Index notation in Lean 4

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Abstract

Index notation is tool commonly used in physics to manipulate tensors. In physics, we use index notation for three different types of tensors: Einstein tensors (e.g., ordinary vectors and matrices), Lorentz tensors, and Van der Waerden tensors. In this paper, we discuss how these are implemented in Lean 4 using a general mathematical theory based on category theory, and related to the notation of an operad.

1. INTRODUCTION

- Notational conventions abound in physics, and non-more so then index notation.
- Notation, in general, is a way for the writer to compactly write down a term in a mathematical expression. The reader can implicitly unfold or elborate the compactly written term back into its full underlying meaning.
- Index notation is a compact way of writing expressions involving tensors.
- With tensors there is a notion of contraction, evaluation and permutation.
- All of these notions can defined independently of index notation.
- Index notation is a way of writing these operations in a compact way.
- In a previous work by the current author, the first foray of formalizing high energy physics in Lean 4 was undertaken, in a project called 'HepLean'.
- One aim of that project is to make Lean easier for the high-energy phsycisists to use.
- Motiviated by this aim, we have implemented index notation in Lean 4.
- In this paper, we will discuss how this is done.
- This is, of course, not the first paper dicussing the implmentation of index notation in a programming language. However, Lean, being a proof assistant, has a different set of requirements.
- In particular, Lean has to be provided a proof of everything.
- We also believe that the underlying mathematics used to implement index notation here is novel.

2. OVERVIEW

3. COLOR

One of the key features of our construction will be the notation of a color. A color is a property associated to an index. To start with an example, Lorentz tensors have two colors up and down, in other words an index can be an up-index or a down-index. Einstien tensors only have one index, and Van der Waerden tensors have six colors; two for left-handed fermions, two for right-handed fermions, and two for four-vector indices.

We generically denote by C the type of colors. As an example, for Lorentz tensors $C = up, down$.

Let $\mathscr{S}$ be the category of types (or sets). The category $\mathscr{S}_{/C}$ is the category of types over C, that is whose objects are maps $X \to C$ and whose morphisms from $X \to C$ to $Y \to C$ are maps $X \to Y$ making the obvious triangle diagram commutes.

The core of $\mathscr{S}_{/C}$, denoted $\mathscr{S}_{/C}^{\times}$, can be thought of as the category of indexing sets of tensors of a given type. We will see this made manifest with a symmetric monoidal functor later.

REFERENCES