Assignment 3

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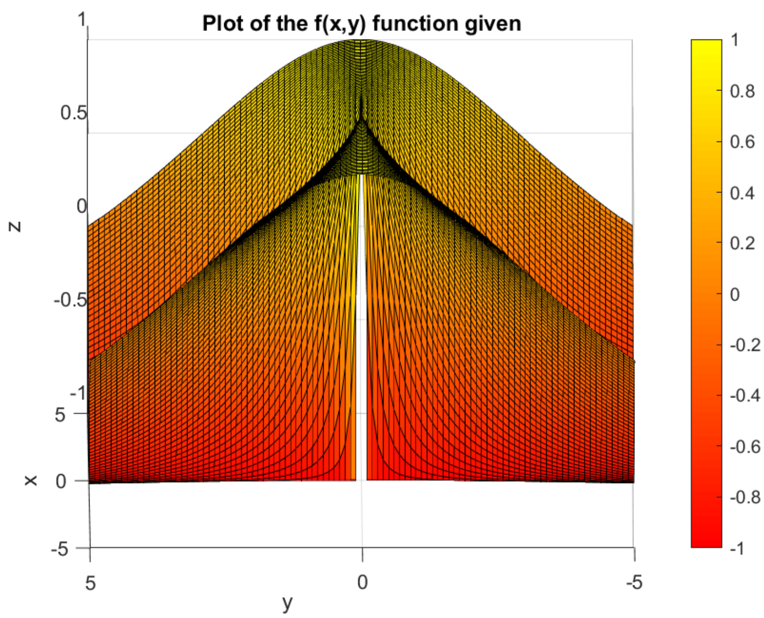
1 3A Surface Plotting

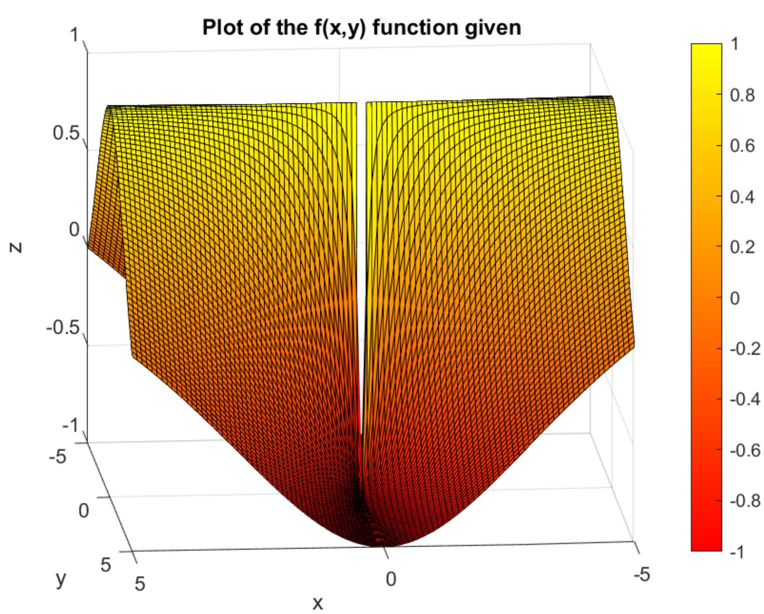
a)
We start by plotting the given function using matlab and surface plots:

```
%This plots f(x,y) as given in the question

[X,Y] = meshgrid(-5:.1:5);
Z = (X.^2 - Y.^2)./(X.^2 + Y.^2);

surf(X,Y,Z)


colorbar
colormap autumn

title('Plot of the f(x,y) function given')
xlabel('x')
ylabel('y')
zlabel('z')
```

This in turn gives us the plot:



and the plot:



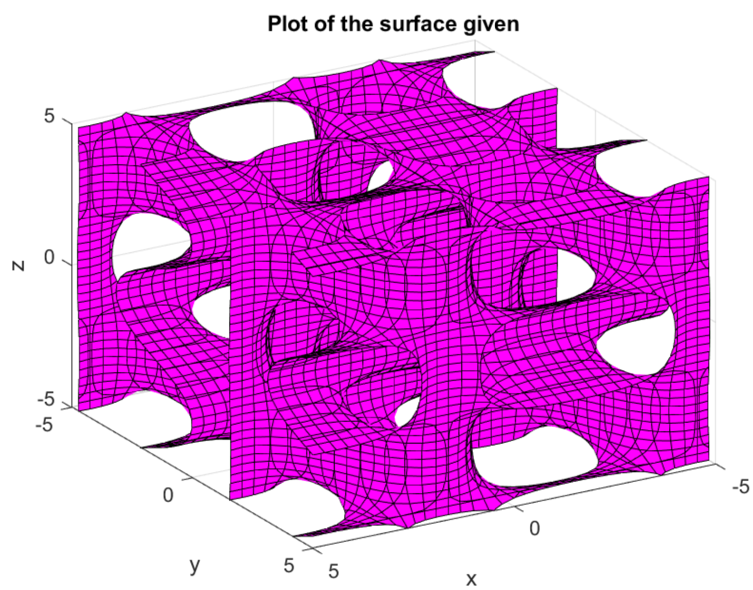
Near the origin ie as x and y tend to 0, the function becomes undefined and tends to infinity, and we observe a gap forming in the surface plot at the origin.

b)

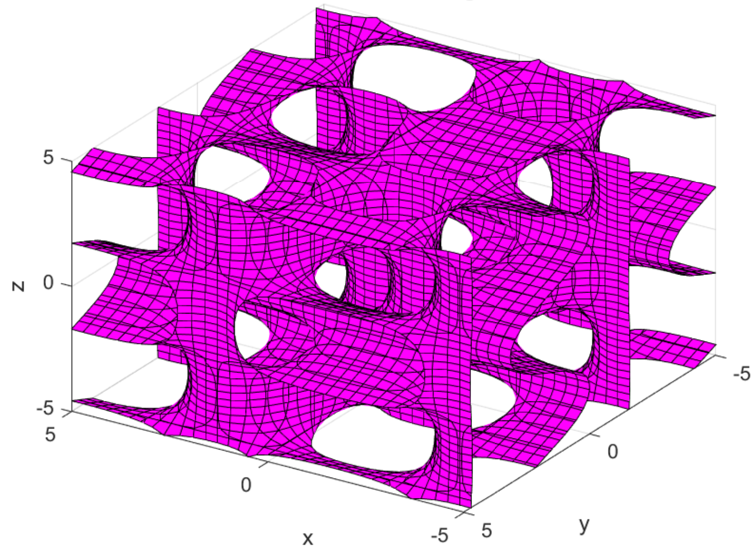
Now we plot the triply periodic surface defined implicitly given in the question using matlab:

```
% This plots the triply periodic surface given in the question


fimplicit3(@(x,y,z) sin(x+y)+sin(x−y)+sin(z+y)+sin(y−z)+sin(x+z)+sin(x−z), 'm' )

title('Plot of the surface given')
xlabel('x')
ylabel('y')
zlabel('z')
```

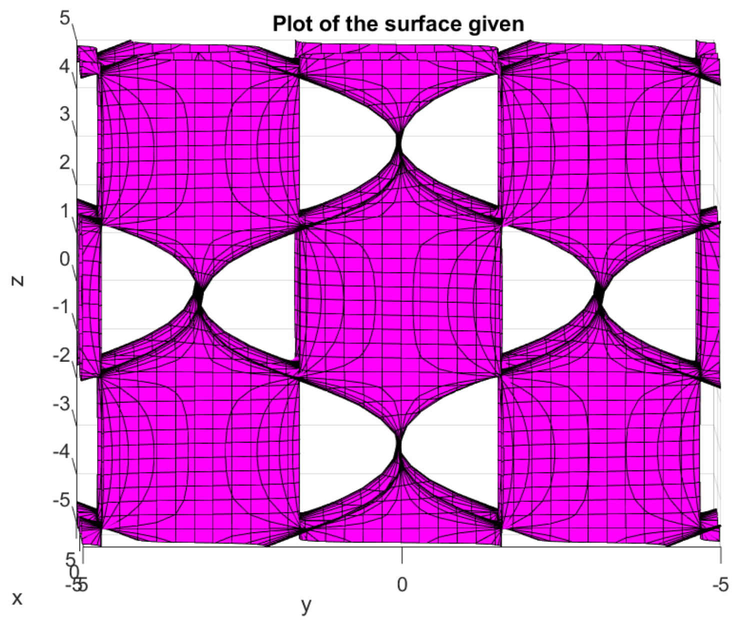
with which we produce the plots:

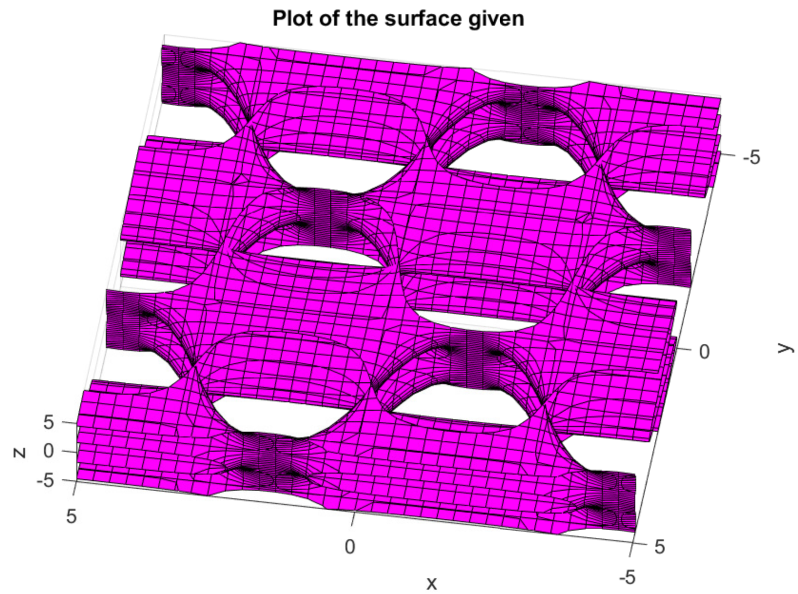


Plot of the surface given



Plot of the surface given





c)
Let us consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$, given by $(x,y) \mapsto 3xe^y - x^3 - e^{3y}$.
Now the critical point of such a function is when :

$$\begin{cases} \dfrac{\partial g}{\partial x} = 0 \\ \dfrac{\partial g}{\partial y} = 0 \end{cases}$$

ie when :

$$\begin{cases} 3e^y - 3x^2 = 0 \\ 3xe^y - 3e^{3y} = 0 \end{cases}$$

ie when:

$$\begin{cases} e^y = x^2 \\ 3xe^y - 3e^{3y} = 0 \end{cases}$$

We substitute $e^y = x^2$ into the second equation.

ie when

$$\begin{cases} e^y = x^2 \\ 3x^3 - 3x^6 = 0 \end{cases}$$

ie when:

$$\begin{cases} e^y = x^2 \\ x^3 = x^6 \end{cases}$$

ie when either of the following occur:

$$\begin{cases} e^y = x^2 \\ x = 1 \end{cases} \qquad \begin{cases} e^y = x^2 \\ x = 0 \end{cases}$$

However, $x \neq 0$ because $x^2 = e^y$, which is always greater than 0 thus the second set of equations is impossible ie when:

ie when:

$$\begin{cases} e^y = x^2 \\ x = 1 \end{cases}$$

ie when:

$$\begin{cases} y = 0 \\ x = 1 \end{cases}$$

Thus g only has one critical point and it is (1,0).
Now we want to show that this critical point is a local maximum.To do this we compute the matrix of second partial derivatives:

$$H = \begin{bmatrix} -6x & 3e^y \\ 3e^y & 3xe^y - 9e^{3y} \end{bmatrix}$$

Now , we apply the second derivative test to the critical point.

Then at the critical point (1,0),

$\frac{\partial^2 g}{\partial x^2}$(1,0)= 6x1= -6<0

and the determinant of the Hessian matrix H at (1,0) is:

$\begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix}$ = 36 - 9 = 7>0

Thus, by the second derivative test, (1,0) is a local maximum of the function g.

However, the global maximum of g is infinity, because if we take y=0 and then the limit of g(x,0) as x tends to minus infinity, we get:

$$\lim_{x \to -\infty} g(x,0).$$

=

$$\lim_{x \to -\infty} 3xe^0 - x^3 - e^0.$$

=

$$\lim_{x \to -\infty} 3x - x^3 - 1.$$

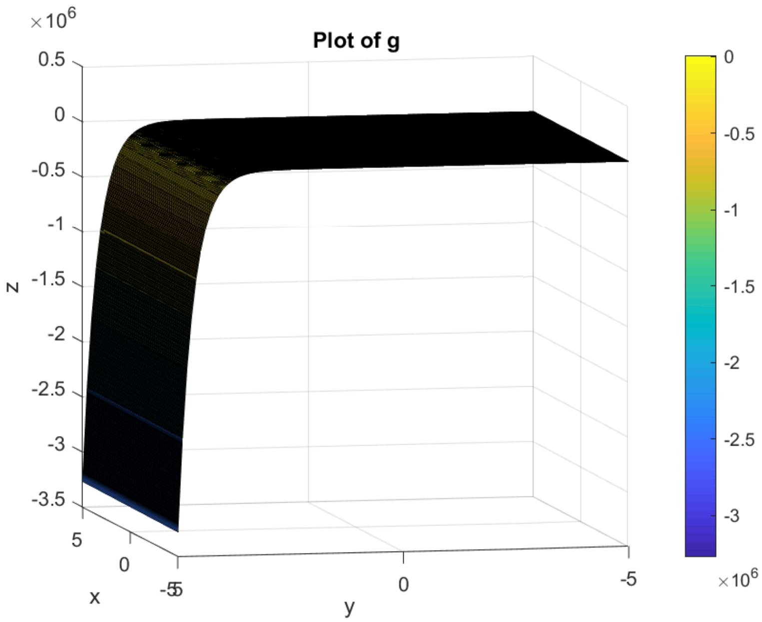
$= \infty$

Thus (1,0) is indeed not a global maximum and the lemma cannot be applied to functions of two variables.

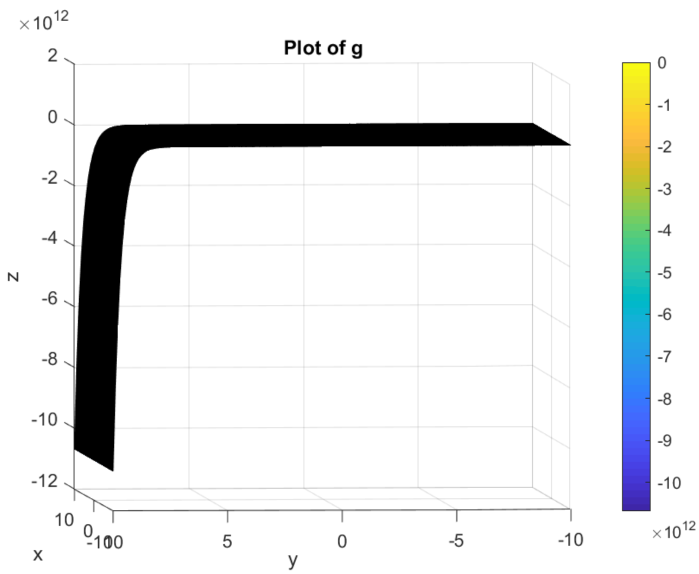
d)
Now we want to plot the function g given in part c).
We plot g on matlab:

```
% We plot the function g given in part c)

[X,Y] = meshgrid(-10:.1:10);
Z = 3*X.*exp(Y)-X.^3-exp(3*Y);
surf(X,Y,Z)
colorbar
title('Plot of g')
xlabel('x')
ylabel('y')
zlabel('z')
```

which produces the plot:



and the plot:

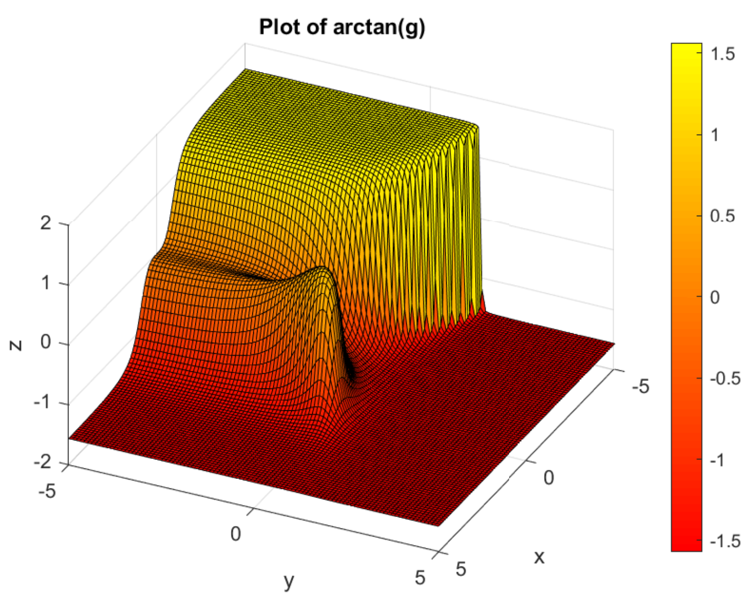


However, this graph is not very lisible as the values of g(x,y) tend to minus infinity. To prevent this and make the plot more lisible and to be able to see the maxima and local behaviour of the function, we try to reduce the values

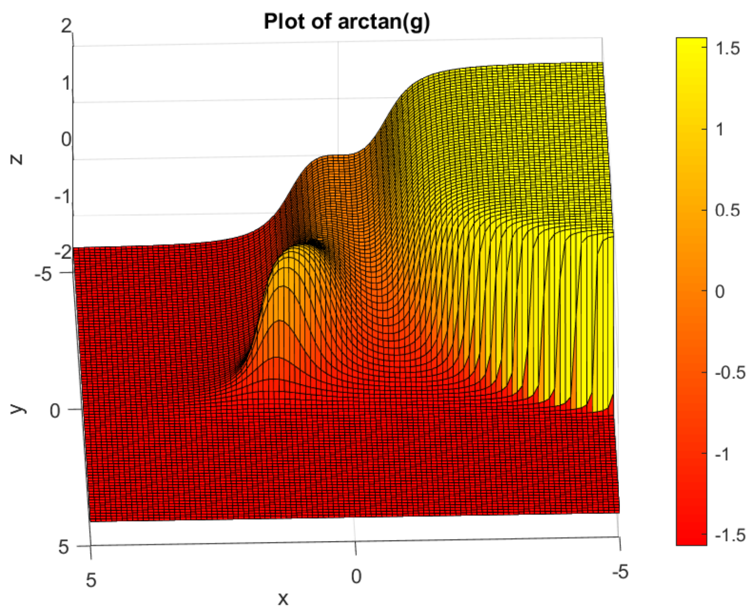
the function can take to a finite interval using the tangent function (ie arctan(g(x,y)) here).
Then we write the m-file:

```
% We plot the function g given in c) composed with the
    function arctan to
% get the values of the function within a finite interval

[X,Y] = meshgrid(-5:.1:5);
Z = 3*X.*exp(Y)-X.^3-exp(3*Y);
W=atan(Z);

surf(X,Y,W)

colorbar
colormap autumn

title('Plot of arctan(g)')
xlabel('x')
ylabel('y')
zlabel('z')
```

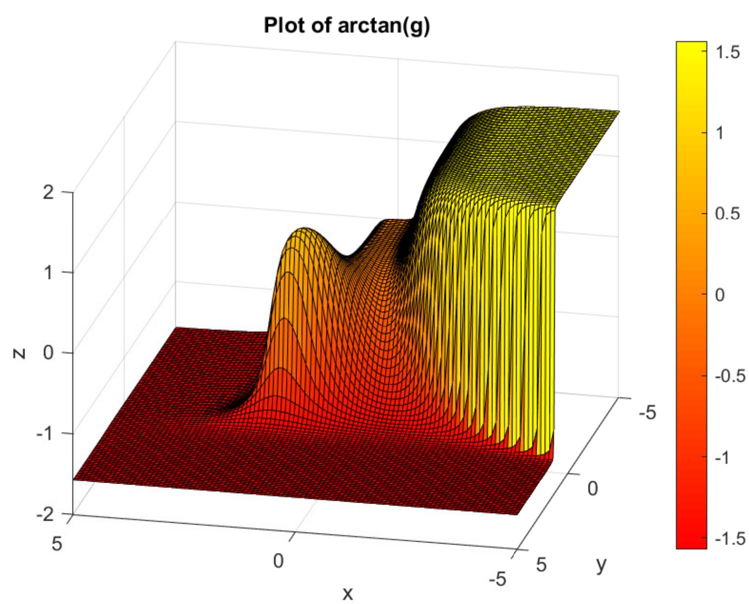
and we then get the plots:



and

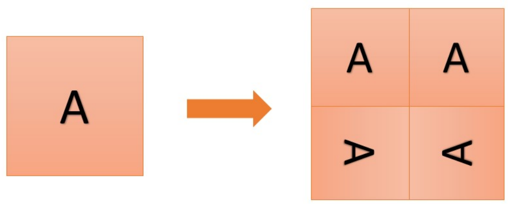

Plot of arctan(g)
z
y
x
2
1
0
-1
-2
-5
0
5
5
0
-5
1.5
1
0.5
0
-0.5
-1
-1.5

and


Plot of arctan(g)
z
x
y
2
1
0
-1
-2
5
0
-5
5
0
-5
1.5
1
0.5
0
-0.5
-1
-1.5

2 3B Recursively defined curves

a)
The pattern looks like:



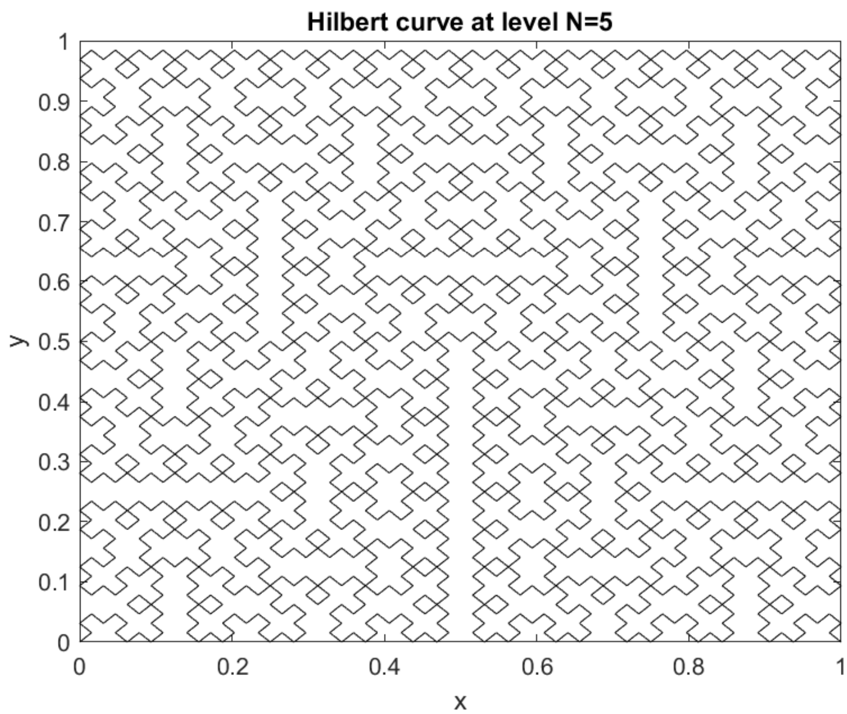
We see that to get the general pattern of the curve we need to rotate the original shape and make it smaller by a factor 2. And repeat this process 4 times and then arrange the resulting 3 shapes in the same manner as in the above diagram.
Then we can adapt the koch.m file given to create the m-file plotting the Hilbert

curve given in the question :

```
function hilbert(l,m,r,level)
%HILBERT     Recursively generated Hilbert curve.
%            hilbert(l, m, r, level) recursively generates a Koch curve, where
%            l is the current left point, m is the currrent middle point, and r is the current right point entered as
%            column vectors and LEVEL is the level of recursion.

if level == 0
  plot([l(1),m(1)],[l(2),m(2)],'k');  % Join l and m.
  hold on
  plot([m(1),r(1)],[m(2),r(2)],'k'); % Join r and m.
```

```
  xlim([0 1]) % Restrict the x axis between 0 and 1
  ylim([0 1]) % Restrict the y axis between 0 and 1
  title('Hilbert curve at level N')

  xlabel('x')
  ylabel('y')
  hold on
else
  A = (1/2)*[cos(-pi/2) -sin(-pi/2); sin(-pi/2) cos(-pi/2)]; % Matrix representing a rotation of 90 degrees clockwise followed by a scaling of 1/2
  B = (1/2)*[cos(pi/2) -sin(pi/2); sin(pi/2) cos(pi/2)]; % Matrix representing a rotation of 90 degrees anti-clockwise followed by a scaling of 1/2
  C = (1/2)*[1 0; 0 1]; % Matrix representing a scaling of 1/2


% First quadrant
  l1=A*l+[0;1/2];
  m1= A*m+[0;1/2];
  r1= A*r+[0;1/2];
hilbert(l1,m1,r1,level-1);

%Second quadrant
   l2=C*l+[0;1/2];
  m2= C*m+[0;1/2];
  r2= C*r+[0;1/2];
hilbert(l2,m2,r2,level-1);

%Third quadrant
l3=C*l+[1/2;1/2];
  m3= C*m+[1/2;1/2];
  r3= C*r+[1/2;1/2];
hilbert(l3,m3,r3,level-1);

%Fourth quadrant
l4=B*l+[1;0];
  m4= B*m+[1;0];
  r4= B*r+[1;0];
hilbert(l4,m4,r4,level-1);

end
end
```

Then at level N= 5, if we insert the command :
hilbert([0;0],[1/2;1/2],[1;0],5)
into matlab, we get the following plot:



b)
Now by observing the curves, we can see that after a certain level N, some points are fixed in the space of the square for a given time t. This is the case for t=1/4.
Indeed, for t=1/4, for N=0, the curve is at (1/4,1/4) and then for all N>0, the curve is at (0,1/2).
As for t=3/16, for N=0, the curve is at (3/16,3/16) and for N=1, the curve is at (1/8, 3/8) and then finally, it becomes a fixed point and for all N>1, the curve is at (1/4,1/2).

c)

We observe that the distance between the closest point to (1/2,0) on the x-axis and (1/2,0) is of $(1/2)^N$.
However, eventually, as $N \to \infty, (1/2)^N \to 0$.
Thus eventually, as N gets bigger the gap closes up between the closest point on the x-axis to (1/2,0) and (1/2,0). In other words, the curve will reach the point (1/2,0) in the limit.
By observing at what t the closest point on the x-axis to (1/2,0) (coming from the left side of the x-axis) occurs for small levels N , we observe that these occur
for t_{N+1}=t_N+$\frac{1}{2^(4+2N)}$ where t_0=0.
So in general, as N tends to infinity,

$$t=\sum_{N=0}^{\infty} \frac{1}{2^(4+2N)} = \frac{1}{2^4} \sum_{N=0}^{\infty} \frac{1}{2^(2N)} = \frac{1}{2^4} \sum_{N=0}^{\infty} \frac{1}{2^(2N)} = \frac{1}{2^4} \frac{1}{1-(1/4)} = \frac{1}{2^4} \frac{4}{3} = \frac{1}{3x4} = \frac{1}{12}$$

Thus we found one time for which, in the limit, the curve touches (1/2,0), and it is t=$\frac{1}{12}$
However, time is symmetric around t=1/2, because the curve is symmetric around the middle of the square at x=1/2.
So the second time at which, in the limit, the curve touches (1/2,0) is

t=1-$\frac{1}{12}$= $\frac{11}{12}$.

Finally, in the limit, the curve touches (1/2,0) at times t=$\frac{1}{12}$ and t=$\frac{11}{12}$.