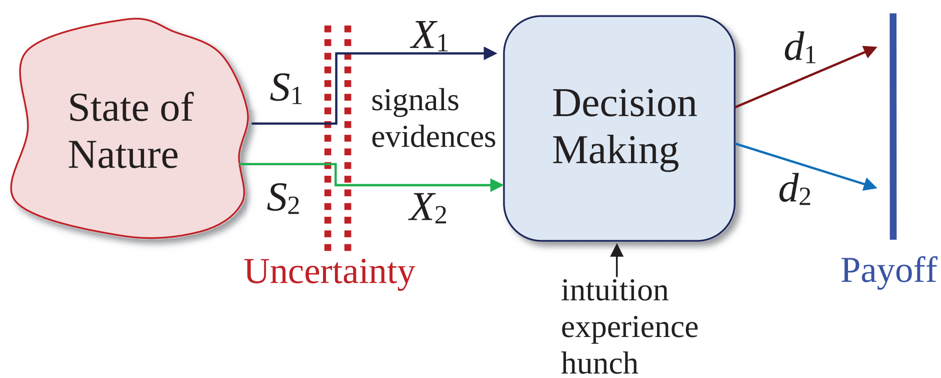
Session 11. Decision Analysis

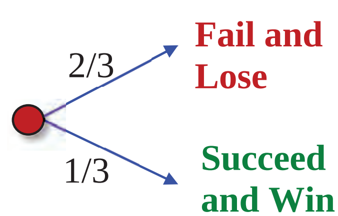
We have all had to make important decisions where we were uncertain about factors that were relevant to the decisions. In this session, we study situations in which decisions are made in an uncertain environment.

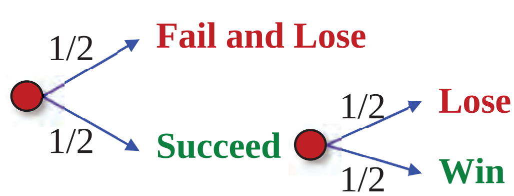


Ex] Buzzer Beater

Suppose a team is down by two points and it has time for one last shot. Let's say there's a 50% chance of scoring on a two-point shot and pushing the game into overtime, but only a 33% chance of making a three-point shot and getting the immediate win. What play should the coach call?"

- **Option #1**: Three-point shot
- **Option #2**: Two-point shot





In such a situation, the rational choice is a

A. Preliminaries

* Decision Analysis Framework

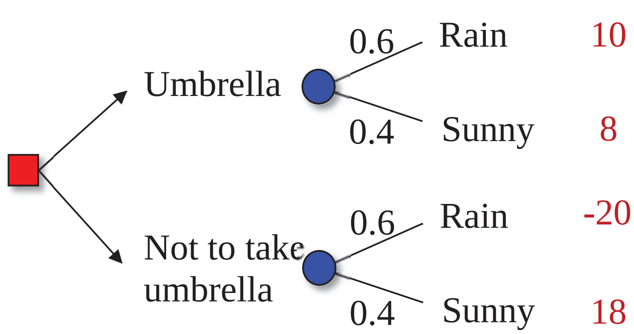
- The decision-maker needs to choose one of the possible *actions* (or decisions) $d_i \in D$. Nature then would choose one of the possible *states of nature*, $s_j \in S$
- Each combination of an action d and state of nature s would result in a *payoff* $V(d, s)$, which is given as one of the entries in a payoff table.
- This payoff table is used to find an *optimal decision* for the decision-maker according to an appropriate criterion.

* Classification

		State of Nature s	
		Discrete	Continuous
Action d	Discrete	**Case A**	Case B
	Continuous	Case C	Case D

Ex] Umbrella problem: Case A

	Rain (s)	Sunny (s_2)
Take (d_1)	No problem	Stylish
Not take (d_2)	Get wet	Reasonable
$P[s_j]$	0.6	0.4



* Various Decision Problems

- **Decision-making under certainty** (*LP*, *IP*, and *NLP*)
  - Decision-maker knows for sure (that is, *with certainty*) outcome or consequence of every decision alternative.
  - Maximize the profit/revenue or Minimize the total cost.

- **Decision-making under uncertainty**

  Decision-maker has *no information* at all about various outcomes or states of nature.

  a. LaPlace criterion
  b. Maximin criterion
  c. Maximax criterion
  d. Minimax regret criterion

- **Decision-making under risk**

  Decision-maker has *some knowledge* regarding probability of occurrence of each outcome or state of nature.

  a. Expected payoff, regret, or utility criteria
  b. Combined expected value and variance

- **Decision-making with multiple objectives**
  - Analytic Hierarchy Process (AHP)

- **Decision-making under conflicts**
  - Game theory

- **Decision-making over time** (*DP*)
  - Sequential decision analysis
  - Optimal stopping rule: When to stop?

* Dominance Relations: *Inferior* in every respect

Ex] Tigers and Saints: *Reduced set* of alternatives

	WW	WL or LW	LL
Option 1	\$100	\$10	-\$50
Option 2	\$50	0	-\$10
Option 3	\$40	-\$10	-\$20

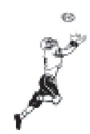
Ex] Application for admission to LSU graduate program:
Multi-objective decision-making

	GPA	GMAT
Applicant 1	4.0	500
Applicant 2	3.0	600
Applicant 3	3.5	700

* Stochastic Dominance

- Minimum variance vs. Expected payoff

- *Return to risk* ratio = $E[X_j]$ / $Stdev[X_j]$



Ex] Simpson's Paradox:

One study indicates treatment A cures 36 percent of its 100 subjects, and treatment B cures 45 percent of its 1,000 subjects. A second study indicates treatment A cures 60 percent of its 1,000 subjects and treatment B cures 65 percent of its 100 subjects. Do you think treatment B *dominates* treatment A?

		Cured	Sample size	Percentage
Study 1	A	36	100	36%
	B	450	1,000	45%
Study 2	A	600	1,000	60%
	B	65	100	65%
Total	A			
	B			

▪ In fact, treatment **A** is *more effective* than treatment **B**!

B. Decision Making Under Risk

* Prior Probabilities

- The decision-maker generally has some information that should be taken into account about the relative likelihood of the possible states of nature.
- Such information can usually be translated to a probability distribution, acting as though the state of nature is a random variable, in which case this distribution is referred to as a prior distribution.
- The individual probabilities for the respective states of nature are called prior probabilities.

I. Expected Value Approach

- If probabilistic information regarding the states of nature is available, one may use the expected monetary value (*EMV*) approach (also known as Expected Value or *EV*).
- Here the expected return for each decision is calculated by summing the products of the payoff under each state of nature and the probability of the respective state of nature occurring.

$$EV(d_i) = \sum_{j=1}^{m} P(s_j)\ V(d_i,\ s_j)$$ where

m = number of states of nature,

$P(s_j)$ = the probability of state of nature s_j,

$V(d_i, s_j)$ = payoff corresponding to desion alaternative d_i and state of nature s_j.

- The decision yielding the best *expected return* is chosen.

II. Expected Regret Approach

- As in the *minimax regret* approach, we need to first construct a regret table (or an *opportunity loss* table).
- This is done by calculating for each state of nature the difference between each payoff and the largest payoff for that state of nature.
- Then, using this regret table, calculate the expected regret for each decision.

- The optimal decision is the one corresponding to the minimum of the expected regret (or opportunity loss).

Ex 1] Marketing Strategy: Consider the following problem with two decision alternatives (d_1 & d_2) and two states of nature s_1 (Market Receptive) and s_2 (Market Unfavorable) with the following payoff table representing profits ($1000). Assume the probability of the market being receptive is known to be 0.8.

(a) Expected monetary value criterion

Payoff		State of nature		Expected payoff
		s_1	s_2	
Decision	d_1	20	6	17.2
	d_2	25	3	
Prior probabilities		0.8	0.2	

(b) Expected regret criterion

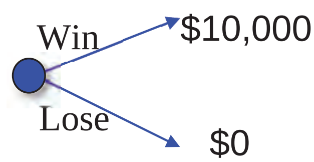
Regret		State of nature		Expected regret
		s_1	s_2	
Decision	d_1			4
	d_2			
Prior probabilities		0.8	0.2	

III. Expected Utility Approach

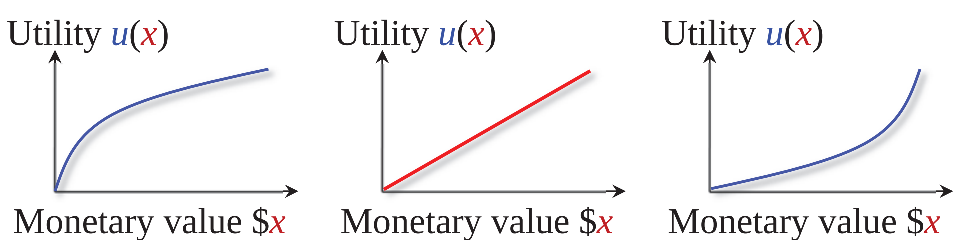
- In many situations, the assumption that the expected payoff in *monetary terms* is the appropriate measure of the consequences of taking an action is inappropriate.
- Certainty equivalent:

  The *certainty equivalent* of a risky option L is an amount $\hat{x}$ such that the decision maker is *indifferent* between the option L and receiving a certain payoff of $\hat{x}$.

  #1. *Certainty* option #2. *Risky* option



- When facing a choice between several options, a decision make should choose the option with the largest expected utility. The utility function $u(x)$ for monetary value x reflects the decision maker's preferences.
- Concave and convex utility functions

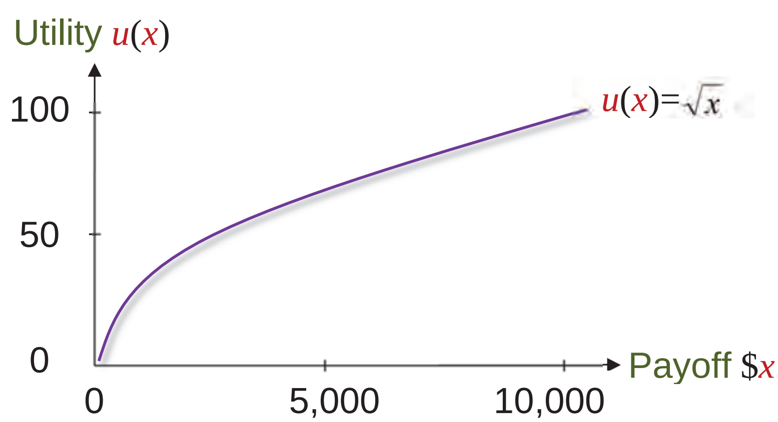


- Risk preference

  If $u(x)$ is *concave*, she is *risk-averse*.
  If $u(x)$ is *linear*, she is *risk-neutral*.
  If $u(x)$ is *convex*, she is *risk-prone*.

Ex 2] Coin Tossing

Suppose that your utility function is $u(x) = \sqrt{x}$. Find the *certainty equivalent* of the following risky option:



- Expected payoff $\bar{x}$ =

- Expected utility EU =

- Certainty equivalent $\hat{x}$ =

  which is the amount of money you should be willing to pay for the risky option.

- Risk premium = $\bar{x} - \hat{x}$=

  which is positive. So you are *risk-averse*.

Not easy to estimate your utility function!

C. Expected Value of Perfect Information

* Perfect Information

- Frequently, an *additional* information is available that can improve the probability estimates for the states of nature.
- The expected value of perfect information (*EVPI*) is the increase in the expected profit that would result if one knew *with certainty* which state of nature would occur.
- The *EVPI* provides an upper bound on the expected value of any sample or survey information.

* EVPI Calculation

- **Step 1**: Determine the maximum payoff corresponding to each state of nature.
- **Step 2**: Compute the expected value of these maximum payoffs.
- **Step 3**: Subtract the *EV* of the optimal decision obtained without the perfect information from the amount determined in Step 2.

Ex 1] Marketing Strategy

Payoff		State of nature		Expected payoff
		s_1	s_2	
Decision	d_1	20	6	17.2
	d_2	25	3	
Prior probabilities		0.8	0.2	
Maximum payoff				

- Expected value *with* perfect information:
- Expected value of perfect information (*EVPI*):

Ex 2] Oil Exploration

An oil drilling company in Lafayette must decide whether to drill a well or not at a site. The geological information of the land indicates that there is a 40% chance of little to no oil, a 10% chance of a wet well, and a 50% chance of a gusher.

The payoff matrix (in $1 million) is as follows:

Payoff	No oil (s_1)	Wet (s_2)	Gusher (s_3)	E[Payoff]
Drill (d_1)	-$90	-$5	+$80	
Not drill (d_2)	$0	$0	$0	
$P[S_i]$	0.4	0.1	0.5	
Payoff w/ PI				

(a) If the company uses the expected *monetary value* (expected *payoff*) criterion, what is its decision?

(b) What is the expected value of perfect information (*EVPI*) if it is available?

EVPI =

(c) If the company uses the expected *opportunity loss* (regret) criterion, what is its decision?

Regret	No oil (s_1)	Wet (s_2)	Gusher (s_3)	E[Regret]
Drill (d_1)				
Not drill (d_2)				
$P[S_i]$	0.4	0.1	0.5	

D. Expected Value of Sample Information

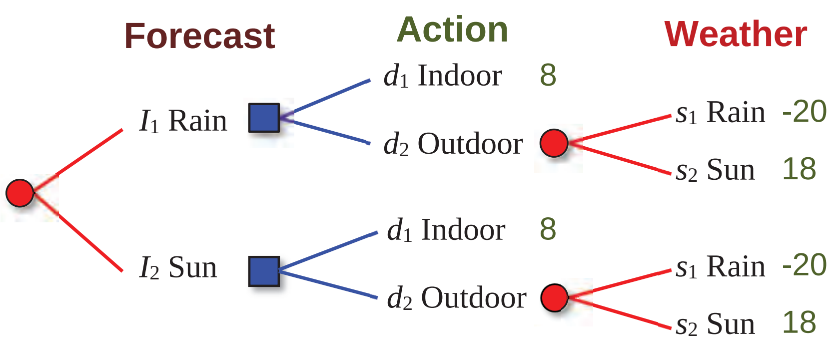
* Bayesian Decision Analysis

Chris is planning to have his birthday party next weekend, but he has been unable to finalize plans due to his indecision over whether to take a chance on it not raining and hold the party outdoors at the park or play it safe and hold the party in-doors.

- Prior information: Having access to past meteorological records for the Baton Rouge area, Chris assessed the following probability distribution for the weather next weekend: P[s_1 *Rain*] = 0.6 and P[s_2 *Sun*] = 0.4
- Payoff matrix

Payoff		Weather	
		s_1 *Rain*	s_2 *Sun*
Action	d_1 Indoor	8	8
	d_2 Outdoor	-20	18
Prior probability, $P[s_i]$		0.6	0.4

- Sample (additional) information I_j: TV weather forecast.



(a) Optimal "if-then" decision rule

If the forecast is I_1 Rain, choose d_1 Indoor or d_2 Outdoor?
If the forecast is I_2 Sun, choose d_1 Indoor or d_2 Outdoor?

(b) What is the expected payoff if Chris follows the meteorologist's forecast (i.e., sample information)?

Ex] Railroad Crossing

On the way to school, Dr. Chun needs to decide whether or not to stop the car at the unmanned railroad crossing with a crossing signal.

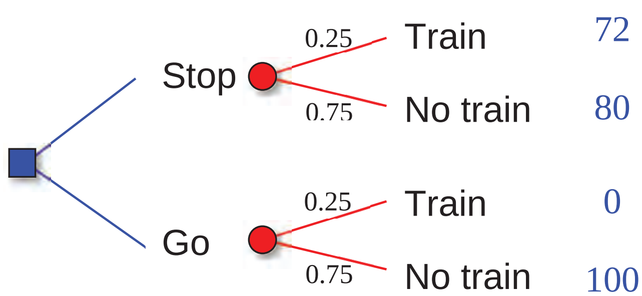
- Prior probabilities (Prior information):

  $P[S_1\text{: Train}] = 0.25$ and $P[S_2\text{: No train}] = 0.75$

- Payoff table:

Payoff, X	S_1: Train	S_2: No train	$E[X]$
d_1: Stop	\$72	\$80	
d_2: Go	\$0	\$100	
$P[S_i]$	0.25	0.75	

(a) ***Case 1***. Optimal strategy with your own prior information:



Always d_1: Stop. Then, the expected payoff is

(b) ***Case 2***. Optimal strategy with the *perfect* information:

If S_1: Train, then d_1: Stop and receive \$72.
If S_2: No Train, then d_2: Go and receive \$100.

Then, the expected payoff is

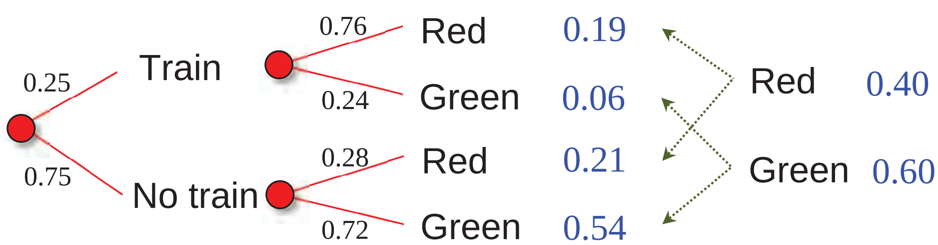
- The expected value of the *perfect* information (*EVPI*) is

  EVPI =

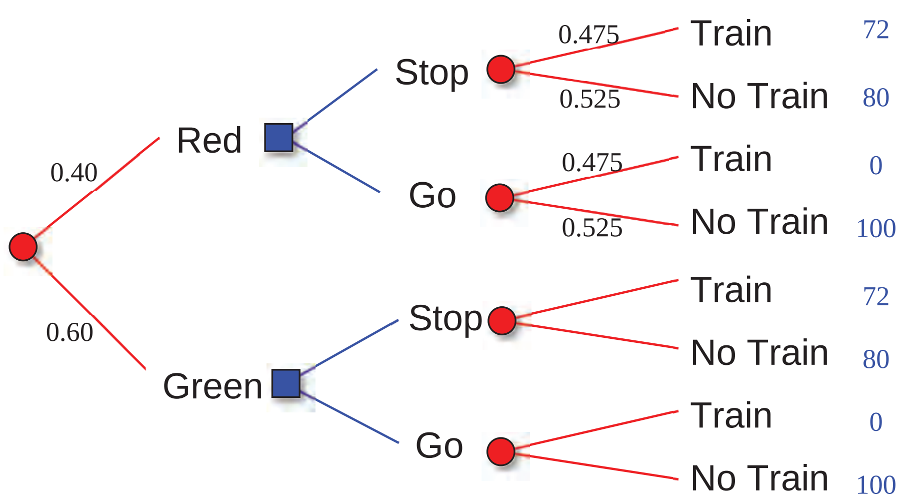
(c) ***Case 3***. Optimal strategy with a *sample* (*imperfect*) information (e.g., railroad crossing signal)

$P[I_j \mid S_i]$	I_1: Red	I_2: Green
S_1: Train	0.76	0.24
S_2: No train	0.28	0.72

- Posterior probabilities, $P[S_i \mid I_j]$



P[Train | Red] = 0.19/0.40 = 0.475
P[No Train | Red] = 0.21/0.40 = 0.525
P[Train | Green] =
P[No Train |Green] =



- Optimal strategy: If Red, then Stop. If Green, then Go.
  Then, the expected payoff is
- Expected value of the *sample* information (*EVSI*) is
- *Efficiency* of the *sample* information = *EVSI*/*EVPI* =

* Matrix Approach

- Information system

$$\mathbf{S} = \begin{array}{r} \\ \textit{train} \\ \textit{no train} \end{array} \begin{array}{c} \begin{array}{cc} \textit{red} & \textit{green} \end{array} \\ \begin{bmatrix} 0.76 & 0.24 \\ 0.28 & 0.72 \end{bmatrix} \end{array}$$

- Decision matrix

$$\mathbf{D} = \begin{array}{r} \\ \textit{red} \\ \textit{green} \end{array} \begin{array}{c} \begin{array}{cc} \textit{stop} & \textit{go} \end{array} \\ \begin{bmatrix} x & 1-x \\ 1-y & y \end{bmatrix} \end{array}$$

- Payoff matrix

$$\mathbf{V} = \begin{array}{r} \\ \textit{stop} \\ \textit{go} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Train} & \textit{No train} \end{array} \\ \begin{bmatrix} 72 & 80 \\ 0 & 100 \end{bmatrix} \end{array}$$

- Prior probability matrix

$$\mathbf{A} = \begin{array}{r} \\ \textit{train} \\ \textit{no train} \end{array} \begin{array}{c} \begin{array}{cc} \textit{train} & \textit{no train} \end{array} \\ \begin{bmatrix} 0.25 & 0 \\ 0 & 0.75 \end{bmatrix} \end{array}$$

- *Expected value* with the information system = $tr(\mathbf{SDVA})$

x	y	Decision	Expected value
0	0	Go if red, stop if green	\$68.52
1	0	Always stop	\$78.00
0	1	Always go	\$75.00
1	1	Go if green, stop if red	

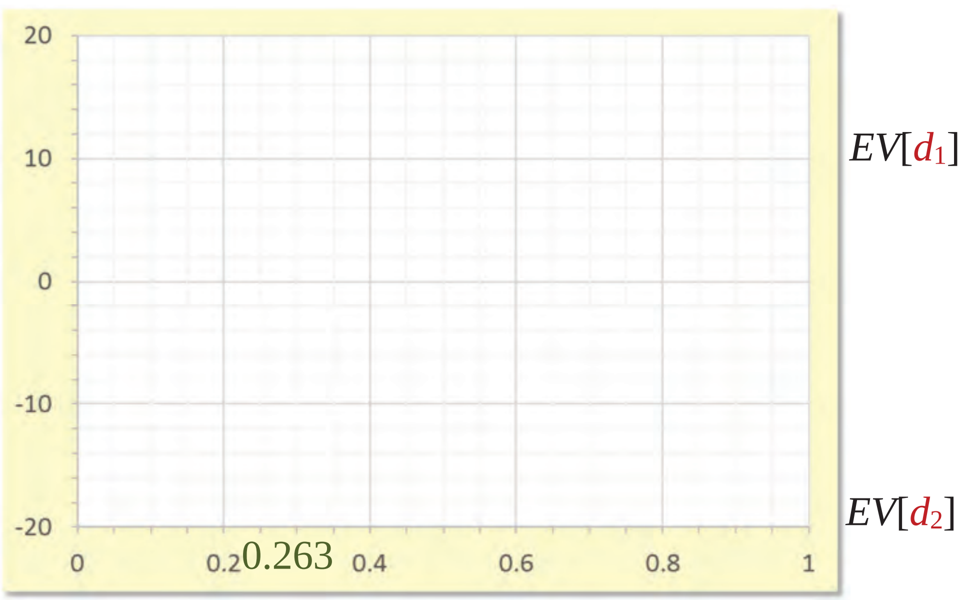
- Comparisons of various information systems

Information system		E[Payoff]	Efficiency	
0.76 0.28	0.24 0.72	\$84.48	43.2%	Current System!
0.80 0.10	0.20 0.90			More Informative?
1.0 0.0	0.0 1.0			Perfect information
0.8 0.8	0.2 0.2			Null information
0.0 1.0	1.0 0.0			Perfect information

E. Sensitivity Analysis

Ex 1] Birthday Party: Chris has not yet decided whether to take a chance on it not raining and hold the party outdoors at the park or play it safe and hold the party in-doors.

Payoff		State of Nature		$EV(p)$
		s_1: Rain	s_2: Sun	
Decision	d_1: Indoor	8	8	
	d_2: Outdoor	-20	18	
$P[S_i]$				
Payoff w/ PI				



(a) Find the optimal decision rule:

If $\leq p \leq$ choose

If $\leq p \leq$ choose

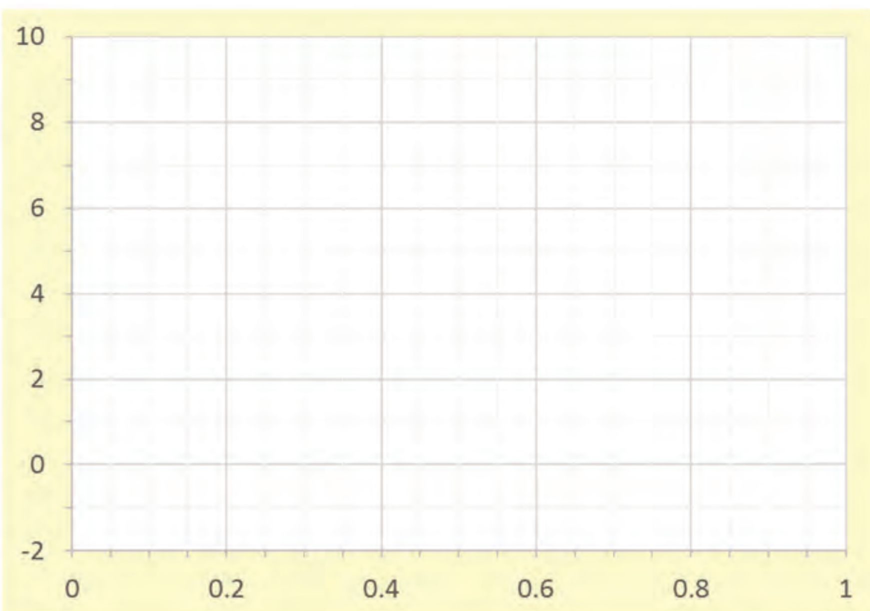
(b) Find the *EVPI* when p=0.6.

If P[Rain] = 0.6, then *EVPI* =

Ex 2] Sensitivity Analysis:

Find the optimal decision rule with the *unknown* probability $P[s_1] = p$.

Payoff	s_1	s_2	$EV(p)$
d_1	-2	10	
d_2	2	8	
d_3	8	-2	
d_4	4	4	
$P[S_i]$	p	$1-p$	
Payoff *w/ PI*			



- Optimal decision rule:

If $0.0 \le p \le$ choose

If $\le p \le$ choose

If $\le p \le 1.0$, choose

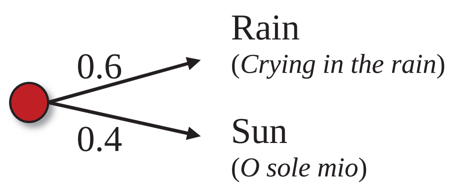
d_4 is *dominated* by a probability combination of d_2 and d_3.

F. Decision Trees

A decision tree is a *chronological* representation of the decision problem. It provides a useful way of visually displaying the problem and then organizing the computational work. For any decision tree, the backward induction procedure always will lead to the *optimal policy*.

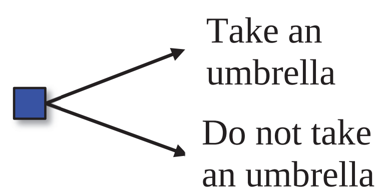
- Each decision tree has two types of nodes; round nodes correspond to the states of nature, while square nodes correspond to the decision alternatives.
- At the end of each limb of a tree are the payoffs attained from the series of branches making up that limb.

A. Chance (or state) node



- Round
- Uncontrollable
- Probabilities on the branches.

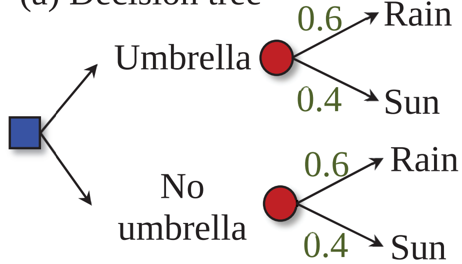
B. Choice (or decision) node



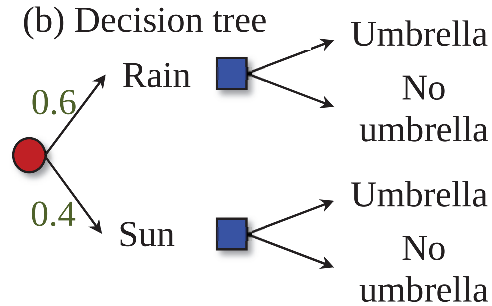
- Square
- Controllable
- No probabilities

Ex] Which decision tree is correct?

(a) Decision tree



(b) Decision tree

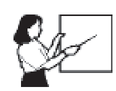


* Recursion to Find the Best Strategy

There is a simple recursion that finds the best strategy without enumerating all the nodes and branches. This recursion consists of two steps:

- **Step 1.** *Evaluating end-points*: Adjacent to each end-point, accumulate the net total profit that accrues if this end-point is reached.
- **Step 2.** *Rolling back*: Then, moving from right to left:
  - At each chance (or state) node, compute the expectation of profit by multiplying the probability of each arc to its right by the expected profit of reaching the node or end-point to which it leads, and summing.
  - At each choice (or decision) node, pick an action whose expectation of profit is largest.

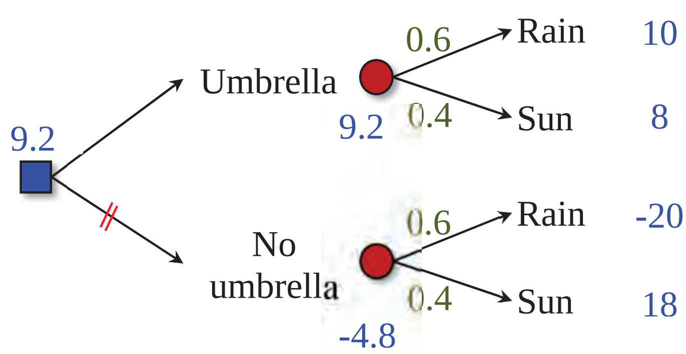
Ex 1] Umbrella Problem



- Expected value criterion

Payoff	Rain s_1	Sun s_2	Expected payoff
Take d_1	10	8	
Not take d_2	-20	18	
$P[S_i]$	0.6	0.4	

- Decision tree



Ex 2] Hoosier Millionaire: "Ask Marilyn," *Parade Magazine* (7/9/1995), 15.

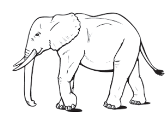
"Our state lottery has a weekly TV show. After elimination rounds, one contestant gets the opportunity to win $1 million. The contestant picks from four hidden windows. Behind each is one of the following: $150,000, $200,000, $1 million, or a "stopper."

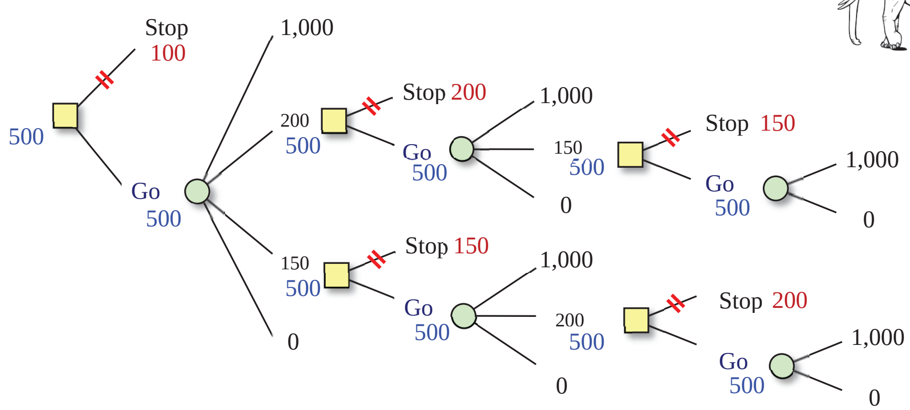
The contestant may keep picking a window until either $1 million is won or the "stopper" is picked - in which case nothing is won. Before playing this final round, the contestant is offered $100,000 cash to stop there. The same applies to the windows; the contestant can either keep the cash contents of the chosen window and stop there or can pick a new window.

Speaking only *statistically*, would a contestant be better off to take the $100,000 and not play the final round at all?"

– Douglas Offutt, Newburgh, Indiana

- Decision tree





Backward induction procedure?

- Optimal decision strategy:

- Expected payoff:

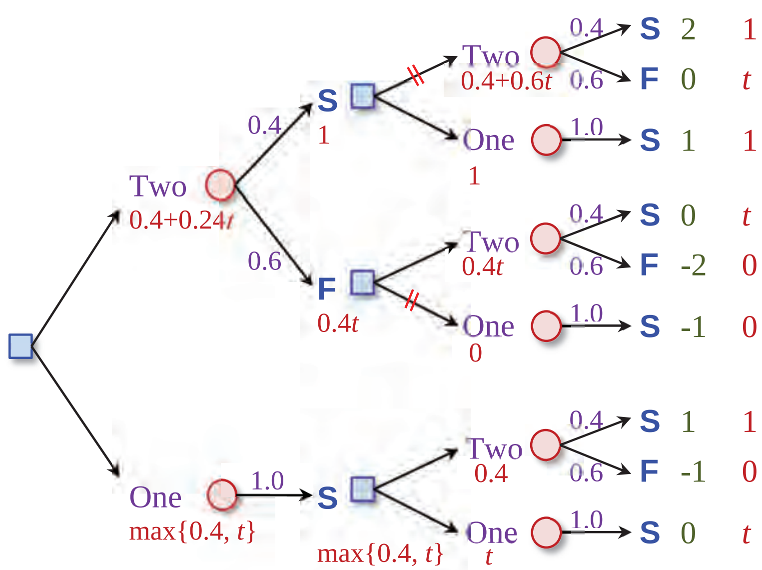
Ex 3] Coach's Dilemma

Suppose that LSU trails by 14 points late in the LSU-Alabama football game. LSU's guardian angel has assured the LSU coach that his team will have the ball two more times during the game and will score a touchdown (worth 6 points) each time it has the ball. The LSU coach has also been assured that Bama will not score any more points.

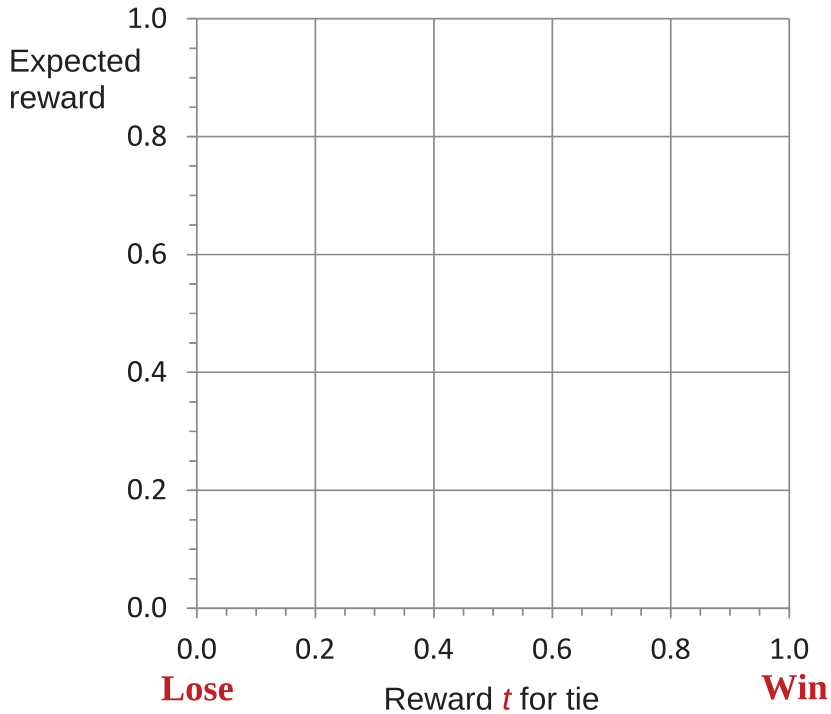
Suppose a win is assigned a value of 1, a tie is assigned a value of t ($0 < t < 1$), and a loss is assigned a value of 0. The coach's problem is to determine whether to go for 1 point or 2 points after each touchdown.

The LSU coach wishes to maximize the expected reward earned from the outcome of the game, assuming that a 1-point conversion is always successful, and a 2-point conversion is successful only 40% of the time.

(a) Draw a decision tree.



(b) Determine the optimal decision rule.



- If 0< t <0.5263, go for two first.
  If it succeeds, then go for one next.
  If it fails, then go for two.

- If 0.5263 < t < 1.0, go for one first.
  Go for one again.

(c) Prove that no matter what value t ($0 < t < 1$) is assigned to a tie, it is never *optimal* to use the following strategy: Go for a 1-point conversion after the *first* touchdown and then go for a 2-point conversion after the *second* touchdown. (Note that this suboptimal strategy is the one most coaches follow!)

- Go for one and then go for two.
  Then, the expected reward is 0.4 for any t.