Answers to the Theory Questions

of the course of Real and Functional Analysis
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Contents

Sheet n. 1

Question 1.1

Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to\infty} E_n$, $\liminf_{n\to\infty} E_n$, $\lim_{n\to\infty} E_n$.

Solution

Let us define the following:

- **Sequence of sets**

  A family (or collection) of sets $\{E_i\}_{i\in I}$ is called a sequence of sets if $I = \mathbb{N}$ (i.e. it is indexed by the set of natural numbers $\mathbb{N}$)

- **Increasing sequence of sets**

  a sequence of sets $\{E_n\}$ is said to be increasing (or ascending) if:
  $$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Decreasing sequence of sets**

  A sequence of sets $\{E_n\}$ is said to be decreasing (or descending) if:
  $$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Limsup for a sequence of sets**

  for a sequence of sets $\{E_n\}$ we define:
  $$\limsup_{n\to\infty} E_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

- **Liminf for a sequence of sets**

  analogously:
  $$\liminf_{n\to\infty} E_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

- **Limit for a sequence of sets**

  as for a sequence of real numbers if the limsup and liminf coincide we may define:

  $$\lim_{n \to \infty} E_n := \liminf_{n \to \infty} E_n = \limsup_{n \to \infty} E_n$$

Question 1.2

Write the definitions of: cover (or covering) of a set; subcover.

Solution

Let us define the following:

- **Cover of a set**

  a family of sets $\{E_i\}_{i \in I}$ is called a cover (or covering) of X if:

  $$X \subseteq \bigcup_{i \in I} E_i$$

- **Subcover**

  a sub-family of a cover $\{E_i\}_{i \in J}$ $(J \subseteq I)$ which forms a cover is called a subcover.

Question 1.3

Write the definitions of: equivalence relation, equivalence class, quotient set.

Solution

Let us define the following:

- **Equivalence relation**

  a relation R in X (i.e. a subset $R \subseteq X \times X$) is an equivalence relation if:

  i) $(x,x) \in R \ \forall x \in X$ (**reflexivity**)
  ii) $(x,y) \in R \implies (y,x) \in R$ (**simmetry**)
  iii) $(x,y) \in R, (y,z) \in R \implies (x,z) \in R$ (**transitivity**)

  Equivalence class

  we define an equivalence class for x w.r.t. R as:

  $$E_x := \{y \in X : yRx\}$$

  i.e. the set of all elements equivalent to x for R

- **Quotient set**

    we define the quotient set of X over R as:

    $$X/R := \{E_x : x \in X\}$$

    i.e. it is the set of all equivalence classes.

Question 1.4

Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

Solution

Let us define the following:

- **Finite sets**

    a set X is finite if $\exists n \in \mathbb{N}$ such that there is a bijection:

    $$f : X \to 1, \dots, n$$

    Example: $\{\frac{1}{1}, \dots, \frac{1}{n}\}$

- **Infinite sets**

    X is infinite if it is not finite.
    Example: $\mathbb{N}$ is clearly infinite

- **Countable sets**

    X is countable if X is equipotent to $\mathbb{N}$
    Example: $\mathbb{Q}$ can be put in bijection with $\mathbb{N}$

- **Uncountable sets**

    X is uncountable if it is infinite and not countable.
    Example: $\mathbb{R}$ is clearly infinite and not countable since it has the cardinality of continuum.

Question 1.5

Write the definitions of: algebra, $\sigma-$algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma-$algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

Solution

Let us define the following:

- **Algebra**

    A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if:
    i) $\emptyset \in \mathcal{A}$
    ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
    iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

$\sigma-$ **algebra**

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma-$ algebra if:

- i) $\emptyset \in \mathcal{A}$

  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$

  iii) $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- ## Measurable space

  The couplet $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma-$ algebra is called a measurable space.

- ## Measurable set

  the elements of the $\sigma-$ algebra of a measurable space are called measurable sets.

Clearly $\bigcap_{k=1}^{+\infty} E_k = \left(\bigcup_{k=1}^{+\infty} E_k^{\mathsf{c}} \right)^{\mathsf{c}}$ and since $\mathcal{A}$ is a $\sigma-$ algebra we have that $\bigcup_{k=1}^{+\infty} E_k^{\mathsf{c}} \in \mathcal{A}$ and thus $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

Question 1.6

State the theorem concerning the existence of the $\sigma-$ algebra generated by a given set. Give an idea of the proof.

Solution

Minimal $\sigma-$ **algebra**

Let $S \subseteq \mathcal{P}(X)$, then there exists a $\sigma-$ algebra $\sigma_0(S)$ such that:

1. $S \subseteq \sigma_0(S)$

2. $\forall \sigma-$ algebra$\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

thus $\sigma_0(S)$ is the minimal $\sigma-$ algebra generated by S.

Sketch of Proof

We construct the set:
$$\mathcal{V} := \{\mathcal{A} \subseteq \mathcal{P}(X) \, \| \, \mathcal{A} \supseteq S, \, \mathcal{A} \quad \sigma-\text{algebra}\}$$
we may define:
$$\sigma_0(S) := \bigcap \{\mathcal{A} : \, \mathcal{A} \in \mathcal{V}\}$$

Question 1.7

Write the definition of the Borel $\sigma-$ algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$.

Solution

Borel $\sigma-$algebra

Let (X, d) be a metric space and let $\mathcal{G}$ be the family of open sets of X, then we define the Borel $\sigma-$algebra as:

$$\mathcal{B}(X) := \sigma_0(\mathcal{G})$$

The elements of $\mathcal{G}$ are called Borel sets, let us enumerate some classes of them:

Classes of Borel sets

i) open sets

ii) closed sets (they are the complementary of open sets and this is a $\sigma-$algebra)

iii) countable intersections of open sets, known as the family G_δ

iv) countable union of closed sets, known as the family F_δ.

Lastly, let us characterize the Borel $\sigma-$algebras $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$:

Characterization of $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$

1. $\mathcal{B}(\mathbb{R}) = \sigma_0(I) = \sigma_0(I_1) = \sigma_0(I_2) = \sigma_0(I_0) = \sigma_0(\hat{I})$
   where:

$$I = \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$$
$$I_1 = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$$
$$I_2 = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$$
$$I_0 = \{(a, b) : -\infty \leq a < b < \infty\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$
$$\hat{I} = \{(a, \infty) : a \in \mathbb{R}\}$$

2. $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I}) = \sigma_0(\tilde{I}_1)$
   where:

$$\tilde{I} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a, +\infty] : a \in \mathbb{R}\}$$
$$\tilde{I}_1 = \{(a, +\infty] : a \in \mathbb{R}\}$$

3. $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_1) = \sigma_0(K_2)$
   where:

$$K_1 = \{\text{n-dimensional closed rectangles}\}$$
$$K_2 = \{\text{n-dimensional open rectangles }\}$$

Question 1.8

Write the definitions of: measure, finite measure, $\sigma-$finite measure, measure space, probability space. Provide some examples of measures.

Solution

Let us define the following:

- **Measure**

  Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$, then a function μ:

  $$\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$$

  is a measure if:

  1. $\mu(\emptyset) = 0$
  2. $\sigma - \mathbf{additivity}$:
     $\forall \{E_n\} \subseteq \mathcal{C}$ disjoint $(E_i \cap E_j \quad \forall i \neq j)$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ we have that:

  $$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k)$$

  ### Finite measure

  a measure μ defined as above is said to be finite if:

  $$\mu(X) < +\infty$$

- $\sigma - \mathbf{finite\ measure}$

  a measure μ is said to be $\sigma - \text{finite}$ if there exists a sequence $\{E_n\}$ such that:

  $$X = \bigcup_{k=1}^{\infty} E_k, \quad \mu(E_k) < +\infty$$

- **Measure space**

  Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma - \text{algebra}$ and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure, then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

- **Probability space**

  if $\mu(X) = 1$ then we say that $(X, \mathcal{A}, \mu)$ is a probability space.

Question 1.9

State and prove the theorem regarding properties of measures. Why the two continuity properties are called in this way? For what concerns continuity w.r.t. a descending sequence E_k, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

Solution

Properties of measures

Let us state and prove the properties of a measure μ on a set X and $\sigma - \text{algebra } \mathcal{A}$:

i) **Additivity:**
   $\forall \{E_1, \ldots, E_n\} \subseteq \mathcal{A}$ disjoint we have:

   $$\mu \left(\bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} \mu(E_k)$$

Proof. indeed if we define a sequence such that:

$$\{E_n\} = \begin{cases} B_k = E_k & \forall k \leq n \\ B_k = \emptyset & \forall k > n \end{cases}$$

this sequence is also disjoint $(E \cap \emptyset = \emptyset \ \forall E \in X)$, thus we may write:

$$\mu \underbrace{\left(\bigcup_{k=1}^{\infty} E_k \right)}_{=\bigcup_{k=1}^{n} E_k \cup \emptyset} = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \sum_{k=n+1}^{\infty} \underbrace{\cancel{\mu(E_k)}}_{=0}$$

$\square$

ii) **Monotonicity:**
$\forall E, F \in \mathcal{A}$ we have:
$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

Proof. We may write F in the following way:

$$F = E \cup (F \setminus E)$$

and since these two sets are obviously disjoint we may use (i) to write:

$$\mu(F) = \mu(E) + \underbrace{\mu(E \setminus F)}_{\geq 0} > \mu(E)$$

$\square$

iii) $\sigma - $**subadditivity:**
$\forall \{E_n\} \subseteq \mathcal{A}$ (**not** disjoint) we have:
$$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. Let us define:

$$\begin{cases} F_1 := E_1 \\ F_n := E_n \setminus \bigcup_{k=1}^{n-1} E_k & \forall n > 1 \end{cases}$$

Clearly $\{F_n\} \subseteq \mathcal{A}$ and $\{F_n\}$ is a disjoint sequence and:

$$F_k \subseteq E_k \quad \forall k \in \mathbb{N} \implies \mu(F_k) \leq \mu(E_k) \text{ by (ii)}$$
$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$$

thus we may write:

$$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \mu \left(\bigcup_{k=1}^{\infty} F_k \right) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

$\square$

iv) **Continuity from below:**
$\forall \{E_n\} \subseteq \mathcal{A}, \ E_k \nearrow$ we have:
$$\mu \left(\lim_{k \to \infty} E_k \right) = \lim_{k \to \infty} \mu(E_k)$$

Proof. Let us define a new sequence $\{F_n\}$ as:

$$\begin{cases} F_k := E_k \setminus E_{k-1} \quad \forall k \in \mathbb{N} \text{ and } E_0 := \emptyset \\ \implies \bigcup_{k=1}^{n} F_k = E_n, \ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

and since $\{F_n\}$ is a disjoint sequence (we may visually think of it as a set of ever increasing rings) we may use (i) to write:

$$\mu \left(\lim_{n\to\infty} E_n \right) = \mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \mu \left(\bigcup_{k=1}^{\infty} F_k \right)$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu \left(\bigcup_{k=1}^{n} F_k \right)$$
$$= \lim_{n\to\infty} \mu(E_n)$$

$\square$

v) **Continuity from above:**
$\forall \{E_n\} \subseteq \mathcal{A},\ E_k \searrow,\ \mu(E_1) < +\infty$ we have:

$$\mu \left(\lim_{n\to\infty} E_n \right) = \lim_{n\to\infty} \mu(E_n)$$

Proof. Like we did above Let us define: a new sequence $\{F_n\}$

$$F_k := E_1 \setminus E_k \quad \forall k \in \mathbb{N}$$

let us note that $\{F_n\}$ is an increasing sequence thus by (iv) we can write:

$$\mu \left(\bigcup_{k=1}^{\infty} F_k \right) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty} (E_k)$$

because by (ii) $\mu(F \setminus E) = \mu(F) - \mu(E)$, moreover:

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (E_1 \cap E_k^{\mathsf{c}}) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^{\mathsf{c}} \right) = E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k \right)$$
$$\implies \mu \left(\bigcup_{k=1}^{\infty} F_k \right) = \mu(E_1) - \mu \left(\bigcap_{k=1}^{\infty} E_k \right)$$

thus combining these two and canceling the $\mu(E_1)$ on both sides we obtain:

$$\lim_{k\to\infty} \mu(E_k) = \mu \left(\bigcap_{k=1}^{\infty} E_k \right)$$

$\square$

let us note that for this last, crucial, step $\mu(E_1)$ must be finite, otherwise we would not be able to cancel it out from both sides.

Sheet n. 2

Question 2.1

Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

Solution

Let us define the following:

- **Sets of zero measure**

  Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $E \subseteq X$ has zero measure if $E \in \mathcal{A}$ and $\mu(E) = 0$. We denote the set of all sets of zero measure by $\mathcal{N}_\mu$

- **Negligible sets**

  a set $E \subseteq X$ is negligible if:
  $$\exists N \in \mathcal{A} \text{ s.t. } E \subseteq N, \ \mu(N) = 0$$

  So any subset of a set of zero measure is negligible, we denote the collection of all negligible sets by τ_μ. Moreover let us note that E doesn't need to be an element of $\mathcal{A}$ ($E \notin \mathcal{A}$)

- **Almost Everywhere**

  a property P on X is said to hold almost everywhere if:
  $$\mu(\{x \in X : P(x) \text{ is false }\}) = 0$$

  We may also say that $\{x \in X : P(x) \text{ is false }\} \in \mathcal{N}_\mu$

  ### Examples

  typical properties that can be true a.e. are: equality, continuity, monotonicity, etc. etc.

Question 2.2

Write the definition of complete measure space. Exhibit an example of a measure space which is not complete.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Counterexample

Let $X = \{a, b, c\}$, $\mathcal{A} = \sigma(\{\emptyset, \{a\}, \{b, c\}, X\})$ and $\mu \equiv 0$, clearly here we have:

$$\tau_\mu \setminus \mathcal{N}_\mu = \{\{b\}, \{c\}\}$$

and clearly $\{b\}, \{c\} \notin \mathcal{A}$. So this measure space is not complete.

Question 2.3

Write the definition of complete measure space. State the theorem concerning the existence of the completion of a measure space. Give just an idea of the proof.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Existence of the completion

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let us define: $\bar{\mathcal{A}}, \bar{\mu}$

$$\bar{\mathcal{A}} = \{E \subseteq X : \exists F, G \in \mathcal{A} \text{ s.t. } F \subseteq E \subseteq G \;\, \mu(G \setminus F) = 0\}$$
$$\bar{\mu} : \bar{\mathcal{A}} \to \overline{\mathbb{R}}_+, \quad \bar{\mu}(E) := \mu(F)$$

then:

1. $\bar{\mathcal{A}}$ is a $\sigma - \text{algebra}$, $\bar{\mathcal{A}} \supseteq \mathcal{A}$

2. $\bar{\mu}$ is a complete measure, $\bar{\mu}|_{\mathcal{A}} = \mu$

and the triplet $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space and is called the completion of $(X, \mathcal{A}, \mu)$, i.e. it the smallest (w.r. to inclusion) complete measure space that cointains $(X, \mathcal{A}, \mu)$

Sketch of proof

[1] We must prove two things:

- **First:** that $\bar{\mathcal{A}}$ is a $\sigma - \text{algebra}$ and that it contains $\mathcal{A}$, the latter is trivial since $\forall A \in \mathcal{A} \quad A \subseteq A \subseteq A \implies A \in \bar{\mathcal{A}}$ while the former is quite hardous so we shall just assume it to be true.

- **Second:** that $\bar{\mu}$ is a complete measure and $\bar{\mu}|_{\mathcal{A}} = \mu$.
  The latter is trivial (see above). We can also easily prove that it is a measure:

  i) $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$ since the only set contained inside $\emptyset$ is $\emptyset$ itself, as the container set we may take any zero set measure inside $\mathcal{A}$.

  ii) that $\sigma - \text{additivity}$ holds is clear since for any disjoint sequence $\{E_n\} \subseteq \bar{\mathcal{A}}$ we may construct two sequences:
  $$\begin{cases} \{F_n\}, \; F_k \subseteq E_k \\ \{G_n\}, \; G_k \supseteq E_k \end{cases} \forall k \in \mathbb{N} \text{ s.t. } \mu(G_k \setminus F_k) = 0$$

  Let us note the following:

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[1]This is a partial proof of my own making. It has been review by the TA and professor Punzo and stated to be correct.

– $\{F_n\}$ is also disjoint because $\{E_n\}$ is disjoint.

– Moreover:

$$\bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} G_k$$

$$\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus F_k)$$

$$\mu \left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k \right) \leq \mu \left(\bigcup_{k=1}^{\infty} (G_k \setminus F_k) \right) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus F_k) = 0$$

The last inequality is true thanks to the $\sigma-$subadditivity and monotonicty of μ.

Thus we can say that:

$$\bar{\mu} \left(\bigcup_{k=1}^{\infty} E_k \right) = \mu \left(\bigcup_{k=1}^{\infty} F_k \right) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$$

thus $\bar{\mu}$ is a measure.

Let us prove that $\bar{\mu}$ is complete. Let $E_1 \in X$ and $E_2 \in \bar{\mathcal{A}}$ such that $\bar{\mu}(E_2) = \mu(F_2) = 0$ and $E_1 \subseteq E_2$, let us note that:

$$\begin{cases} \mu(G_2) = \overcancel{\mu(G_2 \setminus F_2)}^{0} + \overcancel{\mu(F_2)}^{0} \\ \mu(G_2 \setminus \emptyset) = \mu(G_2) - 0 \\ \emptyset \subseteq E_1 \subseteq G_2 \end{cases} \implies E_1 \in \bar{\mathcal{A}}, \ \bar{\mu}(E_1) = \mu(\emptyset) = 0$$

thus any negligible set is also a zero measure set and $\bar{\mu}$ is complete.

Question 2.4

Write the definition of outer measure. State and prove the theorem concerning generation of outer measure on a general set X, starting from a set $K \in \mathcal{P}(X)$, containing $\emptyset$, and a function $\nu : K \to \overline{\mathbb{R}}_+$, $\nu(\emptyset) = 0$. Intuitively, which is the meaning of (K, ν)?

Solution

Outer measure

We say that a function: $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ (where X is any set) is an outer measure if:

i) $\mu^*(\emptyset) = 0$

ii) $E_1 \subseteq E_2 \implies \mu^*(E_1) \leq \mu^*(E_2)$

iii) $\mu^* \left(\bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

Generation of an outer measure

Let $K \subseteq \mathcal{P}(X)$, $\emptyset \in K$, $\nu : K \to \overline{\mathbb{R}}_+$, $\nu(\emptyset) = 0$, then we can generate an outer measure μ^* on X defined as:

$$\begin{cases} \mu^*(E) := \inf \left\{ \sum_{k=1}^{\infty} \nu(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k, \ \{I_n\} \subseteq K \right\}, & \text{if } E \text{ can be covered by a countable union of sets } I_n \in K. \\ \mu^*(E) := +\infty, & \text{otherwise.} \end{cases}$$

Proof. Let us verify that such a μ^* meets the definition of outer measure (2.4.1):

i) $\emptyset \in K$, $0 \leq \mu^*(\emptyset) \leq \nu(\emptyset) = 0$ by the definition of μ^*.

ii) $E_1 \subseteq E_2$, we have two possible cases

• if there exists a countable covering of E_2 then it is also a covering of E_1 and from the definitio of μ^* it follows that:

$$\mu^*(E_1) \leq \mu^*(E_2)$$

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- if there is no countable covering of E_2 then:

$$\mu^*(E_1) \le \mu^*(E_2) = +\infty$$

iii) this condition is obviously met if:

$$\sum_{k=1}^{\infty} \mu^*(E_k) = +\infty$$

otherwise if we suppose that:

$$\sum_{k=1}^{\infty} \mu^*(E_k) < +\infty$$

thus $\mu^*(E_k) < +\infty \; \forall k \in \mathbb{N}$, by the definition of μ^* and inf:

$$\forall \varepsilon > 0, \; \forall n \in \mathbb{N} \quad \exists \{I_{n,k}\} \subseteq K$$

such that:

$$E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n,k})$$

Now, since:

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}, \quad \{I_{n,k}\} \subseteq K$$

it clearly follows that:

$$\mu^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \nu(I_{n,k}) < \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon \cdot \sum_{n=1}^{\infty} \frac{\cancel{1}}{\cancel{2^n}}^{1}$$

because ε is arbitrary, we have the cocnlusion.

$\square$

The intuitive meaning (K, ν) is that K is a special class of sets in X and ν is a function that assigns a value to each set in K. On the other hand ν can be any real valued positive function, thus it is not necessary to be a measure.

Question 2.5

What is the Caratheodory condition? How can it be stated in an equivalent way? Prove it.

Solution

Caratheodory condition

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \subset X$$

Equivalent statement

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if and only if:

$$\mu^*(Z) \ge \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \subset X$$

Proof. It is enough to note that $\forall E \subseteq X$ we have:

$$Z = (Z \cap E) \cup (Z \cap E^c) \quad \forall Z \subset X$$

and thus by the subadditivity of μ^* (iii) we get:

$$\mu^*(Z) \le \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \subset X$$

and we may combine this inequality with the other to yield an equality.

$\square$

Question 2.6

Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

Solution

All zero measure sets are in $\mathcal{L}$

There cannot exist such a set E because all sets of zero aouter measure meet the Caratheodory Inequality (2.5.2).

Proof. Indeed $\forall Z \subseteq X$ by the monotonicty of μ^* (ii) we have:

$$\mu^*(\underbrace{Z \cap E}_{\subseteq E}) + \mu^*(\underbrace{Z \setminus E}_{\subseteq Z}) \leq \overset{0}{\cancel{\mu^*(E)}} + \mu^*(Z)$$

$\square$

Question 2.7

State the theorem concerning generation of a measure as a restriction of an outer measure.

Solution

Generation of a measure from an outer measure

Let us define $\mathcal{L}$ as:
$$\mathcal{L} := \{E \subseteq X : \ E \text{ is } \mu^* - \text{measurable }\}$$
where μ^* is an outer measure on X, then:

i) the collection $\mathcal{L}$ is a $\sigma - $ algebra

ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

Question 2.8

Show that the measure induced by an outer measure on the $\sigma - $ algebra of all sets fulfilling the Caratheodory condition is complete.

Solution

Generation of a measure from an outer measure (proof of completeness)

Let us see that such a measure as the one described in the previous question is complete. Let μ^* be an outer measure on X and $\mathcal{L}$ the $\sigma - $ algebra of all sets fulfilling the Caratheodory condition. Let μ be the measure induced by μ^* on $\mathcal{L}$ ($\mu = \mu^*|_{\mathcal{L}}$).

Proof. Let $N \in \mathcal{L}$ such that $\mu(N) = \mu^*(N) = 0$ and let $E \subseteq N$.
By monotonicty of μ^* (ii):
$$0 \leq \mu^*(E) \leq \mu^*(N) = 0 \implies \mu^*(E) = 0$$

thus by the lemma seen in 2.6.1 we get that $E \in \mathcal{L}$ and so $\mathcal{L}$ is complete. $\square$

Question 2.9

Describe the construction of the Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^n$.

Solution

Construction of the Lebesgue measure on $\mathbb{R}$

Let I be a family of open, bounded intervals in $\mathbb{R}$:

$$I := \{(a,b) : a,b \in \mathbb{R}, a \leq b\}$$

Let us note that $\emptyset \in I$.
Now let us consider a function λ_0:

$$\lambda_0 : I \to \mathbb{R}_+$$
$$\lambda_0(\emptyset) = 0$$
$$\lambda_0((a,b)) = b - a$$

Here we take $X = \mathbb{R}$, $(K, \nu) = (I, \lambda_0)$ and construct the outer Lebesgue measure λ^* as seen above (2.4.2):

$$\lambda^*(E) := \begin{cases} \inf\left\{\sum_{n=1}^{\infty} \lambda_0(I_n) : \quad E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \subseteq I\right\}, & \forall E \subseteq \mathbb{R} \text{ s.t. } E \text{ has a countable covering } \{I_n\} \subseteq I \\ +\infty, \text{ otherwise} \end{cases}$$

The corresponding $\sigma-$algebra is the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R})$ and now we define the Lebesgue measure λ as the measure generated by the outer Lebesgue measure (as seen in 2.7.1):

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})}$$

Construction of the Lebesgue measure on $\mathbb{R}^n$

Analogously to what we have seen above we first define an outer measure and then a (complete) measure but we take:

$$I^n = \left\{ \bigtimes_{k=1}^{n}(a_k, b_k) : \ a_k, b_k \in \mathbb{R}, \ a_k \leq b_k \right\}$$

and accordingly we define:

$$\lambda_0^n : I^n \to \mathbb{R}_+$$
$$\lambda_0^n(\emptyset) = 0$$
$$\lambda_0^n\left(\bigtimes_{k=1}^{n}(a_k, b_k)\right) = \prod_{k=1}^{n}(b_k - a_k)$$

and therefore we take $X = \mathbb{R}^n$ and $(K, \nu) = (I^n, \lambda_0^n)$, we define the outer Lebesgue measure $\lambda^{*,n}$ on $\mathbb{R}^n$ and the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R}^n)$ and finally we construct the n-dimensional Lebesgue measure as:

$$\lambda^n := \lambda^{*,n}|_{\mathcal{L}(\mathbb{R}^n)}$$

Question 2.10

Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

Solution

All countable sets are $\mathcal{L}$-measurable and $\lambda(E) = 0$

Any countable subset $E \subset \mathbb{R}$ is $\mathcal{L}$-measurable and $\lambda(E) = 0$

Proof. Let $a \in \mathbb{R}$, clearly $\{a\} \subseteq (a - \varepsilon, a] \ \forall \varepsilon > 0$, thus by the definition of λ^*:

$$\lambda^*(\{a\}) \leq \lambda^*((a - \varepsilon, a]) = \varepsilon \to 0 \implies \{a\} \in \mathcal{L}$$

Now if E is countable we may write as follows:

$$E = \bigcup_{n=1}^{\infty} \{a_n\} \quad a_n \in \mathbb{R}, \ n \in \mathbb{N}$$

and so by monotonicty (ii):

$$0 \leq \lambda^*(E) = \lambda^* \left(\bigcup_{n=1}^{\infty} \{a_n\} \right) \leq \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$$

thus $\lambda^*(E) = 0 \implies E \in \mathcal{L}$ by the lemma seen above (2.6.1) $\qquad \square$

Question 2.11

Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. Is the inclusion strict? Which is the relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$?

Solution

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty))$ it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$. We already know from above that all bounded intervals belong to $\mathcal{L}(\mathbb{R})$.
Now, let $A \subseteq \mathbb{R}$ be any set. We assume $a \notin A$, otherwise we would replace A with $A \setminus \{a\}$ and this would leave the Lebesgue outer measure unchanged. Furthermore $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \iff (a, +\infty)$ satisfies the Caratheodory Condition (2.5.2):

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$$

where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, +\infty)$.
Since $\lambda^*(A)$ is defined as an inf, to verify the above, it is necessary and sufficient to show that for **any countable collection** $\{I_n\}$ of **open bounded** intervals that **covers** A we have that:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k)$$

For every $k \in \mathbb{N}$ we define:

$$I_k' := I_k \cap (-\infty, a)$$
$$I_k'' := I_k \cap (a, +\infty)$$

then:

$$I_k' \cap I_k'' = \emptyset (\text{disjoint}) \implies \lambda_o(I_k) = \lambda_0(I_k') + \lambda_0(I_k'')$$

Let us note that $\{I_n'\}$ is a countable cover for A_1 and $\{I_n''\}$ is a countable cover for A_2. Hence:

$$\lambda^*(A_1) = \sum_{k=1}^{\infty} \lambda_0(I_k')$$

$$\lambda^*(A_2) = \sum_{k=1}^{\infty} \lambda_0(I_k'')$$

therefore:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k') + \sum_{k=1}^{\infty} \lambda_0(I_k'') = \sum_{k=1}^{\infty} \lambda_0(I_k)$$

which equivalento to the condition above. $\qquad \square$

$$\mathcal{B}\left(\mathbb{R}\right) \subsetneq \mathcal{L}(\mathbb{R})$$

The inclusion demonstrated above can be shown to be strict. A counterexample can be produced (see here) but it is quite pathological.

Relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}\left(\mathbb{R}\right), \lambda)$

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}\left(\mathbb{R}\right), \lambda|_{\mathcal{B}(\mathbb{R})})$. Indeed as we have shown above $\mathcal{B}\left(\mathbb{R}\right)$ is not a complete $\sigma-$algebra while $\mathcal{L}(\mathbb{R})$ is.

Sheet n. 3

Question 3.1

Write the excision property and prove it. Write and prove (partially) the theorem concerning the regularity of the Lebesgue measure on $\mathbb{R}$.

Solution

Excision property

If $A \in \mathcal{L}(\mathbb{R})$, $\lambda^*(A) \leq +\infty$ and $A \subseteq B$, then:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

Proof. Since $A \in \mathcal{L}(\mathbb{R})$ we can use the Caratheodory equality (2.5.1) using $Z = B$, $E = A$:

$$\lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A\ (A \subseteq B)}) + \lambda^*(B \setminus A)$$

so, since $\lambda^*(A) \leq +\infty$ we may write:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

$\square$

Regularity of the Lebesgue Measure

Let $E \subseteq \mathbb{R}$, the following are equal:

i) $E \in \mathcal{L}(\mathbb{R})$

ii) $\forall \varepsilon > 0 \; \exists A \subseteq \mathbb{R}$ open s.t.
$$E \subseteq A \quad \lambda^*(A \setminus E) < \varepsilon$$

iii) $\exists G \subseteq \mathbb{R}$ in the class G_δ (countable intersections of open sets) s.t.
$$E \subseteq G \quad \lambda^*(G \setminus E) = 0$$

iv) $\forall \varepsilon > 0 \; \exists C \subseteq \mathbb{R}$ closed s.t.
$$C \subseteq E \quad \lambda^*(E \setminus C) < \varepsilon$$

v) $\exists F \subseteq \mathbb{R}$ in the class F_δ (countable unions of closed sets) s.t.
$$F \subseteq E \quad \lambda^*(E \setminus F) = 0$$

Proof. Let us give a (partial) proof:

- $(i) \implies (ii)$: if $E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < +\infty$ then by definition of outer measure (2.4.1):

$$\forall \varepsilon > 0 \; \exists \{I_n\} \text{ that covers } E \text{ and } \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$

Let us now define the set O:

$$O := \bigcup_{k=1}^{\infty} I_k, \; O \text{ is open}, \; E \subseteq O$$

and so we may write:

$$\lambda^*(O) \overset{sub-add\,(iii)}{\leq} \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$
$$\implies \lambda^*(O) - \lambda^*(E) < \varepsilon$$

and by the Excision property (3.1.1) ($E \in \mathcal{L}(\mathbb{R})$, $\lambda^*(E) < +\infty$):

$$\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$$

and so we have obtained the second statement (ii).

- $(ii) \implies (iii)$, $\forall k \in \mathbb{N}$ we choose $O_k \supseteq E$ open for which:

$$\lambda^*(O_k \setminus E) < \frac{1}{k}$$

and then define:

$$G = \bigcap_{k=1}^{\infty} O_k \implies G \in G_\delta, \; G \supseteq E$$

Moreover $\forall k \in \mathbb{N}$:

$$G \setminus E \subseteq O_k \setminus E$$

so by monotonicty (ii):

$$\lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$$

let us apply a limit $k \to \infty$ to both sides:

$$\lambda^*(G \setminus E) = 0$$

- $(iii) \implies (i)$, let us note that $G \setminus E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G \setminus E) = 0$ by lemma 2.6.1 and:

$$G \in \mathcal{L}(\mathbb{R}) \text{ since } G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$$
$$\implies E = \underset{\in \mathcal{L}}{G} \cap \underset{\in \mathcal{L}}{(G \setminus E)^{\mathsf{c}}} \in \mathcal{L}$$

$\square$

Question 3.2

Is it true that any subset $E \subseteq \mathbb{R}$ is $\mathcal{L}$-measurable? Is it possibile to find two disjoint sets $A, B \subset \mathbb{R}$ for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Why?

Solution

Vitali's non-measurable sets

Any measurable set $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$ contains a subset that fails to be measurable.
Therefore there exist subsets of $\mathbb{R}$ that are not $\mathcal{L}$-measurable.

Disjoints sets for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$

There are disjoint sets $A, B \subseteq \mathbb{R}$ for which:

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$

Proof. Assume by contradiction that:

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \quad \forall A, B \subseteq \mathbb{R}, \ A \cap B = \emptyset$$

Now $\forall E, Z \subseteq \mathbb{R}$ we write:

$$\lambda^*(\underbrace{Z \cap E}_{=A}) + \lambda^*(\underbrace{Z \cap E^{\mathsf{c}}}_{=B}) = \lambda^*(\underbrace{Z}_{=A \cup B})$$

thus any set E would satisfy the Caratheodory condition (2.5.1) and be $\mathcal{L}$-measurable which is absurd since we know that Vitali's sets exist. $\square$

Question 3.3

Write the definition of measurable function. Show the measurability of the composite function.

Solution

Measurable function

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and f a function:

$$f : X \to X'$$

f is said to be measurable if:

$$f^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{A}'$$

Measurability of the composite function

Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ and $(X'', \mathcal{A}'')$ be three measurable spaces and $f : X \to X'$ and $g : X' \to X''$ two measurable functions. Then the composite function $g \circ f : X \to X''$ is measurable.

Proof.

$$\forall E \in \mathcal{A}' \quad f^{-1}(E) \in \mathcal{A}$$
$$\forall F \in \mathcal{A}'' \quad g^{-1}(F) \in \mathcal{A}'$$

thus:

$$\forall F \in \mathcal{A}'' \quad (g \circ f)^{-1}(F) = f^{-1}\left[\underbrace{g^{-1}(F)}_{:= E \in \mathcal{A}'}\right] \in \mathcal{A}$$

$\square$

Question 3.4

Characterize measurability of functions and prove it.

Solution

Characterization of Measurability

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$ then:

$$f : X \to X' \text{ measurable} \iff f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- ($\implies$): Suppose f be measurable $\implies \mathcal{C}' \subseteq \mathcal{A}'$ and so we get the thesis.

- ($\impliedby$): Let us define the following:

$$\Sigma := \{E \subseteq X' : f^{-1}(E) \in \mathcal{A}\}$$

We can easily see that Σ is a $\sigma - $ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

$$\mathcal{A}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

and we get the thesis.

$\square$

Question 3.5

Write the definitions of:

a) Borel measurable functions;

b) Lebesgue measurable functions.

Solution

a) Borel measurable functions

Let $(X, d), (X, \mathcal{B})$ and $(X', d'), (X', \mathcal{B}')$ be couples of metric spaces and measurable spaces. A function f:

$$f : X \to X' \text{ measurable}$$

is called Borel-measurable or $\mathcal{B}$-meaurable.

b) Lebesgue measurable functions

Let $(X, \mathcal{L})$ be a measurable space and (X', d') a metric space, $(X', \mathcal{B}')$ a measurable space, then:

$$f : X \to X' \text{ measurable}$$

is called Lebesgue-measurable or $\mathcal{L}$-measurable.

Question 3.6

Prove that continuous functions are both Borel and Lebesgue measurable.

Solution

Continuous functions are $\mathcal{B}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{B}$-measurable.

Proof. Let $\mathcal{C}'$ be the class of open sets of X' and $\mathcal{C}$ the class of open sets of X. We have:

$$\forall E \in \mathcal{C}' \quad f^{-1}(E) \in \mathcal{C} \subseteq \mathcal{B} \text{ (by definition of continuity)}$$

and $\mathcal{B}' = \sigma_0(\mathcal{C}')$ so we get the thesis.

$\square$

Continuous functions are $\mathcal{L}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{L}$-measurable.

Proof. Since $\mathcal{B} \subset \mathcal{L}$ and the previous statement has been proven true, the thesis follows trivially. $\qquad\square$

Question 3.7

Characterize Lebesgue measurability of functions and prove it.

Solution

Characterization of Lebesgue measurability

All we must do is apply the Characterization of Measurability (3.4.1) taking $(X, \mathcal{A} = \mathcal{L})$, $(X', \mathcal{A}' = \mathcal{B}')$ and $\mathcal{C}'$ the class of open sets of X', since $\mathcal{B}' = \sigma_0(\mathcal{C}')$. We then can write:

$$f : X \to X' \text{ Lebesgue measurable} \iff f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- ($\implies$): Suppose f be Lebesgue measurable $\implies \mathcal{C}' \subseteq \mathcal{B}'$ and so we get the thesis.

- ($\impliedby$): Let us define the following:

$$\Sigma := \{E \subseteq X' : \ f^{-1}(E) \in \mathcal{L}\}$$

  We can easily see that Σ is a $\sigma -$ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

$$\mathcal{B}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

  and we get the thesis.

$\qquad\square$

Question 3.8

Establish and show all equivalent statements to the fact that $f : X \to \overline{\mathbb{R}}$ is measurable.

Solution

Equivalent statements of measurability

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a function, the following are equal:

  i) f is measurable;

 ii) $\{f > \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

iii) $\{f \geq \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

 iv) $\{f < \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

  v) $\{f \leq \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$.

Proof. Let us prove all the coimplications:

(i) $\iff$ **(iii):**

$$\mathcal{A}' = \mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\overbrace{\{(\alpha, +\infty] : \alpha \in \mathbb{R}\}}^{\mathcal{C}'})$$
$$f \text{ is measurable} \iff f^{-1}(\underbrace{(\alpha, +\infty]}_{E}) \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$$

(ii) $\implies$ **(iii):**

$$\{f \geq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f > \alpha - \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(iii) $\implies$ **(iv):**

$$\{f < \alpha\} = \{f \geq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

(iv) $\implies$ **(v):**

$$\{f \leq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f < \alpha + \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(v) $\implies$ **(ii):**

$$\{f > \alpha\} = \{f \leq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

$\square$

Question 3.9

Let $f, g \in \mathcal{M}(X, \mathcal{A})$. What can we say about measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$? Justify the answer.

Solution

Measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$

Let $f, g \in \mathcal{M}(X, \mathcal{A})$, we have:

  i) $\{f < g\} \in \mathcal{A}$

 ii) $\{f \leq g\} \in \mathcal{A}$

iii) $\{f = g\} \in \mathcal{A}$

Proof.

  i) $\{f < g\} = \bigcup_{r \in \mathbb{Q}} \left[\underbrace{\overbrace{\{f < r\}}^{\in \mathcal{A}} \cap \overbrace{\{r < g\}}^{\in \mathcal{A}}}_{\in \mathcal{A}} \right]$

 ii) $\{f \leq g\} = \{f > g\}^{\mathsf{c}} \in \mathcal{A}$ by the previous point.

iii) $\{f = g\} = \underset{\in \mathcal{A}}{\{f \leq g\}} \cap \underset{\in \mathcal{A}}{\{f \geq g\}} \in \mathcal{A}$

$\square$

Question 3.10

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$. Show that $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$. Can there exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$? Why?

Solution

Measurability of $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$, we have:

  i) $\sup_n f_n \in \mathcal{M}(X, \mathcal{A})$

  ii) $\inf_n f_n \in \mathcal{M}(X, \mathcal{A})$

  iii) $\limsup_n f_n \in \mathcal{M}(X, \mathcal{A})$

  iv) $\liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$

Proof.

  i) $\forall \alpha \in \mathbb{R} \quad \{\sup_{n \in \mathbb{N}} f_n > \alpha\} = \bigcup_{n=1}^{\infty} \{f_n > \alpha\} \in \mathcal{A} \implies \sup_{n \in \mathbb{N}} f_n \in \mathcal{M}$

  ii) $\inf_n f_n = -\sup_{n \in \mathbb{N}}(-f_n) \in \mathcal{M}(X, \mathcal{A})$

  iii) $\limsup_n f_n = \inf_{k \geq 1} \sup_{n \geq k} f_n \in \mathcal{M}(X, \mathcal{A})$

  iv) $\liminf_n f_n = \sup_{k \geq 1} \inf_{n \geq k} f_n \in \mathcal{M}(X, \mathcal{A})$

$\square$

Corollary, $\max\{f, g\} \in \mathcal{M}$

There cannot exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$. Indeed we can write:

$$\max(f, g) = f \cdot \chi_{\{f \geq g\}} + g \cdot \chi_{\{g > f\}}$$

In other words the max can be written as the sum and product of measurable functions, hence it is measurable itself.

Question 3.11

Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Show that $f + g, f \cdot g \in \mathcal{M}(X, \mathcal{A})$.

Solution

Measurability of $f + g, f \cdot g$

Let $f, g : X \to \mathbb{R}$ and $f, g \in \mathcal{M}(X, \mathcal{A})$, we have that $f + g, f \cdot g \in \mathcal{M}(X, \mathcal{A})$.

Proof. Let us define a few new functions φ, ψ and χ:

$$\begin{cases} \varphi(x) = X \to \mathbb{R}^2 & \varphi(x) := (f(x), g(x)) \\ \psi(x) = \mathbb{R}^2 \to \mathbb{R} & \psi(s, t) := s + t \\ \chi(x) = \mathbb{R}^2 \to \mathbb{R} & \chi(s, t) := s \cdot t \end{cases} \implies \begin{cases} \psi \circ \varphi = f + g \\ \chi \circ \varphi = f \cdot g \end{cases}$$

Now, clearly $\psi, \chi \in C^0(\mathbb{R}^2)$ (hence measurable), let us prove that φ is also measurable. We use the Characterization of Measurability (3.4.1):

$$\varphi : (X, \mathcal{A}) \to (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \text{ is measurable} \iff \forall E \subseteq \mathbb{R}^2 \text{ open } \varphi^{-1}(E) \in \mathcal{A}$$

We take:

$$E = R := (a, b) \times (c, d)$$
$$\varphi^{-1}(R) = \{x \in X : (f(x), g(x)) \in R\}$$
$$= \{x \in X : f(x) \in (a, b)\} \cap \{x \in X : g(x) \in (c, d)\}$$
$$= f^{-1}(a, b) \cap g^{-1}(c, d) \in \mathcal{A}$$

Thus $\forall E \subseteq \mathbb{R}^2$ open, we may write:

$$E = \bigcup_{k=1}^{\infty} R_k, \quad R_k = (a_k, b_k) \times (c_k, d_k)$$

$$\varphi^{-1} = \bigcup_{k=1}^{\infty} \varphi^{-1}(R_k) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k) \cap g^{-1}(c_k, d_k) \in \mathcal{A}$$

Hence $\varphi \in \mathcal{M}(X, \mathcal{A})$, and we have:

$$\psi \circ \varphi, \ \chi \circ \varphi \in \mathcal{M}(X, \mathcal{A})$$

$\square$

Question 3.12

Prove that A is measurable if and only if χ_A is a measurable function.

Solution

A is measurable if and only if χ_A is a measurable function

Let $A \subseteq X$ and χ_A be the indicator function of A. We have:

$$\chi_A \in \mathcal{M}(X, \mathcal{A}) \iff A \in \mathcal{A}$$

Proof.

$$\{\chi_A > \alpha\} = \begin{cases} X & \alpha < 0 \\ A & 1 > \alpha \geq 0 \\ \emptyset & \alpha \geq 1 \end{cases}$$

Now, $X, \emptyset \in \mathcal{A}$ by definition, so:

$$A \in \mathcal{A} \iff \chi_A \in \mathcal{M}$$

$\square$

Question 3.13

Prove or disprove the following statements:

a) $f \in \mathcal{M}(X, \mathcal{A}) \iff f_{\pm} \in \mathcal{M}_+(X, \mathcal{A})$;

b) $f \in \mathcal{M}(X, \mathcal{A}) \iff |f| \in \mathcal{M}(X, \mathcal{A})$.

Solution

Measurability of $f_{\pm}$ and $|f|$

Let $f : X \to \mathbb{R}$, we have:

  i) $f \in \mathcal{M}(X, \mathcal{A}) \iff f_{\pm} \in \mathcal{M}_+(X, \mathcal{A})$

  ii) $f \in \mathcal{M}(X, \mathcal{A}) \iff |f| \in \mathcal{M}(X, \mathcal{A})$

Proof.

  i)   • ($\implies$): if $f \in \mathcal{M}(X, \mathcal{A})$, then we define f_+ as:

$$f_+(x) = \max\{f(x), 0\} \geq 0 \quad \forall x \in X$$

and since $f, 0 \in \mathcal{M}(X, \mathcal{A})$ and max is a measurable function we have that $f_+ = \max \circ (f, 0) \in \mathcal{M}_+(X, \mathcal{A})$ by (3.3.2). We may analogously prove the same for f_-.

- ($\Longleftarrow$): if $f_+ \in \mathcal{M}_+(X, \mathcal{A})$, then we define $f = f_+ - f_-$, and since $f_+, f_-, f \in \mathcal{M}(X, \mathcal{A})$ we have that $f \in \mathcal{M}(X, \mathcal{A})$ by (3.11.1).

ii) $f \in \mathcal{M} \implies f_+, f_- \in \mathcal{M}$ by the previous point $\implies |f| = f_+ + f_- \in \mathcal{M}$ by (3.11.1).

$\square$

Question 3.14

Write the definition of simple function. What is its canonical form? How can we characterize measurability of a simple function? Write the definition of step function.

Solution

Definition of simple function

Let X be a set and $s : X \to \mathbb{R}$ a function. We say that s is a simple function if $s(X)$ is a finite set. Furthermore we define the two sets:

$$\mathcal{S}(X, \mathcal{A}) := \{ \text{ measurable simple functions} \}$$
$$\mathcal{S}_+(X, \mathcal{A}) := \{ \text{ measurable simple functions with non-negative values} \}$$

Canonical form of simple function

The canonical form of a simple function is:

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$

where:

$$c_i \in \mathbb{R} \; \forall i = 1, \ldots, n$$
$$E_i = \{x \in X : \; s(x) = c_i\} \; \forall i = 1, \ldots, n$$
$$X = \bigcup_{i=1}^{n} E_i, \; E_k \cap E_l = \emptyset \; \forall k \neq l$$

i.e. E_i is a partition of X.

Measurability of simple function

A simple function is measurable if and only if we have the following:

$$E_i \in \mathcal{A} \; \forall i = 1, \ldots, n$$

i.e. :

$$s(x) \in \mathcal{M}(X, \mathcal{A}) \iff E_i \in \mathcal{A} \; \forall i = 1, \ldots, n$$

this is because $s(x)$ is a linear combination of indicator functions.

Step Functions

Let $I = [a_0, a_1)$ be an interval and $P = \{a_o \equiv x_0 < x_1 < \cdots < x_n \equiv a_1\}$ a partition of I. A function $f : I \to \mathbb{R}$ is a step function if:

$$f := \sum_{k=0}^{n-1} c_k \chi_{[x_k, x_{k+1})}(x)$$

Question 3.15

State and give a sketch of the proof of the Simple Approximation Theorem.

Solution

Simple Approximation Theorem

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence of simple functions $\{s_n\}$ such that:

$$s_n \xrightarrow{n \to \infty} f \text{ in } X \text{ (pointwise)}$$

Furthermore:

i) if $f \in \mathcal{M}(X, \mathcal{A})$, then $\{s_n\} \subseteq \mathcal{S}(X, \mathcal{A})$;

ii) if $f \geq 0 \implies \{s_n\} \uparrow, 0 \leq s_n \leq f$;

iii) f bounded $\implies s_n \xrightarrow{n \to \infty} f$ uniformly in X.

Sketch of proof

Let $f \geq 0$, bounded and $0 \leq f \leq 1 \; \forall x \in X$.

$$f : X \to [0, 1]$$

Let us divide $[0, 1]$ in 2^n intervals of equal length $\forall n \in \mathbb{N}$, then we define:

$$E_k^{(n)} := \left\{ x \in X : \; \frac{k}{2^n} \leq f(x) \leq \frac{k+1}{2^n} \right\} \quad k = 0, \ldots, 2^n - 1$$

$$s_n := \sum_{k=0}^{2^n - 1} \frac{k}{2^n} \chi_{E_k^{(n)}}(x) \quad \forall n \in \mathbb{N}$$

Clearly $\{s_n\}$ has the desired properties.

Question 3.16

Write the definitions of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$. State their properties and prove some of them.

Solution

Definition of $\operatorname{ess\,sup}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and f a function on X. We define:

$$\operatorname{ess\,sup}_X f(x) := \inf \left\{ \sup_{x \in N^c} f(x) : \; N \in \mathcal{N}_\mu \right\}$$

Definition of $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measurable space and f a function on X. We define:

$$\operatorname{ess\,inf}_X f(x) := \sup \left\{ \inf_{x \in N^c} f(x) : \; N \in \mathcal{N}_\mu \right\}$$

Properties of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g \in \mathcal{M}(x, \mathcal{A})$ two functions on X. We have that:

i) $\exists N \in \mathcal{N}_\mu$ such that $\operatorname{ess\,sup}_X f = \sup_{x \in N^c} f$ and $f \leq \operatorname{ess\,sup}_X f$ almost surely $x \in X$;

ii) $\operatorname{ess\,sup}_X f = - \operatorname{ess\,inf}_X -f$;

iii) $\operatorname{ess\,sup}_X k \cdot f = k \cdot \operatorname{ess\,sup}_X f$;

iv) $f \leq g$ a.e. in $X \implies \operatorname{ess\,sup}_X f \leq \operatorname{ess\,sup}_X g$;

v) $\operatorname{ess\,sup}_X(f+g) \le \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$;

vi) $f = g$ almost everywhere in $X \implies \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$;

vii) $g \ge 0$ almost everywhere in $X \implies f \cdot g \le (\operatorname{ess\,sup}_X f) \cdot g$ almost everywhere in X.

Proof. Let us give a partial proof:

i) Suppose $\operatorname{ess\,sup}_X f < +\infty$, $\forall k \in \mathbb{N}\ \exists N_k \in \mathcal{N}_\mu$ such that:

$$\sup_{x \in N_k} f < \operatorname{ess\,sup}_X f + \frac{1}{k}$$

We define $N := \bigcup_{k=1}^{\infty} N_k$. Then $N \in \mathcal{N}_\mu$ and:

$$N^{\mathsf{c}} = \bigcap_{k=1}^{\infty} N_k^{\mathsf{c}} \subseteq N_k^{\mathsf{c}} \quad \forall k \in \mathbb{N}$$

$$\implies \operatorname{ess\,sup}_X f \le \sup_{N^{\mathsf{c}}} f \le \sup_{N_k^{\mathsf{c}}} f < \operatorname{ess\,sup}_X f + \frac{1}{k} \quad \forall k \in \mathbb{N}$$

Now we pass apply a limit $k \to +\infty$ and we get:

$$\sup_{N^{\mathsf{c}}} f = \operatorname{ess\,sup}_X f$$
$$N \supseteq \bar{N} := \{x \in X :\ f(x) > \operatorname{ess\,sup}_X f(x)\} \in \mathcal{A}$$
$$\implies \bar{N} \in \mathcal{N}_\mu \implies f \le \operatorname{ess\,sup}_X f \text{ almost everywhere in } X$$

$\square$

Question 3.17

What is $\mathcal{L}^\infty$? Which is the relation between functions finite a.e. and essentially bounded functions? Justify the answer.

Solution

Definition of $\mathcal{L}^\infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f \in \mathcal{M}(X, \mathcal{A})$ is said to be essentially bounded if:

$$\operatorname{ess\,sup}_X f < +\infty$$

and we define the set of essentially bounded functions as:

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} :\ \text{f is essentially bounded }\}$$

Relation between functions finite a.e. and essentially bounded functions

We have that:

1. $f \in \mathcal{L}^\infty \implies f$ is finite a.e. in X;

2. in general if f is finite a.e. in $X \centernot\implies f \in \mathcal{L}^\infty$.

Proof. 1. We can easily see that:

$$|f| \le \operatorname{ess\,sup} |f| < +\infty \text{ almost everywhere in } X$$

thus f is finite almost everywhere in X;

2. Let us assume that:
$$f \text{ is finite a.e. in } X \implies f \in \mathcal{L}^\infty$$
and let us see a clear counterexample of this, take:
$$f(x) : \mathbb{R} \to \overline{\mathbb{R}} := \begin{cases} \frac{1}{|x|} & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

Clearly f is finite in $E = \mathbb{R} \setminus \{0\}$, i.e. f is finite a.e. in $\mathbb{R}$. Let us note that $\lambda(\{0\}) = 0$. Thus:
$$\operatorname*{ess\,sup}_X |f| = +\infty \implies f \notin \mathcal{L}^\infty$$

$\square$

Question 3.18

Write the definitions of the Lebesgue integral of a nonnegative measurable simple function over X and over a measurable subset $E \subseteq X$. Write the main properties of the integral and prove some of them.

Solution

Lebesgue integral of nonnegative simple functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function with canonical form as in (3.14.2). We define the Lebesgue integral of s over X as:
$$\int_X s \, d\mu := \sum_{k=1}^n c_k \mu(E_k)$$

And its integral over a measurable subset $E \in \mathcal{A}$ as:
$$\int_E s \, d\mu := \int_X s \cdot \chi_E \, d\mu = \sum_{k=1}^n c_k \mu(E_k \cap E)$$

Properties of the Lebesgue integral

i)
$$\int_X \chi_E \, d\mu = \mu(E) \quad \forall E \in \mathcal{A}$$

ii)
$$\int_N s \, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$$

iii) Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $c \geq 0$, then:
$$\int_X c \cdot s \, d\mu = c \cdot \int_X s \, d\mu$$

iv) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, then:
$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

v) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, such that $s \leq t$ then:
$$\int_X s \, d\mu \leq \int_X t \, d\mu$$

vi) $s \in \mathcal{S}_+(X, \mathcal{A})$, $E \subseteq F \in \mathcal{A}$ then:
$$\int_E s \, d\mu \leq \int_F s \, d\mu$$

Proof.

i) We may write:
$$\chi_E = \sum_{k=1}^{2} c_k \chi_{E_k} = \begin{cases} c_1 = 1 & E_1 = E \\ c_2 = 0 & E_2 = E^{\mathsf{c}} \end{cases}$$

thus by applying the definition of the Lebesgue integral we get:

$$\int_X \chi_E \, d\mu = \sum_{k=1}^{2} c_k \mu(E_k) = 1 \cdot \mu(E) + 0 \cdot \mu(E^{\mathsf{c}}) = \mu(E)$$

ii) Let us apply the definition:
$$\int_N s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap N)$$

but by the monotonicity of μ (ii) we have:

$$E_k \cap N \subseteq N \implies \mu(E_k \cap N) \le \mu(N) = 0$$

so the previous sum is equal to 0.

$\square$

Question 3.19

Let $s \in \mathcal{S}_+(X, \mathcal{A})$. For any $E \in \mathcal{A}$, let $\varphi(E) := \int_E s \, d\mu$. Prove that φ is a measure.

Solution

Measure induced by a function

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function. We define the measure φ induced by s as:
$$\varphi(E) := \int_E s \, d\mu \quad \forall E \in \mathcal{A}$$

Proof. Let us see that φ meets the definition of a measure:
Clearly:
$$\varphi : \mathcal{A} \to \overline{\mathbb{R}}_+$$

Furthermore:

i) $\varphi(\emptyset) = 0$ by property 2 of the Lebesgue integral (ii).

ii) Let $\{E_n\} \subseteq \mathcal{A}$ disjoint and $E = \bigcup_{k=1}^{\infty} E_k$, let us write:
$$s := \sum_{l=1}^{m} d_l \chi_{F_l} \quad F_l \in \mathcal{A}$$

thus:

$$\begin{aligned} \varphi(E) &= \int_E s \, d\mu = \sum_{l=1}^{m} d_l \mu(F_l \cap E) \\ &= \sum_{l=1}^{m} d_l \sum_{k=1}^{\infty} \mu(F_l \cap E_k) \text{ by the } \sigma - \text{additivity of } \mu \text{ (ii)} \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{m} d_l \mu(F_l \cap E_k) \\ &= \sum_{k=1}^{\infty} \int_{E_k} s \, d\mu = \sum_{k=1}^{\infty} \varphi(E_k) \end{aligned}$$

$\square$

Question 3.20

Write the two possible equivalent definitions of Lebesgue integral of a measurable nonnegative function.

Solution

Let $f : X \to \overline{\mathbb{R}}_+$ be a measurable nonnegative function ($f \in \mathcal{M}_+(X, \mathcal{A})$). Let us define the set $\mathcal{S}_f$:

$$\mathcal{S}_f = \{s \in \mathcal{S}_+ : \ s \leq f \text{ in } X\}$$

We then have two possible and equivalent definitions of the Lebesgue integral of f.

Definition by $\sup$

We define the integral of f as:

$$\int_X f \, d\mu = \sup_{s \in \mathcal{S}_f} \int_X s \, d\mu$$

Definition by $\lim$

Thanks to the Simple Approximation Theorem (3.15.1) we know:

$$\exists \{s_n\} \subseteq \mathcal{S}_f \quad s_n \leq s_{n+1} \ n \in \mathbb{N}, \ s_n \xrightarrow{n \to \infty} s \text{ in } X$$

So we can define the integral as:

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X s_n \, d\mu$$

Let us note that the integral must be independent of the choice of the sequence $\{s_n\}$.

Question 3.21

State and prove the Chebychev inequality.

Solution

Chebychev inequality

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ then $\forall c > 0$ we have:

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

Proof. Clearly

$$E_C := \{f \geq c\} \in \mathcal{A} \text{ since } f \in \mathcal{M}_+(X, \mathcal{A}) \text{ (see (iii))}$$

and we have that:

$$c \cdot \chi_{E_C} \leq f \cdot \chi_{E_C}$$

thus by the monotonicity of the integral for functions (v) and for sets (vi) we have:

$$c \cdot \mu(E_C) = \int_X c \cdot \chi_{E_C} \, d\mu \leq \int_X f \cdot \chi_{E_C} \, d\mu = \int_{E_C} f \, d\mu \leq \int_X f \, d\mu$$

so we have the Chebychev inequality. $\square$

Question 3.22

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$. Show that f is finite a.e. in X.

Solution

f is finite a.e. in X if $\int_X f d\mu < +\infty$

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$, then f is finite a.e. in X.

Proof. Let us note that the thesis is equivalent to $\mu(\{f = +\infty\}) = 0$. Let us define:

$$\{f = +\infty\} = \bigcap_{n=1}^{\infty} \{f > n\}$$

Clearly we have that:

a) $\{f > n\} \downarrow \{f = +\infty\}$

b) $\mu(\{f > n\}) \leq \frac{1}{n} \cdot \int_X f \, d\mu \; \forall n \in \mathbb{N}$ by the Chebychev inequality (3.21.1).

So since $\mu(\{f > 1\}) \leq \frac{1}{1} \cdot \int_X f \, d\mu < +\infty$ we may apply the continuity from above of μ (v):

$$\mu(\{f = +\infty\}) = \mu\left(\bigcap_{n=1}^{\infty}\{f > n\}\right) = \lim_{n \to \infty} \mu(\{f > n\}) = \lim_{n \to \infty} \frac{1}{n} \cdot \underbrace{\int_X f \, d\mu}_{<+\infty} \longrightarrow 0$$

$\square$