SudoCode: an Analysis of 2 Sudoku Solving Algorithms

Jordan Stebner-Hoang
https://github.com/jstebner/sudocode

Introduction

This project compares methods for solving sudoku puzzles of arbitrary order N, such that the grid that makes up the puzzle is a N*N grid of N*N sub grids, resulting in a N^2*N^2 grid of numbers (e.g., an "order 3 puzzle" would be a classic 9x9 sudoku puzzle). Given an incomplete, solvable, order N sudoku puzzle, the algorithm should return a completed puzzle such that:

- Every symbol on the grid is one of N^2 unique symbols
- Every row of the grid contains N^2 unique symbols
- Every column of the grid contains N^2 unique symbols
- Every N*N sub grid of the grid contains N^2 unique symbols

The algorithms will be measured by comparing the time taken to solve each puzzle in a set of incomplete order N sudoku puzzles (though, N will likely not exceed 6 due to computing constraints) with M missing symbols (blank spaces in the grid). The 2 algorithms I will compare are:

a. Backtracking in row-wise order. This approach will iterate over every cell in the grid filling in empty cells with the first possible symbol it can be (not violating any rules stated above) until the board is filled and valid. If a cell has no possible symbol, then the algorithm will "backtrack" and update previous "guesses" in LIFO order until the board is completed. I could perform this algorithm with either an iterative or recursive approach, but I have not yet decided which I would like to do (although I'm slightly more attracted to a recursive approach due to simplicity of implementation).

b. Constraint propagation with wave function collapse. This approach will first iterate over each unfilled cell in the grid and record the set of possible states it can exhibit. After doing so, it will iterate over each cell existing in a "superposition" of states and filter down its possible states by propagating the puzzle's constraints (listed above in (1)), reducing the "entropy" (number of possible states) each cell exhibits, collapsing the cell to a definite value if it can only exist in a single possible state. If, when filtering, the algorithm makes a full pass over the grid without reducing the entropy of any cell, the algorithm will collapse the state of the cell with lowest entropy using the first possible state and resume searching for the final grid state like backtracking but applying constraint propagation at each guess made.

Design

**Note: for the algorithms:*

- *Purple indicates functions*
  - *All external function calls are defined below*
- *Red indicates variables*
- *Blue indicates references to variables*
- *Bold and capitalized indicates operations*
- *Green indicates comments*

Supporting algorithms:

```
Function possible, accepting parameter grid, row, col, value: // O(n^2)
        IF value is in grid at (row, :) THEN
                RETURN False
        IF value is in grid at (:, col) THEN
                RETURN False
        IF value is in the group of grid at (row, col) THEN
                RETURN False
        RETURN True
```

```
Function filled, accepting parameter grid: // O(n^4)
        FOR each row index in the grid DO
        .       FOR each col index in the grid DO
        .       .       IF grid at (row, col) is 0 THEN
        .       .               RETURN False
        .       .       IF grid at (row, col) is a set THEN
        .       .               RETURN False
        .       END FOR
        END FOR
        RETURN True
```

a. Backtracking

```
Function solve_BT, accepting parameter grid:
        FOR each row index in the grid DO
        .       FOR each col index in the grid DO
        .       .       IF the value of the grid at (row, col) isn't 0 THEN
        .       .               SKIP this col and continue to the next col
        .       .       FOR each symbol in the set of possible symbols DO
        .       .       .       IF it is possible to set grid at (row, col) to symbol THEN
        .       .       .       .       SET grid at (row, col) to symbol
        .       .       .       .       CALL solve_BT(grid)        // recursive call
        .       .       .       .       IF grid is not filled with valid symbols THEN
        .       .       .       .               SET grid at (row, col) to 0
        .       .       .       END IF
        .       .       END FOR
        .       .       RETURN
        .       END FOR
        END FOR
```

b. Constraint Propagation + Backtracking

```
Function solve_CP, accepting parameter grid:
    IF there are any 0s in grid THEN                // create states
    .       FOR each row in the grid DO
    .       .       FOR each col in the grid DO
    .       .       .       IF the value of the grid at (row, col) isn't 0 THEN
    .       .       .               SKIP this col and continue to the next col
    .       .       .       CREATE empty set named states
    .       .       .       FOR each symbol in the set of possible symbols DO
    .       .       .       .       IF it is possible to set grid at (row, col) to symbol THEN
    .       .       .       .               ADD symbol to states
    .       .       .       END FOR
    .       .       .       SET grid at (row, col) to states
    .       .       END FOR
    .       END FOR
    END IF
    CREATE a number named updates starting at 1
    WHILE updates is greater than 0 DO              // prune search space
    .       SET updates to 0
    .       FOR each row in the grid DO
    .       .       FOR each col in the grid DO
    .       .       .       IF the value of the grid at (row, col) isn't a set THEN
    .       .       .               SKIP this col and continue to the next col
    .       .       .       IF the set in the grid at (row, col) is empty THEN
    .       .       .               RETURN
    .       .       .       IF the set in the grid at (row, col) has 1 element THEN          // singleton collapse
    .       .       .       .       FOR each cell in the same row, col, and group as the cell at (row, col) DO
    .       .       .       .       .       IF the value in the cell is a set THEN
    .       .       .       .       .       .       REMOVE the element from the set stored in cell
    .       .       .       .       .       END IF
    .       .       .       .       END FOR
    .       .       .       .       INCREMENT updates
    .       .       .       .       SET grid at (row, col) to the element in the set
    .       .       .       .       SKIP this col and continue to the next col
    .       .       .       END IF
    .       .       .       ELSE    // polyzygotic propagation
    .       .       .       .       FOR each subset of [row, col, and group] that the cell at (row, col) is part of DO
    .       .       .       .       .       CREATE a count of cells in the subset matching the cell at (row, col)
    .       .       .       .       .       IF count is less than the length of the set in the cell at (row, col) THEN
    .       .       .       .       .       .       SKIP this col and continue to the next col
    .       .       .       .       .       END IF
    .       .       .       .       .       ELSE IF count is greater than the length of the set in the cell at (row, col) THEN
    .       .       .       .       .       .       RETURN
    .       .       .       .       .       END IF
    .       .       .       .       .       FOR each cell in the same subset as the cell at (row, col) DO
    .       .       .       .       .       .       IF the cell has a set that isn't equal to the set in the cell at (row, col) THEN
    .       .       .       .       .       .       .       REMOVE elements from the cell at (row, col) from cell
    .       .       .       .       .       .       .       INCREMENT updates
    .       .       .       .       .       .       END IF
    .       .       .       .       .       END FOR
    .       .       .       .       END FOR
    .       .       .       END ELSE
    .       .       .       // more propagation checks can be added in series here
```

```
.         .         .         FOR each symbol in the set in the grid at (row, col) DO          // elimination collapse
.         .         .         .         FOR each subset of [row, col, and group] that the cell at (row, col) is part of DO
.         .         .         .         .         IF the symbol isn't in any set in the current subset THEN
.         .         .         .         .         .         FOR each cell in the same row, col, and group as the cell at (row, col) DO
.         .         .         .         .         .         .         IF the value in the cell is a set THEN
.         .         .         .         .         .         .         .         REMOVE the element from the set stored in cell
.         .         .         .         .         .         .         END IF
.         .         .         .         .         .         END FOR
.         .         .         .         .         .         SET grid at (row, col) to the element in the set
.         .         .         .         .         .         INCREMENT updates
.         .         .         .         .         END IF
.         .         .         .         END FOR
.         .         .         END FOR
.         .         END FOR
.         END FOR
END WHILE
// backtrack
CREATE r_min, c_min both starting with Null
FOR each row index in the grid DO
.         FOR each col index in the grid DO
.         .         IF the value of the grid at (row, col) isn't a set THEN
.         .                   SKIP this col and continue to the next col
.         .         IF r_min and c_min are Null THEN
.         .                   SET r_min to row and c_min to col
.         .         ELSE IF the size of the set in grid at (row, col) is smaller than the set in grid at (r_min, c_min) THEN
.         .                   SET r_min to row and c_min to col
.         END FOR
END FOR
CREATE copy of grid named backup
IF r_min, c_min are both not Null THEN
.         FOR each symbol in the set in grid at (r_min, c_min) DO
.         .         SET grid at (r_min, c_min) to symbol
.         .         CALL solve_CP(grid)          // recursive call
.         .         IF grid is not filled with valid symbols THEN
.         .         .         SET grid to a copy of backup
.         .         END IF
.         .         ELSE
.         .         .         RETURN
.         .         END ELSE
.         END FOR
END IF
```

Analysis

N: "order" of puzzle (classic 9x9 would be N=3)
k: number of elements removed

	Time complexity	Space Complexity	Basic operations	Input Size Consideration
Algo1: Backtrack	O(N^{2k})	O(k)	Hash map search	Input must be a N^2-by- N^2 grid, with k removed elements
Algo2: Const Prop	O(N^{12k})	O($k^2 N^2$)	Hash map search, set difference	Input must be a N^2-by- N^2 grid, with k removed elements

Hypothesis

I believe Algorithm 2 will run faster mostly since it aims to heavily reduce the number of recursive calls by pruning the search space repeatedly (and partially because I'm biased towards it since I designed it). While on paper it appears as though it will perform worse in time and space, I am most interested in seeing how many fewer recursive calls it will make than Algorithm 1 (i.e., fewer guesses). I believe that with enough constraint checks, a sudoku solving algorithm can solve a board with 0 guesses, and Algorithm 2 is a step in that direction.

Test plan

There are 3 files (n=2,3,4) that contain sudoku puzzles with k removed elements, where the largest k depends on n:

- n=2: max(k)=12
- n=3: max(k)=64
- n=4: max(k)=200

Each (n, k) has 40 randomly generated puzzles whose times, space, recursive calls, and validity will be averaged when comparing results. The expected results will be a valid puzzle created by each solving algorithm (instead of using an expected result puzzle, as some solvable puzzles have multiple valid solutions).

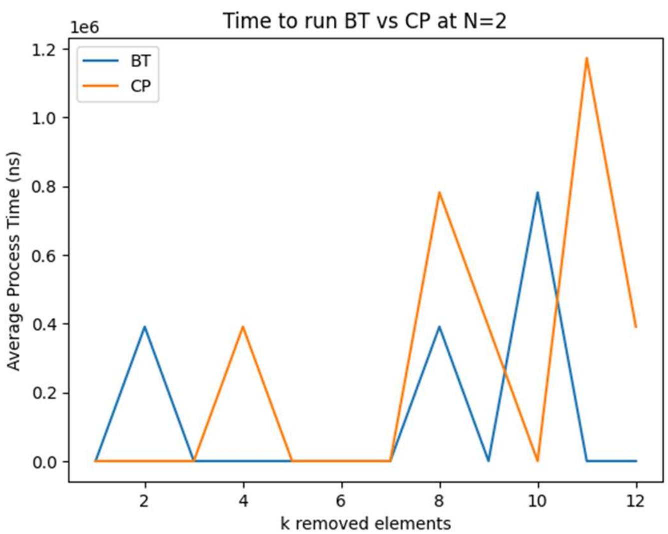
(*I wanted to have up to n=7, but my computer was unable to generate a single n=5 puzzle so I'm mostly going to be comparing changing values of k across 3 classes of n)*

All data can be found in sudocode/data/*.csv

Results

** BT refers to Algorithm 1 (backtracking), while CP refers to Algorithm 2 (constraint propagation)*

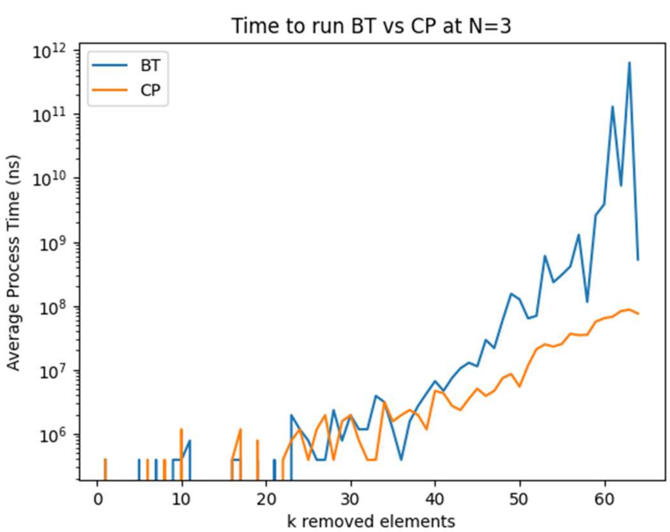
Time



The graph to the left shows the average time taken to run each algorithm across a range of k removed values for order 2 puzzles.

Since order 2 puzzles have few elements, the range of possible removed values is small and doesn't show a clear trend between number of removed values and time taken.

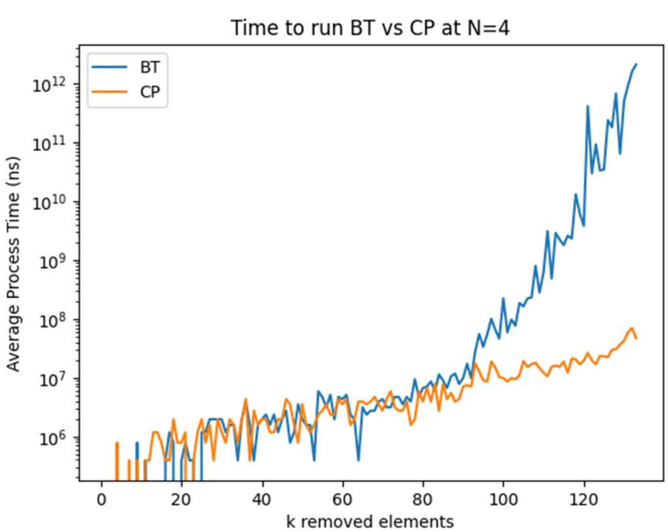
We can see, however, that the time taken on order 2 puzzles is generally higher for CP than BT.



The graph to the left shows the average time taken to run each algorithm across a range of k removed values for order 3 puzzles.

The vertical axis is on a logarithmic scale to show the exponential growth of time taken as k increases. At lower k values (up to ~23), the time taken varies around 0 nano seconds, however at higher k values we can see an upward trend for both BT and CP.

We can see that as k grows for order 3 puzzles, the time taken by BT is significantly larger than CP.

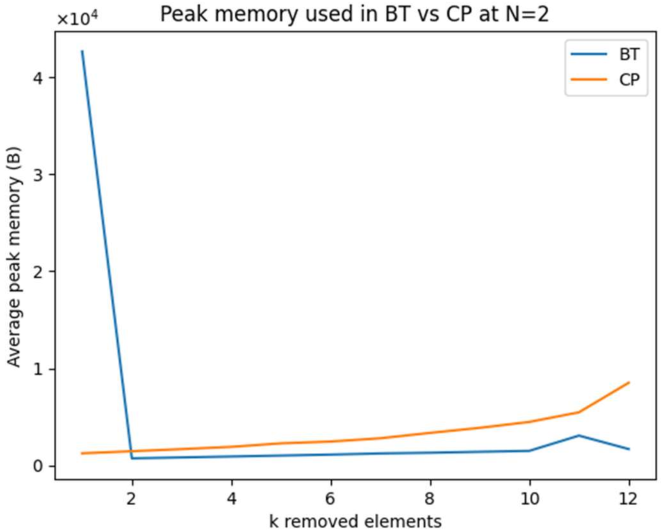


The graph to the left shows the average time taken to run each algorithm across a range of k removed values for order 4 puzzles.

The vertical axis is once again on a logarithmic scale, as the difference between BT and CP is drastic on a linear scale.

We can see that at lower values of k (<90), BT and CP perform similarly on order 4 puzzles, taking roughly 3 ms with a slight upward trend. However, for higher values of k, the difference between BT and CP becomes large.

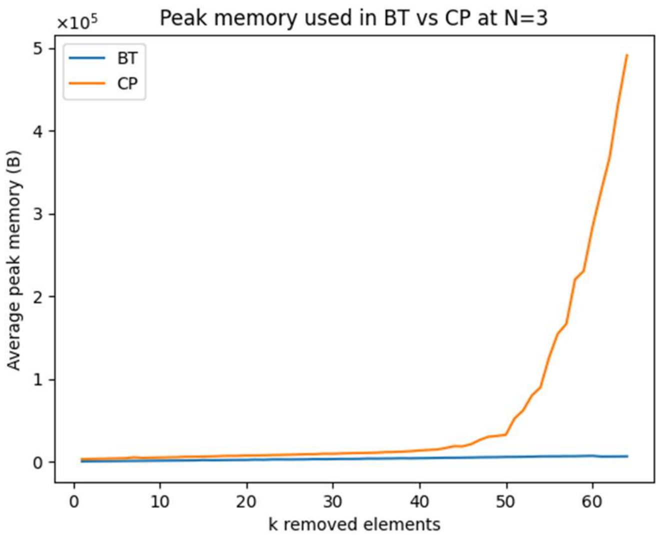
Space



The graph to the left shows the average peak memory when running each algorithm across a range of k removed values for order 2 puzzles.

Again, since order 2 puzzles are small, we can see odd trends in the graph for BT and CP.

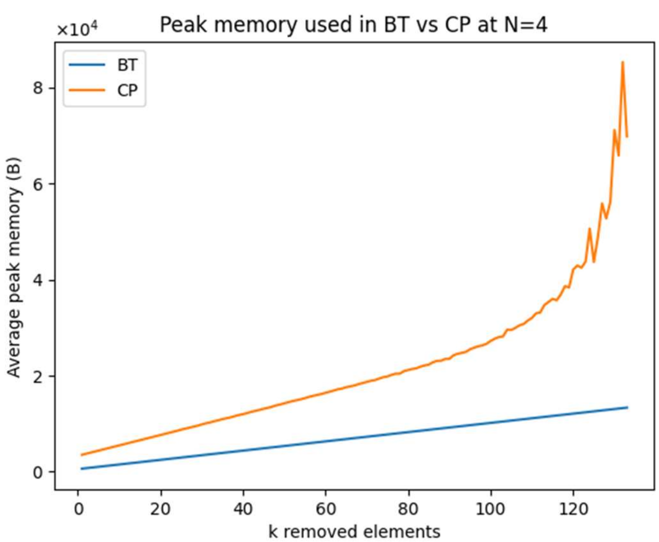
A possible trend we can see from this graph is the slight incline of CP, while BT stays very close to 0.



The graph to the left shows the average peak memory when running each algorithm across a range of k removed values for order 3 puzzles.

This graph offers much more information on the trend of space usage by each algorithm, showing that peak memory is exponentially related to k in CP, while BT once again stays very close to 0.

While this trend is drastic, we should keep in perspective that the highest peak memory used by CP is ~$5*10^5$ Bytes, or 500 KB of memory.



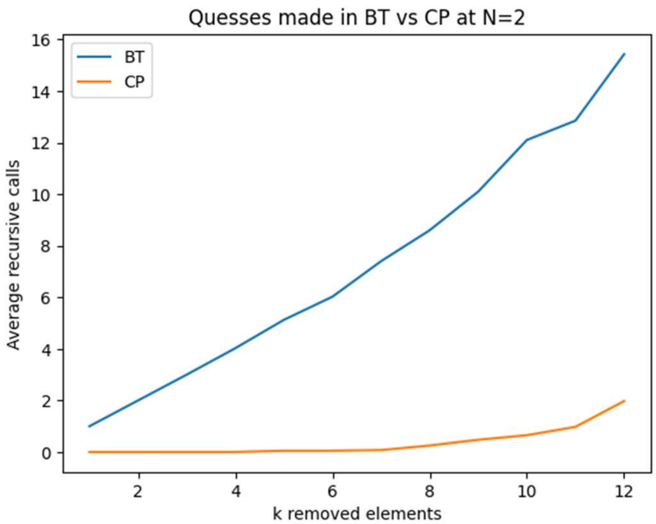
The graph to the left shows the average peak memory when running each algorithm across a range of k removed values for order 2 puzzles.

This graph shows that the peak memory usage as k increased is nonlinear for CP and linear for BT in order 4 puzzles which roughly matches what was predicted in the analysis section.

I like this graph a lot.

Recursive Calls

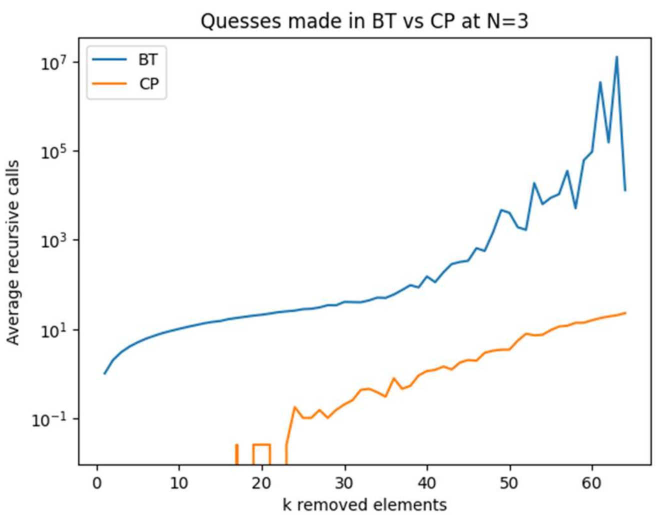
*(TOTAL recursive calls, not depth of recursion)**



The graph to the left shows the average number of recursive calls when running each algorithm across a range of k removed values for order 2 puzzles.

We can see that the number of recursive calls grows linearly as k increase for BT and stays close to 0 for CP.

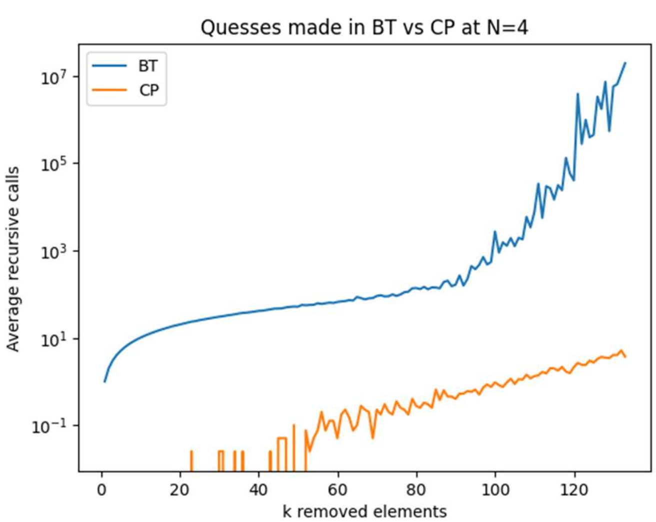
While this is for order 2 puzzles and therefore lacks many data points, I am happy to see that the number of recursive calls for CP stays very low, which was the intention when designing the algorithm.



The graph to the left shows the average number of recursive calls when running each algorithm across a range of k removed values for order 3 puzzles.

The vertical axis is once again on a logarithmic scale, as the difference between BT and CP is drastic on a linear scale.

We can see that the number of recursive calls for BT begins to grow quickly at higher values of k, while CP gets to at most ~10.



The graph to the left shows the average number of recursive calls when running each algorithm across a range of k removed values for order 4 puzzles

The vertical axis is once again on a logarithmic scale, as the difference between BT and CP is drastic on a linear scale.

This graph looks very similar to the order 3 graph on average recursive calls, and shows in higher detail the difference between BT and CP.

Conclusion

Overall, I would say that my hypothesis was true that the constraint propagation algorithm took less process time and recursive calls than the backtracking algorithm. For both metrics on higher values of k, CP performed much better than BT, even though the time complexity for each algorithm suggested otherwise (in the Analysis section). I somewhat expected this, however, since according to the worst case the CP algorithm would perform many more steps than BT within the same possibility space, even though the extra steps performed are to prune that space and reach a goal state more quickly.

The space complexity estimates roughly matched the actual test results, however the memory used by each algorithm was still relatively small (staying under a MB). As my hypothesis was focused on the speed of the solving algorithms, I will turn a blind eye to the fact that backtracking outperformed my constraint propagation algorithm.

I initially wanted to go up to N=7, but I couldn't generate an order 5 puzzle and instead stayed with order 2, 3, and 4. I also created puzzles of order 4 up to k=200, but I was only able to run tests up to k=132 as each test took on average ~20 minutes to run and I was running out of time.

An additional metric was collected for each test, the validity of a solution to the puzzle returned by the algorithm (can be seen in sudocode/out/results.csv). This was to ensure that the algorithms didn't return and invalid puzzle; however, all solutions were valid.

References

https://www.youtube.com/watch?v=G_UYXzGuqvM&t=1s&ab_channel=Computerphile

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