Index notation in Lean 4

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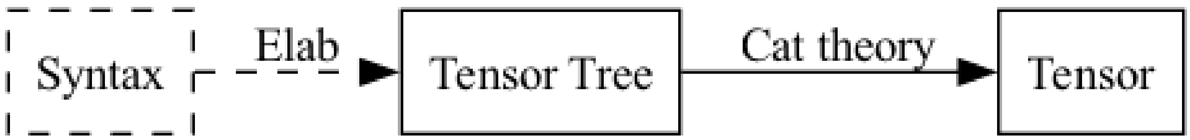
Abstract

Index notation is tool commonly used in physics to manipulate tensors. In physics, we use index notation for three different types of tensors: Einstein tensors (e.g., ordinary vectors and matrices), Lorentz tensors, and Van der Waerden tensors. In this paper, we discuss how these are implemented in Lean 4 using a general mathematical theory based on category theory, and related to the notation of an operad.

1. OVERVIEW



Index notation in Lean, and really in any language including pen and paper, involves three steps:

2. COLOR

One of the key features of our construction will be the notation of a color. A color is a property associated to an index. To start with an example, Lorentz tensors have two colors up and down, in other words an index can be an up-index or a down-index. Einstien tensors only have one index, and Van der Waerden tensors have six colors; two for left-handed fermions, two for right-handed fermions, and two for four-vector indices.

We generically denote by C the type of colors. As an example, for Lorentz tensors $C = up, down$.

Let $\mathscr{S}$ be the category of types (or sets). The category $\mathscr{S}_{/C}$ is the category of types over C, that is whose objects are maps $X \to C$ and whose morphisms from $X \to C$ to $Y \to C$ are maps $X \to Y$ making the obvious triangle diagram commutes.

The core of $\mathscr{S}_{/C}$, denoted $\mathscr{S}^{\times}_{/C}$, can be thought of as the category of indexing sets of tensors of a given type. We will see this made manifest with a symmetric monoidal functor later.

REFERENCES