Answers to the Theory Questions

of the course of Real and Functional Analysis of prof. Fabio Punzo

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The source code for this document can be found at:
https://github.com/jstringara/Latex-projects/tree/master/ARF

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Sheet n. 1

Question 1.1

Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to\infty} E_n$, $\liminf_{n\to\infty} E_n$, $\lim_{n\to\infty} E_n$.

Solution

Let us define the following:

- **Sequence of sets**

  A family (or collection) of sets $\{E_i\}_{i\in I}$ is called a sequence of sets if $I = \mathbb{N}$ (i.e. it is indexed by the set of natural numbers $\mathbb{N}$)

- **Increasing sequence of sets**

  a sequence of sets $\{E_n\}$ is said to be increasing (or ascending) if:

  $$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Decreasing sequence of sets**

  A sequence of sets $\{E_n\}$ is said to be decreasing (or descending) if:

  $$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Limsup for a sequence of sets**

  for a sequence of sets $\{E_n\}$ we define:

  $$\limsup_{n\to\infty} E_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

- **Liminf for a sequence of sets**

  analogously:

  $$\limsup_{n\to\infty} E_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

- **Limit for a sequence of sets**

  as for a sequence of real numbers if the limsup and liminf coincide we may define:

$$\lim_{n\to\infty} E_n := \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n$$

Question 1.2

Write the definitions of: cover (or covering) of a set; subcover.

Solution

Let us define the following:

- **Cover of a set**

  a family of sets $\{E_i\}_{i\in I}$ is called a cover (or covering) of X if:

$$X \subseteq \bigcup_{i\in I} E_i$$

- **Subcover**

  a sub-family of a cover $\{E_i\}_{i\in J}$ $(J \subseteq I)$ which forms a cover is called a subcover.

Question 1.3

Write the definitions of: equivalence relation, equivalence class, quotient set.

Solution

Let us define the following:

- **Equivalence relation**

  a relation R in X (i.e. a subset $R \subseteq X \times X$) is an equivalence relation if:

  i) $(x, x) \in R \; \forall x \in X$ (**reflexivity**)
  ii) $(x, y) \in R \implies (y, x) \in R$ (**simmetry**)
  iii) $(x, y) \in R$, $(y, z) \in R \implies (x, z) \in R$ (**transitivity**)

Equivalence class

we define an equivalence class for x w.r.t. R as:

$$E_x := \{y \in X : yRx\}$$

i.e. the set of all elements equivalent to x for R

- **Quotient set**

  we define the quotient set of X over R as:

  $$X/R := \{E_x : x \in X\}$$

  i.e. it is the set of all equivalence classes.

Question 1.4

Write the definition of equipotent sets. Write the definition of cardinality of a set.

Solution

Let us define the following:

- **Equipotent sets**

  Two sets X and Y are called equipotent if there exists a bijections, that is, a function:

  $$f : X \to Y$$

  that is both injective and surjective.

- **Cardinality of a set**

  the cardinality of a set X is the collection of all sets equipotent to X.

Question 1.5

Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

Solution

Let us define the following:

- **Finite sets**

  a set X is finite if $\exists n \in \mathbb{N}$ such that there is a bijection:

  $$f : X \to 1, \ldots, n$$

  Example: $\{\frac{1}{1}, \ldots, \frac{1}{n}\}$

- **Infinite sets**

  X is infinite if it is not finite.
  Example: $\mathbb{N}$ is clearly infinite

- **Countable sets**

  X is countable if X is equipotent to $\mathbb{N}$
  Example: $\mathbb{Q}$ can be put in bijection with $\mathbb{N}$

- **Uncountable sets**

  X is uncountable if it is infinite and not countable.
  Example: $\mathbb{R}$ is clearly infinite and not countable since it has the cardinality of continuum.

Question 1.6

Write the definitions of: algebra, $\sigma-$ algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma-$ algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

Solution

Let us define the following:

- **Algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

- **$\sigma-$ algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma-$ algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- **Measurable space**

  The couplet $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma-$ algebra is called a measurable space.

- **Measurable set**

  the elements of the $\sigma-$ algebra of a measurable space are called measurable sets.

Question 1.7

State the theorem concerning the existence of the $\sigma-$ algebra generated by a given set. Give an idea of the proof.

Solution

Minimal $\sigma-$algebra

Let $S \subseteq \mathcal{P}(X)$, then there exists a $\sigma-$algebra $\sigma_0(S)$ such that:

1. $S \subseteq \sigma_0(S)$
2. $\forall \sigma-$algebra$\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

thus $\sigma_0(S)$ is the minimal $\sigma-$algebra generated by S.

Sketch of Proof

We construct the set:

$$\mathcal{V} \coloneqq \{\mathcal{A} \subseteq \mathcal{P}(X) \,\|\mathcal{A} \supseteq S,\ \mathcal{A} \quad \sigma-\text{algebra}\}$$

we may define:

$$\sigma_0(S) \coloneqq \bigcap\{\mathcal{A} :\ \mathcal{A} \in \mathcal{V}\}$$

Question 1.8

Write the definition of the Borel $\sigma-$algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$.

Solution

Borel $\sigma-$algebra

Let (X, d) be a metric space and let $\mathcal{G}$ be the family of open sets of X, then we define the Borel $\sigma-$algebra as:

$$\mathcal{B}(X) \coloneqq \sigma_0(\mathcal{G})$$

The elements of $\mathcal{G}$ are called Borel sets, let us enumerate some classes of them:

Classes of Borel sets

i) open sets

ii) closed sets (they are the complementary of open sets and this is a $\sigma-$algebra)

iii) countable intersections of open sets, known as the family G_δ

iv) countable union of closed sets, known as the family F_δ.

Lastly, let us characterize the Borel $\sigma-$algebras $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$:

Characterization of $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$

1. $\mathcal{B}(\mathbb{R}) = \sigma_0(I) = \sigma_0(I_1) = \sigma_0(I_2) = \sigma_0(I_0) = \sigma_0(\hat{I})$
   where:
$$\begin{aligned} I &= \{(a,b) : a,b \in \mathbb{R}, a \leq b\} \\ I_1 &= \{[a,b] : a,b \in \mathbb{R}, a \leq b\} \\ I_2 &= \{(a,b] : a,b \in \mathbb{R}, a \leq b\} \\ I_0 &= \{(a,b) : -\infty \leq a < b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\} \\ \hat{I} &= \{(a,\infty) : a \in \mathbb{R}\} \end{aligned}$$
2. $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I}) = \sigma_0(\tilde{I}_1)$
   where:
$$\begin{aligned} \tilde{I} &= \{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{[-\infty,b) : b \in \mathbb{R}\} \cup \{(a,+\infty] : a \in \mathbb{R}\} \\ \tilde{I}_1 &= \{(a,+\infty] : a \in \mathbb{R}\} \end{aligned}$$
3. $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_1) = \sigma_0(K_2)$
   where:
$$\begin{aligned} K_1 &= \{\text{n-dimensional closed rectangles}\} \\ K_2 &= \{\text{n-dimensional open rectangles }\} \end{aligned}$$

Question 1.9

Write the definitions of: measure, finite measure, $\sigma-$finite measure, measure space, probability space. Provide some examples of measures.

Solution

Let us define the following:

- ## Measure

  Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$, then a function μ:
$$\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$$
  is a measure if:

  1. $\mu(\emptyset) = 0$
  2. $\sigma -$ **additivity:**
     $\forall \{E_n\} \subseteq \mathcal{C}$ disjoint $(E_i \cap E_j \quad \forall i \neq j)$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ we have that:
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

  ### Finite measure

  a measure μ defined as above is said to bw finite if:
$$\mu(X) < +\infty$$

- **$\sigma-$ finite measure**

  a measure μ is said to be $\sigma-$ finite if there exists a sequence $\{E_n\}$ such that:

$$X = \bigcup_{k=1}^{\infty} E_k, \quad \mu(E_k) < +\infty$$

- **Measure space**

  Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma-$ algebra and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure, then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

- **Probability space**

  if $\mu(X) = 1$ then we say that $(X, \mathcal{A}, \mu)$ is a probability space.

Question 1.10

State and prove the theorem regarding properties of measures. Why the two continuity properties are called in this way? For what concerns continuity w.r.t. a descending sequence E_k, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

Solution

Properties of measures

Let us state and prove the properties of a measure μ on a set X and $\sigma-$ algebra $\mathcal{A}$:

i) **Additivity:**
$\forall \{E_1, \ldots, E_n\} \subseteq \mathcal{A}$ disjoint we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. indeed if we define a sequence such that:

$$\{E_n\} = \begin{cases} B_k = E_k & \forall k \leq n \\ B_k = \emptyset & \forall k > n \end{cases}$$

this sequence is also disjoint ($\mathcal{A} \cap \emptyset = \emptyset \ \forall \mathcal{A} \in X$), thus we may write:

$$\underbrace{\mu\left(\bigcup_{k=1}^{\infty} E_k\right)}_{=\bigcup_{k=1}^{n} E_k \cup \emptyset} = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \sum_{k=n+1}^{\infty} \underbrace{\cancel{\mu(E_k)}}_{=0}$$

□

ii) **Monotonicity:**
$\forall E, F \in \mathcal{A}$ we have:

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

Proof. We may write F in the following way:

$$F = E \cup (F \setminus E)$$

and since these two sets are obviously disjoint we may use (i) to write:

$$\mu(F) = \mu(E) + \underbrace{\mu(E \setminus F)}_{\geq 0} > \mu(E)$$

□

iii) $\sigma -$ **subadditivity:**
$\forall \{E_n\} \subseteq \mathcal{A}$ (**not** disjoint) we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. Let us define:

$$\begin{cases} F_1 \coloneqq E_1 \\ F_n \coloneqq E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad \forall n > 1 \end{cases}$$

Clearly $\{F_n\} \subseteq \mathcal{A}$ and $\{F_n\}$ is a disjoint sequence and:

$$\begin{array}{l} F_k \subseteq E_k \quad \forall k \in \mathbb{N} \implies \mu(F_k) \leq \mu(E_k) \text{ by (ii)} \\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{array}$$

thus we may write:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

□

iv) **Continuity from below:**
$\forall \{E_n\} \subseteq \mathcal{A},\ E_k \nearrow$ we have:

$$\mu\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$$

Proof. Let us define a new sequence $\{F_n\}$ as:

$$\begin{cases} F_k \coloneqq E_k \setminus E_{k-1} \quad \forall k \in \mathbb{N} \text{ and } E_0 \coloneqq \emptyset \\ \implies \bigcup_{k=1}^{n} F_k = E_n,\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

and since $\{F_n\}$ is a disjoint sequence (we may visually think of it as a set of ever increasing rings) we may use (i) to write:

$$\begin{aligned} \mu\left(\lim_{n\to\infty} E_n\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

□

v) **Continuity from above:**
$\forall \{E_n\} \subseteq \mathcal{A}$, $E_k \searrow$, $\mu(E_1) < +\infty$ we have:

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof. Like we did above Let us define: a new sequence $\{F_n\}$

$$F_k := E_1 \setminus E_k \quad \forall k \in \mathbb{N}$$

let us note that $\{F_n\}$ is an increasing sequence thus by (iv) we can write:

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty} (E_k)$$

because by (ii)
$mu(F \setminus E) = \mu(F) - \mu(E)$, moreover:

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (E_1 \cap E_k^{\mathsf{c}}) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^{\mathsf{c}}\right) = E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

thus combining these two and canceling the $\mu(E_1)$ on both sides we obtain:

$$\lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

□

let us note that for this last, crucial, step $\mu(E_1)$ must be finite, otherwise we would not be able to cancel it out from both sides.

Question 1.11

Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

Solution

Let us define the following:

- ### Sets of zero measure

  Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $E \subseteq X$ has zero measure if $E \in \mathcal{A}$ and $\mu(E) = 0$. We denote the set of all sets of zero measure by $\mathcal{N}_\mu$

- ### Negligible sets

  a set $E \subseteq X$ is negligible if:

  $$\exists N \in \mathcal{A} \text{ s.t. } E \subseteq N,\ \mu(N) = 0$$

  So any subset of a set of zero measure is negligible, we denote the collection of all negligible sets by τ_μ. Moreover let us note that E doesn't need to be an element of $\mathcal{A}$ ($E \notin \mathcal{A}$)

- **Almost Everywhere**

  a property P on X is said to hold almost everywhere if:

  $$\mu(\{x \in X : P(x) \text{ is false }\}) = 0$$

  We may also say that $\{x \in X : P(x) \text{ is false }\} \in \mathcal{N}_\mu$

  ### Examples

  typical properties that can be true a.e. are: equality, continuity, monotonicity, etc. etc.

Question 1.12

Write the definition of complete measure space. Exhibit an example of a measure space which is not complete.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Counterexample

Let $X = \{a, b, c\}$, $\mathcal{A} = \sigma(\{\emptyset, \{a\}, \{b, c\}, X\})$ and $\mu \equiv 0$, clearly here we have:

$$\tau_\mu \setminus \mathcal{N}_\mu = \{\{b\}, \{c\}\}$$

and clearly $\{b\}, \{c\} \notin \mathcal{A}$. So this measure space is not complete.

Sheet n. 2

Question 2.1

Write the definition of complete measure space. State the theorem concerning the existence of the completion of a measure space. Give just an idea of the proof.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Existence of the completion

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let us define: $\bar{\mathcal{A}}, \bar{\mu}$

$$\bar{\mathcal{A}} = \{E \subseteq X : \exists F, G \in \mathcal{A} \text{ s.t. } F \subseteq E \subseteq G \ \mu(G \setminus F) = 0\}$$
$$\bar{\mu} : \bar{\mathcal{A}} \to \overline{\mathbb{R}}_+, \quad \bar{\mu}(E) := \mu(F)$$

then:

1. $\bar{\mathcal{A}}$ is a $\sigma -$ algebra , $\bar{\mathcal{A}} \supseteq \mathcal{A}$
2. $\bar{\mu}$ is a complete measure, $\bar{\mu}|_{\mathcal{A}} = \mu$

and the triplet $(X, \bar{A}, \bar{\mu})$ is a complete measure space and is called the completion of $(X, \mathcal{A}, \mu)$, i.e. it the smallest (w.r. to inclusion) complete measure space that cointains $(X, \mathcal{A}, \mu)$

Sketch of proof

We must prove two things:

- **First:** that $\bar{\mathcal{A}}$ is a $\sigma -$ algebra and that it contains $\mathcal{A}$, the latter is trivial since $\forall A \in \mathcal{A} \quad A \subseteq A \subseteq A \implies A \in \bar{\mathcal{A}}$ while the former is quite hardous so we shall just assume it to be true.
- **Second:** that $\bar{\mu}$ is a complete measure and $\bar{\mu}|_{\mathcal{A}} = \mu$.
  The latter is trivial (see above). We can also easily prove that it is a measure:
  i) $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$ since the only set contained inside $\emptyset$ is $\emptyset$ itself, as the container set we may take any zero set measure inside $\mathcal{A}$.

ii) that $\sigma-$additivity holds is clear since for any disjoint sequence $\{E_n\} \subseteq \bar{\mathcal{A}}$ we may construct two sequences:

$$\begin{cases} \{F_n\},\ F_k \subseteq E_k \\ \{G_n\},\ G_k \supseteq E_k \end{cases} \forall k \in \mathbb{N} \text{ s.t. } \mu(G_k \setminus F_k) = 0$$

Let us note the following:

- $\{F_n\}$ is also disjoint because $\{E_n\}$ is disjoint.
- Moreover:

$$\bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} G_k$$
$$\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus F_k)$$
$$\mu\left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k\right) \leq \mu\left(\bigcup_{k=1}^{\infty} (G_k \setminus F_k)\right) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus F_k) = 0$$

The last inequality is true thanks to the $\sigma-$subadditivity and monotonicty of μ.

Thus we can say that:

$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$$

thus $\bar{\mu}$ is a measure.
Let us prove that $\bar{\mu}$ is complete. Let $E_1 \in X$ and $E_2 \in \bar{\mathcal{A}}$ such that $\bar{\mu}(E_2) = \mu(F_2) = 0$ and $E_1 \subseteq E_2$, let us note that:

$$\begin{cases} \mu(G_2) = \cancel{\mu(G_2 \setminus F_2)}^0 + \cancel{\mu(F_2)}^0 \\ \mu(G_2 \setminus \emptyset) = \mu(G_2) - 0 \\ \emptyset \subseteq E_1 \subseteq G_2 \end{cases} \implies E_1 \in \bar{\mathcal{A}},\ \bar{\mu}(E_1) = \mu(\emptyset) = 0$$

thus any negligible set is also a zero measure set and $\bar{\mu}$ is complete.

Question 2.2

Write the definition of outer measure. State and prove the theorem concerning generation of outer measure on a general set X, starting from a set $K \in \mathcal{P}(X)$, containing $\emptyset$, and a function $\nu : K \to \overline{\mathbb{R}}_+,\ \nu(\emptyset) = 0$. Intuitively, which is the meaning of (K, ν)?

Solution

Outer measure

We say that a function: $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ (where X is any set) is an outer measure if:

i) $\mu^*(\emptyset) = 0$

ii) $E_1 \subseteq E_2 \implies \mu^*(E_2) \leq \mu^*(E_2)$

iii) $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

Generation of an outer measure

Let $K \subseteq \mathcal{P}(X)$, $\emptyset \in K$, $\nu : K \to \overline{\mathbb{R}}_+$, $\nu(\emptyset) = 0$, then we can generate an outer measure μ^* on X defined as:

$$\begin{cases} \mu^*(E) := \inf\left\{\sum_{k=1}^{\infty} \nu(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k,\ \{I_n\} \subseteq K\right\}, \text{ if } E \text{ can be covered by a countable union of sets } I_n \in K. \\ \mu^*(E) := +\infty, \text{ otherwise.} \end{cases}$$

Proof. Let us verify that such a μ^* meets the definition of outer measure (2.2.1):

i) $\emptyset \in K$, $0 \leq \mu^*(\emptyset) \leq \nu(\emptyset) = 0$ by the definition of μ^*.

ii) $E_1 \subseteq E_2$, we have two possible cases

- if there exists a countable covering of E_2 then it is also a covering of E_1 and from the definitio of μ^* it follows that:

$$\mu^*(E_1) \leq \mu^*(E_2)$$

- if there is no countable covering of E_2 then:

$$\mu^*(E_1) \leq \mu^*(E_2) = +\infty$$

iii) this condition is obviously met if:

$$\sum_{k=1}^{\infty} \mu^*(E_k) = +\infty$$

otherwise if we suppose that:

$$\sum_{k=1}^{\infty} \mu^*(E_k) < +\infty$$

thus $\mu^*(E_k) < +\infty\ \forall k \in \mathbb{N}$, by the definition of μ^* and inf:

$$\forall \varepsilon > 0,\ \forall n \in \mathbb{N} \quad \exists \{I_{n,k}\} \subseteq K$$

such that:

$$E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n,k})$$

Now, since:

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}, \quad \{I_{n,k}\} \subseteq K$$

it clearly follows that:

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \nu(I_{n,k}) < \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon \cdot \cancelto{1}{\sum_{n=1}^{\infty} \frac{1}{2^n}}$$

because ε is arbitrary, we have the cocnlusion.

$\square$

The intuitive meaning (K, ν) is that K is a special class of sets in X and ν is a function that assigns a value to each set in K. On the other hand ν can be any real valued positive function, thus it is not necessary to be a measure.

Question 2.3

What is the Caratheodory condition? How can it be stated in an equivalent way? Prove it.

Solution

Caratheodory condition

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

Equivalent statement

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if:

$$\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

Proof. It is enough to note that $\forall E \subseteq X$ we have:

$$Z = (Z \cap E) \cup (Z \cap E^{\mathsf{c}}) \quad \forall Z \in X$$

and thus by the subadditivity of μ^* (iii) we get:

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

and we may combine this inequality with the other to yield an equality. □

Question 2.4

Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

Solution

All zero measure sets are in $\mathcal{L}$

There cannot exist such a set E because all sets of zero aouter measure meet the Caratheodory Inequality (2.3.2).

Proof. Indeed $\forall Z \subseteq X$ by the monotonicty of μ^* (ii) we have:

$$\mu^*(\underbrace{Z \cap E}_{\subseteq E}) + \mu^*(\underbrace{Z \setminus E}_{\subseteq Z}) \leq \cancel{\mu^*(E)}^{0} + \mu^*(Z)$$

□

Question 2.5

State the theorem concerning generation of a measure as a restriction of an outer measure.

Solution

Generation of a measure from an outer measure

Let us define $\mathcal{L}$ as:

$$\mathcal{L} := \{E \subseteq X : \ E \text{ is } \mu^* - \text{measurable } \}$$

where μ^* is an outer measure on X, then:

i) the collection $\mathcal{L}$ is a σ − algebra

ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

Question 2.6

Show that the measure induced by an outer measure on the σ − algebra of all sets fulfilling the Caratheodory condition is complete.

Solution

Generation of a measure from an outer measure (proof of completeness)

Let us see that such a measure as the one described in the previous question is complete. Let μ^* be an outer measure on X and $\mathcal{L}$ the σ − algebra of all sets fulfilling the Caratheodory condition. Let μ be the measure induced by μ^* on $\mathcal{L}$ ($\mu = \mu^*|_{\mathcal{L}}$).

Proof. Let $N \in \mathcal{L}$ such that $\mu(N) = \mu^*(N) = 0$ and let $E \subseteq N$.
By monotonicty of μ^* (ii):

$$0 \leq \mu^*(E) \leq \mu^*(N) = 0 \implies \mu^*(E) = 0$$

thus by the lemma seen in 2.4.1 we get that $E \in \mathcal{L}$ and so $\mathcal{L}$ is complete. □

Question 2.7

Describe the construction of the Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^n$.

Solution

Construction of the Lebesgue measure on $\mathbb{R}$

Let I be a family of open, bounded intervals in $\mathbb{R}$:

$$I := \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$$

Let us note that $\emptyset \in I$.
Now let us consider a function λ_0:

$$\lambda_0 : I \to \mathbb{R}_+$$
$$\lambda_0(\emptyset) = 0$$
$$\lambda_0((a, b)) = b - a$$

Here we take $X = \mathbb{R}$, $(K, \nu) = (I, \lambda_0)$ and construct the outer Lebesgue measure λ^* as seen above (2.2.2):

$$\lambda^*(E) := \begin{cases} \inf \left\{ \sum_{n=1}^{\infty} \lambda_0(I_n) : \quad E \subseteq \bigcup_{n=1}^{\infty} I_n, \ \{I_n\} \subseteq I \right\}, & \forall E \subseteq \mathbb{R} \text{ s.t. } E \text{ has a countable covering } \{I_n\} \subseteq I \\ +\infty, \ \text{otherwise} \end{cases}$$

The corresponding $\sigma-$algebra is the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R})$ and now we define the Lebesgue measure λ as the measure generated by the outer Lebesgue measure (as seen in 2.5.1):

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})}$$

Construction of the Lebesgue measure on $\mathbb{R}^n$

Analogously to what we have seen above we first define an outer measure and then a (complete) measure but we take:

$$I^n = \left\{ \bigtimes_{k=1}^{n} (a_k, b_k) : \ a_k, b_k \in \mathbb{R}, \ a_k \leq b_k \right\}$$

and accordingly we define:

$$\begin{aligned} &\lambda_0^n : I^n \to \mathbb{R}_+ \\ &\lambda_0^n(\emptyset) = 0 \\ &\lambda_0^n \left(\bigtimes_{k=1}^{n} (a_k, b_k) \right) = \prod_{k=1}^{n} (b_k - a_k) \end{aligned}$$

and therefore we take $X = \mathbb{R}^n$ and $(K, \nu) = (I^n, \lambda_0^n)$, we define the outer Lebesgue measure $\lambda^{*,n}$ on $\mathbb{R}^n$ and the Lebesgue $\sigma-$ algebra $\mathcal{L}(\mathbb{R}^n)$ and finally we construct the n-dimensional Lebesgue measure as:

$$\lambda^n := \lambda^{*,n}|_{\mathcal{L}(\mathbb{R}^n)}$$

Question 2.8

Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

Solution

All countable sets are $\mathcal{L}$-measurable and $\lambda(E) = 0$

Any countable subset $E \subset \mathbb{R}$ is $\mathcal{L}$-measurable and $\lambda(E) = 0$

Proof. Let $a \in \mathbb{R}$, clearly $\{a\} \subseteq (a - \varepsilon, a]$ $\forall \varepsilon > 0$, thus by the definition of λ^*:

$$\lambda^*(\{a\}) \leq \lambda^*((a - \varepsilon, a]) = \varepsilon \to 0 \implies \{a\} \in \mathcal{L}$$

Now if E is countable we may write as follows:

$$E = \bigcup_{n=1}^{\infty} \{a_n\} \quad a_n \in \mathbb{R}, \ n \in \mathbb{N}$$

and so by monotonicty (ii):

$$0 \leq \lambda^*(E) = \lambda^* \left(\bigcup_{n=1}^{\infty} \{a_n\} \right) \leq \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$$

thus $\lambda^*(E) = 0 \implies E \in \mathcal{L}$ by the lemma seen above (2.4.1) $\square$

Question 2.9

Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. Is the inclusion strict? Which is the relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$?

Solution

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty))$ it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$. We already know from above that all bounded intervals belong to $\mathcal{L}(\mathbb{R})$.
Now, let $A \subseteq \mathbb{R}$ be any set. We assume $a \notin A$, otherwise we would replace A with $A \setminus \{a\}$ and this would leave the Lebesgue outer measure unchanged. Furthermore $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \iff (a, +\infty)$ satisfies the Caratheodory Condition (2.3.2):

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$$

where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, +\infty)$.
Since $\lambda^*(A)$ is defined as an inf, to verify the above, it is necessary and sufficient to show that for **any countable collection** $\{I_n\}$ of **open bounded** intervals that **covers** A we have that:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k)$$

For every $k \in \mathbb{N}$ we define:

$$I'_k := I_k \cap (-\infty, a)$$
$$I''_k := I_k \cap (a, +\infty)$$

then:

$$I'_k \cap I''_k = \emptyset(\text{disjoint}) \implies \lambda_o(I_k) = \lambda_0(I'_k) + \lambda_0(I''_k)$$

Let us note that $\{I'_n\}$ is a countable cover for A_1 and $\{I''_n\}$ is a countable cover for A_2. Hence:

$$\lambda^*(A_1) = \sum_{k=1}^{\infty} \lambda_0(I'_k)$$
$$\lambda^*(A_2) = \sum_{k=1}^{\infty} \lambda_0(I''_k)$$

therefore:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) + \sum_{k=1}^{\infty} \lambda_0(I''_k) = \sum_{k=1}^{\infty} \lambda_0(I_k)$$

which equivalento to the condition above. □

$\mathcal{B}(\mathbb{R}) \subsetneqq \mathcal{L}(\mathbb{R})$

The inclusion demonstrated above can be shown to be strict. A counterexample can be produced (see here) but it is quite pathological.

Relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$. Indeed as we have shown above $\mathcal{B}(\mathbb{R})$ is not a complete $\sigma -$ algebra while $\mathcal{L}(\mathbb{R})$ is.

Question 2.10

Is the translate of a measurable set measurable?

Solution

The translate of a measurable set is measurable

The translate of a measurable set is also measurable.
Let us see a simple example: let (a, b) be an interval and $(a + h, b + h)$ its translate.

$$\lambda((a, b)) = b - a$$
$$\lambda((a + h, b + h)) = (b + h) - (a + h) = b - a$$

Question 2.11

Write the excision property and prove it. Write and prove (partially) the theorem concerning the regularity of the Lebesgue measure on $\mathbb{R}$.

Solution

Excision property

If $A \in \mathcal{L}(\mathbb{R})$, $\lambda^*(A) \leq +\infty$ and $A \subseteq B$, then:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

Proof. Since $A \in \mathcal{L}(\mathbb{R})$ we can use the Caratheodory equality (2.3.1) using $Z = B$, $E = A$:

$$\lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A\ (A \subseteq B)}) + \lambda^*(B \setminus A)$$

so, since $\lambda^*(A) \leq +\infty$ we may write:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

□

Regularity of the Lebesgue Measure

Let $E \subseteq \mathbb{R}$, the following are equal:

i) $E \in \mathcal{L}(\mathbb{R})$

ii) $\forall \varepsilon > 0\ \exists A \subseteq \mathbb{R}$ open s.t.
$$E \subseteq A \quad \lambda^*(A \setminus E) < \varepsilon$$

iii) $\exists G \subseteq \mathbb{R}$ in the class G_δ (countable intersections of open sets) s.t.
$$E \subseteq G \quad \lambda^*(G \setminus E) = 0$$

iv) $\forall \varepsilon > 0\ \exists C \subseteq \mathbb{R}$ closed s.t.
$$C \subseteq E \quad \lambda^*(E \setminus C) < \varepsilon$$

v) $\exists F \subseteq \mathbb{R}$ in the class F_δ (countable unions of closed sets) s.t.

$$F \subseteq E \quad \lambda^*(E \setminus F) = 0$$

Proof. Let us give a (partial) proof:

- $(i) \implies (ii)$: if $E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < +\infty$ then by definition of outer measure (2.2.1):

$$\forall \varepsilon > 0 \; \exists \{I_n\} \text{ that covers } E \text{ and } \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$

  Let us now define the set O:

$$O \coloneqq \bigcup_{k=1}^{\infty} I_k, \; O \text{ is open}, \; E \subseteq O$$

  and so we may write:

$$\lambda^*(O) \overset{sub-add\ (iii)}{\leq} \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$
$$\implies \lambda^*(O) - \lambda^*(E) < \varepsilon$$

  and by the Excision property (2.11.1) ($E \in \mathcal{L}(\mathbb{R}), \; \lambda^*(E) < +\infty$):

$$\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$$

  and so we have obtained the second statement (ii).

- $(ii) \implies (iii)$, $\forall k \in \mathbb{N}$ we choose $O_k \supseteq E$ open for which:

$$\lambda^*(O_k \setminus E) < \frac{1}{k}$$

  and then define:

$$G = \bigcap_{k=1}^{\infty} O_k \implies G \in G_\delta, \; G \supseteq E$$

  Moreover $\forall k \in \mathbb{N}$:

$$G \setminus E \subseteq O_k \setminus E$$

  so by monotonicty (ii):

$$\lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$$

  let us apply a limit $k \to \infty$ to both sides:

$$\lambda^*(G \setminus E) = 0$$

- $(iii) \implies (i)$, let us note that $G \setminus E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G \setminus E) = 0$ by lemma 2.4.1 and:

$$G \in \mathcal{L}(\mathbb{R}) \text{ since } G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$$
$$\implies E = \underset{\in \mathcal{L}}{G} \cap (\underset{\in \mathcal{L}}{G \setminus E})^{\mathsf{c}} \in \mathcal{L}$$

□

Question 2.12

Is it true that any subset $E \subseteq \mathbb{R}$ is $\mathcal{L}$-measurable? Is it possibile to find two disjoint sets $A, B \subset \mathbb{R}$ for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Why?

Solution

Vitali's non-measurable sets

Any measurable set $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$ contains a subset that fails to be measurable. Therefore there exist subsets of $\mathbb{R}$ that are not $\mathcal{L}$-measurable.

Disjoints sets for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$

There are disjoint sets $A, B \subseteq \mathbb{R}$ for which:

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$

Proof. Assume by contradiction that:

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \quad \forall A, B \subseteq \mathbb{R},\ A \cap B = \emptyset$$

Now $\forall E, Z \subseteq \mathbb{R}$ we write:

$$\lambda^*(\underbrace{Z \cap E}_{=A}) + \lambda^*(\underbrace{Z \cap E^{\mathsf{c}}}_{=B}) = \lambda^*(\underbrace{Z}_{=A \cup B})$$

thus any set E would satisfy the Caratheodory condition (2.3.1) and be $\mathcal{L}$-measurable which is absurd since we know that Vitali's sets exist. $\square$

Sheet n. 3

Question 3.1

Write the definition of measurable function. Show the measurability of the composite function.

Solution

Measurable function

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and f a function:

$$f : X \to X'$$

f is said to be measurable if:

$$f^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{A}'$$

Measurability of the composite function

Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ and $(X'', \mathcal{A}'')$ be three measurable spaces and $f : X \to X'$ and $g : X' \to X''$ two measurable functions. Then the composite function $g \circ f : X \to X''$ is measurable.

Proof.

$$\begin{aligned} &\forall E \in \mathcal{A}' \quad f^{-1}(E) \in \mathcal{A} \\ &\forall F \in \mathcal{A}'' \quad g^{-1}(F) \in \mathcal{A}' \end{aligned}$$

thus:

$$\forall F \in \mathcal{A}'' \quad (g \circ f)^{-1}(F) = f^{-1}\left[\underbrace{g^{-1}(F)}_{:=E \in \mathcal{A}'}\right] \in \mathcal{A}$$

□

Question 3.2

Characterize measurability of functions and prove it.

Solution

Characterization of Measurability

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$ then:

$$f : X \to X' \text{ measurable} \iff f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- **($\implies$):** Suppose f be measurable $\implies \mathcal{C}' \subseteq \mathcal{A}'$ and so we get the thesis.
- **($\impliedby$):** Let us define the following:

  $$\Sigma := \{E \subseteq X' : \ f^{-1}(E) \in \mathcal{A}\}$$

  We can easily see that Σ is a $\sigma -$ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

  $$\mathcal{A}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

  and we get the thesis.

□

Question 3.3

Write the definitions of:

a) Borel measurable functions;

b) Lebesgue measurable functions.

Solution

a) Borel measurable functions

Let $(X, d), (X, \mathcal{B})$ and $(X', d'), (X', \mathcal{B}')$ be couples of metric spaces and measurable spaces. A function f:

$$f : X \to X' \text{ measurable}$$

is called Borel-measurable or $\mathcal{B}$-meaurable.

b) Lebesgue measurable functions

Let $(X, \mathcal{L})$ be a measurable space and (X', d') a metric space, $(X', \mathcal{B}')$ a measurable space, then:

$$f : X \to X' \text{ measurable}$$

is called Lebesgue-measurable or $\mathcal{L}$-measurable.

Question 3.4

Prove that continuous functions are both Borel and Lebesgue measurable.

Solution

Continous functions are $\mathcal{B}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{B}$-measurable.

Proof. Let $\mathcal{C}'$ be the class of open sets of X' and $\mathcal{C}$ the class of open sets of X. We have:

$$\forall E \in \mathcal{C}' \quad f^{-1}(E) \in \mathcal{C} \subseteq \mathcal{B} \text{ (by definition of continuity)}$$

and $\mathcal{B}' = \sigma_0(\mathcal{C}')$ so we get the thesis. □

Continous functions are $\mathcal{L}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{L}$-measurable.

Proof. Since $\mathcal{B} \subset \mathcal{L}$ and the previous statement has been proven true, the thesis follows trivially. □

Question 3.5

Characterize Lebesgue measurability of functions and prove it.

Solution

Characterization of Lebesgue measurability

All we must do is apply the Characterization of Measurability (3.2.1) taking $(X, \mathcal{A} = \mathcal{L})$, $(X', \mathcal{A}' = \mathcal{B}')$ and $\mathcal{C}'$ the class of open sets of X', since $\mathcal{B} = \sigma_0(\mathcal{C}')$. We then can write:

$$f : X \to X' \text{ Lebesgue measurable } \iff f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- **($\Longrightarrow$):** Suppose f be Lebesgue measurable $\implies$ $\mathcal{C}' \subseteq \mathcal{B}$ and so we get the thesis.
- **($\Longleftarrow$):** Let us define the following:

  $$\Sigma := \{E \subseteq X' : \ f^{-1}(E) \in \mathcal{L}\}$$

  We can easily see that Σ is a $\sigma -$ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

  $$\mathcal{B} = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

  and we get the thesis.

□

Question 3.6

Establish and show all equivalent statements to the fact that $f : X \to \overline{\mathbb{R}}$ is measurable.

Solution

Equivalent statements of measurability

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a function, the following are equal:

i) f is measurable;

ii) $\{f > \alpha\} \in \mathcal{A}\ \forall \alpha \in \mathbb{R}$;

iii) $\{f \geq \alpha\} \in \mathcal{A}\ \forall \alpha \in \mathbb{R}$;

iv) $\{f < \alpha\} \in \mathcal{A}\ \forall \alpha \in \mathbb{R}$;

v) $\{f \leq \alpha\} \in \mathcal{A}\ \forall \alpha \in \mathbb{R}$.

Proof. Let us prove all the coimplications:
(i) $\iff$ (iii):

$$\begin{aligned} &\mathcal{A}' = \mathcal{B}\left(\overline{\mathbb{R}}\right) = \sigma_0(\overbrace{\{(\alpha, +\infty] :\ \alpha \in \mathbb{R}\}}^{\mathcal{C}'}) \\ &f \text{ is measurable} \quad \iff f^{-1}(\underbrace{(\alpha, +\infty]}_{E}) \in \mathcal{A} \quad \forall \alpha \in \mathbb{R} \end{aligned}$$

(ii) $\implies$ (iii):

$$\{f \geq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f > \alpha - \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(iii) $\implies$ (iv):

$$\{f < \alpha\} = \{f \geq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

(iv) $\implies$ (v):

$$\{f \leq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f < \alpha + \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(v) $\implies$ (ii):

$$\{f > \alpha\} = \{f \leq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

□

Question 3.7

Let $f, g \in \mathcal{M}(X, \mathcal{A})$. What can we say about measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$? Justify the answer.

Solution

Measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$

Let $f, g \in \mathcal{M}(X, \mathcal{A})$, we have:

i) $\{f < g\} \in \mathcal{A}$

ii) $\{f \leq g\} \in \mathcal{A}$

iii) $\{f = g\} \in \mathcal{A}$

Proof.

i) $\{f < g\} = \bigcap_{r\in\mathbb{Q}} \left[\underbrace{\overbrace{\{f < r\}}^{\in\mathcal{A}} \cap \overbrace{\{r < g\}}^{\in\mathcal{A}}}_{\in\mathcal{A}} \right]$

ii) $\{f \le g\} = \{f > g\}^{\mathsf{c}} \in \mathcal{A}$ by the previous point.

iii) $\{f = g\} = \underbrace{\{f \le g\}}_{\in\mathcal{A}} \cap \underbrace{\{f \ge g\}}_{\in\mathcal{A}} \in \mathcal{A}$

□

Question 3.8

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$. Show that $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$. Can there exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$? Why?

Solution

Measurability of $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$, we have:

i) $\sup_n f_n \in \mathcal{M}(X, \mathcal{A})$

ii) $\inf_n f_n \in \mathcal{M}(X, \mathcal{A})$

iii) $\limsup_n f_n \in \mathcal{M}(X, \mathcal{A})$

iv) $\liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$

Proof.

i) $\forall \alpha \in \mathbb{R} \quad \{\sup_{n\in\mathbb{N}} f_n > \alpha\} = \bigcup_{n=1}^{\infty} \{f_n > \alpha\} \in \mathcal{A} \implies \sup_{n\in\mathbb{N}} f_n \in \mathcal{M}$

ii) $\inf_n f_n = -\sup_{n\in\mathbb{N}}(-f_n) \in \mathcal{M}(X, \mathcal{A})$

iii) $\limsup_n f_n = \inf_{k\ge1} \sup_{n\ge k} f_n \in \mathcal{M}(X, \mathcal{A})$

iv) $\liminf_n f_n = \sup_{k\ge1} \inf_{n\ge k} f_n \in \mathcal{M}(X, \mathcal{A})$

□

Question 3.9

Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Show that $f + g, f \cdot g \in \mathcal{M}(X, \mathcal{A})$.

Solution

Measurability of $f+g, f\cdot g$

Let $f,g : X \to \mathbb{R}$ and $f,g \in \mathcal{M}(X,\mathcal{A})$, we have that $f+g, f\cdot g \in \mathcal{M}(X,\mathcal{A})$.

Proof. Let us define a few new functions φ, ψ and χ:

$$\begin{cases} \varphi(x) = X \to \mathbb{R}^2 & \varphi(x) := (f(x), g(x)) \\ \psi(x) = \mathbb{R}^2 \to \mathbb{R} & \psi(s,t) := s+t \\ \chi(x) = \mathbb{R}^2 \to \mathbb{R} & \chi(s,t) := s\cdot t \end{cases} \implies \begin{cases} \psi\circ\varphi = f+g \\ \chi\circ\varphi = f\cdot g \end{cases}$$

Now, clearly $\psi, \chi \in C^0(\mathbb{R}^2)$ (hence measurable), let us prove that φ is also measurable. We use the Characterization of Measurability (3.2.1):

$$\varphi : (X,\mathcal{A}) \to (\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right)) \text{ is measurable } \iff \forall E \subseteq \mathbb{R}^2 \text{ open } \varphi^{-1}(E) \in \mathcal{A}$$

We take:

$$\begin{aligned} E = R &:= (a,b)\times(c,d) \\ \varphi^{-1}(R) &= \{x \in X : \ (f(x), g(x)) \in R\} \\ &= \{x \in X : \ f(x) \in (a,b)\} \cap \{x \in X : \ g(x) \in (c,d)\} \\ &= f^{-1}(a,b) \cap g^{-1}(c,d) \in \mathcal{A} \end{aligned}$$

Thus $\forall E \subseteq \mathbb{R}^2$ open, we may write:

$$E = \bigcup_{k=1}^{\infty} R_k, \quad R_k = (a_k, b_k)\times(c_k, d_k)$$

$$\varphi^{-1} = \bigcup_{k=1}^{\infty} \varphi^{-1}(R_k) = \bigcup_{k=1}^{\infty} f^{-1}(a_k,b_k) \cap g^{-1}(c_k,d_k) \in \mathcal{A}$$

Hence $\varphi \in \mathcal{M}(X,\mathcal{A})$, and we have:

$$\psi\circ\varphi, \ \chi\circ\varphi \in \mathcal{M}(X,\mathcal{A})$$

□

Question 3.10

Prove that A is measurable if and only if χ_A is a measurable function.

Solution

A is measurable if and only if χ_A is a measurable function

Let $A \subseteq X$ and χ_A be the indicator function of A. We have:

$$\chi_A \in \mathcal{M}(X,\mathcal{A}) \iff \quad A \in \mathcal{A}$$

Proof.

$$\{\chi_A > \alpha\} = \begin{cases} X & \alpha < 0 \\ A & 1 > \alpha \geq 0 \\ \emptyset & \alpha \geq 1 \end{cases}$$

Now, $X, \emptyset \in \mathcal{A}$ by definition, so:

$$A \in \mathcal{A} \iff \chi_A \in \mathcal{M}$$

□

Question 3.11

Prove or disprove the following statements:

a) $f \in \mathcal{M}(X,\mathcal{A}) \iff f_\pm \in \mathcal{M}_+(X,\mathcal{A})$;

b) $f \in \mathcal{M}(X,\mathcal{A}) \iff |f| \in \mathcal{M}(X,\mathcal{A})$.

Solution

Measurability of $f_\pm$ and $|f|$

Let $f : X \to \mathbb{R}$, we have:

i) $f \in \mathcal{M}(X,\mathcal{A}) \iff f_\pm \in \mathcal{M}_+(X,\mathcal{A})$

ii) $f \in \mathcal{M}(X,\mathcal{A}) \iff |f| \in \mathcal{M}(X,\mathcal{A})$

Proof.

i)
- **($\implies$):** if $f \in \mathcal{M}(X,\mathcal{A})$, then we define f_+ as:

$$f_+(x) = \max\{f(x), 0\} \geq 0 \quad \forall x \in X$$

  and since $f, 0 \in \mathcal{M}(X,\mathcal{A})$ and max is a measurable function we have that $f_+ = \max \circ (f,0) \in \mathcal{M}_+(X,\mathcal{A})$ by (3.1.2). We may analogously prove the same for f_-.
- **($\impliedby$):** if $f_+ \in \mathcal{M}_+(X,\mathcal{A})$, then we define $f = f_+ - f_-$, and since $f_+, f_-, f \in \mathcal{M}(X,\mathcal{A})$ we have that $f \in \mathcal{M}(X,\mathcal{A})$ by (3.9.1).

ii) $f \in \mathcal{M} \implies f_+, f_- \in \mathcal{M}$ by the previous point $\implies |f| = f_+ + f_- \in \mathcal{M}$ by (3.9.1).

□

Question 3.12

Write the definition of simple function. What is its canonical form? How can we characterize measurability of a simple function? Write the definition of step function.

Solution

Definition of simple function

Let X be a set and $s : X \to \mathbb{R}$ a function. We say that s is a simple function if $s(X)$ is a finite set.
Furthermore we define the two sets:

$$\mathcal{S}(X,\mathcal{A}) := \{\text{ measurable simple functions}\}$$
$$\mathcal{S}_+(X,\mathcal{A}) := \{\text{ measurable simple functions with non-negative values}\}$$

Canonical form of simple function

The canonical form of a simple function is:

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$

where:

$$\begin{aligned}
&c_i \in \mathbb{R} \ \forall i = 1, \dots, n \\
&E_i = \{x \in X : \ s(x) = c_i\} \ \forall i = 1, \dots, n \\
&X = \bigcup_{i=1}^{n} E_i, \ E_k \cap E_l = \emptyset \ \forall k \neq l
\end{aligned}$$

i.e. E_i is a partition of X.

Measurability of simple function

A simple function is measurable if and only if we have the following:

$$E_i \in \mathcal{A} \ \forall i = 1, \dots, n$$

i.e. :

$$s(x) \in \mathcal{M}(X, \mathcal{A}) \iff E_i \in \mathcal{A} \ \forall i = 1, \dots, n$$

this is because $s(x)$ is a linear combination of indicator functions.

Step Functions

Let $I = [a_0, a_1)$ be an interval and $P = \{a_o \equiv x_0 < x_1 < \cdots < x_n \equiv a_1\}$ a partition of I. A function $f : I \to \mathbb{R}$ is a step function if:

$$f := \sum_{k=0}^{n-1} c_k \chi_{[x_k, x_{k+1})}(x)$$

Question 3.13

State and give a sketch of the proof of the Simple Approximation Theorem.

Solution

Simple Approximation Theorem

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence of simple functions $\{s_n\}$ such that:

$$s_n \xrightarrow{n \to \infty} f \text{ in } X \text{ (pointwise)}$$

Furthermore:

i) if $f \in \mathcal{M}(X, \mathcal{A})$, then $\{s_n\} \subseteq \mathcal{S}(X, \mathcal{A})$;

ii) if $f \geq 0 \implies \{s_n\} \uparrow, 0 \leq s_n \leq f$;

iii) f bounded $\implies s_n \xrightarrow{n \to \infty} f$ uniformly in X.

Sketch of proof

Let $f \leq 0$, bounded and $0 \leq f \leq 1 \ \forall x \in X$.

$$f : X \to [0, 1]$$

Let us divide $[0, 1]$ in 2^n intervals of equal length $\forall n \in \mathbb{N}$, then we define:

$$E_k^{(n)} := \left\{ x \in X : \ \frac{k}{2^n} \leq f(x) \leq \frac{k+1}{2^n} \right\} \quad k = 0, \dots, 2^n - 1$$
$$s_n := \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_k^{(n)}}(x) \quad \forall n \in \mathbb{N}$$

Clearly $\{s_n\}$ has the desired properties.

Question 3.14

Write the definitions of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$. State their properties and prove some of them.

Solution

Definition of $\operatorname{ess\,sup}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and f a function on X. We define:

$$\operatorname*{ess\,sup}_X f(x) := \inf \left\{ \sup_{x \in N^c} f(x) : \ N \in \mathcal{N}_\mu \right\}$$

Definition of $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measurable space and f a function on X. We define:

$$\operatorname*{ess\,inf}_X f(x) := \sup \left\{ \inf_{x \in N^c} f(x) : \ N \in \mathcal{N}_\mu \right\}$$

Properties of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g \in \mathcal{M}(x, \mathcal{A})$ two functions on X. We have that:

i) $\exists N \in \mathcal{N}_\mu$ such that $\operatorname{ess\,sup}_X f = \sup_{x \in N^c} f$ and $f \leq \operatorname{ess\,sup}_X f$ almost surely $x \in X$;

ii) $\operatorname{ess\,sup}_X f = -\operatorname{ess\,inf}_X -f$;

iii) $\operatorname{ess\,sup}_X k \cdot f = k \cdot \operatorname{ess\,sup}_X f$;

iv) $f \leq g \implies \operatorname{ess\,sup}_X f \leq \operatorname{ess\,sup}_X g$;

v) $\operatorname{ess\,sup}_X (f + g) \leq \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$;

vi) $f = g$ almost everywhere in $X \implies \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$;

vii) $g \geq 0$ almost everywhere in $X \implies f \cdot g \leq (\operatorname{ess\,sup}_X f) \cdot g$ almost everywhere in X.

Proof. Let us give a partial proof:

i) Suppose $\operatorname{ess\,sup}_X f < +\infty$, $\forall k \in \mathbb{N}\ \exists N_k \in \mathcal{N}_\mu$ such that:

$$\sup_{x \in N_k} f < \operatorname*{ess\,sup}_X f + \frac{1}{k}$$

We define $N := \bigcup_{k=1}^{\infty} N_k$. Then $N \in \mathcal{N}_\mu$ and:

$$N^{\mathsf{c}} = \bigcap_{k=1}^{\infty} N_k^{\mathsf{c}} \subseteq N_k^{\mathsf{c}} \quad \forall k \in \mathbb{N}$$
$$\implies \operatorname*{ess\,sup}_X f \le \sup_{N^{\mathsf{c}}} f \le \sup_{N_k^{\mathsf{c}}} f < \operatorname*{ess\,sup}_X f + \frac{1}{k} \quad \forall k \in \mathbb{N}$$

Now we pass apply a limit $k \to +\infty$ and we get:

$$\sup_{N^{\mathsf{c}}} f = \operatorname*{ess\,sup}_X f$$
$$N \supseteq \bar{N} := \{x \in X : \ f(x) > \operatorname*{ess\,sup}_X f(x)\} \in \mathcal{A}$$
$$\implies \bar{N} \in \mathcal{N}_\mu \implies f \le \operatorname*{ess\,sup}_X f \text{ almost everywhere in } X$$

□

Question 3.15

What is $\mathcal{L}^\infty$? Which is the relation between functions finite a.e. and essentially bounded functions? Justify the answer.

Solution

Definition of $\mathcal{L}^\infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f \in \mathcal{M}(X, \mathcal{A})$ is said to be essentially bounded if:

$$\operatorname*{ess\,sup}_X f < +\infty$$

and we define the set of essentially bounded functions as:

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : \ \text{f is essentially bounded }\}$$

Relation between functions finite a.e. and essentially bounded functions

We have that:

1. $f \in \mathcal{L}^\infty \implies f$ is finite a.e. in X;
2. in general if f is finite a.e. in $X \;\not\Longrightarrow\; f \in \mathcal{L}^\infty$.

Proof. 1. We can easily see that:

$$|f| \le \operatorname{ess\,sup} |f| < +\infty \text{ almost everywhere in } X$$

thus f is finite almost everywhere in X;

2. Let us assume that:

$$f \text{ is finite a.e. in } X \implies f \in \mathcal{L}^\infty$$

and let us see a clear counterexample of this, take:

$$f(x) : \mathbb{R} \to \overline{\mathbb{R}} := \begin{cases} \frac{1}{|x|} & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

Clearly f is finite in $E = \mathbb{R} \setminus \{0\}$, i.e. f is finite a.e. in $\mathbb{R}$. Let us note that $\lambda(\{0\}) = 0$. Thus:

$$\operatorname*{ess\,sup}_X |f| = +\infty \implies f \notin \mathcal{L}^\infty$$

$\square$

Sheet n. 4

Question 4.1

Define the Cantor set. State its main properties and prove some of them.

Solution

Definition of the Cantor set

The Cantor set is defined iteratively, let us illustrate the first two steps:

Step 1: We start with the interval $[0, 1]$ and remove from it the open interval $(1/3, 2/3)$. We define the following sets:

$$I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right) \quad J_{1,1} = \left[0, \frac{1}{3}\right] \quad J_{1,2} = \left[\frac{2}{3}, 1\right]$$

and:

$$C_1 = \bigcup_{k=1}^{2} J_{1,k} \quad \lambda(C_1) = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

Step 2: We now remove the open set $(1/9,\ 2/9)$ from $J_{1,1}$ and the open set $(7/9,\ 8/9)$ from $J_{1,2}$. We define the following sets:

$$\begin{aligned} I_{2,1} &= \left(\frac{1}{9}, \frac{2}{9}\right) \quad J_{2,1} = \left[0, \frac{1}{9}\right] \quad J_{2,2} = \left[\frac{2}{9}, \frac{1}{3}\right] \\ I_{2,2} &= \left(\frac{7}{9}, \frac{8}{9}\right) \quad J_{2,3} = \left[\frac{2}{3}, \frac{7}{9}\right] \quad J_{2,4} = \left[\frac{8}{9}, 1\right] \end{aligned}$$

and:

$$C_2 = \bigcup_{k=1}^{4} J_{2,k} \quad \lambda(C_2) = 4 \cdot \frac{1}{9} = \frac{4}{9}$$

So at the n-th step we will have:

$$C_n = \bigcup_{k=1}^{2^n} J_{n,k} \quad \lambda(C_n) = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$$

Thus we can finally define the Cantor set $\mathcal{C}$ as:

$$\mathcal{C} = \bigcup_{n=1}^{\infty} C_n$$

let us note that since the endpoints of all the closed intervals are always preserved at each step we have that $C_n \supseteq C_{n+1}$ and thus $C_n \downarrow \mathcal{C}$.

Properties of the Cantor set

i) $\mathcal{C}$ is closed since it is the countable intersection of closed sets (C_n closed $\forall n \in \mathbb{N}$);

ii) $\mathcal{C} \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$ by virtue of its closedness;

iii) $\lambda(\mathcal{C}) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty}(2/3)^n = 0$ since $\lambda(C_1) = 1/3 < +\infty$ and λ is continous from above (v).

iv) $int(\mathcal{C}) = \emptyset$

Proof.

$$int(\mathcal{C}) \subseteq \mathcal{C} \quad \lambda(\mathcal{C}) = 0 \implies \lambda(int(\mathcal{C})) = 0$$

by the monotonicity of λ (ii). Now, since $int(\mathcal{C})$ is open ($\mathcal{C}$ is closed) it must contain an interval, but intervals have positive measure (this holds true only in $\mathcal{L}(\mathbb{R})$) and thus $int(\mathcal{C}) = \emptyset$. $\square$

Alternatively:

Proof. Let us assume that $int(\mathcal{C}) \neq \emptyset$, then:

$$\exists J \text{ open } \subseteq int(\mathcal{C})$$

now, since $\lambda(J) = l > 0$ we may write that:

$$\lambda(J) = l > \left(\frac{2}{3}\right)^n = \lambda(C_n) \quad \exists n \in \mathbb{N}$$

in other words $\exists n \in \mathbb{N}$ such that $J \supseteq C_n \implies J \not\subseteq C_n$ which is absurd since we assumed that $J \subseteq int(\mathcal{C}) \implies J \subseteq C_n\ \forall n \in \mathbb{N}$. Thus $int(\mathcal{C}) = \emptyset$. $\square$

v) $\mathcal{C}$ is uncountable, indeed each of its elements can be written as an alternating series of 0s and 2s divided by 3^n. This would be equal to approximanting each element by going right or left through the sets $J_{n,k}$ where 0 represents a choice to go right and 2 a choice to go left. We can write this as follows:

$$\mathcal{C} = \left\{ x \in [0,1] \middle| \, x = \sum_{n=1}^{\infty} \frac{x_n}{3^n},\ x_n \in \{0,2\} \right\}$$

and thus $\mathcal{C}$ can be put into a bijection with $\{0,2\}^{\mathbb{N}}$ which is uncountable.

Question 4.2

Define the Vitali-Lebesgue function. State its main properties and prove some of them.

Solution

Vitali-Lebesgue Function

As for the cantor set we shall define Vitali's function iteratively as a sequence of functions $\{f_n\}$. This sequence is defined as follows:

$$f_0(x) = 0 \quad x \in [0,1]$$

$$f_1(x) = \begin{cases} \frac{3}{2}t & t \in [0,1/3] \\ \frac{1}{2} & t \in (1/3,2/3) \\ \frac{3}{2}t - \frac{1}{2} & t \in [1/3,2/3] \end{cases}$$

$$\vdots$$

$$f_n(x) = \begin{cases} \frac{1}{2}f_{n-1}(3t) & t \in [0,1/3] \\ f_{n-1}(t) & t \in (1/3,2/3) \\ \frac{1}{2}f_{n-1}(3t-2) & t \in [1/3,2/3] \end{cases}$$

and we define Vitali's function V as:

$$f_n \to V \in C([0,1])$$

let us prove that such a function exists and is unique.

Proof. Let us prove that f_n is a Cauchy sequence in $C([0,1])$, we may prove that:

$$||f||_\infty = \max_{t\in[0,1]} |f(t)| \to ||f_n - f_{n-1}||_\infty < \frac{1}{2^n}$$

let us assume this to be true, for now, then to prove that $\{f_n\}$ is Cauchy we have to prove that:

$$||f_m - f_n||_\infty < \varepsilon \quad \forall m > n \in \mathbb{N},\ \exists \varepsilon > 0$$

indeed we may write:

$$\begin{aligned} ||f_m - f_n||_\infty &= ||f_m - f_{n+1} + f_{n+1} - f_n||_\infty \\ &\leq ||f_m - f_{n+1}||_\infty + ||f_{n+1} - f_n||_\infty \text{ by the triangular inequality} \\ &\leq \sum_{k=n}^{m} ||f_{k+1} - f_k||_\infty \text{by repeating the previous step} \\ &\leq \sum_{k=n}^{m-1} \frac{1}{2^{k+1}} < \varepsilon \text{ since the series is convergent} \end{aligned}$$

thus the limit exists and is unique and we have a function:

$$V : [0,1] \to [0,1]$$

□

Properties of Vitali's function

Vitali's function has the following properties:

i) $V(0) = 0$, $V(1) = 1$ and V is continuous since it is the uniform limit of continuous functions.

ii) V is non-decreasing in $[0,1]$ since f_n is non-decreasing for all $n \in \mathbb{N}$.

Proof. Let $0 \le x < y \le 1$ then:

$$V(x) = \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} f_n(y) = V(y)$$

□

iii) $V([0,1]) = [0,1]$ since $V \in C([0,1])$ and $V(0) = 0$ and $V(1) = 1$. Thus by Intermediate value theorem V must cross all the values in between.

iv) $V' = 0$ almost everywhere since V is constant on $[0,1] \setminus \mathcal{C}$ and $\lambda(\mathcal{C}) = 0$.

Question 4.3

Write the definitions of the Lebesgue integral of a nonnegative measurable simple function over X and over a measurable subset $E \subseteq X$. Write the main properties of the integral and prove some of them.

Solution

Lebesgue integral of nonnegative simple functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function with canonical form as in (3.12.2). We define the Lebesgue integral of s over X as:

$$\int_X s\, d\mu := \sum_{k=1}^{n} c_k \mu(E_k)$$

And its integral over a measurable subset $E \in \mathcal{A}$ as:

$$\int_E s\, d\mu := \int_X s \cdot \chi_E\, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap E)$$

Properties of the Lebesgue integral

i)
$$\int_X \chi_E\, d\mu = \mu(E) \quad \forall E \in \mathcal{A}$$

ii)
$$\int_N s\, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$$

iii) Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $c \ge 0$, then:

$$\int_X c \cdot s\, d\mu = c \cdot \int_X s\, d\mu$$

iv) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, then:

$$\int_X (s + t)\, d\mu = \int_X s\, d\mu + \int_X t\, d\mu$$

v) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, such that $s \le t$ then:

$$\int_X s\, d\mu \le \int_X t\, d\mu$$

vi) $s \in \mathcal{S}_+(X, \mathcal{A})$, $E \subseteq F \in \mathcal{A}$ then:

$$\int_E s\, d\mu \leq \int_F s\, d\mu$$

Proof.

i) We may write:

$$\chi_E = \sum_{k=1}^{2} c_k \chi_{E_k} = \begin{cases} c_1 = 1 & E_1 = E \\ c_2 = 0 & E_2 = E^{\mathsf{c}} \end{cases}$$

thus by applying the definition of the Lebesgue integral we get:

$$\int_X \chi_E\, d\mu = \sum_{k=1}^{2} c_k \mu(E_k) = 1 \cdot \mu(E) + 0 \cdot \mu(E^{\mathsf{c}}) = \mu(E)$$

ii) Let us apply the definition:

$$\int_N s\, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap N)$$

but by the monotonicity of μ (ii) we have:

$$E_k \cap N \subseteq N \implies \mu(E_k \cap N) \leq \mu(N) = 0$$

so the previous sum is equal to 0.

□

Question 4.4

Let $s \in \mathcal{S}_+(X, \mathcal{A})$. For any $E \in \mathcal{A}$, let $\varphi(E) \coloneqq \int_E s d\mu$. Prove that φ is a measure.

Solution

Measure induced by a function

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function. We define the measure φ induced by s as:

$$\varphi(E) \coloneqq \int_E s\, d\mu \quad \forall E \in \mathcal{A}$$

Proof. Let us see that φ meets the definition of a measure:
Clearly:

$$\varphi : \mathcal{A} \to \overline{\mathbb{R}}_+$$

Furthermore:

i) $\varphi(\emptyset) = 0$ by property 2 of the Lebesgue integral (ii).

ii) Let $\{E_n\} \subseteq \mathcal{A}$ disjoint and $E = \bigcup_{k=1}^{\infty} E_k$, let us write:

$$s \coloneqq \sum_{l=1}^{m} d_l \chi_{F_l} \quad F_l \in \mathcal{A}$$

thus:

$$\begin{aligned}
\varphi(E) &= \int_E s\,d\mu = \sum_{l=1}^{m} d_l \mu(F_l \cap E) \\
&= \sum_{l=1}^{m} d_l \sum_{k=1}^{\infty} \mu(F_l \cap E_k) \text{ by the } \sigma - \text{additivity of } \mu \text{ (ii)} \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{m} d_l \mu(F_l \cap E_k) \\
&= \sum_{k=1}^{\infty} \int_{E_k} s\,d\mu = \sum_{k=1}^{\infty} \varphi(E_k)
\end{aligned}$$

□

Question 4.5

Write the two possible equivalent definitions of Lebesgue integral of a measurable nonnegative function.

Solution

Let $f : X \to \overline{\mathbb{R}}_+$ be a measurable nonnegative function ($f \in \mathcal{M}_+(X, \mathcal{A})$). Let us define the set $\mathcal{S}_f$:

$$\mathcal{S}_f = \{s \in \mathcal{S}_+ :\ s \leq f \text{ in } X\}$$

We then have two possible and equivalent definitions of the Lebesgue integral of f.

Definition by sup

We define the integral of f as:

$$\int_X f\,d\mu = \sup_{s \in \mathcal{S}_f} \int_X s\,d\mu$$

Definition by lim

Thanks to the Simple Approximation Theorem (3.13.1) we know:

$$\exists \{s_n\} \subseteq \mathcal{S}_f \quad s_n \leq s_{n+1}\ n \in \mathbb{N},\ s_n \xrightarrow{n\to\infty} s \text{ in } X$$

So we can define the integral as:

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X s_n\,d\mu$$

Let us note that the integral must be independent of the choice of the sequence $\{s_n\}$.

Question 4.6

State and prove the Chebychev inequality.

Solution

Chebychev inequality

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ then $\forall c > 0$ we have:

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

Proof. Clearly

$$E_C \coloneqq \{f \geq c\} \in \mathcal{A} \text{ since } f \in \mathcal{M}_+(X, \mathcal{A}) \text{ (see (iii))}$$

and we have that:

$$c \cdot \chi_{E_C} \leq f \cdot \chi_{E_C}$$

thus by the monotonicity of the integral for functions (v) and for sets (vi) we have:

$$c \cdot \mu(E_C) = \int_X c \cdot \chi_{E_C} \, d\mu \leq \int_X f \cdot \chi_{E_C} \, d\mu = \int_{E_C} f \, d\mu \leq \int_X f \, d\mu$$

so we have the Chebychev inequality. □

Question 4.7

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$. Show that f is finite a.e. in X.

Solution

f is finite a.e. in X if $\int_X f d\mu < +\infty$

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$, then f is finite a.e. in X.

Proof. Let us note that the thesis is equivalent to $\mu(\{f = +\infty\}) = 0$. Let us define:

$$\{f = +\infty\} = \bigcap_{n=1}^{\infty} \{f > n\}$$

Clearly we have that:

a) $\{f > n\} \downarrow \{f = +\infty\}$

b) $\mu(\{f > n\}) \leq \frac{1}{n} \cdot \int_X f \, d\mu \ \forall n \in \mathbb{N}$ by the Chebychev inequality (4.6.1).

So since $\mu(\{f > 1\}) \leq \frac{1}{1} \cdot \int_X f \, d\mu < +\infty$ we may apply the continuity from above of μ (v):

$$\mu(\{f = +\infty\}) = \mu\left(\bigcap_{n=1}^{\infty} \{f > n\}\right) = \lim_{n \to \infty} \mu(\{f > n\}) = \lim_{n \to \infty} \frac{1}{n} \cdot \underbrace{\int_X f \, d\mu}_{<+\infty} \longrightarrow 0$$

□

Question 4.8

State and prove the vanishing lemma for functions $f \in \mathcal{M}_+(X, \mathcal{A})$.

Solution

Vanishing lemma for $f \in \mathcal{M}_+(X, \mathcal{A})$

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu = 0$, then we have that $f = 0$ a.e. in X.

Proof. Let us note that the thesis is equivalent to $\mu(\{f > 0\}) = 0$. Let us define:

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\}$$

Clearly we have that:

a) $\left\{ f > \frac{1}{n} \right\} \uparrow \{f > 0\}$

b) $\frac{1}{n} \cdot \chi_{\left\{f > \frac{1}{n}\right\}} \leq f \cdot \chi_{\left\{f > \frac{1}{n}\right\}}$

by Chebychev inequality (4.6.1) we have:

$$\mu\left(\left\{ f > \frac{1}{n} \right\} \right) \leq \frac{1}{1/n} \cdot \int_X f\, d\mu = 0 \quad \forall n \in \mathbb{N}$$

thus by the continuity from below of μ (iv) we have:

$$\mu(\{f > 0\}) = \mu\left(\bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\} \right) = \lim_{n\to\infty} \mu\left(\left\{ f > \frac{1}{n} \right\} \right) == 0$$

$\square$

Question 4.9

State and prove the Monotone Convergence Theorem (or Beppo Levi Theorem).

Solution

Monotone Convergence Theorem

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ and $f : X \to \overline{\mathbb{R}}_+$ be such that:

i) $f_n \leq f_{n+1}$ in X $\forall n \in \mathbb{N}$

ii) $f_n \xrightarrow{n\to\infty} f$ pointwise in X

then:

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X \lim_{n\to\infty} f_n\, d\mu = \int_X f\, d\mu$$

Proof. $f \in \mathcal{M}_+(X, \mathcal{A})$
by monotonicity of the integral for functions (v) we have:

$$\alpha := \int_X f_n\, d\mu \leq \int_X f_{n+1}\, d\mu \leq \int_X f\, d\mu \longrightarrow \alpha \leq \int_X f\, d\mu$$

now, we have to prove that $\alpha \geq \int_X f\, d\mu$. Indeed $\forall \varepsilon \in (0, 1)$, $\forall s \in \mathcal{S}_f$ let:

$$E_n := \{(1 - \varepsilon)s \leq f_n\} \quad n \in \mathbb{N}$$

Let us note that:

a) $\{E_n\} \subseteq \mathcal{A}$;

b) $\{E_n\} \uparrow$, since $\{f_n\} \uparrow$;

c) $X = \bigcup_{n=1}^{\infty} E_n$.

Clearly $\bigcup_{n=1}^{\infty} E_n \subseteq X$, we have to show that $X \subseteq \bigcup_{n=1}^{\infty} E_n$. Now, let us fix $x \in X$, we have two possibilities:

- **$f(x) = +\infty$:** then $\exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$(1-\varepsilon)s(x) < f_n(x) \implies x \in E_n \ \forall n > \bar{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

- **$f(x) < +\infty$:** then $\exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$(1-\varepsilon)s(x) \leq (1-\varepsilon)f(x) < f_n(x) \implies x \in E_n \ \forall n > \bar{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

Thus we have that $X \subseteq \bigcup_{n=1}^{\infty} E_n$ and $\bigcup_{n=1}^{\infty} E_n \subseteq X \implies X = \bigcup_{n=1}^{\infty} E_n$.
It clearly follows that:

$$(1-\varepsilon) \cdot \int_{E_n} s \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f \, d\mu$$

now let $n \to \infty$ ($E_n \xrightarrow{n\to\infty} X$):

$$(1-\varepsilon) \cdot \int_X s \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu = \alpha$$

but since $\varepsilon \in (0,1)$ can be arbitrarily small we have:

$$\int_X s \, d\mu \leq \alpha \implies \sup_{s \in \mathcal{S}_f} \int_X s \, d\mu = \int_X f \, d\mu \leq \alpha$$

thus we have proved that $\int_X f \, d\mu = \alpha$. □

Question 4.10

State and prove Fatou's Lemma.

Solution

Fatou's lemma

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, then:

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \geq \int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu$$

Proof. We already know that $\liminf_{n\to\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$ by (3.8.1).
Let us a define a new sequence $\{g_n\}$ such that:

$$g_k : X \to \overline{\mathbb{R}}_+ \quad g_k := \inf_{n \geq k} f_n$$

We can clearly see that:

a) $\{g_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, $\{g_n\} \uparrow$;

b) $g_k \leq f_k$ for all $k \in \mathbb{N}$;

c) $\lim_{k\to\infty} g_k = \sup_{k\geq 1} g_k = \sup_{k\geq 1} \inf_{n\geq k} f_n = \liminf_{n\to\infty} f_n$.

thus by monotonicty of the integral for functions (v) and (b) we have:

$$\int_X g_k \, d\mu \leq \int_X f_k \, d\mu \quad \forall k \in \mathbb{N}$$

Now, since $\{g_n\}$ is an increasing sequence so is $\int_X g_k \, d\mu$ and thus it admits a limit (which coincides with its lim inf), thus, if we apply the lim inf to both sides, we have:

$$\liminf_{k\to\infty} \int_X g_k \, d\mu = \lim_{k\to\infty} \int_X g_k \, d\mu \leq \liminf_{k\to\infty} \int_X f_k \, d\mu$$

Now let us apply the Monotone Convergence Theorem (4.9.1) to the right hand side:

$$\int_X \lim_{k\to\infty} g_k \, d\mu \overset{(c)}{=} \int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu \leq \liminf_{k\to\infty} \int_X f_k \, d\mu$$

and so we have obtained our thesis. $\square$