Answers to the Theory Questions

of the course of Real and Functional Analysis of prof. Fabio Punzo

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Contents

Sheet n. 1

Question 1.1

Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to\infty} E_n$, $\liminf_{n\to\infty} E_n$, $\lim_{n\to\infty} E_n$.

Solution

Let us define the following:

- **Sequence of sets**

  A family (or collection) of sets $\{E_i\}_{i\in I}$ is called a sequence of sets if $I = \mathbb{N}$ (i.e. it is indexed by the set of natural numbers $\mathbb{N}$)

- **Increasing sequence of sets**

  a sequence of sets $\{E_n\}$ is said to be increasing (or ascending) if:

  $$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Decreasing sequence of sets**

  A sequence of sets $\{E_n\}$ is said to be decreasing (or descending) if:

  $$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Limsup for a sequence of sets**

  for a sequence of sets $\{E_n\}$ we define:

  $$\limsup_{n\to\infty} E_n \coloneqq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

- **Liminf for a sequence of sets**

  analogously:

  $$\limsup_{n\to\infty} E_n \coloneqq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

- **Limit for a sequence of sets**

  as for a sequence of real numbers if the limsup and liminf coincide we may define:

  $$\lim_{n\to\infty} E_n := \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n$$

Question 1.2

Write the definitions of: cover (or covering) of a set; subcover.

Solution

Let us define the following:

- **Cover of a set**

  a family of sets $\{E_i\}_{i\in I}$ is called a cover (or covering) of X if:

  $$X \subseteq \bigcup_{i\in I} E_i$$

- **Subcover**

  a sub-family of a cover $\{E_i\}_{i\in J}$ $(J \subseteq I)$ which forms a cover is called a subcover.

Question 1.3

Write the definitions of: equivalence relation, equivalence class, quotient set.

Solution

Let us define the following:

- **Equivalence relation**

  a relation R in X (i.e. a subset $R \subseteq X \times X$) is an equivalence relation if:

  i) $(x, x) \in R \; \forall x \in X$ (**reflexivity**)
  ii) $(x, y) \in R \implies (y, x) \in R$ (**simmetry**)
  iii) $(x, y) \in R$, $(y, z) \in R \implies (x, z) \in R$ (**transitivity**)

  ### Equivalence class

  we define an equivalence class for x w.r.t. R as:

  $$E_x := \{y \in X : yRx\}$$

  i.e. the set of all elements equivalent to x for R

- **Quotient set**

  we define the quotient set of X over R as:

  $$X/R \coloneqq \{E_x : x \in X\}$$

  i.e. it is the set of all equivalence classes.

Question 1.4

Write the definition of equipotent sets. Write the definition of cardinality of a set.

Solution

Let us define the following:

- **Equipotent sets**

  Two sets X and Y are called equipotent if there exists a bijections, that is, a function:

  $$f : X \to Y$$

  that is both injective and surjective.

- **Cardinality of a set**

  the cardinality of a set X is the collection of all sets equipotent to X.

Question 1.5

Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

Solution

Let us define the following:

- **Finite sets**

  a set X is finite if $\exists n \in \mathbb{N}$ such that there is a bijection:

  $$f : X \to 1, \ldots, n$$

  Example: $\{\frac{1}{1}, \ldots, \frac{1}{n}\}$

- **Infinite sets**

  X is infinite if it is not finite.
  Example: $\mathbb{N}$ is clearly infinite

- **Countable sets**

  X is countable if X is equipotent to $\mathbb{N}$
  Example: $\mathbb{Q}$ can be put in bijection with $\mathbb{N}$

- **Uncountable sets**

  X is uncountable if it is infinite and not countable.
  Example: $\mathbb{R}$ is clearly infinite and not countable since it has the cardinality of continuum.

Question 1.6

Write the definitions of: algebra, $\sigma-$ algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma-$ algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

Solution

Let us define the following:

- **Algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

- **$\sigma-$ algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma-$ algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^{\mathsf{c}} \in \mathcal{A}$
  iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- **Measurable space**

  The couplet $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma-$ algebra is called a measurable space.

- **Measurable set**

  the elements of the $\sigma-$ algebra of a measurable space are called measurable sets.

Question 1.7

State the theorem concerning the existence of the $\sigma-$ algebra generated by a given set. Give an idea of the proof.

Solution

Minimal $\sigma-$algebra

Let $S \subseteq \mathcal{P}(X)$, then there exists a $\sigma-$algebra $\sigma_0(S)$ such that:

1. $S \subseteq \sigma_0(S)$
2. $\forall \sigma-$algebra$\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

thus $\sigma_0(S)$ is the minimal $\sigma-$algebra generated by S.

Sketch of Proof

We construct the set:

$$\mathcal{V} := \{\mathcal{A} \subseteq \mathcal{P}(X) \,\|\mathcal{A} \supseteq S,\ \mathcal{A} \quad \sigma-\text{algebra}\}$$

we may define:

$$\sigma_0(S) := \bigcap \{\mathcal{A} :\ \mathcal{A} \in \mathcal{V}\}$$

Question 1.8

Write the definition of the Borel $\sigma-$algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$.

Solution

Borel $\sigma-$algebra

Let (X, d) be a metric space and let $\mathcal{G}$ be the family of open sets of X, then we define the Borel $\sigma-$algebra as:

$$\mathcal{B}(X) := \sigma_0(\mathcal{G})$$

The elements of $\mathcal{G}$ are called Borel sets, let us enumerate some classes of them:

Classes of Borel sets

i) open sets

ii) closed sets (they are the complementary of open sets and this is a $\sigma-$algebra)

iii) countable intersections of open sets, known as the family G_δ

iv) countable union of closed sets, known as the family F_δ.

Lastly, let us characterize the Borel $\sigma-$algebras $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$:

Characterization of $\mathcal{B}(\mathbb{R}), \mathcal{B}\left(\overline{\mathbb{R}}\right)$ and $\mathcal{B}\left(\mathbb{R}^N\right)$

1. $\mathcal{B}(\mathbb{R}) = \sigma_0(I) = \sigma_0(I_1) = \sigma_0(I_2) = \sigma_0(I_0) = \sigma_0(\hat{I})$
   where:
$$\begin{aligned}
I &= \{(a,b) : a,b \in \mathbb{R}, a \leq b\} \\
I_1 &= \{[a,b] : a,b \in \mathbb{R}, a \leq b\} \\
I_2 &= \{(a,b] : a,b \in \mathbb{R}, a \leq b\} \\
I_0 &= \{(a,b) : -\infty \leq a < b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\} \\
\hat{I} &= \{(a,\infty) : a \in \mathbb{R}\}
\end{aligned}$$
2. $\mathcal{B}\left(\overline{\mathbb{R}}\right) = \sigma_0(\tilde{I}) = \sigma_0(\tilde{I}_1)$
   where:
$$\begin{aligned}
\tilde{I} &= \{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a, +\infty] : a \in \mathbb{R}\} \\
\tilde{I}_1 &= \{(a, +\infty] : a \in \mathbb{R}\}
\end{aligned}$$
3. $\mathcal{B}\left(\mathbb{R}^N\right) = \sigma_0(K_1) = \sigma_0(K_2)$
   where:
$$\begin{aligned}
K_1 &= \{\text{n-dimensional closed rectangles}\} \\
K_2 &= \{\text{n-dimensional open rectangles }\}
\end{aligned}$$

Question 1.9

Write the definitions of: measure, finite measure, $\sigma-$finite measure, measure space, probability space. Provide some examples of measures.

Solution

Let us define the following:

- ### Measure

  Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$, then a function μ:
$$\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$$
  is a measure if:

  1. $\mu(\emptyset) = 0$
  2. $\sigma-$ **additivity:**
     $\forall \{E_n\} \subseteq \mathcal{C}$ disjoint $(E_i \cap E_j \quad \forall i \neq j)$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ we have that:
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

  ### Finite measure

  a measure μ defined as above is said to bw finite if:
$$\mu(X) < +\infty$$

- **$\sigma -$ finite measure**

  a measure μ is said to be $\sigma -$ finite if there exists a sequence $\{E_n\}$ such that:

$$X = \bigcup_{k=1}^{\infty} E_k, \quad \mu(E_k) < +\infty$$

- **Measure space**

  Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma -$ algebra and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure, then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

- **Probability space**

  if $\mu(X) = 1$ then we say that $(X, \mathcal{A}, \mu)$ is a probability space.

Question 1.10

State and prove the theorem regarding properties of measures. Why the two continuity properties are called in this way? For what concerns continuity w.r.t. a descending sequence E_k, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

Solution

Properties of measures

Let us state and prove the properties of a measure μ on a set X and $\sigma -$ algebra $\mathcal{A}$:

i) **Additivity:**
$\forall \{E_1, \dots, E_n\} \subseteq \mathcal{A}$ disjoint we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. indeed if we define a sequence such that:

$$\{E_n\} = \begin{cases} B_k = E_k & \forall k \leq n \\ B_k = \emptyset & \forall k > n \end{cases}$$

this sequence is also disjoint ($\mathcal{A} \cap \emptyset = \emptyset \ \forall \mathcal{A} \in X$), thus we may write:

$$\underbrace{\mu\left(\bigcup_{k=1}^{\infty} E_k\right)}_{=\bigcup_{k=1}^{n} E_k \cup \emptyset} = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \sum_{k=n+1}^{\infty} \underbrace{\cancel{\mu(E_k)}}_{=0}$$

□

ii) **Monotonicity:**
$\forall E, F \in \mathcal{A}$ we have:

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

Proof. We may write F in the following way:

$$F = E \cup (F \setminus E)$$

and since these two sets are obviously disjoint we may use (i) to write:

$$\mu(F) = \mu(E) + \underbrace{\mu(E \setminus F)}_{\geq 0} > \mu(E)$$

$\square$

iii) $\sigma -$ **subadditivity:**
$\forall \{E_n\} \subseteq \mathcal{A}$ (**not** disjoint) we have:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. Let us define:

$$\begin{cases} F_1 \coloneqq E_1 \\ F_n \coloneqq E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad \forall n > 1 \end{cases}$$

Clearly $\{F_n\} \subseteq \mathcal{A}$ and $\{F_n\}$ is a disjoint sequence and:

$$\begin{array}{l} F_k \subseteq E_k \quad \forall k \in \mathbb{N} \implies \mu(F_k) \leq \mu(E_k) \text{ by (ii)} \\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{array}$$

thus we may write:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

$\square$

iv) **Continuity from below:**
$\forall \{E_n\} \subseteq \mathcal{A},\ E_k \nearrow$ we have:

$$\mu\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$$

Proof. Let us define a new sequence $\{F_n\}$ as:

$$\begin{cases} F_k \coloneqq E_k \setminus E_{k-1} \quad \forall k \in \mathbb{N} \text{ and } E_0 \coloneqq \emptyset \\ \implies \bigcup_{k=1}^{n} F_k = E_n,\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

and since $\{F_n\}$ is a disjoint sequence (we may visually think of it as a set of ever increasing rings) we may use (i) to write:

$$\begin{aligned} \mu\left(\lim_{n\to\infty} E_n\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

$\square$

v) **Continuity from above:**
$\forall \{E_n\} \subseteq \mathcal{A}$, $E_k \searrow$, $\mu(E_1) < +\infty$ we have:

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof. Like we did above Let us define: a new sequence $\{F_n\}$

$$F_k := E_1 \setminus E_k \quad \forall k \in \mathbb{N}$$

let us note that $\{F_n\}$ is an increasing sequence thus by (iv) we can write:

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty} (E_k)$$

because by (ii)
$mu(F \setminus E) = \mu(F) - \mu(E)$, moreover:

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (E_1 \cap E_k^{\mathsf{c}}) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^{\mathsf{c}}\right) = E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

thus combining these two and canceling the $\mu(E_1)$ on both sides we obtain:

$$\lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

$\square$

let us note that for this last, crucial, step $\mu(E_1)$ must be finite, otherwise we would not be able to cancel it out from both sides.

Question 1.11

Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

Solution

Let us define the following:

- ### Sets of zero measure

  Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $E \subseteq X$ has zero measure if $E \in \mathcal{A}$ and $\mu(E) = 0$. We denote the set of all sets of zero measure by $\mathcal{N}_\mu$

- ### Negligible sets

  a set $E \subseteq X$ is negligible if:

  $$\exists N \in \mathcal{A} \text{ s.t. } E \subseteq N,\ \mu(N) = 0$$

  So any subset of a set of zero measure is negligible, we denote the collection of all negligible sets by τ_μ. Moreover let us note that E doesn't need to be an element of $\mathcal{A}$ ($E \notin \mathcal{A}$)

- **Almost Everywhere**

  a property P on X is said to hold almost everywhere if:

  $$\mu(\{x \in X : P(x) \text{ is false }\}) = 0$$

  We may also say that $\{x \in X : P(x) \text{ is false }\} \in \mathcal{N}_\mu$

  ### Examples

  typical properties that can be true a.e. are: equality, continuity, monotonicity, etc. etc.

Question 1.12

Write the definition of complete measure space. Exhibit an example of a measure space which is not complete.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Counterexample

Let $X = \{a, b, c\}$, $\mathcal{A} = \sigma(\{\emptyset, \{a\}, \{b, c\}, X\})$ and $\mu \equiv 0$, clearly here we have:

$$\tau_\mu \setminus \mathcal{N}_\mu = \{\{b\}, \{c\}\}$$

and clearly $\{b\}, \{c\} \notin \mathcal{A}$. So this measure space is not complete.