# Index notation in Lean 4

Joseph Tooby-Smith Reykjavik University

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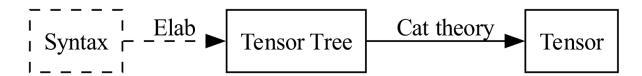
#### **Abstract**

Index notation is tool commonly used in physics to manipulate tensors. In physics, we use index notation for three different types of tensors: Einstein tensors (e.g., ordinary vectors and matrices), Lorentz tensors, and Van der Waerden tensors. In this paper, we discuss how these are implemented in Lean 4 using a general mathematical theory based on category theory, and related to the notation of an operad.

### 1. INTRODUCTION

- Notational conventions abound in physics, and non-more so then index notation.
- Notation, in general, is a way for the writer to compactly write down a term in a mathematical expression. The reader can implicitly unfold or elborate the compactly written term back into its full underlying meaning.
- Index notation is a compact way of writing expressions involving tensors.
- With tensors there is a notion of contraction, evaluation and permutation.
- All of these notions can defined independently of index notation.
- Index notation is a way of writing these operations in a compact way.
- In a previous work by the current author, the first foray of formalizing high energy physics in Lean 4 was undertaken, in a project called 'HepLean'.
- One aim of that project is to make Lean easier for the high-energy phsycisists to use.
- Motiviated by this aim, we have implemented index notation in Lean 4.
- In this paper, we will discuss how this is done.
- This is, of course, not the first paper dicussing the implementation of index notation in a programming language. However, Lean, being a proof assistant, has a different set of requirements.
- In particular, Lean has to be provided a proof of everything.
- We also believe that the underlying mathematics used to implement index notation here is novel.

## 2. OVERVIEW



## 3. COLOR

One of the key features of our construction will be the notation of a color. A color is a property associated to an index. To start with an example, Lorentz tensors have two colors up and down, in other words an index can be an up-index or a down-index. Einstien tensors only have one index, and Van der Waerden tensors have six colors; two for left-handed fermions, two for right-handed fermions, and two for four-vector indices.

We generically denote by C the type of colors. As an example, for Lorentz tensors C = up, down.

Let  $\mathscr{S}$  be the category of types (or sets). The category  $\mathscr{S}_{/C}$  is the category of types over C, that is whose objects are maps  $X \to C$  and whose morphisms from  $X \to C$  to  $Y \to C$  are maps  $X \to Y$  making the obvious triangle diagram commutes.

The core of  $\mathscr{S}_{/C}$ , denoted  $\mathscr{S}_{/C}^{\times}$ , can be thought of as the category of indexing sets of tensors of a given type. We will see this made manifest with a symmetric monoidal functor later.

### REFERENCES