Session 3. Linear Programming

* Linear Programming (LP)

- LP is a tool for solving *optimization* problems. It has been widely applied to the general problem of allocating *limited resources* among *competing activities* in a best possible (i.e., optimal) way.
- In a survey of Fortune 500 firms, 85% of those responding said that they had used LP.
- The adjective *linear* means that all the mathematical functions in the model are required to be *linear functions*. The word *programming* does not refer here to computer programming; rather, it is essentially a synonym for *planning*.

* Construction of the Optimization Model

- *Decision variables*

  What does the model seek to determine?
  What are the unknown *variables* of the problem?

- *Objective function* (*Max* or *Min*)

  What is the objective that needs to be achieved to determine the optimum solution from among all the feasible values of the variables?

- *Constraints*

  What constraints must be imposed on the variables to satisfy the *limitations* of the model system?

A. Introduction

A furniture maker in Houma has 3 hours of free time and 4 units of wood in which he will make chairs and tables. He estimates that a chair requires 1 hour of time and 1 unit of wood, while a table requires 1 hour of time and 2 units of wood.

The profit margin is $5 for a chair and $7 for a table. Determine how many chairs and tables should be made to maximize the profit.

		Output		
		Chair	Table	
Input	Time	1 hour	1 hour	3 hours
	Wood	1 unit	2 unit	4 units
Revenue		$5	$7	

(a) **Method 1**: Rule of thumb

Consider all the feasible combinations of chairs and tables!

Table	3	21	26	31	36
	2	14	19	24	29
	1		12		22
	0	0		10	
		0	1	2	3
		Chair			

- Optimal solution is

- Maximum profit is

(b) **Method 2**: Linear programming

- Model formulation:

1. Variables

x_1 =

x_2 =

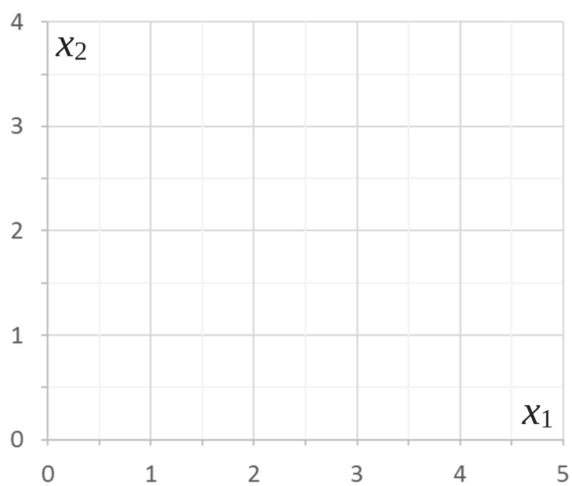
z =

2. Objective function

Max z =

3. Constraints

- Model solution: Graphical method



Extreme points (x_1, x_2)	Profit $z = 5x_1 + 7x_2$

▪ Optimal solution:

(c) **Method 3**: Simplex method

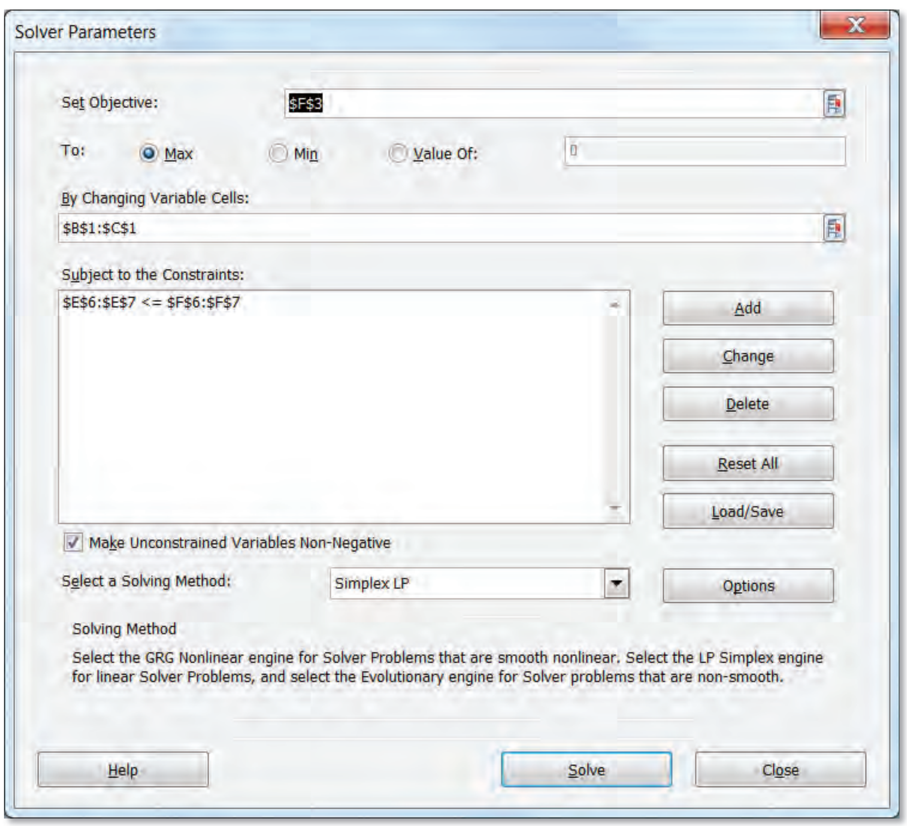
The feasible region is a polyhedron and the objective function is linear! Then, we have a very efficient algorithm for solving LP problems developed by George Dantzig in 1947.

(d) **Method 4**: Microsoft Excel - Solver

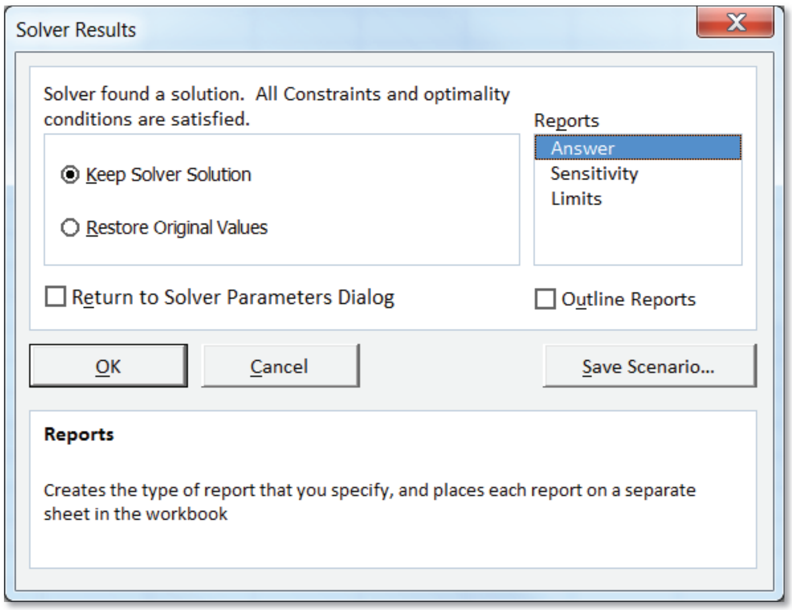
	A	B	C	D	E	F
1	x_j =	2	1			
2						
3	c_j =	5	7		z =	17
4						
5						b_j
6	a_{ij} =	1	1		3	3
7		1	2		4	4

Cell **F3** =SUMPRODUCT(B1:C1,B3:C3)
Cell **E6** =SUMPRODUCT(B1:C1,B6:C6)
Cell **E7** =SUMPRODUCT(B1:C1,B7:C7)

- Solver parameters menu:



- Excel printout (Solver):



Microsoft Excel 14.0 Answer Report
Worksheet: [Book1]Sheet1
Report Created: 7/18/2021 8:37:28 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Solver Options

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$3	z =	0	17

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$1	xj =	0	2	Contin
$C$1	xj =	0	1	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$6	aij = z =	3	$E$6<=$F$6	Binding	0
$E$7	z =	4	$E$7<=$F$7	Binding	0

* Assumptions of Linear Programming

1. Proportionality assumption:

The objective function must be a *linear* function of the decision variables.

- The contribution to the objective function z from each decision variable x_i is *proportional* to the value of the decision variable.
- The contribution to the objective function for any variable is *independent* of the values of the other decision variables.

2. Additivity assumption:

Each LP constraint must be a *linear* inequality or linear equation.

- The contribution of each variable x_i to the left-hand side of each constraint b_i is *proportional* to the value of the variable.
- The contribution of a variable to the left-hand side of each constraint is *independent* of the values of the other variables.

3. Divisibility assumption:

Each decision variable x_i is allowed to assume *fractional* values.

4. Certainty assumption:

Each parameter is known with *certainty*.

B. Graphical Method

* Procedure

- **Step 1**. Formulate the resource allocation problem as a LP.
  - Variables (*two* variables!)
  - Objective function
  - Constraints

- **Step 2**. Plot each of the *constraint inequalities* on the graph.
  - Convert the inequality to an *equality* and plot the straight line that represents this equation.
  - Choose a *trial point*, e.g., (0, 0).
  - Determine if the *trial point* satisfies the inequality.

    If yes => all points on the *same* side of the line as the trial point also satisfy the inequality.

    If no => all points on the *opposite* side satisfy the inequality.

- **Step 3**. Identify the *feasible region*.

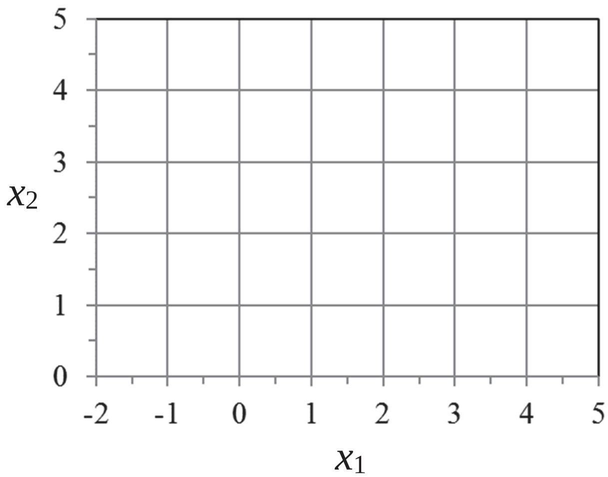
- **Step 4**. Solve the problem using the *extreme point method*.
  - Identify each extreme (or corner) point of the feasible region.
  - For each extreme point, evaluate its objective function value.
  - Identify an *optimal solution* by choosing the extreme point with the largest (or smallest) objective function value.

Ex 1] Minimization

Min $z = 2x_1 + x_2$

$x_1 + x_2 \geq 4$

$-x_1 + x_2 \leq 2 \qquad x_1 \geq 0$ and $x_2 \geq 0$



(x_1, x_2)	$z=2x_1+x_2$

- Optimal solution:

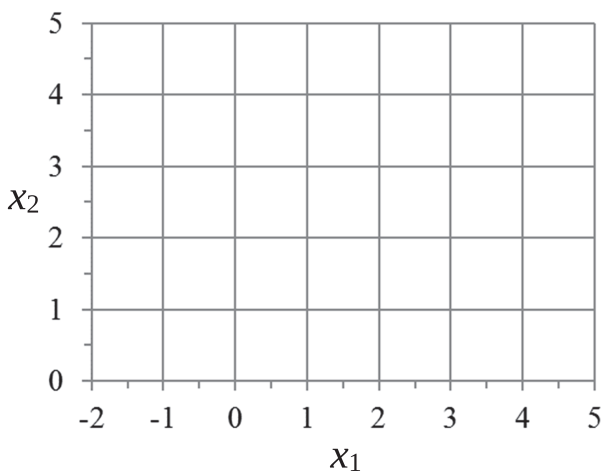
Ex 2] Equality

Max $z = x_1 + 2x_2$

$x_1 + x_2 = 4$

$x_1 \leq 2$

$x_2 \leq 3 \qquad x_1 \geq 0$ and $x_2 \geq 0$

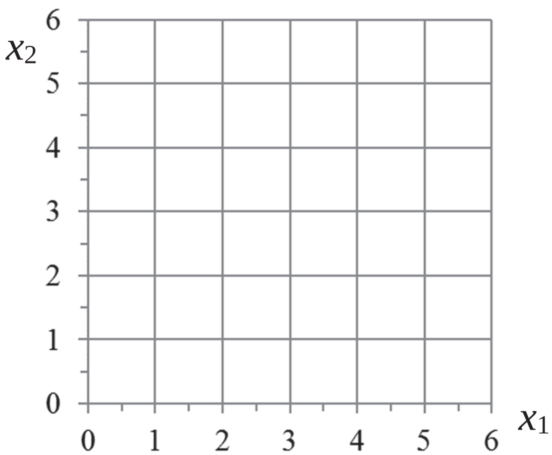


(x_1, x_2)	$z=x_1+2x_2$

- Optimal solution:

Ex 3] Primal LP *Max* $z = 9\,x_1 + 6\,x_2$

$$2\,x_1 + 1\,x_2 \le 6 \qquad (1)$$
$$3\,x_1 + 6\,x_2 \le 18 \qquad (2)$$
$$\text{and } x_1 \ge 0 \text{ and } x_2 \ge 0$$

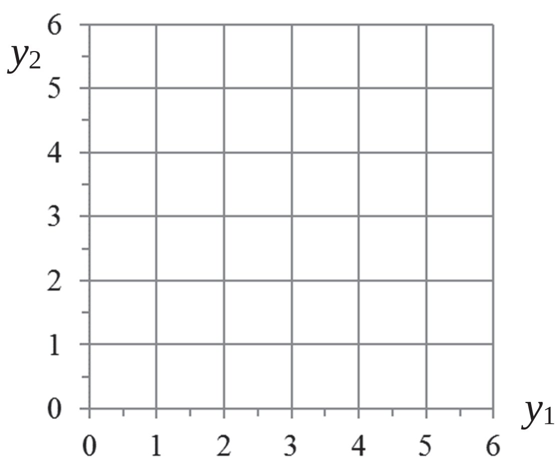


(x_1, x_2)	$z = 9x_1 + 6x_2$
(6, 0)	54
(0, 6)	36

- Optimal solution:

Ex 4] Dual LP *Min* $w = 6\,y_1 + 18\,y_2$

$$2\,y_1 + 3\,y_2 \ge 9$$
$$1\,y_1 + 6\,y_2 \ge 6$$
$$\text{and } y_1 \ge 0 \text{ and } y_2 \ge 0$$



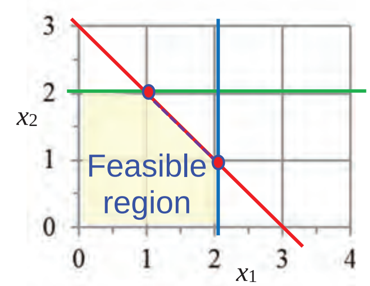
(y_1, y_2)	$w = 6y_1 + 18y_2$
(0, 0)	0
(4.5, 0)	27
(0, 1)	18

- Optimal solution:

* Special Cases of LP

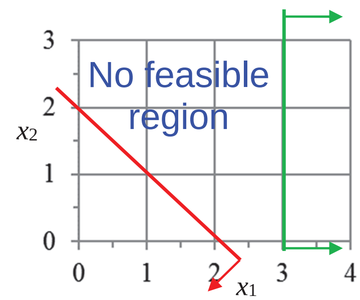
1. Alternative (or multiple) optimal solutions

$$\begin{aligned} Max\ \ z &= x_1 + x_2 \\ x_1 + x_2 &\le 3 \\ x_1 &\le 2 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$



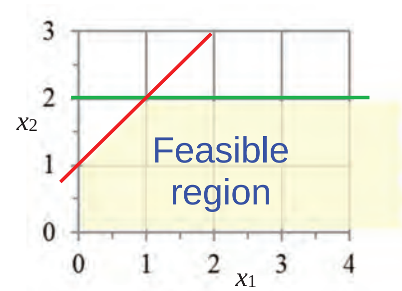
2. Infeasible solution

$$\begin{aligned} Max\ \ z &= x_1 + x_2 \\ x_1 + x_2 &\le 2 \\ x_1 &\ge 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$



3. Unbounded solution

$$\begin{aligned} Max\ \ z &= x_1 + x_2 \\ -x_1 + x_2 &\le 1 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$



If max $z = -x_1+2x_2$, we have a *bounded* solution, (1, 2).

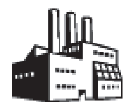
Ex] Why don't we allow an LP to have $<$ or $>$ constraints?

Max $z = x$ subject to $x < 1$

C. Word Problems

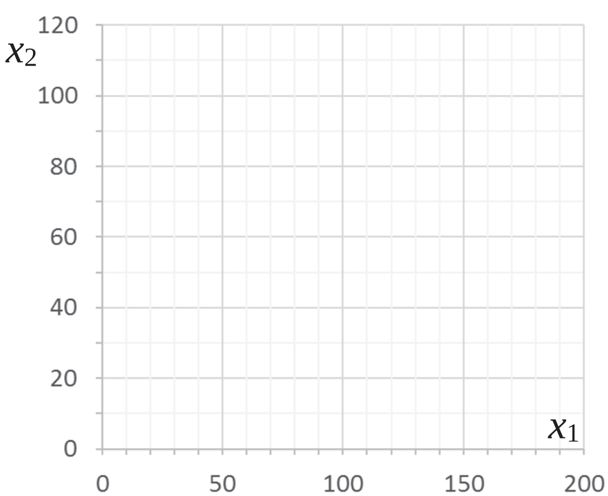
Ex 1] Unique Feasible Solution

Tigers Skinco is a small leather goods firm. The firm produces two types of goods: a *leather jacket* which requires 3 square yards of leather and a *leather handbag* which requires 4 square yard of leather to manufacture. Each *jacket* can be sold for $50 apiece and each *handbag* can be sold for $60 apiece to a small New Orleans boutique. The boutique has agreed that it will purchase up to 100 jackets and 90 handbags per month. Skinco has just signed a contract with a leather supplier where 480 square yards will be delivered to the firm each month. The firm's objective is to *maximize* total revenue.



- Model formulation

- Graphical solution



(x_1, x_2)	$z=50x_1+60x_2$

- Optimal solution:

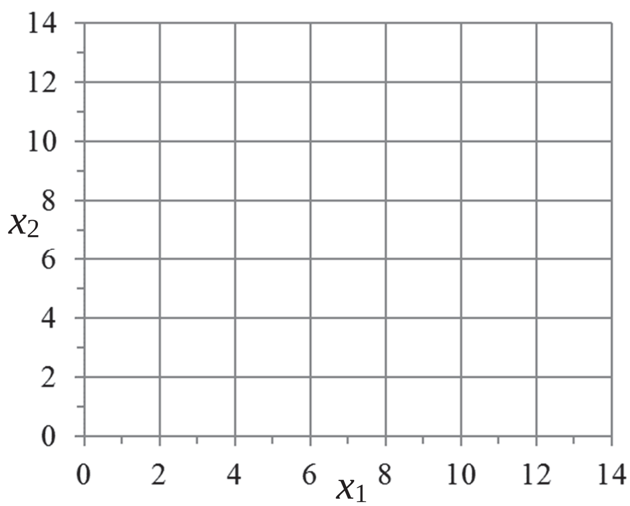
Ex 2] Unbounded Feasible Region

GM manufactures luxury cars and trucks. GM has embarked on an ambitious TV advertising campaign and has decided to purchase several one-minute commercial spots on two types of programs: comedy shows and football game. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men.

A one-minute comedy ad costs \$50,000, and a one-minute football ad costs \$100,000. GM would like the commercials to be seen by at least 28 million women and at least 24 million men. Use linear programming to determine how GM can meet its advertising requirements at minimum cost.

- Model formulation

- Graphical solution



(x_1, x_2)	z

- Optimal solution:

- *Unbounded* feasible region?

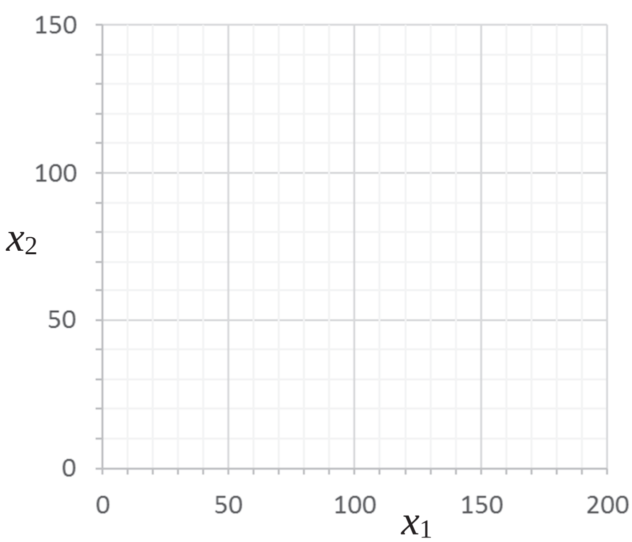
Ex 3] Ratio Constraint

Tiger Taxi Co. in Shreveport can purchase two types of gasoline - *standard* which has an octane rating of 86 and *premium* which has an octane rating 92. The company requires at least 120 gallons of gasoline and, for maintenance purposes, has decided that the average octane rating of the gasoline purchase should be at least 88. Standard gas costs $1.00 a gallon and premium gas costs $1.20 a gallon.

Formulate a linear program to *minimize* the company's gas costs.

- Model formulation

- Graphical solution

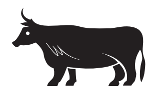


(x_1, x_2)	z

- Optimal solution:

Ex 4] Equality Constraint

Village Butcher Shop in Houma traditionally makes its meat loaf from a combination of ground beef and ground pork. The ground beef contains 80% meat and 20% fat, and costs the shop 90 cents per pound; the ground pork contains 68% meat and 32% fat, and costs 60 cents per pound. How much of each kind of meat should the shop use in each pound of meat loaf if it wants to *minimize* its cost and to keep the fat content of the meat loaf to no more than 24%?



- Variables

  $z =$

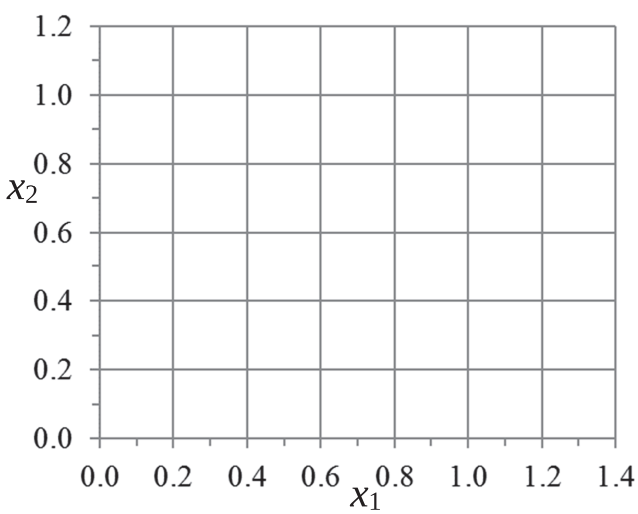
  $x_1 =$

  $x_2 =$

- Model formulation

  $Min\ \ z\ =$

- Graphical solution



(x_1, x_2)	z

- Optimal solution:

D. Formulations and Applications

Ex 1] Portfolio Management

A brokerage firm in Covington manages stock portfolios for a number of clients. A new client has requested that the firm handle an $80,000 investment portfolio. As an initial investment strategy, the client would like to restrict the portfolio to a mix of the following two stocks:

Stock	Price per share	Annual return per share	Risk index per share
Canadian Oil	$25	$3	0.50
Star Properties	$50	$5	0.25

The *risk index* for the stock is a rating of the relative risk of the two investment alternatives. For the current portfolio, an upper limit of 700 has been set for the total risk index of all investments. In addition, the firm has set an upper limit of 1,000 shares for the riskier Canadian Oil stock. Formulate this decision problem as an LP model.

- Variables

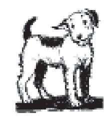
- Objective function

- Constraints

Ex 2] Product Mix Problem

The Red Rover pet food company is planning to launch a new type of dog food. A can of dog food must contain protein, carbohydrate and fat in at least the following amounts: protein, 3 ounces; carbohydrate, 5 ounces; fat, 4 ounces.

Four types of gruel are to be blended together in various proportions to produce a least-cost can of dog food satisfying these requirements. The contents (in ounces) and price of the gruels are given below:



Gruel	Protein	Carbohydrate	Fat	Price ($)
1	3	7	5	3
2	5	4	6	4
3	2	2	3	1.5
4	3	8	2	2
Minimum	3	5	4	

Formulate an LP to determine the least-cost blend for the Red Rover company.

- Decision variables

  x_i =

- Objective function

- Constraints

\# Matrix form?

Ex 3] Blending Problem

Juicy Lucy in Denham Spring blends apple, cranberry, and grape juices into consumer beverage products. The three products and their specifications include:

	Product	Specifications	Selling price
1	Apple-Cranberry Drink	Not more than 80% apple Not less than 25% cranberry	$1.85/gal
2	Apple-Grape Drink	Not more than 70% apple Not less than 40% grape	$1.90/gal
3	Grape Juice	100% grape juice	$2.50/gal

The *maximum* amounts available (in gallons) for this period from the suppliers are:

Apple: 1,100, Cranberry: 200, Grape: 600.

Formulate a revenue-maximizing LP for Juicy Lucy.

- Variables

- Objective function

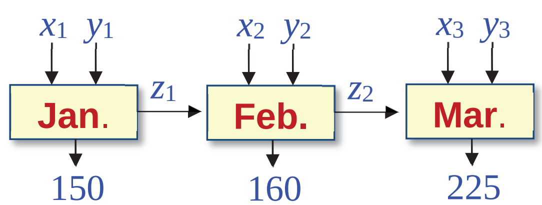
- Constraints

Ex 4] Multi-Period Production Planning

Brite Electronics in Central has just signed a contract to supply searchlights to the US Army. During the next 3 months, the firm must meet the following delivery schedule.

Month, i	Jan.	Feb.	Mar.
Quantity, d_i	150 units	160 units	225 units

Brite's factory can produce searchlights at a cost of $35 per unit. However, the factory can produce at most 160 units per month. Additional lights can be purchased from a Miami factory at the cost of $50 per lamp. Searchlights can be stored for an inventory cost of $5 per unit per month. Formulate a linear program to determine the optimal production plan.



- Variables

- Objective function

- Constraints

Appendix: SAS/OR for Linear Programming

* Proc OptModel

- A general purpose optimization modeling language.
- It provides a modeling syntax designed specifically for building and working with optimization models including linear, mixed-integer, quadratic, and general nonlinear optimization.
- Solvers in **proc optmodel**
  - LP for linear programming
  - MILP for mixed integer linear programming
  - QP for quadratic programming
  - NLP for general nonlinear programming

* Proc OptLP

- It solves linear programming problems that are submitted in a SAS data set that uses a mathematical programming system (MPS) format.
- Solvers in **proc optlp**
  - Primal simplex solver
  - Dual simplex solver
  - Network simplex solver
  - Interior point solver

*** Proc OptMILP**: Mixed integer linear programming

*** Proc OptQP**: Quadratic programming

Ex 1] Consider the following LP problem:

$$\text{Max } z = x_1 + x_2 + x_3$$

Subject to

$$3\, x_1 + 2\, x_2 - 1\, x_3 \le 1$$
$$-2\, x_1 - 3\, x_2 + 2\, x_3 \le 1$$
$$x_i \ge 0$$

▪ SAS Input

```
proc optmodel;
   var x{i in 1..3} >= 0;
   max f =    x[1] +   x[2] +   x[3];
   con c1:  3*x[1] + 2*x[2] -   x[3] <= 1;
   con c2: -2*x[1] - 3*x[2] + 2*x[3] <= 1;
   solve with lp;
   print x;
quit;
```

▪ SAS Output

Problem Summary	
Objective Sense	Maximization
Objective Function	f
Objective Type	Linear
Number of Variables	3
Bounded Above	0
Bounded Below	3
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	2
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	6

Solution Summary	
Solver	Primal Simplex
Objective Function	f
Solution Status	Optimal
Objective Value	8
Iterations	2
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0

[1]	x
1	0
2	3
3	5

Ex 2] Consider the following LP problem:

$$\text{Min } z = 2x_1 - 3x_2 - 4x_3$$

$$\begin{array}{lll} \text{Subject to} & -2x_2 - 3x_3 \geq -5 & \text{(R1)} \\ & x_1 + x_2 + 2x_3 \leq 4 & \text{(R2)} \\ & x_1 + 2x_2 + 3x_3 \leq 7 & \text{(R3)} \\ & x_i \geq 0 & \end{array}$$

▪ MPS-format SAS data

```
data example;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME         .       EXAMPLE    .   .     .
ROWS         .         .        .   .     .
N            COST      .        .   .     .
G            R1        .        .   .     .
L            R2        .        .   .     .
L            R3        .        .   .     .
COLUMNS      .         .        .   .     .
.            X1      COST       2   R2    1
.            X1      R3         1   .     .
.            X2      COST      -3   R1   -2
.            X2      R2         1   R3    2
.            X3      COST      -4   R1   -3
.            X3      R2         2   R3    3
RHS          .       .          .   .     .
.            RHS     R1        -5   R2    4
.            RHS     R3         7   .     .
ENDATA       .       .          .   .     .
```

▪ OptLP procedure

```
title1 'The OPTLP
Procedure';
proc optlp data = example
  objsense   = min
  presolver = automatic
  solver     = primal
  primalout = expout
  dualout    = exdout;
run;
```

▪ SAS output

```
title2 'Primal Solution';
proc print data=expout
label;
run;

title2 'Dual Solution';
proc print data=exdout
label;
run;
```

▪ SAS printout

Problem Summary	
Problem Name	EXAMPLE
Objective Sense	Minimization
Objective Function	COST
RHS	RHS
Number of Variables	3
Bounded Above	0
Bounded Below	3
Bounded Above and Below	0
Free	0
Fixed	0
Number of Constraints	3
LE (<=)	2
EQ (=)	0
GE (>=)	1
Range	0
Constraint Coefficients	8

Solution Summary	
Solver	Primal simplex
Objective Function	COST
Solution Status	Optimal
Objective Value	-7.5
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	1
Presolve Time	0.00
Solution Time	0.00

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	COST	RHS	X1	N	2	0	1.7977E308	0.0	L	2.0
2	COST	RHS	X2	N	-3	0	1.7977E308	2.5	B	0.0
3	COST	RHS	X3	N	-4	0	1.7977E308	0.0	L	0.5

Dual Solution

Obs	Objective Function ID	RHS ID	Constraint Name	Constraint Type	Constraint RHS	Constraint Lower Bound	Constraint Upper Bound	Dual Variable Value	Constraint Status	Constraint Activity
1	COST	RHS	R1	G	-5	.	.	1.5	U	-5.0
2	COST	RHS	R2	L	4	.	.	0.0	B	2.5
3	COST	RHS	R3	L	7	.	.	0.0	B	5.0

▪ Excel output

1. Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$5	x1	0	2	2	1E+30	2
$E$5	x2	2.5	0	-3	0.333333333	1E+30
$F$5	x3	0	0.5	-4	1E+30	0.5

2. Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$J$9	y1	-5	1.5	-5	5	2
$J$10	y2	2.5	0	4	1E+30	1.5
$J$11	y3	5	0	7	1E+30	2