A Dirichlet Process Mixture Model for Spherical Data

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Massachusetts Institute of Technology

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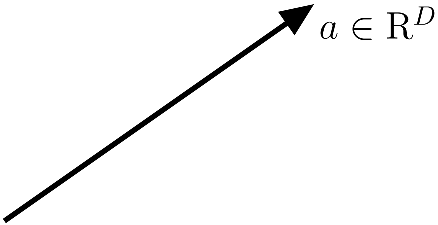
CSAIL

Data on the Sphere = Directional Data



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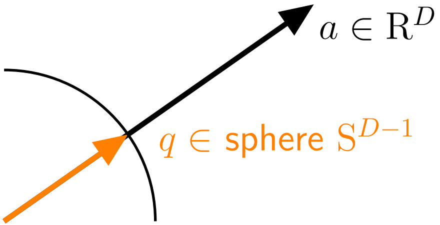

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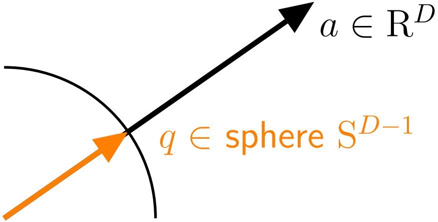


$$a = q||a||_2$$

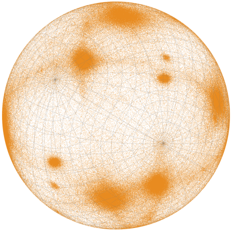
Interested in:

directional data = data on the sphere S^{D-1}
$\Rightarrow$ all information contained in direction q

Native Directional Data

Examples of directional data ($||a||_2 = 1$):

- surface normals [Furukawa 2009, Holz 2011, Straub 2014]



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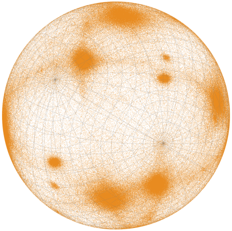

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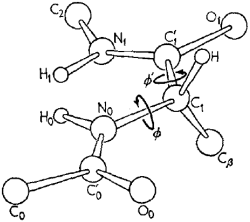

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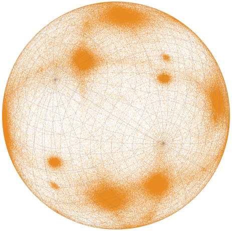

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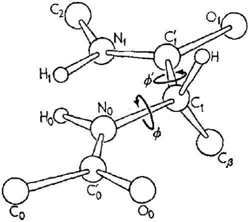

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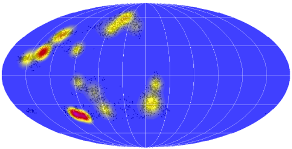

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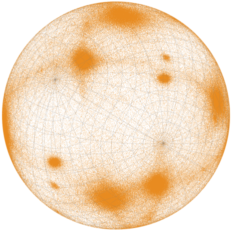
- quaternions representing 3D rotations [Choe 2006]



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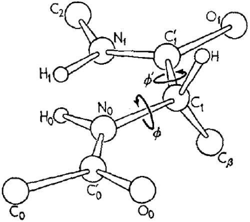

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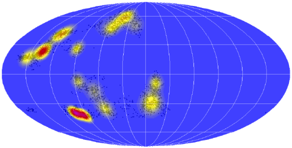
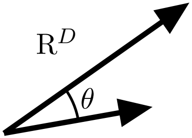


[Bangert 2010]

Euclidean Data Treated as Directional Data

have Euclidean data and use angular deviation for comparison
(cosine distance $\cos(\theta) = \frac{a^T}{||a||_2} \frac{b}{||b||_2}$)



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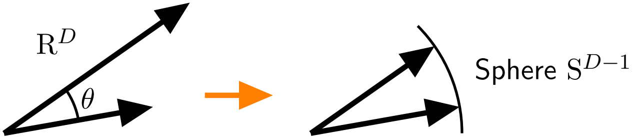

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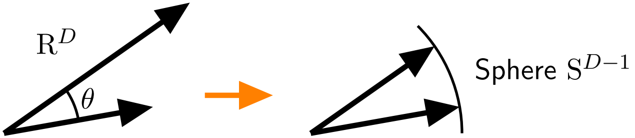

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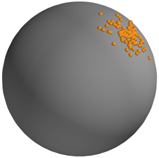
Representative examples:

- semantic word vectors (word2vec) [Mikolov 2013]
- word frequency counts [Dhillon 2001, Strehl 2000]
- gene expression data [Banerjee 2005]

Distributions on the Sphere



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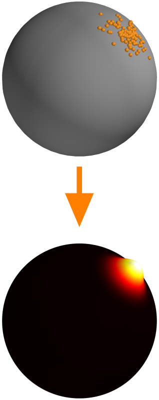

von-Mises-Fisher (vMF)
Kent, Bingham

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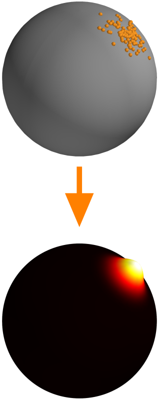

+ no linearization
− difficult normalizers
− posterior sampling

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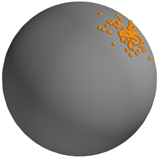
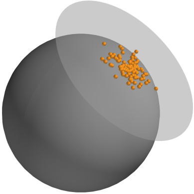
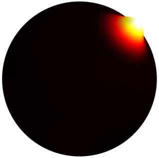


Linearization using

Riemannian Geometry





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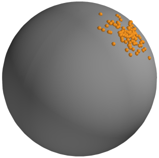

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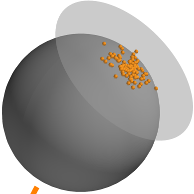

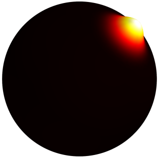

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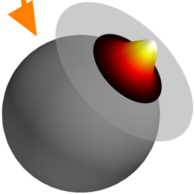

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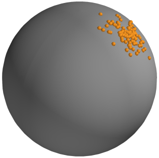

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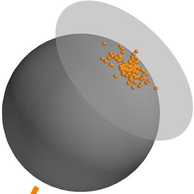

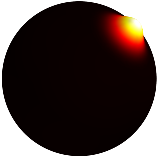

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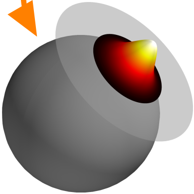

– linearization errors
+ established inference
+ generalization to other manifolds

von-Mises-Fisher (vMF)
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Tangent space Gaussian (TG)

Tangent Space Gaussian via Riemannian Geometry

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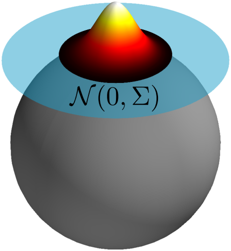
(see Pennec 1999)



mean $\mu \in \mathrm{S}^{D-1}$

covariance Σ in tangent space

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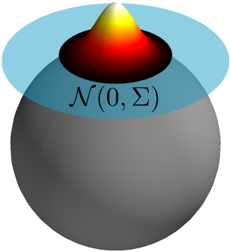

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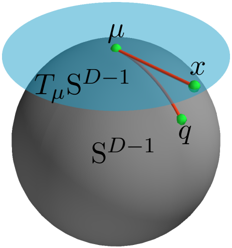

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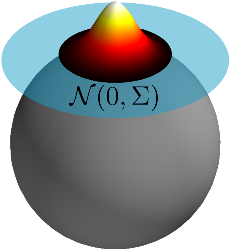

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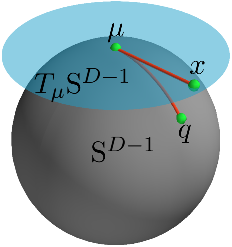

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$$q \sim \mathcal{N}\left(\mathrm{Log}_\mu(q); 0, \Sigma\right)$$

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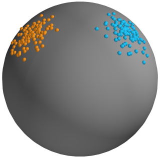

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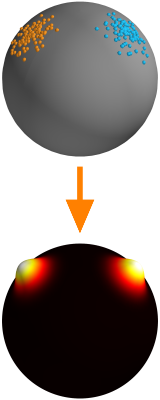

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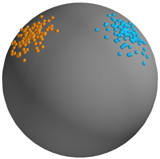

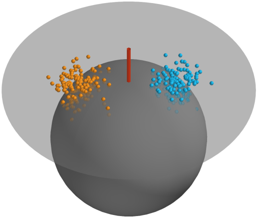

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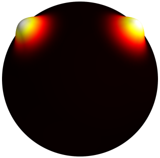

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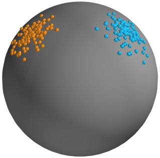

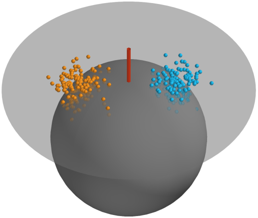

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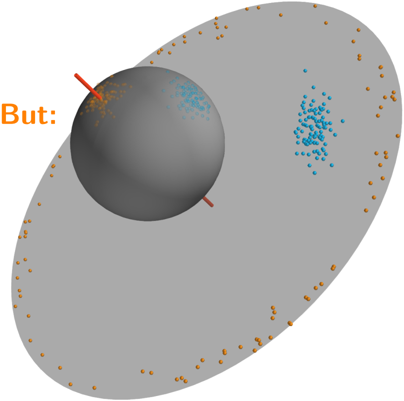

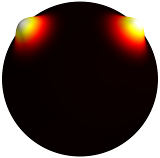

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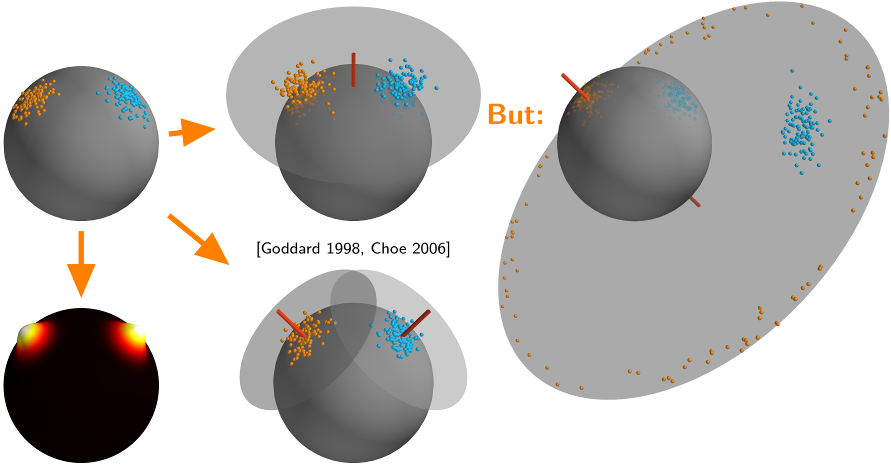

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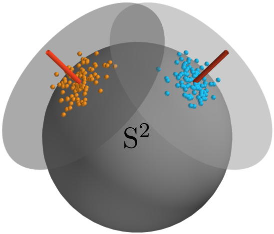
Dirichlet Process Tangent Gaussian Mixture Model

Goals:

- flexibel mixture model for directional data
- anisotropic component distributions
- efficient manifold-respecting inference



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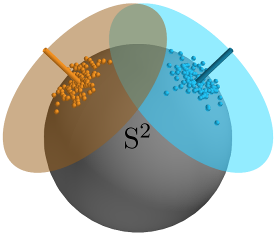

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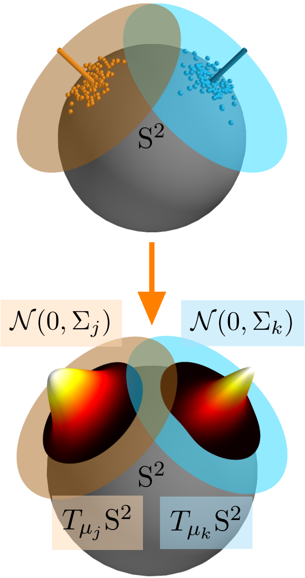

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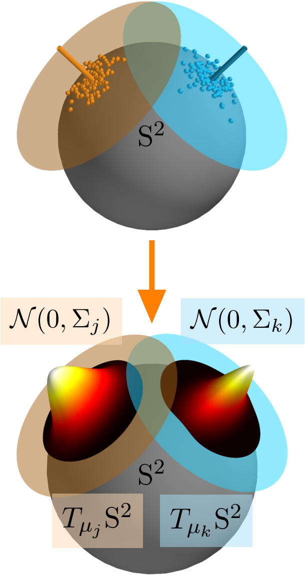
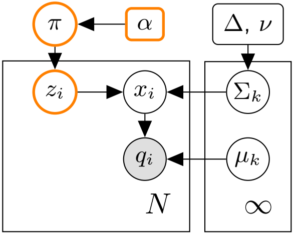
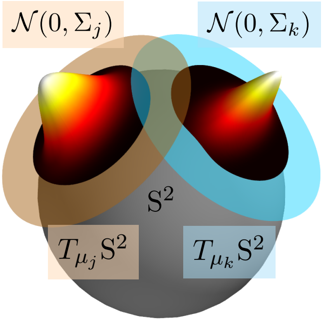


DP-TGMM Generative Model





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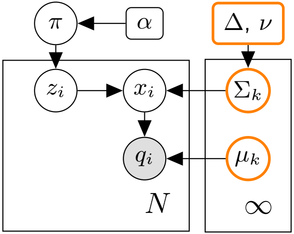

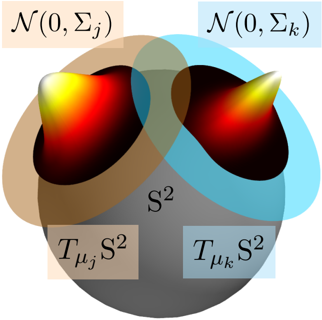

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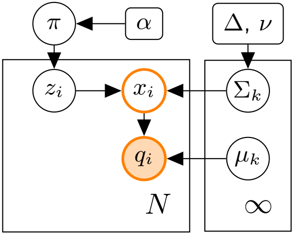

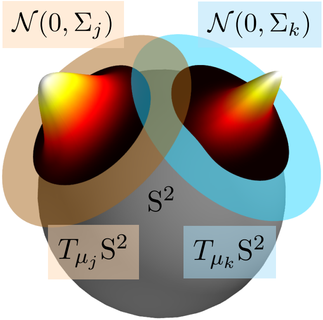

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$$x_i \sim \mathcal{N}(0, \Sigma_{z_i}) \qquad q_i = \mathrm{Exp}_{\mu_{z_i}}(x_i)$$

DP-TGMM Inference

restricted Gibbs sampler with Metropolis-Hastings split/merge proposals

z $\leftarrow 1\,, K \leftarrow 1$
for $t \in \{1, \ldots, T\}$ **do**
    sample parameters
    sample labels
    propose splits of all clusters
    propose merges between all pairs of clusters
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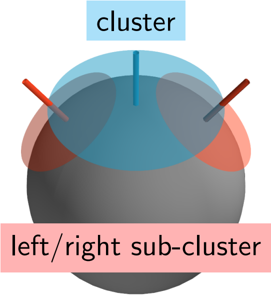

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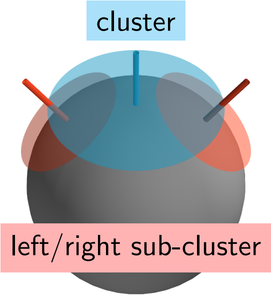

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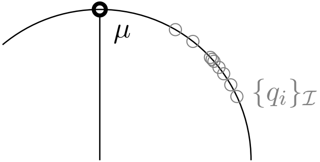
- efficient manifold-aware inference

Efficient Manifold-aware Inference

Problem: inference requires frequent computation of sufficient statistics of $\{q_i\}_{\mathcal{I}}$ in $T_\mu \mathrm{S}^{D-1}$, where μ changes each iteration.

$$S_\mu = \sum_{i \in \mathcal{I}} \mathrm{Log}_\mu(q_i) \mathrm{Log}_\mu(q_i)^T$$



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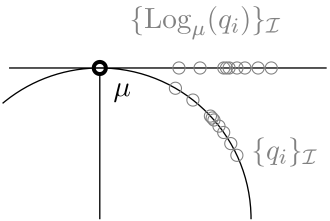

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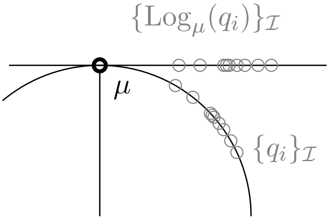

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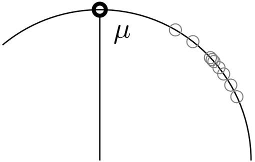

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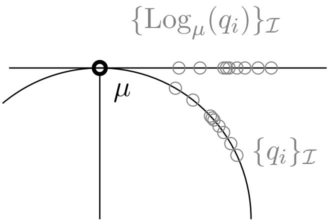

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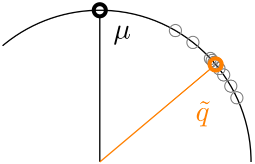

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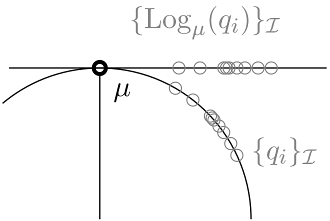

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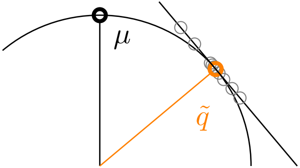

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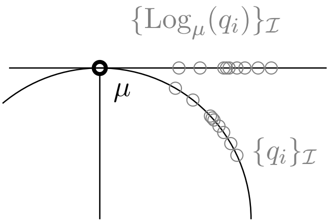

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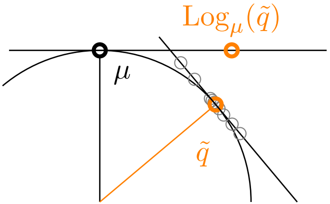
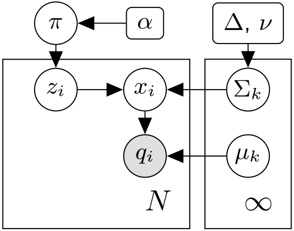
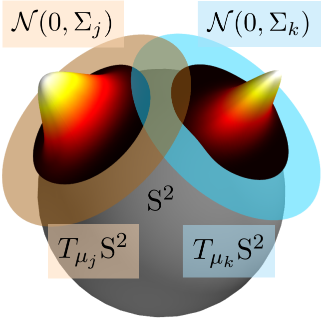


DP-TGMM Recap

Approach and Contributions:

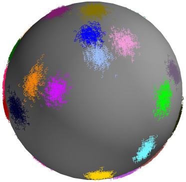
- BNP Dirichlet process mixture model for directional data
- Gaussian distributions in cluster-dependent tangent spaces
- manifold-aware sub-cluster split/merge MCMC sampler





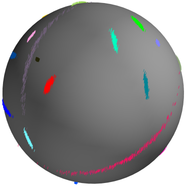
Synthetic Data – Number of Clusters

isotropic



Truth

anisotropic



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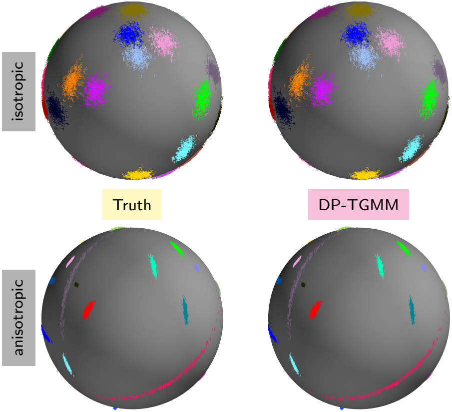

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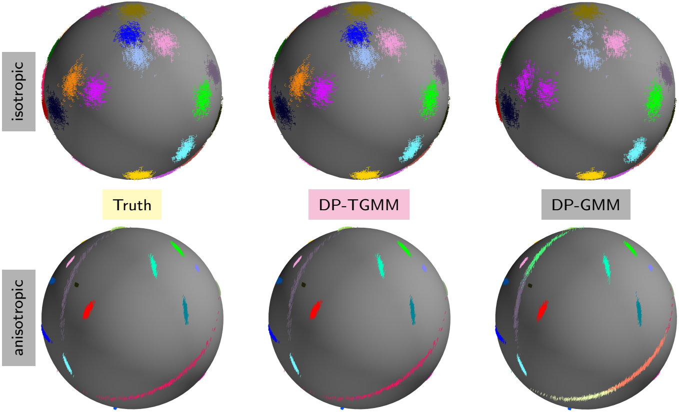

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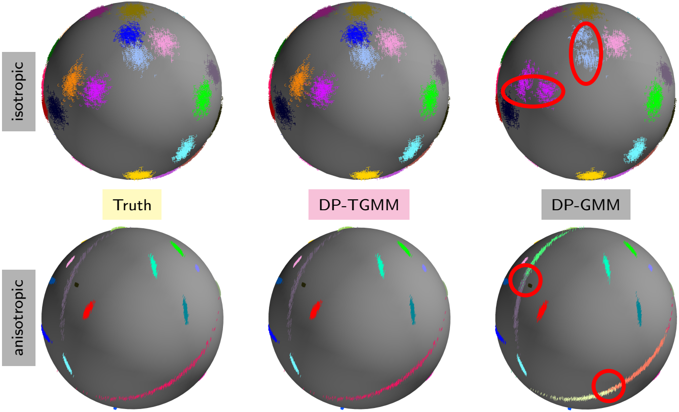

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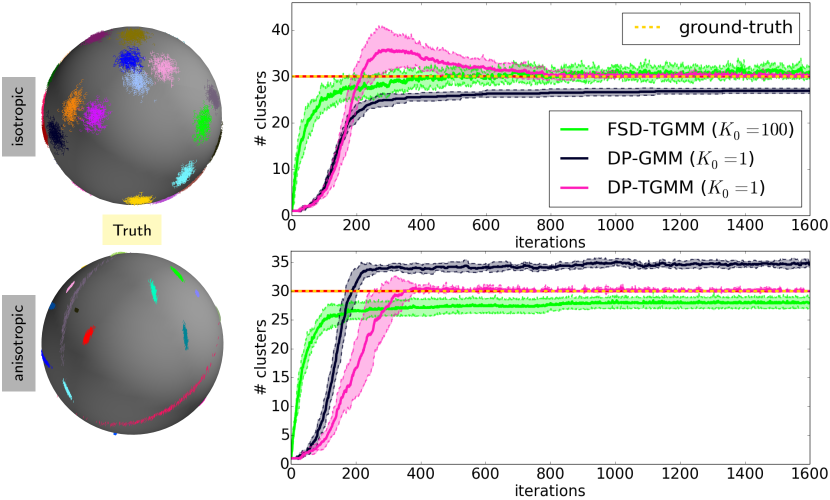

Semantic Word Vectors (word2vec)

word $\overset{\text{word2vec}}{\Longrightarrow}$ embedding space (here 20D) [Mikolov et al. 2013]

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semantically similar words have small angular deviation

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word $\overset{\text{word2vec}}{\Longrightarrow}$ embedding space (here 20D) [Mikolov et al. 2013]

semantically similar words have small angular deviation

"finance"	**"music"**	**"religion "**	**"leisure"**	**"government"**	**"food"**
funding	symphonic	orthodoxy	malls	parliamentary	tomatoes
prospective	operatic	orthodox	hotels	enacted	edible
loans	soloists	evangelical	dining	parliament	fruit
financing	orchestral	christians	nightlife	delegation	meats
funds	music	primacy	outdoor	unanimously	meat
contracts	waltz	preaching	upscale	granting	vegetables
compensation	trios	doctrines	shopping	mandate	juice
regulations	lute	rabbis	restaurants	constitutional	baked
assets	soloist	clergy	taverns	citizenship	corn
investors	flute	catholicism	shops	committee	tasting

Semantic Word Vectors (word2vec) – Covariances

condition number $\kappa = \frac{\max \operatorname{eig}(\Sigma)}{\min \operatorname{eig}(\Sigma)}$ for each inferred cluster

$\kappa > 1 \Rightarrow$ anisotropic covariance

$\kappa = 1 \Rightarrow$ isotropic covariance

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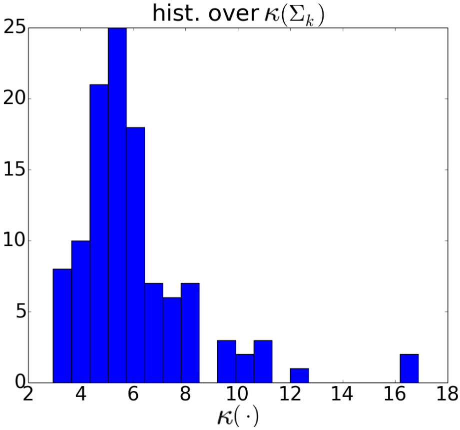

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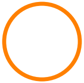

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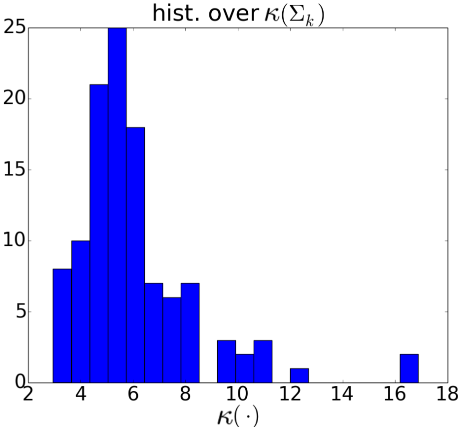
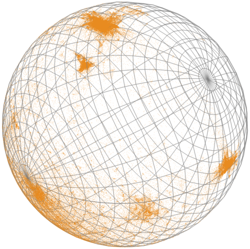


$\Rightarrow$ inferred clusters are anisotropic

Directional Segmentation

directional scene segmentation = clustering of scene's surface normals

image of scene



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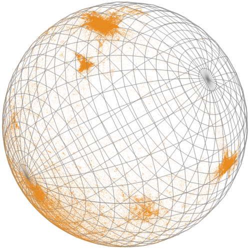

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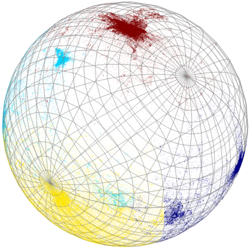

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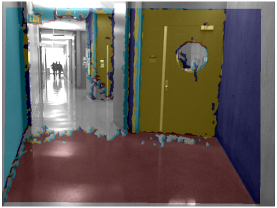

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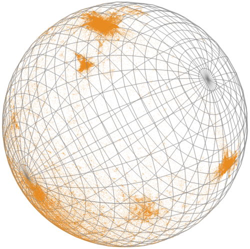

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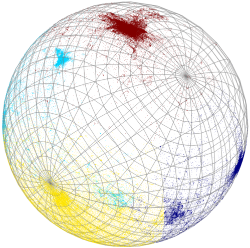

surface normal clustering

Directional Segmentation of NYU RGB-D Dataset

RGB

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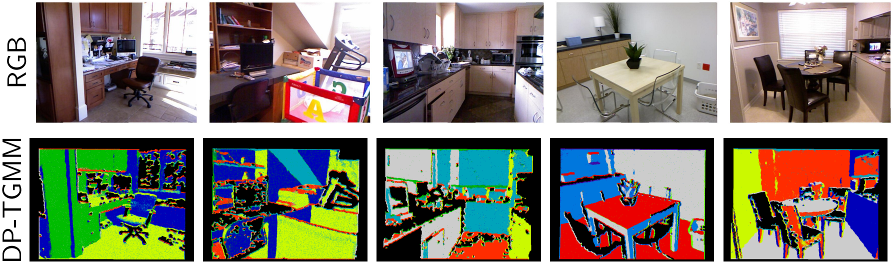

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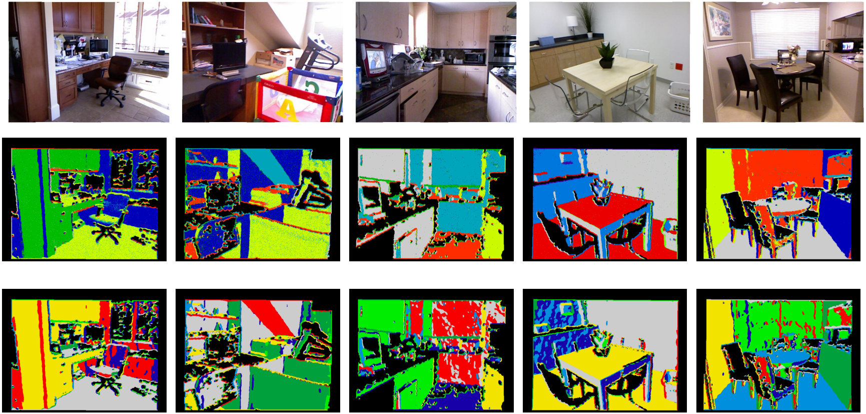

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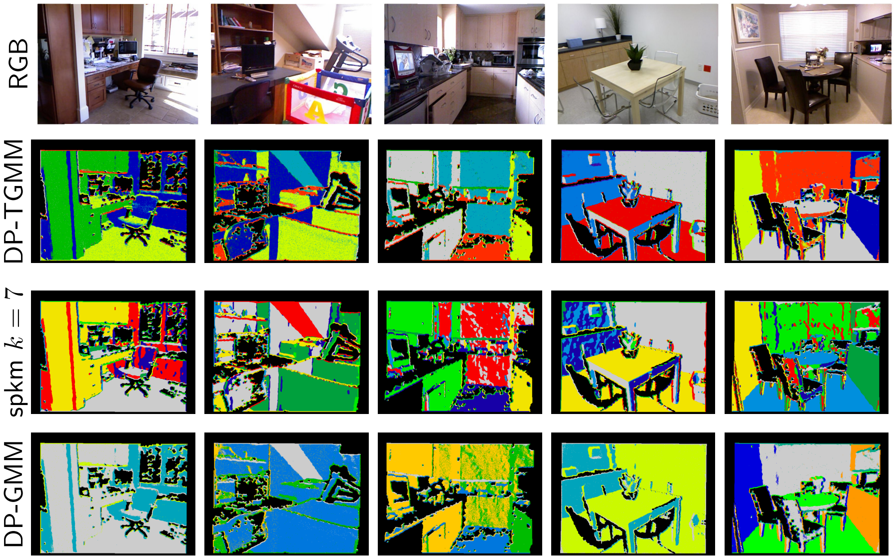
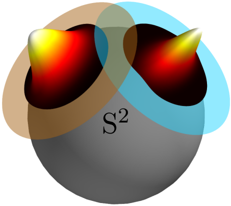
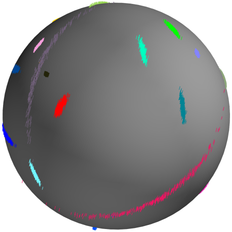
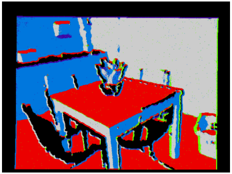


Conclusion







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- anisotropic component distributions on the sphere
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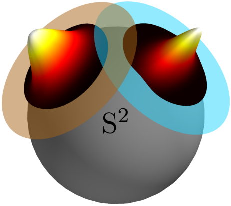

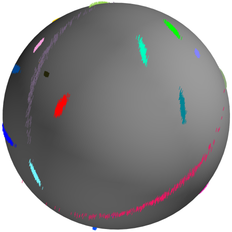

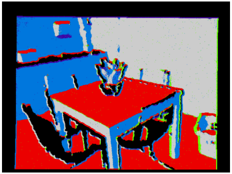

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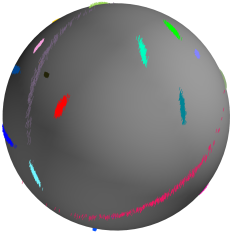

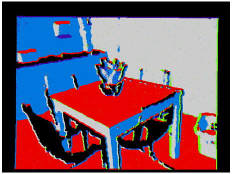

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links to paper, slides and code are at http://people.csail.mit.edu/jstraub/

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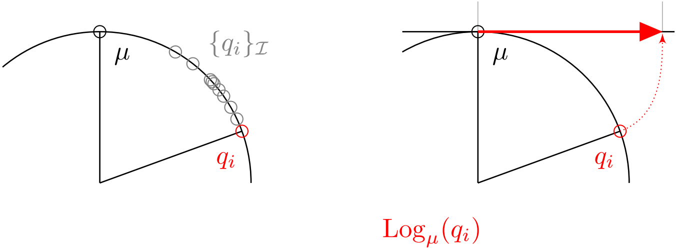

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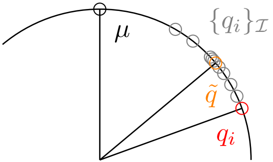

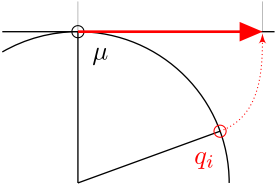

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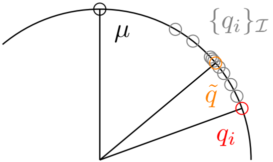

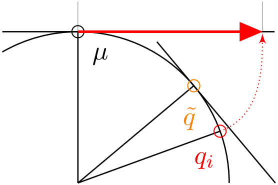

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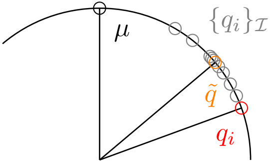

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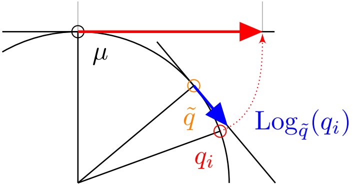

$\mathrm{Log}_\mu(q_i)$

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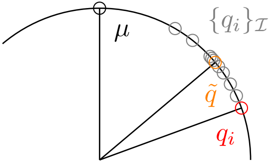

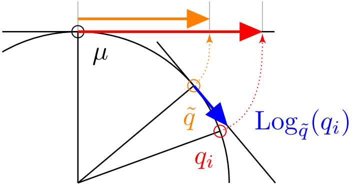

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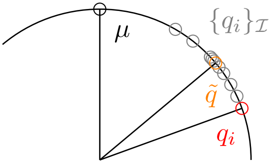

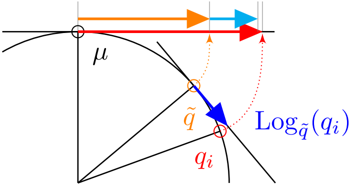

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