Session 12. Advanced Decision Models

* Game Theory

- Game theory attempts to mathematically capture behavior in *strategic situations*, in which an individual's success in making choices depends on the choices of others.

  Ex] Soccer penalty kicks

P[Scoring a goal]		Goalie	
		b_1: Dive left	b_2: Dive right
Right-footed Kicker	a_1: Kick left	0.637	0.944
	a_2: Kick right	0.893	0.437

  - The *optimal* strategy for the kicker?
  - The *optimal* strategy for the goalie?
  - The *value* of the game?

* Multi-Objective Decision Making

- Many real-world decision problems have multiple objectives. For example, when choosing a medical treatment plan, we want to maximize the efficacy of the treatment, but also minimize the side effects.
- These objectives typically conflict, e.g., we can often increase the efficacy of the treatment, but at the cost of more severe side effects.
- Multi-objective methods
  - Analytical hierarchical process (*AHP*)
  - Multi-objective linear programming (*MOLP*)
  - Multi-attribute utility theory

A. Game Theory*

* Game Theory

- Two or more decision-makers have *conflicting* interests.
- Game theory attempts to mathematically capture behavior in *strategic situations*, in which an individual's success in making choices depends on the choices of others.
- Game theory has been widely recognized as an important tool in many fields. Eight game theorists have won *Nobel prizes* in economics.

* Basic Assumptions

1. *Both* players are *rational*.
2. *Both* players choose their strategies solely to *promote their own welfare* (no compassion for the opponent).

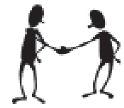
* Payoff Table

- It shows the gains (or payoffs) for Player A that would result from each combination of strategies for the two players.

Payoff Table		Player B			
		b_1	b_2	…	b_n
Player A	a_1	r_{11}	r_{12}	…	r_{1n}
	a_2	r_{21}	r_{22}	…	r_{2n}
	…	…	…	…	…
	a_m	r_{m1}	r_{m2}	…	r_{mn}

* Classifications

- Two person game
  - Zero-sum game
  - Constant-sum game
  - Non-constant-sum game
- *n*-person game



* Solving Two-Person Zero-Sum Games

- **Step 1**. Games with dominated strategies:

  Eliminate any of Player A's dominated strategies. Looking at the *reduced* payoff table, eliminate any of Player B's dominated strategies. Now eliminate any of Player A's dominated strategies. Continue in this fashion until no more dominated strategies can be found.

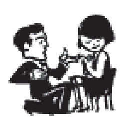
- **Step 2**. Games with pure strategies:

  Find the maximin strategy for Player A that maximizes the minimum payoff. Find the minimax strategy for Player B that minimizes the maximum loss. If the maximin value equals to the minimax value, then we find the *stable* solution (*saddle point*) for both players. Players A and B should exclusively use their maximin and minimax strategies, respectively.

- **Step 3**. Games with mixed strategies:

  Introduce variables that represent the probabilities of selecting each strategy for the players. Find the maximin mixed strategy for Player A that maximizes the *expected* minimum payoff. For Player B, find the minimax mixed strategy that minimizes the *expected* maximum loss.

* Solution Methods for Mixed Strategies



- Graphical solution (or calculus):
  If one of the players has only *two* strategies, we can find the optimal mixed strategies from a 2-dimensional plane.
- Linear programming (*LP*) formulation
  Use Microsoft Excel to solve the LP model.

Ex 1] Games with Dominated Strategies

The head football coach of LSU is attempting to come up with a strategy to deal with Alabama. LSU is on offense, and Alabama is on defense. The LSU coach has 5 preferred plays, but is not sure which to select. He knows, however, that Alabama usually employs one of 3 defensive strategies. Over the years, he has diligently recorded the average yardage gained by his team for each combination of strategies used:

Yardage gained		Alabama		
		b_1	b_2	b_3
LSU	a_1	0	-1	5
	a_2	7	5	10
	a_3	15	-4	-5
	a_4	5	0	10
	a_5	-5	-10	10

Which of the five plays should the LSU coach select?

- Step 1:
- Step 2:
- Step 3:
- Step 4:
- The optimal strategies are

Ex 2] Games with Pure Strategies: Players A and B simultaneously call out one of the numbers, one or two.

		Player B		MAX
		b_1: One	b_2 Two	*min*
Player A	a_1: One	+\$3	**+\$2**	
	a_2: Two	+\$1	-\$6	
MIN	*max*			

- The *optimal* strategies =

Ex 3] Games with Mixed Strategies

Suppose that **Player A** chooses a_1 with probability x and chooses a_2 with probability $1-x$, and **Player B** chooses b_1 with probability y and chooses b_2 with probability $1-y$.

Player A's Payoff v_{ij}		**Player B**		$P[a_i]$
		b_1	b_2	
Player A	a_1	2	4	x
	a_2	3	1	$1-x$
$P[b_j]$		y	$1-y$	1.0

Joint Probabilities		**Player B**		$P[a_i]$
		b_1	b_2	
Player A	a_1	xy	$x(1-y)$	x
	a_2	$(1-x)y$	$(1-x)(1-y)$	$1-x$
$P[b_j]$		y	$1-y$	1.0

- The expected value v of the game to Player A is

$$v = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \; r_{ij} \; y_j =$$

- **Player A**'s *maximin* strategy, $(x^*, 1-x^*)$

Find the first-order derivative with respect to y and set it equal to 0. Solve the equation for x.

$$\frac{d}{dy} v = \frac{d}{dy}(-4xy + 3x + 2y + 1) =$$

Thus, the optimal strategy is x^* = and $1-x^*$ =

- **Player B**'s *minimax* strategy, $(y^*, 1-y^*)$

Find the first-order derivative with respect to x and set it equal to 0. Solve the equation for y.

$$\frac{d}{dx} v = \frac{d}{dx}(-4xy + 3x + 2y + 1) =$$

Thus, the optimal strategy is y^*= and $1-y^*$=

- Value of the game, v^*

$$v^* = -4\,x^* y^* + 3\,x^* + 2\,y^* + 1 =$$

Ex 4] Pure strategies: Consider the following payoff table, which represents player A's gain. Is this a fair game?

Payoff Table for A		Player B				*Maxi min*
		b_1	b_2	b_3	b_4	
Player A	a_1	3	2	4	2	
	a_2	6	-4	-8	-3	
	a_3	4	2	3	2	
	a_4	-5	-3	7	-4	
Minimax						

- Dominated strategies?
- Optimal strategies?
- Value of the game?

Ex 5] Mixed Strategies: Suppose that **Player A** chooses a_1 with probability x and chooses a_2 with probability $1\text{-}x$, and **Player B** chooses b_1 with probability y and chooses b_2 with probability $1\text{-}y$.

Player A's Payoff r_{ij}'		Player B b_1	b_2	P[a_i]
Player A	a_1	9	13	x
	a_2	11	7	$1\text{-}x$
P[b_j]		y	$1\text{-}y$	1.0

Joint Probabilities		Player B b_1	b_2	P[a_i]
Player A	a_1	xy	$x(1\text{-}y)$	x
	a_2	$(1\text{-}x)y$	$(1\text{-}x)(1\text{-}y)$	$1\text{-}x$
P[b_j]		y	$1\text{-}y$	1.0

- Player A's *maximin* strategy, $(x^*, 1\text{-}x^*)$

  The optimal strategy is x^* = and $1\text{-}x^*$ =

- Player B's *minimax* strategy, $(y^*, 1\text{-}y^*)$

  The optimal strategy is y^*= and $1\text{-}y^*$=

- Value of the game, v^*

  $$v^* = 6\,x^* + 4\,y^* - 8\,x^*y^* + 7 =$$

Invariance under the change of *location* and *scale*: $r_{ij}' = 2r_{ij}+5$

B. LP Formulation for Mixed Strategies*

Any game with mixed strategies can be solved by transforming the problem to a linear programming problem.

Payoff Table		Player B				P[a_i]	w_i
		b_1	b_2	…	b_n		
Player A	a_1	r_{11}	r_{12}	…	r_{1n}	x_1	w_1
	a_2	r_{21}	r_{22}	…	r_{2n}	x_2	w_2
	…	…	…	…	…		
	a_m	r_{m1}	r_{m2}	…	r_{mn}	x_m	w_m
P[b_j]		y_1	y_2	…	y_n	1.0	$max\{w_j\}$
v_j		v_1	v_2	…	v_n	$min\{v_i\}$	v^*

(a) **Player A**: Let v be the Player A's *minimum* expected gain. Since Player A is trying to maximize the minimum expected gain v, the LP model is

$$Max\ z = v$$

$$\sum_{i=1}^{m} x_i r_{ij} \geq v \quad \text{for } j = 1, 2, \dots, n \text{ and}$$

$$\sum_{i=1}^{m} x_i = 1 \quad \text{where } x_i \geq 0 \text{ and } v \text{ is } \textit{unrestricted} \text{ in sign.}$$

(b) **Player B**: Let w be the Player B's *maximum* expected loss. Since Player B is trying to minimize the maximum expected profit w, the LP model is

$$Min\ z = w$$

$$\sum_{j=1}^{n} r_{ij} y_j \leq w \quad \text{for } i = 1, 2, \dots, m \text{ and}$$

$$\sum_{j=1}^{n} y_j = 1 \quad \text{where } y_j \geq 0 \text{ and } w \text{ is } \textit{unrestricted} \text{ in sign.}$$

(c) **Combined LP formulation**

$$\min_{1\le j\le n} \sum_{i=1}^{m} x_i r_{ij} = v \;\le\; w = \max_{1\le i\le m} \sum_{j=1}^{n} r_{ij} y_j$$

Player A's *maximin* value ≤ Player B's *minimax* value

- It can be shown from the dual theorem that the *optimal* objective function values v and w are equal.
- Thus, player A's *floor* (maximin value) equals player B's *ceiling* (minimax value).
- This result is often known as the *minimax theorem*.
- Objective function:

$$Min\ z = w - v$$

- Constraints

$$\sum_{i=1}^{m} x_i r_{ij} \ge v \qquad \text{for } j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} r_{ij} y_j \le w \qquad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_i = 1$$

$$\sum_{j=1}^{n} y_j = 1$$

where $0 \le x_i < 1$ and $0 \le y_j \le 1$ and

v and w are *unrestricted* in sign.

Ex 1] Penalty Kicks in Soccer

Suppose that the kicker can only shoot either right or left. Similarly, the goalie must jump either right or left. (For simplicity, assume that kicking the ball in the center or staying in the center is not a viable option for either the kicker or the goalie.) A kicker has a higher probability of scoring on his natural side (i.e. on the left side for a right-footed kicker) and the goalie has a higher probability of saving the goal if he guesses the correct side.

Right-footed kicker's payoff		Goalie		Kicker's strategy	Goalie's loss
		b_1: Left	b_2 Right		
Kicker	a_1: Left	63.7%	94.4%	x_1	$\$w_1$
	a_2: Right	89.3%	43.7%	x_2	$\$w_2$
Goalie's strategy		y_1	y_2	Sum = 1	$w=\max\{w_1, w_2\}$
Kicker's gain		$\$v_1$	$\$v_2$	$v=\min\{v_1, v_2\}$	*Min* $z=w-v$

- **LP model**

  - Objective function

    Minimize $z = w - v$

  - Constraints

$$0.637\, x_1 + 0.893\, x_2 = v_1 \qquad 0.637\, y_1 + 0.944\, y_2 = w_1$$

$$0.944\, x_1 + 0.437\, x_2 = v_2 \qquad 0.893\, y_1 + 0.437\, y_2 = w_2$$

$$v_1 \geq v \qquad w_1 \leq w$$

$$v_2 \geq v \qquad w_2 \leq w$$

$$x_1 + x_2 = 1 \qquad y_1 + y_2 = 1$$

$$0 \leq x_1 \text{ and } x_2 \leq 1 \qquad 0 \leq y_1 \text{ and } y_2 \leq 1$$

v and w are *unrestricted* in sign.

  - Decision variables

    (x_1, x_2, y_1, y_2) and (v, w).

▪ Various strategies

Right-footed kicker’s payoff		Goalie	
		Randomly	*Strategically*
Kicker	*Randomly*	Case 1	Case 2
	Strategically	Case 3	Case 4

- Case #1

  - Mixed strategy: **x**= (0.5, 0.5) and **y**= (0.5, 0.5).
  - The kicker’s average success rate is **72.8**%.

- Case #2
  - Mixed strategy: **x**= (0.5, 0.5) and **y**= (0.664, 0.336).
  - The kicker’s average success rate is **69.1**%.

- Case #3
  - Mixed strategy: **x**= (0.598, 0.402) and **y**= (0.5, 0.5).
  - The kicker’s average success rate is **79.1**%.

- Case #4
  - Mixed strategy: **x**= (0.598, 0.402) and **y**= (0.664, 0.336).
  - The kicker’s average success rate is **74.0**%.

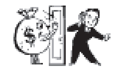
▪ **Kicker’s success rate** for each case

Right-footed kicker’s payoff		Goalie	
		Randomly	*Strategically*
Kicker	*Randomly*	72.8%	69.1%
	Strategically	79.1%	74.0%

Their mixed strategies help them maximize their expected equilibriums, thereby providing them with the best response for each other’s actions.

Ex 2] Search Game

Dr. Chun writes down one of the numbers (**1**, **2**, **3**) and you must repeatedly guess this number until you get it, losing $1 for each wrong guess. After each guess, Dr. Chun will say whether your guess is too high, too low, or correct.



- Game theory
  - Dr. Chun's strategy a_i: How to pick his number i.
  - Your strategy b_j: How to sequence your guesses

- **Case #1**. Dr. Chun *randomly* picks the number.
  - *Pure* strategy: you always choose the strategy, (2, 1 or 3).
  - The value of the game is $0.67.

		Your strategy					$P[a_i]$
		1,2,3	1,3,2	2,1or3	3,1,2	3,2,1	
His strategy	1	0	0	1	1	2	1/3
	2	1	2	0	2	1	1/3
	3	2	1	1	0	0	1/3
E[*Payoff*]		$1	$1	$0.67	$1	$1	$0.67

- **Case #2**. Dr. Chun *strategically* picks the number.
  - *Mixed* strategies
  - The value of the game is $0.80.

		Your strategy					$P[a_i]$
		1,2,3	1,3,2	2,1or3	3,1,2	3,2,1	
His strategy	1	0	0	1	1	2	0.4
	2	1	2	0	2	1	0.2
	3	2	1	1	0	0	0.4
$P[b_j]$		0	0.2	0.6	0.2	0	$0.80

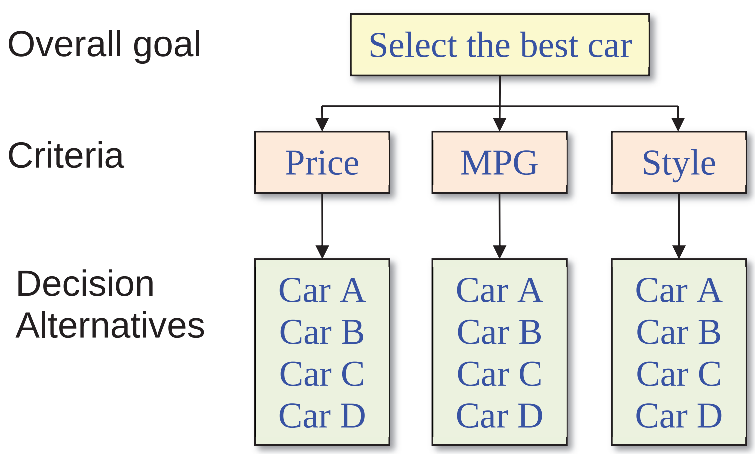
C. Analytic Hierarchy Process*

* Analytic Hierarchy Process

- The Analytic Hierarchy Process (AHP) is a structured technique for organizing and analyzing complex decisions.
- Based on mathematics and psychology, it was developed by Thomas L. Saaty in the 1970s and has been extensively studied and refined since then.
- Users of the AHP first decompose their decision problem into a hierarchy of more easily comprehended sub-problems, each of which can be analyzed independently.

* Developing the Hierarchy

1. Overall goal
2. Criteria
3. Decision alternatives



* Applications

Choice, ranking, prioritization, resource allocation, benchmarking, quality management, conflict resolution, etc.

* Procedures

- **Step 1**. **Criteria**: The decision maker specifies judgments about the relative importance of each criterion in terms of its contribution to the achievement of the overall goal.
- **Step 2**. **Alternatives**: The decision maker indicates a preference or priority for each decision alternatives in terms of how it contributes to each criterion.
- **Step 3**. **Overall preference**: The output of AHP is a prioritized ranking indicating the overall preference for each decision alternative.

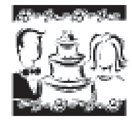
* Measurement Scale

Criteria →	Score	**← Alternatives**
Absolutely more important	**9**	Extremely preferred
Very strongly more important	**7**	Very strongly preferred
Strongly more important	**5**	Strongly preferred
Weakly more important	**3**	Moderately preferred
Of equal importance	**1**	Equally preferred

* Criticisms

- While the general consensus is that it is both technically valid and practically useful, the AHP does have its critics. Most of the criticisms involve a phenomenon called *rank reversal*.
- Decision-making involves ranking alternatives in terms of criteria or attributes of those alternatives. It is an axiom of some decision theories that when new alternatives are added to a decision problem, the ranking of the old alternatives must not change — that "rank reversal" must not occur.

Ex] Soulmate: Kevin is ready to select his mate for life and has determined that beauty, intelligence, and personality are the key factors in selecting a mate.



▪ **Step 1. Criteria**: Obtain the weight for each criterion.

Criteria	Beauty	Intelligence	Personality
Beauty	1		
Intelligence	**4**	1	
Personality	**8**	**4**	1
Total			

Criteria	Beauty	Intelligence	Personality	Total	**Weight**
Beauty	0.077	0.048	0.091	0.216	
Intelligence				0.680	
Personality	0.615	0.762	0.727	2.104	
Total	1.000	1.000	1.000	3.000	1.000

▪ **Step 2. Alternatives**: Three girlfriends (Mary, Melanie, and Molly) are begging to be the Kevin's mate.

(a) ***Beauty***

	Mary	Melanie	Molly
Mary		**5**	**3**
Melanie			
Molly		**2**	
Total			

	Mary	Melanie	Molly	Total	***Score***
Mary	0.652	0.625	0.667	1.944	
Melanie				0.366	
Molly	0.217	0.250	0.222	0.689	
Total	1.000	1.000	1.000	3.000	1.000

(b) ***Intelligence***

	Mary	Melanie	Molly
Mary			
Melanie	**6**		**2**
Molly	**4**		
Total			

	Mary	Melanie	Molly	Total	*Score*
Mary	0.091	0.100	0.077	0.268	
Melanie					
Molly	0.364	0.300	0.308	0.972	
Total	1.000	1.000	1.000	3.000	1.000

(c) ***Personality***

	Mary	Melanie	Molly
Mary	1	**4**	
Melanie		1	
Molly	**4**	**9**	1
Total			

	Mary	Melanie	Molly	Total	*Score*
Mary	0.190	0.286	0.184	0.660	
Melanie					
Molly	0.762	0.643	0.725	2.130	
Total	1.000	1.000	1.000	3.000	1.000

- **Step 3. Overall preference**:

	Beauty	Intelligence	Personality	*Weighted Score*
Mary	**0.648**	**0.089**	**0.220**	
Melanie				
Molly	**0.230**	**0.324**	**0.713**	
Weight	**0.072**	**0.227**	**0.701**	1.000

- Thus, Kevin should choose Molly with **0.590**.

D. Multi-Objective Linear Programming**

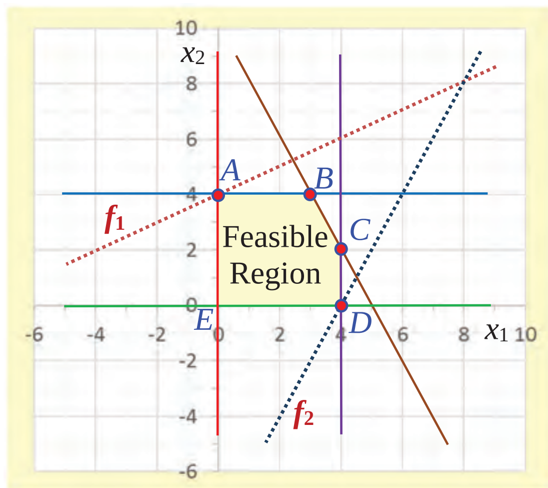
* Original MOLP problem in decision space (x_1, x_2)

- Objective functions

  $Max \quad f_1(\mathbf{x}) = -x_1 + 2x_2$
  $Max \quad f_2(\mathbf{x}) = 2x_1 - x_2$

- Constraints

  (1) $x_1 \le 4$
  (2) $x_2 \le 4$
  (3) $2x_1 + x_2 \le 10$
  (4) $x_1 \ge 0$
  (5) $x_2 \ge 0$



- Efficient set={A, B, C, D}
  - Point A = (0, 4) $f_1(A) = 8$ $f_2(A) = -4$
  - Point B = (3, 4) $f_1(B) = 5$ $f_2(B) = 2$
  - Point C = (4, 2) $f_1(C) = 0$ $f_2(C) = 6$
  - Point D = (4, 0) $f_1(D) = -4$ $f_2(D) = 8$

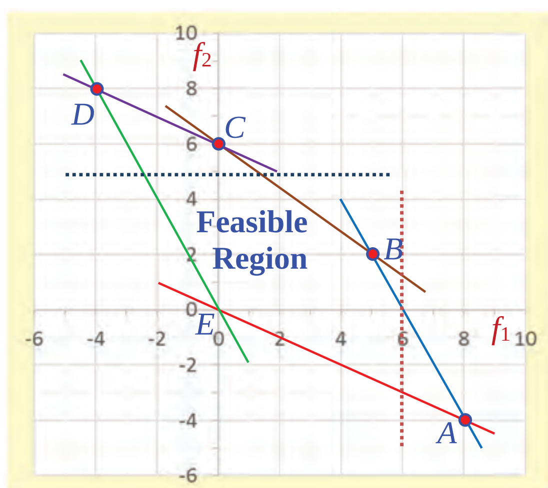
* Reformulated MOLP problem in criterion space (f_1, f_2)

- Objective functions

  $Max \quad f_1$
  $Max \quad f_2$

- Constraints

  (1) $f_1 + 2f_2 \le 12$
  (2) $2f_1 + f_2 \le 12$
  (3) $4f_1 + 5f_2 \le 30$
  (4) $f_1 + 2f_2 \ge 0$
  (5) $2f_1 + f_2 \ge 0$



- Efficient set={A, B, C, D}

* Solving MOLP

- **Method 1**: Weighted sum with the weight w.
  - Two objective functions in the original LP are $f_1(\mathbf{x}) = -x_1 + 2x_2$ and $f_2(\mathbf{x}) = 2x_1 - x_2$.

  Thus, the weighted sum is $f_c(\mathbf{x}) = w\, f_1(\mathbf{x}) + (1- w)\, f_2(\mathbf{x})$

Max $f_c(\mathbf{x})$	$f_1(A)= 8$ $f_2(A)= -4$	$f_1(B)= 5$ $f_2(B)= 2$	$f_1(C)= 0$ $f_2(C)= 6$	$f_1(D)= -4$ $f_2(D)= 8$
w=0.3	-0.4	2.9	4.2	**4.4**
w=0.4	0.8	3.2	**3.6**	3.2
w=0.5	2.0	**3.5**	3.0	2.0
w=0.8	**5.6**	4.4	1.2	-1.6

- **Method 2**: Goal programming

  - Each objective is viewed as a "goal".
  - Deviation variables, a_i and b_i, are the amounts a targeted goal i is underachieved or overachieved, respectively.
  - The goals themselves are added to the constraint set with a_i and b_i, acting as the *slack* and *surplus* variables.
  - Goal constraints:
    - Goal 1: $f_1(\mathbf{x}) = -x_1 + 2x_2 > s_1 \quad \rightarrow -x_1 + 2x_2 + a_1 - b_1 = s_1$
    - Goal 2: $f_2(\mathbf{x}) = 2x_1 - x_2 > s_2 \quad \rightarrow 2x_1 - x_2 + a_2 - b_2 = s_2$
  - Assume that s_1=6 and s_2=5, and the new objective function is *Min* z = \$2 a_1 +\$1 a_2.

Min $z = 2a_1 + 1a_2$	$f_1(A)= 8$ $f_2(A)= -4$	$f_1(B)= 5$ $f_2(B)= 2$	$f_1(C)= 0$ $f_2(C)= 6$	$f_1(D)= -4$ $f_2(D)= 8$
(a_1, b_1) for Goal 1	(0, 2)		(6, 0)	(10, 0)
(a_2, b_2) for Goal 2	(9, 0)		(0, 1)	(0, 3)
Objective function z	\$9		\$12	\$20