Mathematics of Program Construction 2019

Porto, Portugal

Verified Self-Explaining Computation

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input state: $[x \mapsto 1, y \mapsto 0, z \mapsto 2]$

```
if (y = 1) then { y := x + 1 }
           else { y := y + 1 };
z := z + 1
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$$\boxed{\text{eval}}$$

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eval

expected: $[x \mapsto 1, y \mapsto 2, z \mapsto 3]$

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$$\Downarrow$$

$$\boxed{\text{eval}}$$

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expected: $[x \mapsto 1, y \mapsto 2, z \mapsto 3]$
actual: $[x \mapsto 1, y \mapsto 1, z \mapsto 3]$

Debugger

Slicing

$$[x \mapsto \square, y \mapsto 1, z \mapsto \square]$$

$$[x \mapsto \Box, y \mapsto 0, z \mapsto \Box]$$

```
if (y = 1) then { □ }
           else { y := y + 1 }; □
```

$$\Uparrow$$

backward slicing

$$\Uparrow$$

$$[x \mapsto \Box, y \mapsto 1, z \mapsto \Box]$$

$[x \mapsto \square, y \mapsto 0, z \mapsto \square]$

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if (y = 1) then { □ }
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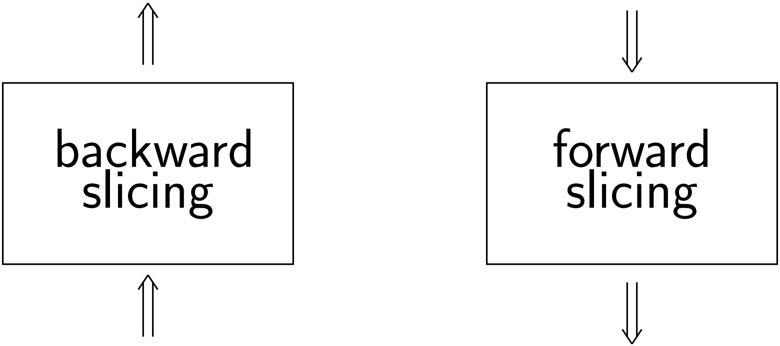

$[x \mapsto \square, y \mapsto 1, z \mapsto \square]$ $\qquad$ $[x \mapsto \square, y \mapsto 1, z \mapsto \square]$

Specifying backward slicing

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- **consistency**: backward slicing retains code required to produce output we are interested in.
- **minimality**: backward slicing produces the smallest partial program and partial input state that suffice to achieve consistency.

Galois connection slicing framework

Slicing based on Galois connections:

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- known from literature [15, 17]

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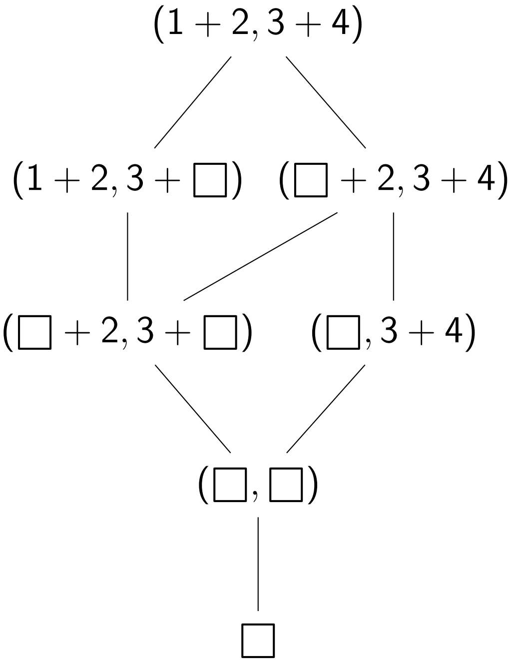
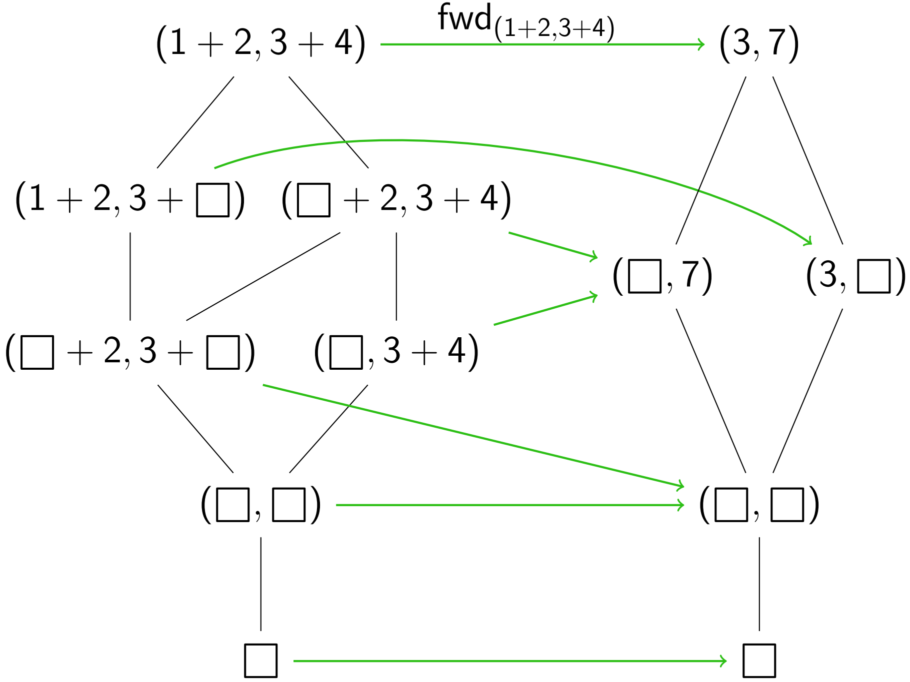
Our contributions:

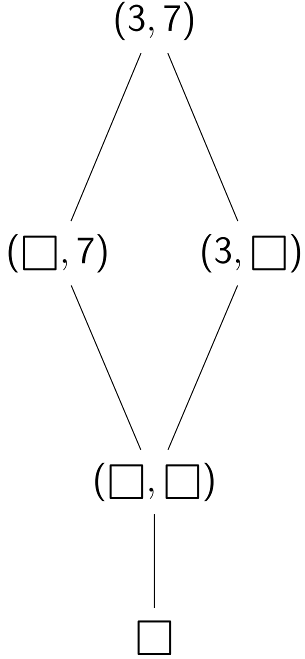
- minimal and consistent slicing for an imperative language
- proofs formalised in Coq
- a different proof strategy



Partial expressions form a *finite lattice*


$\mathrm{fwd}_{(1+2,3+4)}$
$(1+2, 3+4)$
$(3, 7)$
$(1+2, 3+\square)$
$(\square+2, 3+4)$
$(\square, 7)$
$(3, \square)$
$(\square+2, 3+\square)$
$(\square, 3+4)$
$(\square, \square)$
$(\square, \square)$
$\square$
$\square$

Partial values form a *finite lattice*



Galois connections

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$$(P, \sqsubseteq_P), \quad (Q, \sqsubseteq_Q), \quad f : P \to Q, \quad g : Q \to P$$

f and g form a Galois connection (written $f \dashv g$) when

$$f(p) \sqsubseteq_Q q \iff p \sqsubseteq_Q g(q)$$

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$$\mathit{bwd} : \mathit{values}_{\Box} \to \mathit{expressions}_{\Box}, \quad \mathit{fwd} : \mathit{expressions}_{\Box} \to \mathit{values}_{\Box}$$

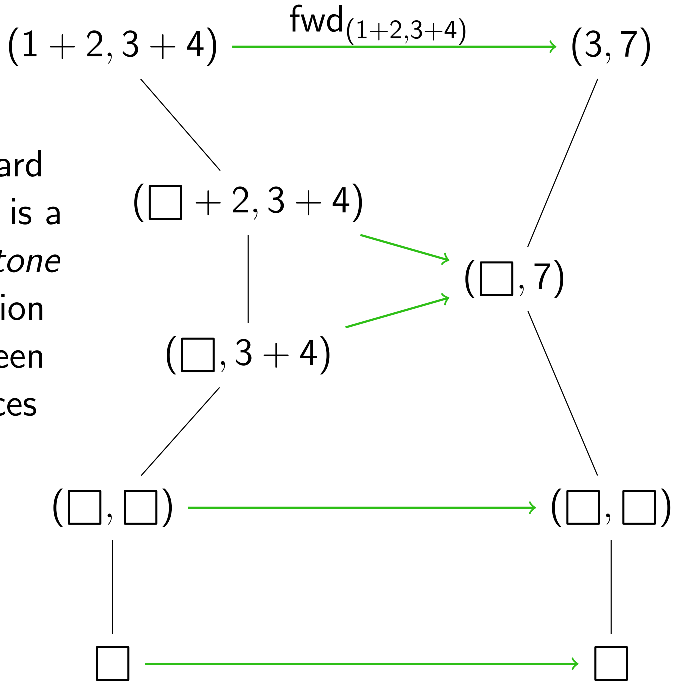
bwd and *fwd* form a Galois connection (written $\mathit{bwd} \dashv \mathit{fwd}$) when

$$\mathit{bwd}(\text{slicing criterion}) \sqsubseteq \text{expr}_{\Box} \iff \text{slicing criterion} \sqsubseteq \mathit{fwd}(\text{expr}_{\Box})$$

Galois connections

Given a monotone forward slicing function fwd there exists a backward slicing function bwd such that bwd $\dashv$ fwd.

Forward slicing is a *monotone* function between lattices



Galois connections

Given a monotone forward slicing function fwd there exists a backward slicing function bwd such that bwd $\dashv$ fwd.

The choice of fwd uniquely determines bwd.

Galois connections

In order to prove that bwd $\dashv$ fwd we need to show that:

1. both fwd and bwd are monotone
2. *deflation* property holds:

$$\forall_{q \in Q} \;\; f(g(q)) \sqsubseteq_Q q$$

3. *inflation* property holds:

$$\forall_{p \in P} \;\; p \sqsubseteq_P g(f(p))$$

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$$\forall\ \mathsf{bwd}(\mathsf{fwd}(\mathsf{expr}_{\Box})) \sqsubseteq \mathsf{expr}_{\Box}$$

3. *inflation* property holds:

$$\forall\ \mathsf{slicing\,criterion} \sqsubseteq \mathsf{fwd}(\mathsf{bwd}(\mathsf{slicing\,criterion}))$$

Our formalisation strategy:

1. prove that any functions fulfilling conditions (1)-(3) form a Galois connection
2. show that our definitions of forward and backward slicing fulfil conditions (1)-(3)

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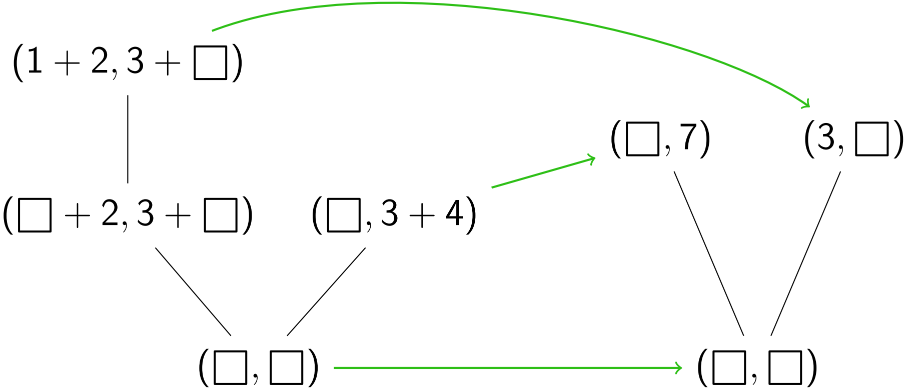
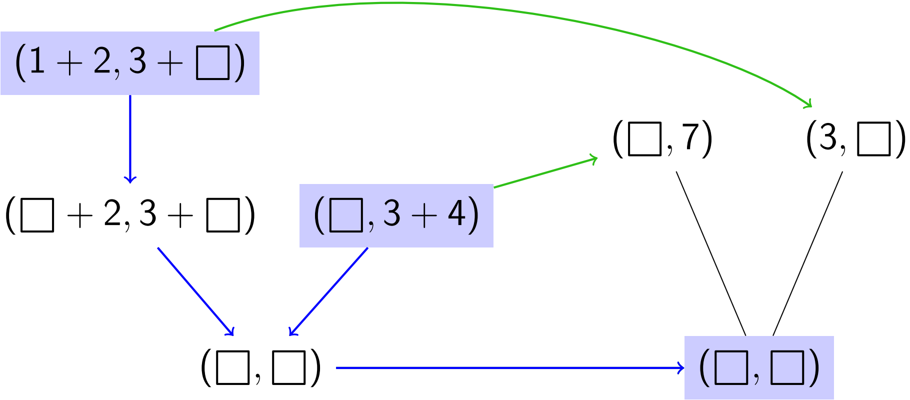
Possible to rely on meet and join preservation instead of monotonicity.

Forward slicing *preserves meets*



$$fwd(x \sqcap y) \;=\; fwd(x) \sqcap fwd(y)$$

Forward slicing *preserves meets*



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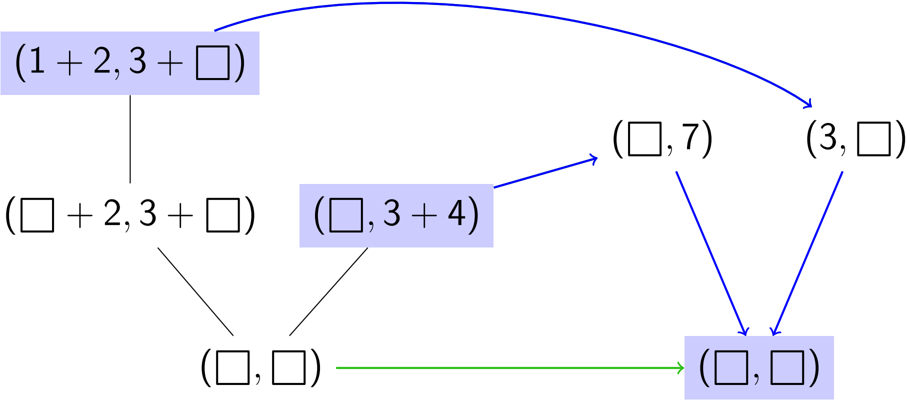

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Syntax

$$a \quad ::= \quad n \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \cdot a_2$$

Syntax

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\begin{array}{lcl}
a & ::= & n \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \cdot a_2 \\
b & ::= & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2
\end{array}
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v & ::= & v_a \mid v_b \\
\mu & ::= & \varnothing \mid \mu, x \mapsto v_a \ (x \text{ fresh})
\end{array}
$$

Forward slicing: notation

$$T :: \mu, e \nearrow v$$

Notation:

- $T ::$ - execution trace
- μ - partial state
- e - partial program (expression or command)
- v - slicing result (μ' for commands)

Forward slicing: expressions

$$\frac{T_1 :: \mu, a_1 \nearrow v_1 \qquad T_2 :: \mu, a_2 \nearrow v_2}{T_1 + T_2 :: \mu, a_1 + a_2 \nearrow v_1 +_{\mathbb{N}} v_2} \quad v_1, v_2 \neq \square$$

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$$\frac{T_1 :: \mu, a_1 \nearrow \Box}{T_1 + T_2 :: \mu, a_1 + a_2 \nearrow \Box} \qquad \frac{T_2 :: \mu, a_2 \nearrow \Box}{T_1 + T_2 :: \mu, a_1 + a_2 \nearrow \Box}$$

Forward slicing: expressions

$$\frac{}{x(v_a) :: \mu, x \nearrow \mu(x)}$$

Important that we read x from μ and not v_a from a trace; we can't assume $\mu(x) = v_a$.

Forward slicing: commands

$$\frac{T_a :: \mu, a \nearrow v_a}{x := T_a :: \mu, x := a \nearrow \mu[x \mapsto v_a]}$$

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$$\frac{T_1 :: \mu, \boxed{} \nearrow \mu' \qquad T_2 :: \mu', \boxed{} \nearrow \mu''}{T_1\,;\ T_2 :: \mu, \boxed{} \nearrow \mu''}$$

Forward slicing: commands

$$\frac{T_1 :: \mu, \Box \nearrow \mu'}{\texttt{if}_{\textbf{true}}\ T_b\ \texttt{then}\ \{\ T_1\ \} :: \mu, \Box \nearrow \mu'}$$

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Backward slicing: notation

$$T :: \mu, v \searrow \mu', e$$

Notation:

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Also:

- $\varnothing_\mu$ - all variables map to $\square$, same domain as μ
- $\varnothing$ - empty state (no variables)

Backward slicing: base cases

Expressions:

$$\frac{}{T :: \mu, \Box \searrow \varnothing_\mu, \Box}$$

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Backward slicing: variable assignments

$$\frac{}{x := T_a :: \mu[x \mapsto \square] \searrow \mu[x \mapsto \square], \square}$$

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$$\frac{T_a :: v_a \searrow \mu_a, a}{x := T_a :: \mu[x \mapsto v_a] \searrow \mu_a \sqcup \mu[x \mapsto \Box], x := a} \quad v_a \neq \Box$$

Backward slicing: conditionals

$$\dfrac{T_1 :: \mu \searrow \mu', c_1 \qquad T_b :: \mathtt{true} \searrow \mu_b, b}{\mathtt{if}_{\mathbf{true}}\ T_b\ \mathtt{then}\ \{\ T_1\ \} :: \mu \searrow \mu' \sqcup \mu_b, \mathtt{if}\ b\ \mathtt{then}\ \{\ c_1\ \}\ \mathtt{else}\ \{\ \square\ \}} \quad c_1 \neq \square$$

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- evaluation defined as inductive data types
- slicing defined as a (dependent) function; takes evaluation evidence; `refine` tactic used to construct evidence
- proofs of monotonicity, inflation, and deflation for defined slicing functions

Summary

We have developed a slicing algorithm for imperative programs based on Galois connections and formalised it in Coq.

More in the paper:

- full rules for forward and backward slicing
- extended example involving loops
- details of Coq formalisation

Future work:

- formalisation of a full-scale language

Implementation available at:

`https://bitbucket.org/jstolarek/gc_imp_slicing`

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Minimality:

$$\mathsf{bwd}_e(v') = \bigsqcap\{e' \mid v' \sqsubseteq \mathsf{fwd}_e(e')\}$$