# Answers to the Theory Questions

of the course of Real and Functional Analysis of prof. Fabio Punzo

by Jacopo Stringara
jstringara@gmail.com

October 1, 2022

The source code for this document can be found at:
https://github.com/jstringara/Latex-projects/tree/master/ARF

# Contents

# Sheet n. 1

# Question 1.1

Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to\infty} E_n$, $\liminf_{n\to\infty} E_n$, $\lim_{n\to\infty} E_n$.

### Solution

Let us define the following:

- **Sequence of sets**

  A family (or collection) of sets $\{E_i\}_{i\in I}$ is called a sequence of sets if $I = \mathbb{N}$ (i.e. it is indexed by the set of natural numbers $\mathbb{N}$)

- **Increasing sequence of sets**

  a sequence of sets $\{E_n\}$ is said to be increasing (or ascending) if:

  $$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Decreasing sequence of sets**

  A sequence of sets $\{E_n\}$ is said to be decreasing (or descending) if:

  $$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Limsup for a sequence of sets**

  for a sequence of sets $\{E_n\}$ we define:

  $$\limsup_{n\to\infty} E_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

- **Liminf for a sequence of sets**

  analogously:

  $$\limsup_{n\to\infty} E_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

- **Limit for a sequence of sets**

  as for a sequence of real numbers if the limsup and liminf coincide we may define:

  $$\lim_{n\to\infty} E_n := \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n$$

# Question 1.2

Write the definitions of: cover (or covering) of a set; subcover.

### Solution

Let us define the following:

- **Cover of a set**

  a family of sets $\{E_i\}_{i\in I}$ is called a cover (or covering) of $X$ if:

  $$X \subseteq \bigcup_{i\in I} E_i$$

- **Subcover**

  a sub-family of a cover $\{E_i\}_{i\in J}$ $(J \subseteq I)$ which forms a cover is called a subcover.

# Question 1.3

Write the definitions of: equivalence relation, equivalence class, quotient set.

### Solution

Let us define the following:

- **Equivalence relation**

  a relation $R$ in $X$ (i.e. a subset $R \subseteq X \times X$) is an equivalence relation if:

  i) $(x, x) \in R \; \forall x \in X$ (**reflexivity**)
  ii) $(x, y) \in R \implies (y, x) \in R$ (**simmetry**)
  iii) $(x, y) \in R$, $(y, z) \in R \implies (x, z) \in R$ (**transitivity**)

  ### Equivalence class

  we define an equivalence class for $x$ w.r.t. $R$ as:

  $$E_x := \{y \in X : yRx\}$$

  i.e. the set of all elements equivalent to $x$ for $R$

- **Quotient set**

  we define the quotient set of $X$ over $R$ as:

  $$X/R \coloneqq \{E_x : x \in X\}$$

  i.e. it is the set of all equivalence classes.

# Question 1.4

Write the definition of equipotent sets. Write the definition of cardinality of a set.

## Solution

Let us define the following:

- **Equipotent sets**

  Two sets $X$ and $Y$ are called equipotent if there exists a bijections, that is, a function:

  $$f : X \to Y$$

  that is both injective and surjective.

- **Cardinality of a set**

  the cardinality of a set $X$ is the collection of all sets equipotent to X.

# Question 1.5

Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

## Solution

Let us define the following:

- **Finite sets**

  a set $X$ is finite if $\exists n \in \mathbb{N}$ such that there is a bijection:

  $$f : X \to 1, \ldots, n$$

  **Example:** $\{\frac{1}{1}, \ldots, \frac{1}{n}\}$

- **Infinite sets**

  $X$ is infinite if it is not finite.
  **Example:** $\mathbb{N}$ is clearly infinite

- **Countable sets**

  $X$ is countable if $X$ is equipotent to $\mathbb{N}$
  **Example:** $\mathbb{Q}$ can be put in bijection with $\mathbb{N}$

- **Uncountable sets**

  $X$ is uncountable if it is infinite and not countable.
  **Example:** $\mathbb{R}$ is clearly infinite and not countable since it has the cardinality of continuum.

# Question 1.6

Write the definitions of: algebra, $\sigma -$ algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma -$ algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

### Solution

Let us define the following:

- **Algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

  **$\sigma -$ algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma -$ algebra if:

- i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- **Measurable space**

  The couplet $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma -$ algebra is called a measurable space.

- **Measurable set**

  the elements of the $\sigma -$ algebra of a measurable space are called measurable sets.

# Question 1.7

State the theorem concerning the existence of the $\sigma -$ algebra generated by a given set. Give an idea of the proof.

### Solution

### Minimal $\sigma-$algebra

Let $S \subseteq \mathcal{P}(X)$, then there exists a $\sigma-$algebra $\sigma_0(S)$ such that:

1. $S \subseteq \sigma_0(S)$
2. $\forall \sigma-$algebra$\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

thus $\sigma_0(S)$ is the minimal $\sigma-$algebra generated by $S$.

### Sketch of Proof

We construct the set:

$$\mathcal{V} := \{\mathcal{A} \subseteq \mathcal{P}(X) \,\|\mathcal{A} \supseteq S,\ \mathcal{A} \quad \sigma-\text{algebra}\}$$

we may define:

$$\sigma_0(S) := \bigcap\{\mathcal{A} :\ \mathcal{A} \in \mathcal{V}\}$$

# Question 1.8

Write the definition of the Borel $\sigma-$algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$.

### Solution

### Borel $\sigma-$algebra

Let $(X, d)$ be a metric space and let $\mathcal{G}$ be the family of open sets of $X$, then we define the Borel $\sigma-$algebra as:

$$\mathcal{B}(X) := \sigma_0(\mathcal{G})$$

The elements of $\mathcal{G}$ are called Borel sets, let us enumerate some classes of them:

### Classes of Borel sets

i) open sets

ii) closed sets (they are the complementary of open sets and this is a $\sigma-$algebra)

iii) countable intersections of open sets, known as the family $G_\delta$

iv) countable union of closed sets, known as the family $F_\delta$.

Lastly, let us characterize the Borel $\sigma-$algebras $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$:

### Characterization of $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$

1. $\mathcal{B}(\mathbb{R}) = \sigma_0(I) = \sigma_0(I_1) = \sigma_0(I_2) = \sigma_0(I_0) = \sigma_0(\hat{I})$
   where:
$$\begin{aligned} I &= \{(a,b) : a,b \in \mathbb{R}, a \leq b\} \\ I_1 &= \{[a,b] : a,b \in \mathbb{R}, a \leq b\} \\ I_2 &= \{(a,b] : a,b \in \mathbb{R}, a \leq b\} \\ I_0 &= \{(a,b) : -\infty \leq a < b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\} \\ \hat{I} &= \{(a,\infty) : a \in \mathbb{R}\} \end{aligned}$$
2. $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I}) = \sigma_0(\tilde{I}_1)$
   where:
$$\begin{aligned} \tilde{I} &= \{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a,+\infty] : a \in \mathbb{R}\} \\ \tilde{I}_1 &= \{(a,+\infty] : a \in \mathbb{R}\} \end{aligned}$$
3. $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_1) = \sigma_0(K_2)$
   where:
$$\begin{aligned} K_1 &= \{\text{n-dimensional closed rectangles}\} \\ K_2 &= \{\text{n-dimensional open rectangles }\} \end{aligned}$$

# Question 1.9

Write the definitions of: measure, finite measure, $\sigma-$finite measure, measure space, probability space. Provide some examples of measures.

### Solution

Let us define the following:

- ### Measure

  Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$, then a function $\mu$:
$$\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$$
  is a measure if:

  1. $\mu(\emptyset) = 0$
  2. $\sigma -$ **additivity:**
     $\forall \{E_n\} \subseteq \mathcal{C}$ disjoint ($E_i \cap E_j \quad \forall i \neq j$) such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ we have that:
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

  ### Finite measure

  a measure $\mu$ defined as above is said to bw finite if:
$$\mu(X) < +\infty$$

- **$\sigma -$ finite measure**

  a measure $\mu$ is said to be $\sigma -$ finite if there exists a sequence $\{E_n\}$ such that:

$$X = \bigcup_{k=1}^{\infty} E_k, \quad \mu(E_k) < +\infty$$

- **Measure space**

  Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma -$ algebra and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure, then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

- **Probability space**

  if $\mu(X) = 1$ then we say that $(X, \mathcal{A}, \mu)$ is a probability space.

# Question 1.10

State and prove the theorem regarding properties of measures. Why the two continuity properties are called in this way? For what concerns continuity w.r.t. a descending sequence $E_k$, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

### Solution

### Properties of measures

Let us state and prove the properties of a measure $\mu$ on a set $X$ and $\sigma -$ algebra $\mathcal{A}$:

i) **Additivity:**
$\forall \{E_1, \ldots, E_n\} \subseteq \mathcal{A}$ disjoint we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

*Proof.* indeed if we define a sequence such that:

$$\{E_n\} = \begin{cases} B_k = E_k & \forall k \leq n \\ B_k = \emptyset & \forall k > n \end{cases}$$

this sequence is also disjoint ($\mathcal{A} \cap \emptyset = \emptyset \ \forall \mathcal{A} \in X$), thus we may write:

$$\underbrace{\mu\left(\bigcup_{k=1}^{\infty} E_k\right)}_{=\bigcup_{k=1}^{n} E_k \cup \emptyset} = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \sum_{k=n+1}^{\infty} \underbrace{\cancel{\mu(E_k)}}_{=0}$$

□

ii) **Monotonicity:**
$\forall E, F \in \mathcal{A}$ we have:

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

*Proof.* We may write $F$ in the following way:

$$F = E \cup (F \setminus E)$$

and since these two sets are obviously disjoint we may use (i) to write:

$$\mu(F) = \mu(E) + \underbrace{\mu(E \setminus F)}_{\geq 0} > \mu(E)$$

$\square$

iii) $\sigma -$ **subadditivity:**
$\forall \{E_n\} \subseteq \mathcal{A}$ (**not** disjoint) we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

*Proof.* Let us define:

$$\begin{cases} F_1 \coloneqq E_1 \\ F_n \coloneqq E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad \forall n > 1 \end{cases}$$

Clearly $\{F_n\} \subseteq \mathcal{A}$ and $\{F_n\}$ is a disjoint sequence and:

$$\begin{array}{l} F_k \subseteq E_k \quad \forall k \in \mathbb{N} \implies \mu(F_k) \leq \mu(E_k) \text{ by (ii)} \\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{array}$$

thus we may write:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

$\square$

iv) **Continuity from below:**
$\forall \{E_n\} \subseteq \mathcal{A},\ E_k \nearrow$ we have:

$$\mu\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$$

*Proof.* Let us define a new sequence $\{F_n\}$ as:

$$\begin{cases} F_k \coloneqq E_k \setminus E_{k-1} \quad \forall k \in \mathbb{N} \text{ and } E_0 \coloneqq \emptyset \\ \implies \bigcup_{k=1}^{n} F_k = E_n,\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

and since $\{F_n\}$ is a disjoint sequence (we may visually think of it as a set of ever increasing rings) we may use (i) to write:

$$\begin{aligned} \mu\left(\lim_{n\to\infty} E_n\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

$\square$

v) **Continuity from above:**
$\forall\{E_n\} \subseteq \mathcal{A}$, $E_k \searrow$, $\mu(E_1) < +\infty$ we have:

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

*Proof.* Like we did above Let us define: a new sequence $\{F_n\}$

$$F_k := E_1 \setminus E_k \quad \forall k \in \mathbb{N}$$

let us note that $\{F_n\}$ is an increasing sequence thus by (iv) we can write:

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty}(E_k)$$

because by (ii)
$mu(F \setminus E) = \mu(F) - \mu(E)$, moreover:

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty}(E_1 \cap E_k^{\mathsf{c}}) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^{\mathsf{c}}\right) = E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

thus combining these two and canceling the $\mu(E_1)$ on both sides we obtain:

$$\lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

□

let us note that for this last, crucial, step $\mu(E_1)$ must be finite, otherwise we would not be able to cancel it out from both sides.

# Question 1.11

Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

## Solution

Let us define the following:

- **Sets of zero measure**

  Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $E \subseteq X$ has zero measure if $E \in \mathcal{A}$ and $\mu(E) = 0$. We denote the set of all sets of zero measure by $\mathcal{N}_\mu$

- **Negligible sets**

  a set $E \subseteq X$ is negligible if:

  $$\exists N \in \mathcal{A} \text{ s.t. } E \subseteq N,\ \mu(N) = 0$$

  So any subset of a set of zero measure is negligible, we denote the collection of all negligible sets by $\tau_\mu$. Moreover let us note that $E$ doesn't need to be an element of $\mathcal{A}$ $(E \notin \mathcal{A})$

- **Almost Everywhere**

  a property $P$ on $X$ is said to hold almost everywhere if:

  $$\mu(\{x \in X : P(x) \text{ is false }\}) = 0$$

  We may also say that $\{x \in X : P(x) \text{ is false }\} \in \mathcal{N}_\mu$

  ### Examples

  typical properties that can be true a.e. are: equality, continuity, monotonicity, etc. etc.

# Question 1.12

Write the definition of complete measure space. Exhibit an example of a measure space which is not complete.

## Solution

## Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

## Counterexample

Let $X = \{a, b, c\}$, $\mathcal{A} = \sigma(\{\emptyset, \{a\}, \{b, c\}, X\})$ and $\mu \equiv 0$, clearly here we have:

$$\tau_\mu \setminus \mathcal{N}_\mu = \{\{b\}, \{c\}\}$$

and clearly $\{b\}, \{c\} \notin \mathcal{A}$. So this measure space is not complete.

# Sheet n. 2

# Question 2.1

Write the definition of complete measure space. State the theorem concerning the existence of the completion of a measure space. Give just an idea of the proof.

### Solution

### Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

### Existence of the completion

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let us define: $\bar{\mathcal{A}}, \bar{\mu}$

$$\bar{\mathcal{A}} = \{E \subseteq X : \exists F, G \in \mathcal{A} \text{ s.t. } F \subseteq E \subseteq G \; \mu(G \setminus F) = 0\}$$
$$\bar{\mu} : \bar{\mathcal{A}} \to \overline{\mathbb{R}}_+, \quad \bar{\mu}(E) := \mu(F)$$

then:

1. $\bar{\mathcal{A}}$ is a $\sigma-$ algebra , $\bar{\mathcal{A}} \supseteq \mathcal{A}$
2. $\bar{\mu}$ is a complete measure, $\bar{\mu}|_{\mathcal{A}} = \mu$

and the triplet $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space and is called the completion of $(X, \mathcal{A}, \mu)$, i.e. it the smallest (w.r. to inclusion) complete measure space that cointains $(X, \mathcal{A}, \mu)$

### Sketch of proof

We must prove two things:

- **First:** that $\bar{\mathcal{A}}$ is a $\sigma-$ algebra and that it contains $\mathcal{A}$, the latter is trivial since $\forall A \in \mathcal{A} \quad A \subseteq A \subseteq A \implies A \in \bar{\mathcal{A}}$ while the former is quite hardous so we shall just assume it to be true.
- **Second:** that $\bar{\mu}$ is a complete measure and $\bar{\mu}|_{\mathcal{A}} = \mu$.
  The latter is trivial (see above). We can also easily prove that it is a measure:
  i) $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$ since the only set contained inside $\emptyset$ is $\emptyset$ itself, as the container set we may take any zero set measure inside $\mathcal{A}$.

ii) that $\sigma-$additivity holds is clear since for any disjoint sequence $\{E_n\} \subseteq \bar{\mathcal{A}}$ we may construct two sequences:

$$\begin{cases} \{F_n\},\ F_k \subseteq E_k \\ \{G_n\},\ G_k \supseteq E_k \end{cases} \quad \forall k \in \mathbb{N} \text{ s.t. } \mu(G_k \setminus F_k) = 0$$

Let us note the following:

- $\{F_n\}$ is also disjoint because $\{E_n\}$ is disjoint.
- Moreover:

$$\bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} G_k$$
$$\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus F_k)$$
$$\mu\left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k\right) \leq \mu\left(\bigcup_{k=1}^{\infty} (G_k \setminus F_k)\right) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus F_k) = 0$$

The last inequality is true thanks to the $\sigma-$subadditivity and monotonicty of $\mu$.

Thus we can say that:

$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$$

thus $\bar{\mu}$ is a measure.
Let us prove that $\bar{\mu}$ is complete. Let $E_1 \in X$ and $E_2 \in \bar{\mathcal{A}}$ such that $\bar{\mu}(E_2) = \mu(F_2) = 0$ and $E_1 \subseteq E_2$, let us note that:

$$\begin{cases} \mu(G_2) = \cancel{\mu(G_2 \setminus F_2)}^0 + \cancel{\mu(F_2)}^0 \\ \mu(G_2 \setminus \emptyset) = \mu(G_2) - 0 \\ \emptyset \subseteq E_1 \subseteq G_2 \end{cases} \implies E_1 \in \bar{\mathcal{A}},\ \bar{\mu}(E_1) = \mu(\emptyset) = 0$$

thus any negligible set is also a zero measure set and $\bar{\mu}$ is complete.

# Question 2.2

Write the definition of outer measure. State and prove the theorem concerning generation of outer measure on a general set $X$, starting from a set $K \in \mathcal{P}(X)$, containing $\emptyset$, and a function $\nu : K \to \overline{\mathbb{R}}_+,\ \nu(\emptyset) = 0$. Intuitively, which is the meaning of $(K, \nu)$?

## Solution

## Outer measure

We say that a function: $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ (where $X$ is any set) is an outer measure if:

i) $\mu^*(\emptyset) = 0$

ii) $E_1 \subseteq E_2 \implies \mu^*(E_2) \leq \mu^*(E_2)$

iii) $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

## Generation of an outer measure

Let $K \subseteq \mathcal{P}(X)$, $\emptyset \in K$, $\nu : K \to \overline{\mathbb{R}}_+$, $\nu(\emptyset) = 0$, then we can generate an outer measure $\mu^*$ on $X$ defined as:

$$\begin{cases} \mu^*(E) \coloneqq \inf \left\{ \sum_{k=1}^{\infty} \nu(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k,\ \{I_n\} \subseteq K \right\},\ \text{if } E \text{ can be covered by a countable union of sets } I_n \in K. \\ \mu^*(E) \coloneqq +\infty,\ \text{otherwise.} \end{cases}$$

*Proof.* Let us verify that such a $\mu^*$ meets the definition of outer measure (2.2.1):

i) $\emptyset \in K$, $0 \leq \mu^*(\emptyset) \leq \nu(\emptyset) = 0$ by the definition of $\mu^*$.

ii) $E_1 \subseteq E_2$, we have two possible cases

- if there exists a countable covering of $E_2$ then it is also a covering of $E_1$ and from the definitio of $\mu^*$ it follows that:

$$\mu^*(E_1) \leq \mu^*(E_2)$$

- if there is no countable covering of $E_2$ then:

$$\mu^*(E_1) \leq \mu^*(E_2) = +\infty$$

iii) this condition is obviously met if:

$$\sum_{k=1}^{\infty} \mu^*(E_k) = +\infty$$

otherwise if we suppose that:

$$\sum_{k=1}^{\infty} \mu^*(E_k) < +\infty$$

thus $\mu^*(E_k) < +\infty\ \forall k \in \mathbb{N}$, by the definition of $\mu^*$ and inf:

$$\forall \varepsilon > 0,\ \forall n \in \mathbb{N} \quad \exists \{I_{n,k}\} \subseteq K$$

such that:

$$E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n,k})$$

Now, since:

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}, \quad \{I_{n,k}\} \subseteq K$$

it clearly follows that:

$$\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \nu(I_{n,k}) < \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon \cdot \cancel{\sum_{n=1}^{\infty} \frac{1}{2^n}}^{1}$$

because $\varepsilon$ is arbitrary, we have the cocnlusion.

□

The intuitive meaning $(K, \nu)$ is that $K$ is a special class of sets in $X$ and $\nu$ is a function that assigns a value to each set in $K$. On the other hand $\nu$ can be any real valued positive function, thus it is not necessary to be a measure.

# Question 2.3

What is the Caratheodory condition? How can it be stated in an equivalent way? Prove it.

### Solution

### Caratheodory condition

Let $\mu^*$ be an outer measure on a set $X$, then we say that $E \subset X$ is $\mu^*$-measurable if:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

### Equivalent statement

Let $\mu^*$ be an outer measure on a set $X$, then we say that $E \subset X$ is $\mu^*$-measurable if:

$$\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

*Proof.* It is enough to note that $\forall E \subseteq X$ we have:

$$Z = (Z \cap E) \cup (Z \cap E^{\mathsf{c}}) \quad \forall Z \in X$$

and thus by the subadditivity of $\mu^*$ (iii) we get:

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

and we may combine this inequality with the other to yield an equality. □

# Question 2.4

Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

### Solution

### All zero measure sets are in $\mathcal{L}$

There cannot exist such a set $E$ because all sets of zero aouter measure meet the Caratheodory Inequality (2.3.2).

*Proof.* Indeed $\forall Z \subseteq X$ by the monotonicty of $\mu^*$ (ii) we have:

$$\mu^*(\underbrace{Z \cap E}_{\subseteq E}) + \mu^*(\underbrace{Z \setminus E}_{\subseteq Z}) \leq \cancelto{0}{\mu^*(E)} + \mu^*(Z)$$

□

# Question 2.5

State the theorem concerning generation of a measure as a restriction of an outer measure.

### Solution

### Generation of a measure from an outer measure

Let us define $\mathcal{L}$ as:

$$\mathcal{L} := \{E \subseteq X : \ E \text{ is } \mu^* - \text{measurable } \}$$

where $\mu^*$ is an outer measure on $X$, then:

i) the collection $\mathcal{L}$ is a $\sigma$ − algebra

ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

# Question 2.6

Show that the measure induced by an outer measure on the $\sigma$ − algebra of all sets fulfilling the Caratheodory condition is complete.

### Solution

### Generation of a measure from an outer measure (proof of completeness)

Let us see that such a measure as the one described in the previous question is complete. Let $\mu^*$ be an outer measure on $X$ and $\mathcal{L}$ the $\sigma$ − algebra of all sets fulfilling the Caratheodory condition. Let $\mu$ be the measure induced by $\mu^*$ on $\mathcal{L}$ ($\mu = \mu^*|_{\mathcal{L}}$).

*Proof.* Let $N \in \mathcal{L}$ such that $\mu(N) = \mu^*(N) = 0$ and let $E \subseteq N$.
By monotonicty of $\mu^*$ (ii):

$$0 \leq \mu^*(E) \leq \mu^*(N) = 0 \implies \mu^*(E) = 0$$

thus by the lemma seen in 2.4.1 we get that $E \in \mathcal{L}$ and so $\mathcal{L}$ is complete. □

# Question 2.7

Describe the construction of the Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^n$.

### Solution

### Construction of the Lebesgue measure on $\mathbb{R}$

Let $I$ be a family of open, bounded intervals in $\mathbb{R}$:

$$I := \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$$

Let us note that $\emptyset \in I$.
Now let us consider a function $\lambda_0$:

$$\begin{aligned}&\lambda_0 : I \to \mathbb{R}_+\\ &\lambda_0(\emptyset) = 0\\ &\lambda_0((a, b)) = b - a\end{aligned}$$

Here we take $X = \mathbb{R}$, $(K, \nu) = (I, \lambda_0)$ and construct the outer Lebesgue measure $\lambda^*$ as seen above (2.2.2):

$$\lambda^*(E) := \begin{cases} \inf \left\{ \sum_{n=1}^{\infty} \lambda_0(I_n) : \quad E \subseteq \bigcup_{n=1}^{\infty} I_n, \ \{I_n\} \subseteq I \right\}, & \forall E \subseteq \mathbb{R} \text{ s.t. } E \text{ has a countable covering } \{I_n\} \subseteq I \\ +\infty, \ \text{otherwise} \end{cases}$$

The corresponding $\sigma-$algebra is the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R})$ and now we define the Lebesgue measure $\lambda$ as the measure generated by the outer Lebesgue measure (as seen in 2.5.1):

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})}$$

### Construction of the Lebesgue measure on $\mathbb{R}^n$

Analogously to what we have seen above we first define an outer measure and then a (complete) measure but we take:

$$I^n = \left\{ \bigtimes_{k=1}^{n} (a_k, b_k) : \ a_k, b_k \in \mathbb{R}, \ a_k \leq b_k \right\}$$

and accordingly we define:

$$\begin{aligned} &\lambda_0^n : I^n \to \mathbb{R}_+ \\ &\lambda_0^n(\emptyset) = 0 \\ &\lambda_0^n \left( \bigtimes_{k=1}^{n} (a_k, b_k) \right) = \prod_{k=1}^{n} (b_k - a_k) \end{aligned}$$

and therefore we take $X = \mathbb{R}^n$ and $(K, \nu) = (I^n, \lambda_0^n)$, we define the outer Lebesgue measure $\lambda^{*,n}$ on $\mathbb{R}^n$ and the Lebesgue $\sigma-$ algebra $\mathcal{L}(\mathbb{R}^n)$ and finally we construct the n-dimensional Lebesgue measure as:

$$\lambda^n := \lambda^{*,n}|_{\mathcal{L}(\mathbb{R}^n)}$$

# Question 2.8

Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

## Solution

### All countable sets are $\mathcal{L}$-measurable and $\lambda(E) = 0$

Any countable subset $E \subset \mathbb{R}$ is $\mathcal{L}$-measurable and $\lambda(E) = 0$

*Proof.* Let $a \in \mathbb{R}$, clearly $\{a\} \subseteq (a - \varepsilon, a] \ \forall \varepsilon > 0$, thus by the definition of $\lambda^*$:

$$\lambda^*(\{a\}) \leq \lambda^*((a - \varepsilon, a]) = \varepsilon \to 0 \implies \{a\} \in \mathcal{L}$$

Now if E is countable we may write as follows:

$$E = \bigcup_{n=1}^{\infty} \{a_n\} \quad a_n \in \mathbb{R}, \ n \in \mathbb{N}$$

and so by monotonicty (ii):

$$0 \leq \lambda^*(E) = \lambda^* \left( \bigcup_{n=1}^{\infty} \{a_n\} \right) \leq \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$$

thus $\lambda^*(E) = 0 \implies E \in \mathcal{L}$ by the lemma seen above (2.4.1) $\square$

# Question 2.9

Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. Is the inclusion strict? Which is the relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$?

## Solution

### $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

*Proof.* Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty))$ it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$. We already know from above that all bounded intervals belong to $\mathcal{L}(\mathbb{R})$.
Now, let $A \subseteq \mathbb{R}$ be any set. We assume $a \notin A$, otherwise we would replace $A$ with $A \setminus \{a\}$ and this would leave the Lebesgue outer measure unchanged. Furthermore $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \iff (a, +\infty)$ satisfies the Caratheodory Condition (2.3.2):

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$$

where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, +\infty)$.
Since $\lambda^*(A)$ is defined as an inf, to verify the above, it is necessary and sufficient to show that for **any countable collection** $\{I_n\}$ of **open bounded** intervals that **covers** $A$ we have that:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k)$$

For every $k \in \mathbb{N}$ we define:

$$I'_k := I_k \cap (-\infty, a)$$
$$I''_k := I_k \cap (a, +\infty)$$

then:

$$I'_k \cap I''_k = \emptyset (\text{disjoint}) \implies \lambda_o(I_k) = \lambda_0(I'_k) + \lambda_0(I''_k)$$

Let us note that $\{I'_n\}$ is a countable cover for $A_1$ and $\{I''_n\}$ is a countable cover for $A_2$. Hence:

$$\lambda^*(A_1) = \sum_{k=1}^{\infty} \lambda_0(I'_k)$$
$$\lambda^*(A_2) = \sum_{k=1}^{\infty} \lambda_0(I''_k)$$

therefore:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) + \sum_{k=1}^{\infty} \lambda_0(I''_k) = \sum_{k=1}^{\infty} \lambda_0(I_k)$$

which equivalento to the condition above. □

### $\mathcal{B}(\mathbb{R}) \subsetneqq \mathcal{L}(\mathbb{R})$

The inclusion demonstrated above can be shown to be strict. A counterexample can be produced (see here) but it is quite pathological.

## Relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$. Indeed as we have shown above $\mathcal{B}(\mathbb{R})$ is not a complete $\sigma -$ algebra while $\mathcal{L}(\mathbb{R})$ is.

# Question 2.10

Is the translate of a measurable set measurable?

### Solution

### The translate of a measurable set is measurable

The translate of a measurable set is also measurable.
Let us see a simple example: let $(a, b)$ be an interval and $(a + h, b + h)$ its translate.

$$\lambda((a, b)) = b - a$$
$$\lambda((a + h, b + h)) = (b + h) - (a + h) = b - a$$

# Question 2.11

Write the excision property and prove it. Write and prove (partially) the theorem concerning the regularity of the Lebesgue measure on $\mathbb{R}$.

### Solution

### Excision property

If $A \in \mathcal{L}(\mathbb{R})$, $\lambda^*(A) \leq +\infty$ and $A \subseteq B$, then:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

*Proof.* Since $A \in \mathcal{L}(\mathbb{R})$ we can use the Caratheodory equality (2.3.1) using $Z = B$, $E = A$:

$$\lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A\ (A \subseteq B)}) + \lambda^*(B \setminus A)$$

so, since $\lambda^*(A) \leq +\infty$ we may write:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

□

### Regularity of the Lebesgue Measure

Let $E \subseteq \mathbb{R}$, the following are equal:

i) $E \in \mathcal{L}(\mathbb{R})$

ii) $\forall \varepsilon > 0\ \exists A \subseteq \mathbb{R}$ open s.t.
$$E \subseteq A \quad \lambda^*(A \setminus E) < \varepsilon$$

iii) $\exists G \subseteq \mathbb{R}$ in the class $G_\delta$ (countable intersections of open sets) s.t.
$$E \subseteq G \quad \lambda^*(G \setminus E) = 0$$

iv) $\forall \varepsilon > 0\ \exists C \subseteq \mathbb{R}$ closed s.t.
$$C \subseteq E \quad \lambda^*(E \setminus C) < \varepsilon$$

v) $\exists F \subseteq \mathbb{R}$ in the class $F_\delta$ (countable unions of closed sets) s.t.

$$F \subseteq E \quad \lambda^*(E \setminus F) = 0$$

*Proof.* Let us give a (partial) proof:

- $(i) \implies (ii)$: if $E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < +\infty$ then by definition of outer measure (2.2.1):

$$\forall \varepsilon > 0 \; \exists \{I_n\} \text{ that covers } E \text{ and } \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$

  Let us now define the set $O$:

$$O \coloneqq \bigcup_{k=1}^{\infty} I_k, \; O \text{ is open}, \; E \subseteq O$$

  and so we may write:

$$\lambda^*(O) \overset{sub-add\ (iii)}{\leq} \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$
$$\implies \lambda^*(O) - \lambda^*(E) < \varepsilon$$

  and by the Excision property (2.11.1) ($E \in \mathcal{L}(\mathbb{R}), \; \lambda^*(E) < +\infty$):

$$\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$$

  and so we have obtained the second statement (ii).

- $(ii) \implies (iii)$, $\forall k \in \mathbb{N}$ we choose $O_k \supseteq E$ open for which:

$$\lambda^*(O_k \setminus E) < \frac{1}{k}$$

  and then define:

$$G = \bigcap_{k=1}^{\infty} O_k \implies G \in G_\delta, \; G \supseteq E$$

  Moreover $\forall k \in \mathbb{N}$:

$$G \setminus E \subseteq O_k \setminus E$$

  so by monotonicty (ii):

$$\lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$$

  let us apply a limit $k \to \infty$ to both sides:

$$\lambda^*(G \setminus E) = 0$$

- $(iii) \implies (i)$, let us note that $G \setminus E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G \setminus E) = 0$ by lemma 2.4.1 and:

$$G \in \mathcal{L}(\mathbb{R}) \text{ since } G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$$
$$\implies E = \underset{\in \mathcal{L}}{G} \cap (\underset{\in \mathcal{L}}{G \setminus E})^{\mathsf{c}} \in \mathcal{L}$$

$\square$

# Question 2.12

Is it true that any subset $E \subseteq \mathbb{R}$ is $\mathcal{L}$-measurable? Is it possibile to find two disjoint sets $A, B \subset \mathbb{R}$ for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Why?

## Solution

## Vitali's non-measurable sets

Any measurable set $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$ contains a subset that fails to be measurable. Therefore there exist subsets of $\mathbb{R}$ that are not $\mathcal{L}$-measurable.

## Disjoints sets for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$

There are disjoint sets $A, B \subseteq \mathbb{R}$ for which:

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$

*Proof.* Assume by contradiction that:

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \quad \forall A, B \subseteq \mathbb{R},\ A \cap B = \emptyset$$

Now $\forall E, Z \subseteq \mathbb{R}$ we write:

$$\lambda^*(\underbrace{Z \cap E}_{=A}) + \lambda^*(\underbrace{Z \cap E^{\mathsf{c}}}_{=B}) = \lambda^*(\underbrace{\ Z\ }_{=A \cup B})$$

thus any set $E$ would satisfy the Caratheodory condition (2.3.1) and be $\mathcal{L}$-measurable which is absurd since we know that Vitali's sets exist. $\square$