(M)$

Proof for 2: Clearly, for all $A, B, C \subset M$,

- $A \subset A$ holds
- $A \subset B$ and $B \subset A$ implies $A = B$
- $A \subset B$ and $B \subset C$ implies $A \subset C$

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A partial order $\preceq$ on P is called a **total order** if

$$\text{either } p \preceq q \text{ or } q \preceq p \text{ for all } p, q \in P$$

Example. $\leqslant$ is a total order on $\mathbb{R}$

Ex. Prove: $\subset$ is not a total order on $\wp(M)$ when $|M| > 1$

Proof: If M has more than two elements, then we can take nonempty $A, B \subset M$ with $A \cup B = \varnothing$

But then $A \subset B$ and $B \subset A$ both fail

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Pointwise Partial Orders

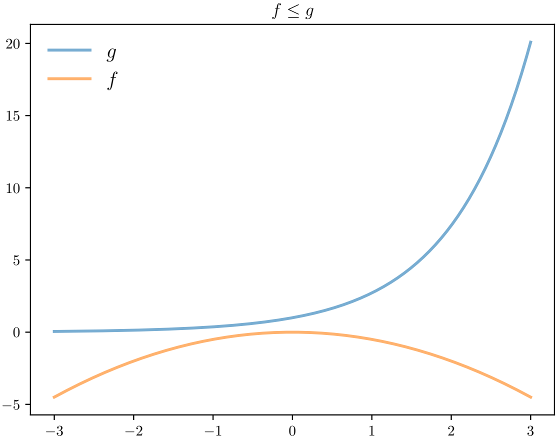
Let

- M be any set and
- let $\mathbb{R}^M$ be all $f\colon M \to \mathbb{R}$

The **pointwise partial order** over $\mathbb{R}^M$ is writen as $\leqslant$ and defined as follows:

- Given f, g in $\mathbb{R}^M$, we set

$$f \leqslant g \iff f(x) \leqslant g(x) \text{ for all } x \in M$$



$f \leq g$
g
f
20
15
10
5
0
−5
−3
−2
−1
0
1
2
3

Ex. Show $\leqslant$ is a partial order on $\mathbb{R}^M$

Proof:

Let's just check antisymmetry

Fix $f, g \in \mathbb{R}^M$ and suppose $f \leqslant g$ and $g \leqslant f$

Pick any $x \in M$

By definition, $f(x) \leqslant g(x)$ and $g(x) \leqslant f(x)$

Therefore, $f(x) = g(x)$

Since x was arbitrary, we have $f = g$

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Let's define the **pointwise partial order for matrices**

Let $\mathbb{M}^{n \times k} :=$ all $n \times k$ matrices

For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{M}^{n \times k}$, we set

$$A \leqslant B \iff a_{ij} \leqslant b_{ij} \text{ for all } i, j$$

Example.

$$\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} \leqslant \begin{pmatrix} 10 & 20 \\ 0 & 10 \end{pmatrix}$$

Ex. Show that $\leqslant$ is a partial order on $\mathbb{M}^{n \times k}$

Special case: **pointwise order for vectors**

Recall $[n] := \{1, \ldots, n\}$

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we write

$$x \leqslant y \quad \iff \quad x_i \leqslant y_i \text{ for all } i \in [n]$$

Pointwise partial order $\leqslant$ on $\mathbb{R}^2$:

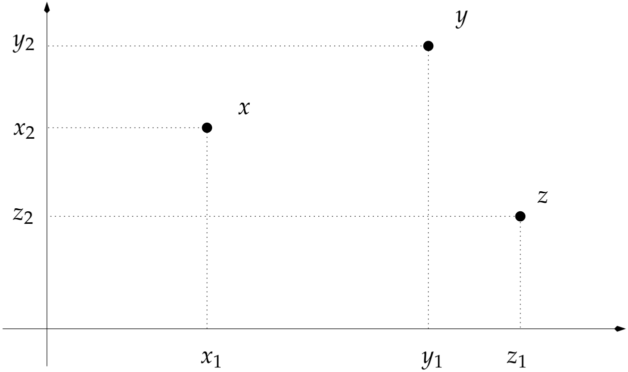

Figure: Pointwise we have $x \leqslant y$ but not $z \leqslant y$

Ex. Prove: for $a, b \in \mathbb{R}^n$ and sequence (x_k) in $\mathbb{R}^n$, we have

$$a \leqslant x_k \leqslant b \text{ for all } k \in \mathbb{N} \text{ and } x_k \to x \text{ implies } a \leqslant x \leqslant b$$

Proof: Fix $i \in [n]$

Let a_i be the i-th element of a, etc.

It suffices to show that

$$a_i \leqslant x_i \leqslant b_i \tag{1}$$

Note $x_k \to x$ implies $x_{i,k} \to x_i$

Moreover, $a_i \leqslant x_{i,k} \leqslant b_i$ for all k

Weak inequalities in $\mathbb{R}$ are preserved under limits, so (1) holds

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Weak inequalities in $\mathbb{R}$ are preserved under limits, so (1) holds

In other words, the pointwise partial order $\leqslant$ is preserved under limits

As a result, these sets are **closed**

- $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : 0 \leqslant x\}$
- $[a, b] := \{x \in \mathbb{R}^n : a \leqslant x \leqslant b\}$
- etc.

A key connection between order and topology!

Ex. Prove: If B is $m \times k$ and $B \geqslant 0$, then

$$|Bx| \leqslant B|x| \text{ for all } k \times 1 \text{ column vectors } x$$

Proof: Fix $B \in \mathbb{M}^{m\times k}$ with $b_{ij} \geqslant 0$ for all i, j

Fix $i \in [m]$ and $x \in \mathbb{R}^k$

By the triangle inequality, we have $|\sum_j b_{ij}x_j| \leqslant \sum_j b_{ij}|x_j|$

Stacking these inequalities yields

$$|Bx| \leqslant B|x|$$

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Lemma. Given a finite set M and f, g in $\mathbb{R}^M$, we have

$$|\max_{x\in M} f(x) - \max_{x\in M} g(x)| \leqslant \max_{x\in M} |f(x) - g(x)|$$

Proof: Fixing $f, g \in \mathbb{R}^M$, we have

$$f = f - g + g \leqslant |f - g| + g \qquad \text{(pointwise)}$$

$$\therefore \quad \max f \leqslant \max(|f - g| + g) \leqslant \max |f - g| + \max g$$

$$\therefore \quad \max f - \max g \leqslant \max |f - g|$$

Reversing the roles of f and g proves the claim

Order-preserving maps

Let

- $(P, \preceq)$ and $(Q, \trianglelefteq)$ be partially ordered sets
- $T \colon P \to Q$

T is called **order-preserving** if, for all $x, y \in P$,

$$x \preceq y \quad \implies \quad Tx \trianglelefteq Ty$$

- Meaning: If x goes up then Tx goes up
- Very important concept for dynamic programming

Example. Let $(P, \preceq) = (\mathcal{C}, \leqslant)$ where

- $\mathcal{C}$ is all continuous functions from $[a, b]$ to $\mathbb{R}$
- $\leqslant$ is the pointwise partial order

If $I \colon \mathcal{C} \to \mathbb{R}$ is defined by

$$Ig := \int_a^b g(x)dx \qquad (g \in \mathcal{C})$$

then I is order-preserving on $\mathcal{C}$

(Larger functions have larger integrals)

Example. Let $\leqslant$ denote the pointwise partial order on $\mathbb{R}^n$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Tx = Ax + b$

If $A \geqslant 0$, then T is order preserving on $\mathbb{R}^n$

Proof: Fix $x \leqslant y$

Then $0 \leqslant y - x$

$$\therefore \quad 0 \leqslant A(y - x) \leqslant Ay - Ax$$

$$\therefore \quad Ax \leqslant Ay$$

$$\therefore \quad Tx \leqslant Ty$$

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Special Case: Real-Valued Functions

Special case: maps from $(P, \preceq)$ into $(\mathbb{R}, \leqslant)$

Then "order-preserving" = "increasing"

In particular, we also call $h \in \mathbb{R}^P$

- **increasing** if $x \preceq y$ implies $h(x) \leqslant h(y)$ and
- **decreasing** if $x \preceq y$ implies $h(x) \geqslant h(y)$

Let P be partially ordered by $\preceq$

We write $i\mathbb{R}^P$ **for the increasing functions in $\mathbb{R}^P$**

Thus,

$$h \in i\mathbb{R}^P \quad \iff \quad x, y \in P \text{ and } x \preceq y \text{ implies } h(x) \leqslant h(y)$$

Example. Let $P = \{1, \ldots, n\}$ and let $\preceq$ be the usual order $\leqslant$ on $\mathbb{R}$

Then

- $x \mapsto 2x$ and $x \mapsto \mathbb{1}\{2 \leqslant x\}$ are in $i\mathbb{R}^P$
- $x \mapsto -x$ and $x \mapsto \mathbb{1}\{x \leqslant 2\}$ are not

Ex. Prove the following:

If $f, g \in i\mathbb{R}^P$, then

- $\alpha f + \beta g \in i\mathbb{R}^P$ when $\alpha, \beta \geqslant 0$
- $f \vee g \in i\mathbb{R}^P$
- $f \wedge g \in i\mathbb{R}^P$

Ex. Given finite P, show that $i\mathbb{R}^P$ is closed in $\mathbb{R}^P$

<u>Proof</u>: Take $(f_k)_{k\geqslant 1}$ in $i\mathbb{R}^P$ and $f \in \mathbb{R}^P$ with $f_k \to f$

Since $f_k \to f$ we have $f_k(z) \to f(z)$ for all $z \in P$

- norm convergence implies pointwise convergence

Fix $x, y \in P$ with $x \preceq y$

From $(f_k) \subset i\mathbb{R}^P$ we have $f_k(x) \leqslant f_k(y)$ for all k

Since weak inequalities are preserved under limits, $f(x) \leqslant f(y)$

Hence $f \in i\mathbb{R}^P$

Strict inequalities

We write

- $f \ll g$ if $f(x) < g(x)$ for all $x \in M$
- $x \ll y$ if $x_i < y_i$ for all $i \in [n]$
- $A \ll B$ if $a_{ij} < b_{ij}$ for all i, j

These are not partial orders

Ex. Why is $f \ll g$ not a partial order on $\mathbb{R}^M$?

Parametric Monotonicity

Let $(P, \preceq)$ be a partially ordered set

Given two self-maps S and T on P, we set

$$S \preceq T \iff Sx \preceq Tx \text{ for every } x \in P$$

We say that T **dominates** S on P

Ex. Show that $\preceq$ is a partial order on

$$\mathcal{S}_P := P^P := \text{ set of all self-maps on } P$$

Proof of antisymmetry of $\preceq$ on $\mathcal{S}_P$:

Let $(P, \preceq)$ and $S, T \in \mathcal{S}_P$ be as defined above

Suppose $S \preceq T$ and $T \preceq S$

Fix any $x \in P$

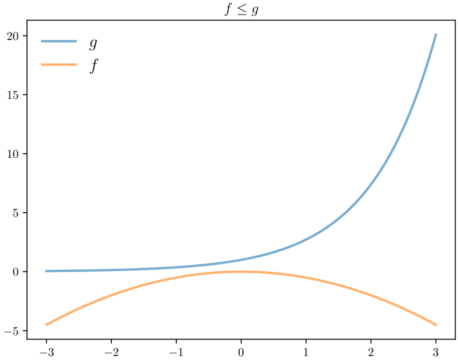
We have $Sx \preceq Tx$ and $Tx \preceq Sx$

Since $\preceq$ is antisymmetric on P, we have $Sx = Tx$

Since p was arbitrary, $S = T$

Hence $\preceq$ is antisymmetric on $\mathcal{S}_P$

Example. If $(\preceq, P) = (\leqslant, \mathbb{R})$, then $\leqslant$ is the pointwise partial order over functions



Example. Consider $\mathbb{R}^n_+$ with the pointwise partial order $\leqslant$

- Called the **positive cone** in $\mathbb{R}^n$

Let

- $Sx = Ax + b$
- $Tx = Bx + b$

Ex. Show that $0 \leqslant A \leqslant B \implies T$ dominates S on $\mathbb{R}^n_+$

Proof: Fixing $x \in \mathbb{R}^n_+$, suffices to show that $Sx \leqslant Tx$

Since $A \leqslant B$ and $x \geqslant 0$, we have $Ax \leqslant Bx$

Hence $Sx \leqslant Tx$

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Conjecture: If $S \preceq T$, then the fixed points of T will be larger

This is not true in general...

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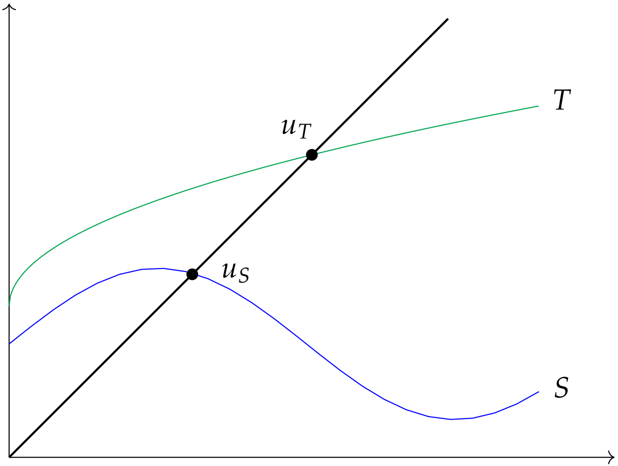
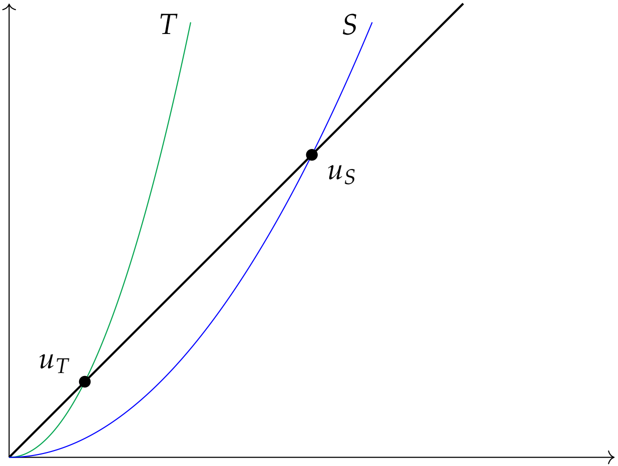
Sometimes true:



And sometimes false:



One difference: in the first case, T is globally stable

This leads us to our next result

Proposition. Let

- S and T be self-maps on $M \subset \mathbb{R}^n$
- $\leqslant$ be the pointwise partial order on M

If

1. T dominates S on M and
2. T is order-preserving and globally stable on M,

then the unique fixed point of T dominates any fixed point of S

Proof: Assume the conditions

Let

- u_T be the unique fixed point of T and
- u_S be any fixed point of S

Since $S \leqslant T$, we have $u_S = S u_S \leqslant T u_S$

Applying T to both sides of $u_S \leqslant T u_S$ gives

$$u_S \leqslant T u_S \leqslant T^2 u_S$$

Continuing in this fashion yields $u_S \leqslant T^k u_S$ for all $k \in \mathbb{N}$

Since $\leqslant$ is preserved under limits and T is globally stable,

$$u_S \leqslant \lim_k T^k u_S = u_T$$

Example. Recall that, in the job search model,

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h^* \right\} \varphi(w')$$

We found h^* as the fixed point of $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(w')$$

In the exercise, you showed that g is a contraction map on $\mathbb{R}_+$

Ex. Prove that the optimal continuation value h^* is increasing in β

Proof: Fix $\beta_1 \leqslant \beta_2$ and let

- $h_i^* :=$ fixed point corresponding to β_i
- $g_i :=$ fixed point map corresponding to β_i

Since $\beta_1 \leqslant \beta_2$, we have $g_1(h) \leqslant g_2(h)$ for all $h \in \mathbb{R}_+$

In addition,

1. g_2 is a contraction (so globally stable) and
2. g_2 is increasing

Hence $h_1^* \leqslant h_2^*$

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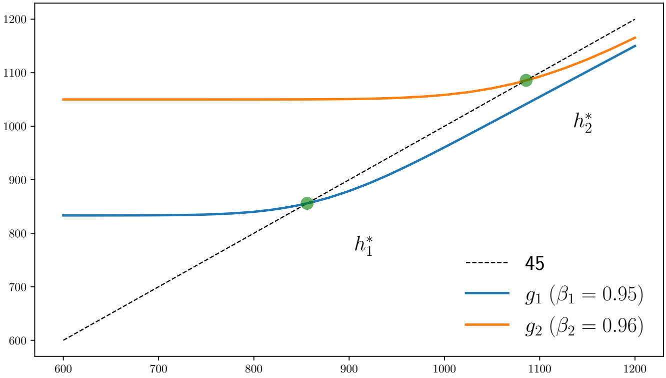
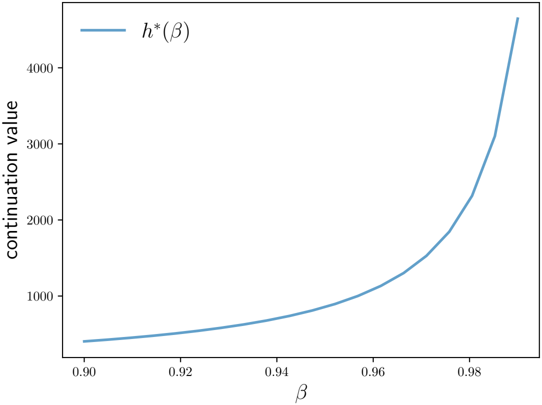


1200
1100
1000
900
800
700
600
600
700
800
900
1000
1100
1200
h_2^*
h_1^*
45
g_1 ($\beta_1 = 0.95$)
g_2 ($\beta_2 = 0.96$)

Ex. Replicate this figure



(First Order) Stochastic Dominance

Partial order over distributions!

Example. Equivalent:

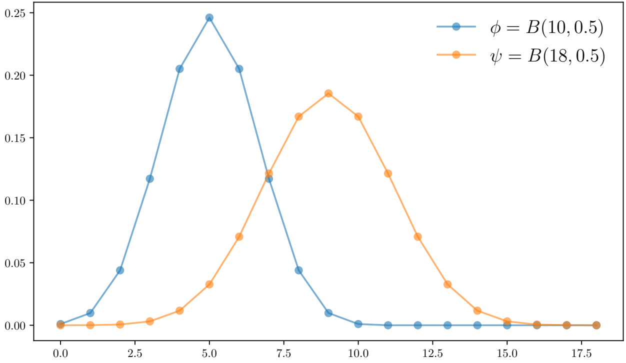
- $X \sim B(n, 0.5)$
- X counts the # of heads in n flips of a fair coin

Suppose $\varphi \stackrel{d}{=} X \sim B(10, 0.5)$ and $\psi \stackrel{d}{=} Y \sim B(18, 0.5)$

- Y counts over more flips, so "larger on average"

Hence we expect that ψ is "larger than" φ in some sense

Distribution ψ seems "larger than" φ — usually produces higher draws



But how can we make this idea precise?

Let X be a finite set partially ordered by $\preceq$

Fix $\varphi, \psi \in \mathcal{D}(\mathsf{X})$

Write $\langle u, \varphi \rangle$ for $\sum_x u(x)\varphi(x)$, etc.

We say that ψ **stochastically dominates** φ and write $\varphi \preceq_{\mathrm{F}} \psi$ if

$$u \in i\mathbb{R}^{\mathsf{X}} \quad \implies \quad \langle u, \varphi \rangle \leqslant \langle u, \psi \rangle$$

Example. If

- $\varphi \stackrel{d}{=} X \sim B(10, 0.5)$ and
- $\psi \stackrel{d}{=} Y \sim B(18, 0.5)$,

then $\varphi \preceq_{\mathrm{F}} \psi$

Proof: Fix $u \in i\mathbb{R}^{\mathsf{X}}$ and let

- $\mathsf{X} = \{0, \dots, 18\}$ and
- $W_1, \dots, W_{18}$ be IID Bernoulli with $\mathbb{P}\{W_i = 1\} = 0.5$ for all i

Then $X := \sum_{i=1}^{10} W_i \stackrel{d}{=} \varphi$ and $Y := \sum_{i=1}^{18} W_i \stackrel{d}{=} \psi$

Clearly $X \leqslant Y$

Hence $u(X) \leqslant u(Y)$

Hence $\mathbb{E}u(X) \leqslant \mathbb{E}u(Y)$

In other words,

$$\langle u, \varphi \rangle \leqslant \langle u, \psi \rangle$$

Example. An agent has preferences over outcomes in X

Preferences are determined by a utility function $u \in \mathbb{R}^{\mathsf{X}}$

The agent prefers more to less, so $u \in i\mathbb{R}^{\mathsf{X}}$

Suppose that the agent ranks lotteries over X according to expected utility

- evaluates $\varphi \in \mathcal{D}(\mathsf{X})$ according to $\sum_x u(x)\varphi(x)$

Then the agent (weakly) prefers ψ to φ whenever $\varphi \preceq_{\mathrm{F}} \psi$

Alternative definition

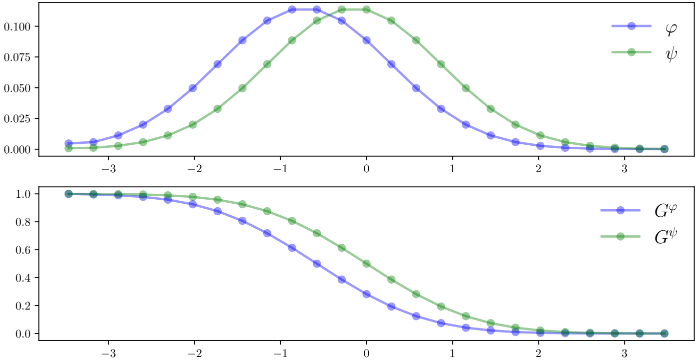
Given $\varphi \in \mathcal{D}(\mathsf{X})$, let

$$G^{\varphi}(y) := \sum_{x \in \mathsf{X}} \mathbb{1}\{y \preceq x\}\varphi(x) \qquad (y \in \mathsf{X})$$

This is the **counter** CDF of φ

Lemma. For each $\varphi, \psi \in \mathcal{D}(\mathsf{X})$, the following statements hold:

1. $\varphi \preceq_{\mathrm{F}} \psi \implies G^{\varphi} \leqslant G^{\psi}$
2. If X is totally ordered by $\preceq$, then $G^{\varphi} \leqslant G^{\psi} \implies \varphi \preceq_{\mathrm{F}} \psi$


0.100
0.075
0.050
0.025
0.000
−3
−2
−1
0
1
2
3
φ
ψ
1.0
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3
G^{φ}
G^{ψ}

Lemma. $\preceq_{\mathrm{F}}$ is a partial order on $\mathcal{D}(\mathsf{X})$

Proof:

Let's just prove transitivity

Suppose $f, g, h \in \mathcal{D}(\mathsf{X})$ with $f \preceq_{\mathrm{F}} g$ and $g \preceq_{\mathrm{F}} h$

Fixing $u \in i\mathbb{R}^{\mathsf{X}}$, we have

$$\langle u, f\rangle \leqslant \langle u, g\rangle \quad \text{and} \quad \langle u, g\rangle \leqslant \langle u, h\rangle$$

Hence $\langle u, f\rangle \leqslant \langle u, h\rangle$

Since u was arbitrary in $i\mathbb{R}^{\mathsf{X}}$, we are done

Monotone Markov Chains

A stochastic matrix P on $\mathsf{X} \times \mathsf{X}$ is called **monotone increasing** if

$$x, y \in \mathsf{X} \text{ and } x \preceq y \quad \Longrightarrow \quad P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$$

Example. Consider the AR(1) model $X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$

Apply Tauchen discretization, mapping to

- $n \times n$ stochastic matrix P on
- state space $\mathsf{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}$

Lemma. If $\rho \geqslant 0$ (+ve autocorrelation), then P is monotone increasing

Ex. Prove that P is monotone increasing if and only if P is invariant on $i\mathbb{R}^{\mathsf{X}}$

Proof of $\implies$

Suppose P is monotone increasing and fix $u \in i\mathbb{R}^{\mathsf{X}}$

We claim that $Pu \in i\mathbb{R}^{\mathsf{X}}$

To see this, pick any $x, y \in \mathsf{X}$ with $x \preceq y$

Since $P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$, we have

$$(Pu)(x) := \sum_{x'} u(x')P(x,x') \leqslant \sum_{x'} u(x')P(y,x') =: (Pu)(y)$$

Hence $Pu \in i\mathbb{R}^{\mathsf{X}}$, as was to be shown

Ex. Prove: If P is monotone increasing then so is P^t for all $t \in \mathbb{N}$

Proof by induction: Clearly true for $t = 1$

Suppose also true for arbitrary t

Then, for any $u \in i\mathbb{R}^{\mathsf{X}}$, we have $P^t u \in i\mathbb{R}^{\mathsf{X}}$

But P is monotone increasing, so this yields

$$P^{t+1} u = P P^t u \in i\mathbb{R}^{\mathsf{X}}$$

Hence P^{t+1} is invariant on $i\mathbb{R}^{\mathsf{X}}$

Hence monotone increasing

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Job Search Revisited

Now we return to the job search problem

Aims:

1. drop some of the restrictive assumptions we made earlier
2. analyze optimality

First extension: change wage draws are to be correlated

- More realistic than the IID setting
- Closer to standard research environments

Assume (W_t) is P-Markov on finite set $\mathsf{W} \subset \mathbb{R}_+$

The value function is denoted v^*

- $v^*(w)$ is maximum lifetime value given current wage offer is w

The value function satisfies the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w') \right\} \qquad (w \in \mathsf{W})$$

The corresponding Bellman operator is

$$(Tv)(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v(w') P(w, w') \right\}$$

Ex. Prove that T is an order-preserving self-map on $\mathcal{V} := \mathbb{R}_+^{\mathsf{W}}$

Proof of the order-preserving property

Given $f, g \in \mathcal{V}$ with $f \leqslant g$, we claim that $Tf \leqslant Tg$

Indeed, if $w \in \mathsf{W}$, then

$$\sum_{w' \in \mathsf{W}} f(w')P(w, w') \leqslant \sum_{w' \in \mathsf{W}} g(w')P(w, w')$$

Hence $(Tf)(w) \leqslant (Tg)(w)$

Since w was arbitrary, we have $Tf \leqslant Tg$

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Since w was arbitrary, we have $Tf \leqslant Tg$

Set

$$\|f - g\|_\infty = \max_{w \in \mathsf{W}} |f(w) - g(w)|$$

Ex. Prove that T is a contraction of modulus β on $\mathcal{V}$ with respect to the norm $\| \cdot \|_\infty$

Proof:

- Similar to the IID case
- Please complete as an exercise

Lemma. v^* is increasing on W whenever P is monotone increasing

Proof: Let $i\mathcal{V} :=$ increasing functions in $\mathcal{V}$

Since $i\mathcal{V}$ is closed, suffices to show that T is invariant on $i\mathcal{V}$

Fix $v \in i\mathcal{V}$

Then

- $h(w) := c + \beta(Pv)(w)$ is in $i\mathcal{V}$ and
- $e(w) := w/(1-\beta)$ is in $i\mathcal{V}$

It follows that $Tv = e \vee h$ is in $i\mathcal{V}$

We use value function iteration to solve for the value function

- Iterate from arbitrary guess v to get $v_k = T^k v$
- Compute the v_k-greedy policy

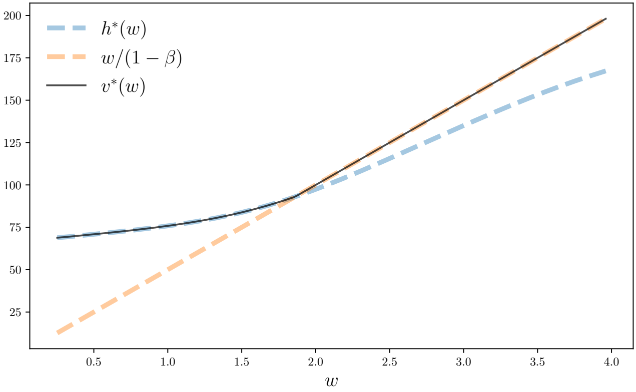
```julia
using QuantEcon, LinearAlgebra
include("s_approx.jl")

"Creates an instance of the job search model with Markov wages."
function create_markov_js_model(;
        n=200,        # wage grid size
        ρ=0.9,        # wage persistence
        ν=0.2,        # wage volatility
        β=0.98,       # discount factor
        c=1.0         # unemployment compensation
    )
    mc = tauchen(n, ρ, ν)
    w_vals, P = exp.(mc.state_values), mc.p
    return (; n, w_vals, P, β, c)
end
```

```julia
" The Bellman operator Tv = max{e, c + β P v} with e(w) = w / (1-β)."
function T(v, model)
    (; n, w_vals, P, β, c) = model
    h = c .+ β * P * v
    e = w_vals ./ (1 - β)
    return max.(e, h)
end

" Get a v-greedy policy."
function get_greedy(v, model)
    (; n, w_vals, P, β, c) = model
    σ = w_vals / (1 - β) .>= c .+ β * P * v
    return σ
end

"Solve the infinite-horizon Markov job search model by VFI."
function vfi(model)
    v_init = zero(model.w_vals)
    v_star = successive_approx(v -> T(v, model), v_init)
    σ_star = get_greedy(v_star, model)
    return v_star, σ_star
end
```

The a **continuation value function** is given by

$$h^*(w) := c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w') \qquad (w \in \mathsf{W}).$$

- depends on w due to correlated wages



$h^*(w)$
$w/(1-\beta)$
$v^*(w)$
200
175
150
125
100
75
50
25
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

Ex. Explain why h^* is increasing in the last figure

Answer Since $\rho > 0$, P is monotone increasing

Hence $v^* \in i\mathcal{V}$

Since $h^* = c + \beta P v^*$, it follows that $h^* \in i\mathcal{V}$

Positive autocorrelation in wages means that

- high current wages predict high future wages
- value of waiting rises with current wages

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Job Search with Separation

Let's now allow for separation

- matches between workers and firms terminate with probability α every period

Other aspects of the problem are unchanged

Conditional on current offer w, let

- $v_u^*(w)$ = max lifetime value for unemployed worker
- $v_e^*(w)$ = max lifetime value for employed worker

We have

$$v_u^*(w) = \max \left\{ v_e^*(w),\, c + \beta \sum_{w' \in \mathsf{W}} v_u^*(w') P(w, w') \right\}$$

and

$$v_e^*(w) = w + \beta \left[\alpha \sum_{w'} v_u^*(w') P(w, w') + (1 - \alpha) v_e^*(w) \right]$$

Proposition When $0 < \alpha, \beta < 1$, these equations both have unique solutions in $\mathcal{V}$

Step one: solve for v_e^* as

$$v_e^*(w) = \frac{1}{1 - \beta(1 - \alpha)} \left(w + \alpha\beta (P v_u^*)(w) \right)$$

Substitute to get

$$v_u^*(w) = \max \left\{ \frac{1}{1 - \beta(1 - \alpha)} \left(w + \alpha\beta (P v_u^*)(w) \right), \, c + \beta \, (P v_u^*)(w) \right\}$$

Ex.

- Prove that $\exists$ a unique $v_u^* \in \mathcal{V}$ that solves this equation
- Propose a convergent method for solving for both v_u^* and v_e^*

The stopping and continuation values are given by

$$s^*(w) := \frac{1}{1 - \beta(1 - \alpha)} (w + \alpha\beta(Pv_u^*)(w))$$

and

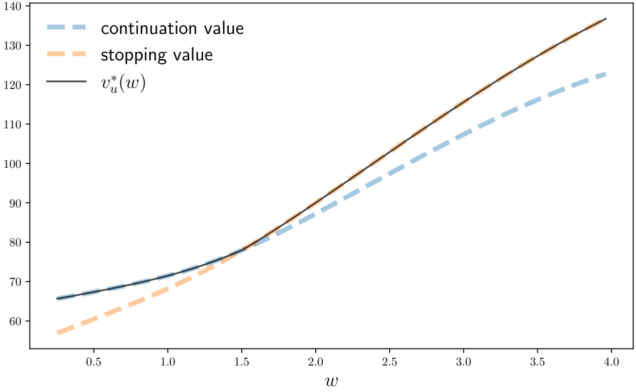
$$h_e^*(w) := c + \beta \, (Pv_u^*)(w)$$

Note $v_u^* = s^* \vee h^*$

Unemployed agent's optimal policy:

$$\sigma^*(w) := \mathbb{1}\{s^*(w) \geqslant h^*(w)\}$$

Reservation wage $w^* := \min\{w \in \mathsf{W} : s^*(w) \geqslant h^*(w)\}$

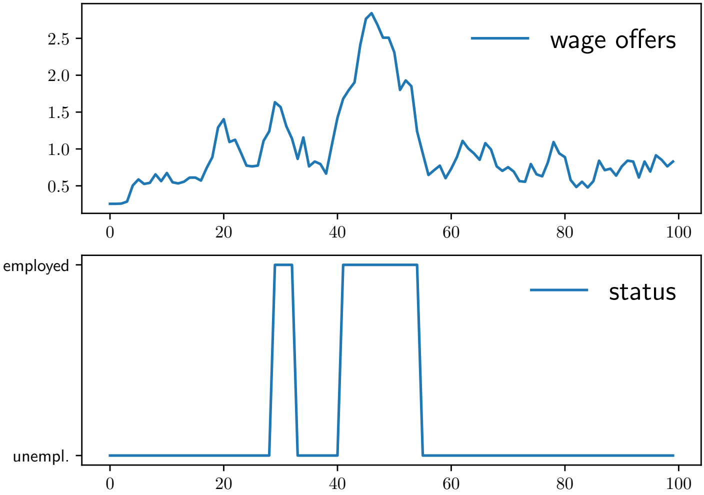


continuation value
stopping value
$v_u^*(w)$
140
130
120
110
100
90
80
70
60
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

```julia
include("markov_js_with_sep.jl")   # Code to solve model
using Distributions

# Create and solve model
model = create_js_with_sep_model()
(; n, w_vals, P, β, c, α) = model
v_star, σ_star = vfi(model)

# Create Markov distributions to draw from
P_dists = [DiscreteRV(P[i, :]) for i in 1:n]

function update_wages_idx(w_idx)
    return rand(P_dists[w_idx])
end
```

```julia
function sim_wages(ts_length=100)
    w_idx = rand(DiscreteUniform(1, n))
    W = zeros(ts_length)
    for t in 1:ts_length
        W[t] = w_vals[w_idx]
        w_idx = update_wages_idx(w_idx)
    end
    return W
end
```

```
function sim_outcomes(; ts_length=100)
    status = 0
    E, W = [], []
    w_idx = rand(DiscreteUniform(1, n))
    ts_length = 100
    for t in 1:ts_length
        if status == 0
            status = σ_star[w_idx] ? 1 : 0
        else
            status = rand() < α ? 0 : 1
        end
        push!(W, w_vals[w_idx])
        push!(E, status)
        w_idx = update_wages_idx(w_idx)
    end
    return W, E
end
```



wage offers
2.5
2.0
1.5
1.0
0.5
0
20
40
60
80
100
employed
unempl.
status
0
20
40
60
80
100

Ex. Here's an open-ended optional exercise

Let $E_t =$ employment status

- Show $X_t = (W_t, E_t)$ is a Markov chain
- Write down the state space and prove irreducibility

Let ψ^* be the unique stationary distribution

Ergodicity: fraction of time a worker spends unemployed should be equal to prob of unemployment under ψ^*

- Check it

Prob of unemployment under ψ^* equals unemployment rate

Adjust model parameters to match current umemployment rate