Answers to the Theory Questions

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The source code for this document can be found at:
https://github.com/jstringara/Latex-projects/tree/master/ARF

Contents

Sheet n. 1

Question 1.1

Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to\infty} E_n$, $\liminf_{n\to\infty} E_n$, $\lim_{n\to\infty} E_n$.

Solution

Let us define the following:

- **Sequence of sets**

  A family (or collection) of sets $\{E_i\}_{i\in I}$ is called a sequence of sets if $I = \mathbb{N}$ (i.e. it is indexed by the set of natural numbers $\mathbb{N}$)

- **Increasing sequence of sets**

  a sequence of sets $\{E_n\}$ is said to be increasing (or ascending) if:

$$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Decreasing sequence of sets**

  A sequence of sets $\{E_n\}$ is said to be decreasing (or descending) if:

$$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

- **Limsup for a sequence of sets**

  for a sequence of sets $\{E_n\}$ we define:

$$\limsup_{n\to\infty} E_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

- **Liminf for a sequence of sets**

  analogously:

$$\limsup_{n\to\infty} E_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

- **Limit for a sequence of sets**

  as for a sequence of real numbers if the limsup and liminf coincide we may define:

$$\lim_{n\to\infty} E_n := \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n$$

Question 1.2

Write the definitions of: cover (or covering) of a set; subcover.

Solution

Let us define the following:

- **Cover of a set**

  a family of sets $\{E_i\}_{i\in I}$ is called a cover (or covering) of X if:

$$X \subseteq \bigcup_{i\in I} E_i$$

- **Subcover**

  a sub-family of a cover $\{E_i\}_{i\in J}$ $(J \subseteq I)$ which forms a cover is called a subcover.

Question 1.3

Write the definitions of: equivalence relation, equivalence class, quotient set.

Solution

Let us define the following:

- **Equivalence relation**

  a relation R in X (i.e. a subset $R \subseteq X \times X$) is an equivalence relation if:

  i) $(x, x) \in R \; \forall x \in X$ (**reflexivity**)
  ii) $(x, y) \in R \implies (y, x) \in R$ (**simmetry**)
  iii) $(x, y) \in R$, $(y, z) \in R \implies (x, z) \in R$ (**transitivity**)

Equivalence class

we define an equivalence class for x w.r.t. R as:

$$E_x := \{y \in X : yRx\}$$

i.e. the set of all elements equivalent to x for R

- **Quotient set**

  we define the quotient set of X over R as:

  $$X/R \coloneqq \{E_x : x \in X\}$$

  i.e. it is the set of all equivalence classes.

Question 1.4

Write the definition of equipotent sets. Write the definition of cardinality of a set.

Solution

Let us define the following:

- **Equipotent sets**

  Two sets X and Y are called equipotent if there exists a bijections, that is, a function:

  $$f : X \to Y$$

  that is both injective and surjective.

- **Cardinality of a set**

  the cardinality of a set X is the collection of all sets equipotent to X.

Question 1.5

Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

Solution

Let us define the following:

- **Finite sets**

  a set X is finite if $\exists n \in \mathbb{N}$ such that there is a bijection:

  $$f : X \to 1, \dots, n$$

  Example: $\{\frac{1}{1}, \dots, \frac{1}{n}\}$

- **Infinite sets**

  X is infinite if it is not finite.
  Example: $\mathbb{N}$ is clearly infinite

- **Countable sets**

  X is countable if X is equipotent to $\mathbb{N}$
  Example: $\mathbb{Q}$ can be put in bijection with $\mathbb{N}$

- **Uncountable sets**

  X is uncountable if it is infinite and not countable.
  Example: $\mathbb{R}$ is clearly infinite and not countable since it has the cardinality of continuum.

Question 1.6

Write the definitions of: algebra, $\sigma-$algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma-$algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

Solution

Let us define the following:

- **Algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^c \in \mathcal{A}$
  iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

- **$\sigma -$ algebra**

  A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma -$ algebra if:

  i) $\emptyset \in \mathcal{A}$
  ii) $E \in \mathcal{A} \implies E^c \in \mathcal{A}$
  iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- **Measurable space**

  The couplet $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma -$ algebra is called a measurable space.

- **Measurable set**

  the elements of the $\sigma -$ algebra of a measurable space are called measurable sets.

Question 1.7

State the theorem concerning the existence of the $\sigma -$ algebra generated by a given set. Give an idea of the proof.

Solution

Minimal $\sigma -$ algebra

Let $S \subseteq \mathcal{P}(X)$, then there exists a $\sigma -$ algebra $\sigma_0(S)$ such that:

1. $S \subseteq \sigma_0(S)$
2. $\forall \sigma -$ algebra$\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

thus $\sigma_0(S)$ is the minimal $\sigma -$ algebra generated by S.

Sketch of Proof

We construct the set:

$$\mathcal{V} := \{\mathcal{A} \subseteq \mathcal{P}(X) \,\|\, \mathcal{A} \supseteq S,\ \mathcal{A} \quad \sigma - \text{algebra}\}$$

we may define:

$$\sigma_0(S) := \bigcap \{\mathcal{A} :\ \mathcal{A} \in \mathcal{V}\}$$

Question 1.8

Write the definition of the Borel $\sigma -$ algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}\left(\overline{\mathbb{R}}\right)$ and $\mathcal{B}\left(\mathbb{R}^N\right)$.

Solution

Borel $\sigma -$ algebra

Let (X, d) be a metric space and let $\mathcal{G}$ be the family of open sets of X, then we define the Borel $\sigma -$ algebra as:

$$\mathcal{B}(X) := \sigma_0(\mathcal{G})$$

The elements of $\mathcal{G}$ are called Borel sets, let us enumerate some classes of them:

Classes of Borel sets

i) open sets

ii) closed sets (they are the complementary of open sets and this is a $\sigma -$ algebra)

iii) countable intersections of open sets, known as the family G_δ

iv) countable union of closed sets, known as the family F_δ.

Lastly, let us characterize the Borel $\sigma -$ algebras $\mathcal{B}(\mathbb{R}), \mathcal{B}\left(\overline{\mathbb{R}}\right)$ and $\mathcal{B}\left(\mathbb{R}^N\right)$:

Characterization of $\mathcal{B}(\mathbb{R}), \mathcal{B}\left(\overline{\mathbb{R}}\right)$ and $\mathcal{B}\left(\mathbb{R}^N\right)$

1. $\mathcal{B}(\mathbb{R}) = \sigma_0(I) = \sigma_0(I_1) = \sigma_0(I_2) = \sigma_0(I_0) = \sigma_0(\hat{I})$
   where:

$$\begin{aligned} I &= \{(a, b) : a, b \in \mathbb{R}, a \leq b\} \\ I_1 &= \{[a, b] : a, b \in \mathbb{R}, a \leq b\} \\ I_2 &= \{(a, b] : a, b \in \mathbb{R}, a \leq b\} \\ I_0 &= \{(a, b) : -\infty \leq a < b < \infty\} \cup \{(a, \infty) : a \in \mathbb{R}\} \\ \hat{I} &= \{(a, \infty) : a \in \mathbb{R}\} \end{aligned}$$

2. $\mathcal{B}\left(\overline{\mathbb{R}}\right) = \sigma_0(\tilde{I}) = \sigma_0(\tilde{I}_1)$
   where:

$$\begin{aligned} \tilde{I} &= \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a, +\infty] : a \in \mathbb{R}\} \\ \tilde{I}_1 &= \{(a, +\infty] : a \in \mathbb{R}\} \end{aligned}$$

3. $\mathcal{B}\left(\mathbb{R}^N\right) = \sigma_0(K_1) = \sigma_0(K_2)$
   where:

$$\begin{aligned} K_1 &= \{\text{n-dimensional closed rectangles}\} \\ K_2 &= \{\text{n-dimensional open rectangles }\} \end{aligned}$$

Question 1.9

Write the definitions of: measure, finite measure, $\sigma -$ finite measure, measure space, probability space. Provide some examples of measures.

Solution

Let us define the following:

- **Measure**

  Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$, then a function μ:

  $$\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$$

  is a measure if:

  1. $\mu(\emptyset) = 0$
  2. $\sigma -$ **additivity:**
     $\forall \{E_n\} \subseteq \mathcal{C}$ disjoint $(E_i \cap E_j \quad \forall i \neq j)$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ we have that:

  $$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

  ### Finite measure

  a measure μ defined as above is said to bw finite if:

  $$\mu(X) < +\infty$$

- **$\sigma -$ finite measure**

  a measure μ is said to be $\sigma -$ finite if there exists a sequence $\{E_n\}$ such that:

  $$X = \bigcup_{k=1}^{\infty} E_k, \quad \mu(E_k) < +\infty$$

- **Measure space**

  Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a $\sigma -$ algebra and $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure, then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

- **Probability space**

  if $\mu(X) = 1$ then we say that $(X, \mathcal{A}, \mu)$ is a probability space.

Question 1.10

State and prove the theorem regarding properties of measures. Why the two continuity properties are called in this way? For what concerns continuity w.r.t. a descending sequence E_k, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

Solution

Properties of measures

Let us state and prove the properties of a measure μ on a set X and $\sigma-$ algebra $\mathcal{A}$:

i) **Additivity:**
$\forall\{E_1,\ldots,E_n\}\subseteq\mathcal{A}$ disjoint we have:

$$\mu\left(\bigcup_{k=1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}\mu(E_k)$$

Proof. indeed if we define a sequence such that:

$$\{E_n\}=\begin{cases}B_k=E_k & \forall k\le n\\ B_k=\emptyset & \forall k>n\end{cases}$$

this sequence is also disjoint ($\mathcal{A}\cap\emptyset=\emptyset\ \forall\mathcal{A}\in X$), thus we may write:

$$\underbrace{\mu\left(\bigcup_{k=1}^{\infty}E_k\right)}_{=\bigcup_{k=1}^{n}E_k\cup\emptyset}=\sum_{k=1}^{\infty}\mu(E_k)=\sum_{k=1}^{n}\mu(E_k)+\sum_{k=n+1}^{\infty}\underbrace{\cancel{\mu(E_k)}}_{=0}$$

□

ii) **Monotonicity:**
$\forall E,F\in\mathcal{A}$ we have:

$$E\subseteq F\implies\mu(E)\le\mu(F)$$

Proof. We may write F in the following way:

$$F=E\cup(F\setminus E)$$

and since these two sets are obviously disjoint we may use (i) to write:

$$\mu(F)=\mu(E)+\underbrace{\mu(E\setminus F)}_{\ge0}>\mu(E)$$

□

iii) $\sigma-$ **subadditivity:**
$\forall\{E_n\}\subseteq\mathcal{A}$ (**not** disjoint) we have:

$$\mu\left(\bigcup_{k=1}^{\infty}E_k\right)\le\sum_{k=1}^{\infty}\mu(E_k)$$

Proof. Let us define:

$$\begin{cases}F_1:=E_1\\ F_n:=E_n\setminus\bigcup_{k=1}^{n-1}E_k & \forall n>1\end{cases}$$

Clearly $\{F_n\}\subseteq\mathcal{A}$ and $\{F_n\}$ is a disjoint sequence and:

$$F_k\subseteq E_k\quad\forall k\in\mathbb{N}\implies\mu(F_k)\le\mu(E_k)\text{ by (ii)}$$
$$\bigcup_{k=1}^{\infty}F_k=\bigcup_{k=1}^{\infty}E_k$$

thus we may write:

$$\mu\left(\bigcup_{k=1}^{\infty}E_k\right)=\mu\left(\bigcup_{k=1}^{\infty}F_k\right)=\sum_{k=1}^{\infty}\mu(F_k)\le\sum_{k=1}^{\infty}\mu(E_k)$$

□

iv) **Continuity from below:**
$\forall\{E_n\}\subseteq\mathcal{A},\ E_k\nearrow$ we have:

$$\mu\left(\lim_{k\to\infty}E_k\right)=\lim_{k\to\infty}\mu(E_k)$$

Proof. Let us define a new sequence $\{F_n\}$ as:

$$\begin{cases} F_k := E_k \setminus E_{k-1} \quad \forall k \in \mathbb{N} \text{ and } E_0 := \emptyset \\ \implies \bigcup_{k=1}^{n} F_k = E_n,\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

and since $\{F_n\}$ is a disjoint sequence (we may visually think of it as a set of ever increasing rings) we may use (i) to write:

$$\begin{aligned} \mu\left(\lim_{n\to\infty} E_n\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\ &= \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

□

v) **Continuity from above:**
$\forall \{E_n\} \subseteq \mathcal{A}$, $E_k \searrow$, $\mu(E_1) < +\infty$ we have:

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof. Like we did above Let us define: a new sequence $\{F_n\}$

$$F_k := E_1 \setminus E_k \quad \forall k \in \mathbb{N}$$

let us note that $\{F_n\}$ is an increasing sequence thus by (iv) we can write:

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty} (E_k)$$

because by (ii)
$mu(F \setminus E) = \mu(F) - \mu(E)$, moreover:

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (E_1 \cap E_k^{\mathsf{c}}) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^{\mathsf{c}}\right) = E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

thus combining these two and canceling the $\mu(E_1)$ on both sides we obtain:

$$\lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

□

let us note that for this last, crucial, step $\mu(E_1)$ must be finite, otherwise we would not be able to cancel it out from both sides.

Question 1.11

Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

Solution

Let us define the following:

- **Sets of zero measure**

  Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $E \subseteq X$ has zero measure if $E \in \mathcal{A}$ and $\mu(E) = 0$. We denote the set of all sets of zero measure by $\mathcal{N}_\mu$

- **Negligible sets**

  a set $E \subseteq X$ is negligible if:
  $$\exists N \in \mathcal{A} \text{ s.t. } E \subseteq N,\ \mu(N) = 0$$
  So any subset of a set of zero measure is negligible, we denote the collection of all negligible sets by τ_μ. Moreover let us note that E doesn't need to be an element of $\mathcal{A}$ ($E \notin \mathcal{A}$)

- **Almost Everywhere**

  a property P on X is said to hold almost everywhere if:
  $$\mu(\{x \in X : P(x) \text{ is false }\}) = 0$$
  We may also say that $\{x \in X : P(x) \text{ is false }\} \in \mathcal{N}_\mu$

  ### Examples

  typical properties that can be true a.e. are: equality, continuity, monotonicity, etc. etc.

Question 1.12

Write the definition of complete measure space. Exhibit an example of a measure space which is not complete.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Counterexample

Let $X = \{a, b, c\}$, $\mathcal{A} = \sigma(\{\emptyset, \{a\}, \{b, c\}, X\})$ and $\mu \equiv 0$, clearly here we have:
$$\tau_\mu \setminus \mathcal{N}_\mu = \{\{b\}, \{c\}\}$$
and clearly $\{b\}, \{c\} \notin \mathcal{A}$. So this measure space is not complete.

Sheet n. 2

Question 2.1

Write the definition of complete measure space. State the theorem concerning the existence of the completion of a measure space. Give just an idea of the proof.

Solution

Complete measure space

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\tau_\mu \subseteq \mathcal{A}$

Existence of the completion

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let us define: $\bar{\mathcal{A}}, \bar{\mu}$

$$\bar{\mathcal{A}} = \{E \subseteq X : \exists F, G \in \mathcal{A} \text{ s.t. } F \subseteq E \subseteq G \; \mu(G \setminus F) = 0\}$$
$$\bar{\mu} : \bar{\mathcal{A}} \to \overline{\mathbb{R}}_+, \quad \bar{\mu}(E) := \mu(F)$$

then:

1. $\bar{\mathcal{A}}$ is a $\sigma -$ algebra , $\bar{\mathcal{A}} \supseteq \mathcal{A}$
2. $\bar{\mu}$ is a complete measure, $\bar{\mu}|_{\mathcal{A}} = \mu$

and the triplet $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space and is called the completion of $(X, \mathcal{A}, \mu)$, i.e. it the smallest (w.r. to inclusion) complete measure space that cointains $(X, \mathcal{A}, \mu)$

Sketch of proof

[1] We must prove two things:

- **First:** that $\bar{\mathcal{A}}$ is a $\sigma -$ algebra and that it contains $\mathcal{A}$, the latter is trivial since $\forall A \in \mathcal{A} \quad A \subseteq A \subseteq A \implies A \in \bar{\mathcal{A}}$ while the former is quite hardous so we shall just assume it to be true.
- **Second:** that $\bar{\mu}$ is a complete measure and $\bar{\mu}|_{\mathcal{A}} = \mu$.
  The latter is trivial (see above). We can also easily prove that it is a measure:
  i) $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$ since the only set contained inside $\emptyset$ is $\emptyset$ itself, as the container set we may take any zero set measure inside $\mathcal{A}$.
  ii) that $\sigma -$ additivity holds is clear since for any disjoint sequence $\{E_n\} \subseteq \bar{\mathcal{A}}$ we may construct two sequences:
  $$\begin{cases} \{F_n\}, \; F_k \subseteq E_k \\ \{G_n\}, \; G_k \supseteq E_k \end{cases} \forall k \in \mathbb{N} \text{ s.t. } \mu(G_k \setminus F_k) = 0$$
  Let us note the following:
  - $\{F_n\}$ is also disjoint because $\{E_n\}$ is disjoint.

[1] This is a partial proof of my own making. It has been review by the TA and professor Punzo and stated to be correct.

– Moreover:

$$\bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} G_k$$
$$\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus F_k)$$
$$\mu\left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} F_k\right) \leq \mu\left(\bigcup_{k=1}^{\infty} (G_k \setminus F_k)\right) \leq \sum_{k=1}^{\infty} \mu(G_k \setminus F_k) = 0$$

The last inequality is true thanks to the $\sigma-$subadditivity and monotonicty of μ.

Thus we can say that:

$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \bar{\mu}(E_k)$$

thus $\bar{\mu}$ is a measure.
Let us prove that $\bar{\mu}$ is complete. Let $E_1 \in X$ and $E_2 \in \bar{\mathcal{A}}$ such that $\bar{\mu}(E_2) = \mu(F_2) = 0$ and $E_1 \subseteq E_2$, let us note that:

$$\begin{cases} \mu(G_2) = \overset{0}{\cancel{\mu(G_2 \setminus F_2)}} + \overset{0}{\cancel{\mu(F_2)}} \\ \mu(G_2 \setminus \emptyset) = \mu(G_2) - 0 \\ \emptyset \subseteq E_1 \subseteq G_2 \end{cases} \implies E_1 \in \bar{\mathcal{A}},\ \bar{\mu}(E_1) = \mu(\emptyset) = 0$$

thus any negligible set is also a zero measure set and $\bar{\mu}$ is complete.

Question 2.2

Write the definition of outer measure. State and prove the theorem concerning generation of outer measure on a general set X, starting from a set $K \in \mathcal{P}(X)$, containing $\emptyset$, and a function $\nu : K \to \mathbb{R}_+,\ \nu(\emptyset) = 0$. Intuitively, which is the meaning of (K, ν)?

Solution

Outer measure

We say that a function: $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ (where X is any set) is an outer measure if:

i) $\mu^*(\emptyset) = 0$

ii) $E_1 \subseteq E_2 \implies \mu^*(E_2) \leq \mu^*(E_2)$

iii) $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$

Generation of an outer measure

Let $K \subseteq \mathcal{P}(X),\ \emptyset \in K,\ \nu : K \to \overline{\mathbb{R}}_+,\ \nu(\emptyset) = 0$, then we can generate an outer measure μ^* on X defined as:

$$\begin{cases} \mu^*(E) := \inf\left\{\sum_{k=1}^{\infty} \nu(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k,\ \{I_n\} \subseteq K\right\}, \text{ if } E \text{ can be covered by a countable union of sets } I_n \in K. \\ \mu^*(E) := +\infty, \text{ otherwise.} \end{cases}$$

Proof. Let us verify that such a μ^* meets the definition of outer measure (2.2.1):

i) $\emptyset \in K$, $0 \leq \mu^*(\emptyset) \leq \nu(\emptyset) = 0$ by the definition of μ^*.

ii) $E_1 \subseteq E_2$, we have two possible cases

- if there exists a countable covering of E_2 then it is also a covering of E_1 and from the definitio of μ^* it follows that:

$$\mu^*(E_1) \leq \mu^*(E_2)$$

- if there is no countable covering of E_2 then:

$$\mu^*(E_1) \leq \mu^*(E_2) = +\infty$$

iii) this condition is obviously met if:

$$\sum_{k=1}^{\infty} \mu^*(E_k) = +\infty$$

otherwise if we suppose that:

$$\sum_{k=1}^{\infty} \mu^*(E_k) < +\infty$$

thus $\mu^*(E_k) < +\infty \ \forall k \in \mathbb{N}$, by the definition of μ^* and inf:

$$\forall \varepsilon > 0,\ \forall n \in \mathbb{N} \quad \exists \{I_{n,k}\} \subseteq K$$

such that:

$$E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n,k})$$

Now, since:

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}, \quad \{I_{n,k}\} \subseteq K$$

it clearly follows that:

$$\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \nu(I_{n,k}) < \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon \cdot \overset{1}{\cancel{\sum_{n=1}^{\infty} \frac{1}{2^n}}}$$

because ε is arbitrary, we have the cocnlusion.

□

The intuitive meaning (K, ν) is that K is a special class of sets in X and ν is a function that assigns a value to each set in K. On the other hand ν can be any real valued positive function, thus it is not necessary to be a measure.

Question 2.3

What is the Caratheodory condition? How can it be stated in an equivalent way? Prove it.

Solution

Caratheodory condition

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

Equivalent statement

Let μ^* be an outer measure on a set X, then we say that $E \subset X$ is μ^*-measurable if:

$$\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

Proof. It is enough to note that $\forall E \subseteq X$ we have:

$$Z = (Z \cap E) \cup (Z \cap E^{\mathsf{c}}) \quad \forall Z \in X$$

and thus by the subadditivity of μ^* (iii) we get:

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \setminus E) \quad \forall Z \in X$$

and we may combine this inequality with the other to yield an equality. □

Question 2.4

Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

Solution

All zero measure sets are in $\mathcal{L}$

There cannot exist such a set E because all sets of zero aouter measure meet the Caratheodory Inequality (2.3.2).

Proof. Indeed $\forall Z \subseteq X$ by the monotonicty of μ^* (ii) we have:

$$\mu^*(\underbrace{Z \cap E}_{\subseteq E}) + \mu^*(\underbrace{Z \setminus E}_{\subseteq Z}) \leq \overset{0}{\cancel{\mu^*(E)}} + \mu^*(Z)$$

$\square$

Question 2.5

State the theorem concerning generation of a measure as a restriction of an outer measure.

Solution

Generation of a measure from an outer measure

Let us define $\mathcal{L}$ as:

$$\mathcal{L} := \{E \subseteq X : \ E \text{ is } \mu^* - \text{measurable } \}$$

where μ^* is an outer measure on X, then:

i) the collection $\mathcal{L}$ is a σ − algebra

ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

Question 2.6

Show that the measure induced by an outer measure on the σ − algebra of all sets fulfilling the Caratheodory condition is complete.

Solution

Generation of a measure from an outer measure (proof of completeness)

Let us see that such a measure as the one described in the previous question is complete. Let μ^* be an outer measure on X and $\mathcal{L}$ the σ − algebra of all sets fulfilling the Caratheodory condition. Let μ be the measure induced by μ^* on $\mathcal{L}$ ($\mu = \mu^*|_{\mathcal{L}}$).

Proof. Let $N \in \mathcal{L}$ such that $\mu(N) = \mu^*(N) = 0$ and let $E \subseteq N$.
By monotonicty of μ^* (ii):

$$0 \leq \mu^*(E) \leq \mu^*(N) = 0 \implies \mu^*(E) = 0$$

thus by the lemma seen in 2.4.1 we get that $E \in \mathcal{L}$ and so $\mathcal{L}$ is complete. $\square$

Question 2.7

Describe the construction of the Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^n$.

Solution

Construction of the Lebesgue measure on $\mathbb{R}$

Let I be a family of open, bounded intervals in $\mathbb{R}$:

$$I \coloneqq \{(a,b) : a, b \in \mathbb{R}, a \leq b\}$$

Let us note that $\emptyset \in I$.
Now let us consider a function λ_0:

$$\begin{aligned}
&\lambda_0 : I \to \mathbb{R}_+ \\
&\lambda_0(\emptyset) = 0 \\
&\lambda_0((a,b)) = b - a
\end{aligned}$$

Here we take $X = \mathbb{R}$, $(K, \nu) = (I, \lambda_0)$ and construct the outer Lebesgue measure λ^* as seen above (2.2.2):

$$\lambda^*(E) \coloneqq \begin{cases} \inf\left\{\sum_{n=1}^{\infty} \lambda_0(I_n) \ : \quad E \subseteq \bigcup_{n=1}^{\infty} I_n, \ \{I_n\} \subseteq I\right\}, & \forall E \subseteq \mathbb{R} \text{ s.t. } E \text{ has a countable covering } \{I_n\} \subseteq I \\ +\infty, \ \text{otherwise} \end{cases}$$

The corresponding $\sigma-$algebra is the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R})$ and now we define the Lebesgue measure λ as the measure generated by the outer Lebesgue measure (as seen in 2.5.1):

$$\lambda \coloneqq \lambda^*|_{\mathcal{L}(\mathbb{R})}$$

Construction of the Lebesgue measure on $\mathbb{R}^n$

Analogously to what we have seen above we first define an outer measure and then a (complete) measure but we take:

$$I^n = \left\{ \bigtimes_{k=1}^{n} (a_k, b_k) : \ a_k, b_k \in \mathbb{R}, \ a_k \leq b_k \right\}$$

and accordingly we define:

$$\begin{aligned}
&\lambda_0^n : I^n \to \mathbb{R}_+ \\
&\lambda_0^n(\emptyset) = 0 \\
&\lambda_0^n\left(\bigtimes_{k=1}^{n} (a_k, b_k)\right) = \prod_{k=1}^{n} (b_k - a_k)
\end{aligned}$$

and therefore we take $X = \mathbb{R}^n$ and $(K, \nu) = (I^n, \lambda_0^n)$, we define the outer Lebesgue measure $\lambda^{*,n}$ on $\mathbb{R}^n$ and the Lebesgue $\sigma-$algebra $\mathcal{L}(\mathbb{R}^n)$ and finally we construct the n-dimensional Lebesgue measure as:

$$\lambda^n \coloneqq \lambda^{*,n}|_{\mathcal{L}(\mathbb{R}^n)}$$

Question 2.8

Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

Solution

All countable sets are $\mathcal{L}$-measurable and $\lambda(E) = 0$

Any countable subset $E \subset \mathbb{R}$ is $\mathcal{L}$-measurable and $\lambda(E) = 0$

Proof. Let $a \in \mathbb{R}$, clearly $\{a\} \subseteq (a-\varepsilon, a]$ $\forall \varepsilon > 0$, thus by the definition of λ^*:

$$\lambda^*(\{a\}) \leq \lambda^*((a-\varepsilon, a]) = \varepsilon \to 0 \implies \{a\} \in \mathcal{L}$$

Now if E is countable we may write as follows:

$$E = \bigcup_{n=1}^{\infty} \{a_n\} \quad a_n \in \mathbb{R},\ n \in \mathbb{N}$$

and so by monotonicty (ii):

$$0 \leq \lambda^*(E) = \lambda^* \left(\bigcup_{n=1}^{\infty} \{a_n\} \right) \leq \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$$

thus $\lambda^*(E) = 0 \implies E \in \mathcal{L}$ by the lemma seen above (2.4.1) $\square$

Question 2.9

Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. Is the inclusion strict? Which is the relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$?

Solution

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty))$ it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$. We already know from above that all bounded intervals belong to $\mathcal{L}(\mathbb{R})$.
Now, let $A \subseteq \mathbb{R}$ be any set. We assume $a \notin A$, otherwise we would replace A with $A \setminus \{a\}$ and this would leave the Lebesgue outer measure unchanged. Furthermore $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \iff (a, +\infty)$ satisfies the Caratheodory Condition (2.3.2):

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$$

where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, +\infty)$.
Since $\lambda^*(A)$ is defined as an inf, to verify the above, it is necessary and sufficient to show that for **any countable collection** $\{I_n\}$ of **open bounded** intervals that **covers** A we have that:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k)$$

For every $k \in \mathbb{N}$ we define:

$$I'_k := I_k \cap (-\infty, a)$$
$$I''_k := I_k \cap (a, +\infty)$$

then:

$$I'_k \cap I''_k = \emptyset(\text{disjoint}) \implies \lambda_o(I_k) = \lambda_0(I'_k) + \lambda_0(I''_k)$$

Let us note that $\{I'_n\}$ is a countable cover for A_1 and $\{I''_n\}$ is a countable cover for A_2. Hence:

$$\lambda^*(A_1) = \sum_{k=1}^{\infty} \lambda_0(I'_k)$$
$$\lambda^*(A_2) = \sum_{k=1}^{\infty} \lambda_0(I''_k)$$

therefore:

$$\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) + \sum_{k=1}^{\infty} \lambda_0(I''_k) = \sum_{k=1}^{\infty} \lambda_0(I_k)$$

which equivalento to the condition above. $\square$

$\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$

The inclusion demonstrated above can be shown to be strict. A counterexample can be produced (see here) but it is quite pathological.

Relation between $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$. Indeed as we have shown above $\mathcal{B}(\mathbb{R})$ is not a complete $\sigma-$algebra while $\mathcal{L}(\mathbb{R})$ is.

Question 2.10

Is the translate of a measurable set measurable?

Solution

The translate of a measurable set is measurable

The translate of a measurable set is also measurable.
Let us see a simple example: let (a, b) be an interval and $(a+h, b+h)$ its translate.

$$\begin{aligned}
&\lambda((a,b)) = b - a \\
&\lambda((a+h, b+h)) = (b+h) - (a+h) = b - a
\end{aligned}$$

Question 2.11

Write the excision property and prove it. Write and prove (partially) the theorem concerning the regularity of the Lebesgue measure on $\mathbb{R}$.

Solution

Excision property

If $A \in \mathcal{L}(\mathbb{R})$, $\lambda^*(A) \leq +\infty$ and $A \subseteq B$, then:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

Proof. Since $A \in \mathcal{L}(\mathbb{R})$ we can use the Caratheodory equality (2.3.1) using $Z = B$, $E = A$:

$$\lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A\ (A \subseteq B)}) + \lambda^*(B \setminus A)$$

so, since $\lambda^*(A) \leq +\infty$ we may write:

$$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$$

□

Regularity of the Lebesgue Measure

Let $E \subseteq \mathbb{R}$, the following are equal:

i) $E \in \mathcal{L}(\mathbb{R})$

ii) $\forall \varepsilon > 0\ \exists A \subseteq \mathbb{R}$ open s.t.

$$E \subseteq A \quad \lambda^*(A \setminus E) < \varepsilon$$

iii) $\exists G \subseteq \mathbb{R}$ in the class G_δ (countable intersections of open sets) s.t.

$$E \subseteq G \quad \lambda^*(G \setminus E) = 0$$

iv) $\forall \varepsilon > 0\ \exists C \subseteq \mathbb{R}$ closed s.t.

$$C \subseteq E \quad \lambda^*(E \setminus C) < \varepsilon$$

v) $\exists F \subseteq \mathbb{R}$ in the class F_δ (countable unions of closed sets) s.t.

$$F \subseteq E \quad \lambda^*(E \setminus F) = 0$$

Proof. Let us give a (partial) proof:

- $(i) \implies (ii)$: if $E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < +\infty$ then by definition of outer measure (2.2.1):

  $$\forall \varepsilon > 0\ \exists \{I_n\} \text{ that covers } E \text{ and } \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$

  Let us now define the set O:

  $$O := \bigcup_{k=1}^{\infty} I_k,\ O \text{ is open},\ E \subseteq O$$

  and so we may write:

  $$\begin{aligned} \lambda^*(O) &\overset{sub-add\ (iii)}{\leq} \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon \\ &\implies \lambda^*(O) - \lambda^*(E) < \varepsilon \end{aligned}$$

  and by the Excision property (2.11.1) $(E \in \mathcal{L}(\mathbb{R}),\ \lambda^*(E) < +\infty)$:

  $$\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$$

  and so we have obtained the second statement (ii).

- $(ii) \implies (iii)$, $\forall k \in \mathbb{N}$ we choose $O_k \supseteq E$ open for which:

  $$\lambda^*(O_k \setminus E) < \frac{1}{k}$$

  and then define:

  $$G = \bigcap_{k=1}^{\infty} O_k \implies G \in G_\delta,\ G \supseteq E$$

  Moreover $\forall k \in \mathbb{N}$:

  $$G \setminus E \subseteq O_k \setminus E$$

  so by monotonicty (ii):

  $$\lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$$

  let us apply a limit $k \to \infty$ to both sides:

  $$\lambda^*(G \setminus E) = 0$$

- $(iii) \implies (i)$, let us note that $G \setminus E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G \setminus E) = 0$ by lemma 2.4.1 and:

  $$\begin{aligned} &G \in \mathcal{L}(\mathbb{R}) \text{ since } G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \\ &\implies E = \underset{\in \mathcal{L}}{G} \cap \underset{\in \mathcal{L}}{(G \setminus E)^{\mathsf{c}}} \in \mathcal{L} \end{aligned}$$

□

Question 2.12

Is it true that any subset $E \subseteq \mathbb{R}$ is $\mathcal{L}$-measurable? Is it possibile to find two disjoint sets $A, B \subset \mathbb{R}$ for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Why?

Solution

Vitali's non-measurable sets

Any measurable set $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$ contains a subset that fails to be measurable.
Therefore there exist subsets of $\mathbb{R}$ that are not $\mathcal{L}$-measurable.

Disjoints sets for which $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$

There are disjoint sets $A, B \subseteq \mathbb{R}$ for which:

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$

Proof. Assume by contradiction that:

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \quad \forall A, B \subseteq \mathbb{R},\ A \cap B = \emptyset$$

Now $\forall E, Z \subseteq \mathbb{R}$ we write:

$$\lambda^*(\underbrace{Z \cap E}_{=A}) + \lambda^*(\underbrace{Z \cap E^{\mathsf{c}}}_{=B}) = \lambda^*(\underbrace{Z}_{=A \cup B})$$

thus any set E would satisfy the Caratheodory condition (2.3.1) and be $\mathcal{L}$-measurable which is absurd since we know that Vitali's sets exist. $\square$

Sheet n. 3

Question 3.1

Write the definition of measurable function. Show the measurability of the composite function.

Solution

Measurable function

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and f a function:

$$f : X \to X'$$

f is said to be measurable if:

$$f^{-1}(A) \in \mathcal{A} \quad \forall A \in \mathcal{A}'$$

Measurability of the composite function

Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ and $(X'', \mathcal{A}'')$ be three measurable spaces and $f : X \to X'$ and $g : X' \to X''$ two measurable functions. Then the composite function $g \circ f : X \to X''$ is measurable.

Proof.

$$\begin{aligned} &\forall E \in \mathcal{A}' \quad f^{-1}(E) \in \mathcal{A} \\ &\forall F \in \mathcal{A}'' \quad g^{-1}(F) \in \mathcal{A}' \end{aligned}$$

thus:

$$\forall F \in \mathcal{A}'' \quad (g \circ f)^{-1}(F) = f^{-1}\left[\underbrace{g^{-1}(F)}_{:=E \in \mathcal{A}'}\right] \in \mathcal{A}$$

□

Question 3.2

Characterize measurability of functions and prove it.

Solution

Characterization of Measurability

Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be two measurable spaces and $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$ then:

$$f : X \to X' \text{ measurable} \iff f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- **($\Longrightarrow$):** Suppose f be measurable $\Longrightarrow$ $\mathcal{C}' \subseteq \mathcal{A}'$ and so we get the thesis.
- **($\Longleftarrow$):** Let us define the following:

$$\Sigma := \{E \subseteq X' : \ f^{-1}(E) \in \mathcal{A}\}$$

We can easily see that Σ is a $\sigma -$ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

$$\mathcal{A}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

and we get the thesis.

□

Question 3.3

Write the definitions of:

a) Borel measurable functions;

b) Lebesgue measurable functions.

Solution

a) Borel measurable functions

Let $(X, d), (X, \mathcal{B})$ and $(X', d'), (X', \mathcal{B}')$ be couples of metric spaces and measurable spaces. A function f:

$$f : X \to X' \text{ measurable}$$

is called Borel-measurable or $\mathcal{B}$-meaurable.

b) Lebesgue measurable functions

Let $(X, \mathcal{L})$ be a measurable space and (X', d') a metric space, $(X', \mathcal{B}')$ a measurable space, then:

$$f : X \to X' \text{ measurable}$$

is called Lebesgue-measurable or $\mathcal{L}$-measurable.

Question 3.4

Prove that continuous functions are both Borel and Lebesgue measurable.

Solution

Continuous functions are $\mathcal{B}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{B}$-measurable.

Proof. Let $\mathcal{C}'$ be the class of open sets of X' and $\mathcal{C}$ the class of open sets of X. We have:

$$\forall E \in \mathcal{C}' \quad f^{-1}(E) \in \mathcal{C} \subseteq \mathcal{B} \text{ (by definition of continuity)}$$

and $\mathcal{B}' = \sigma_0(\mathcal{C}')$ so we get the thesis. □

Continuous functions are $\mathcal{L}$-measurable

A continuous function $f : X \to X'$ is $\mathcal{L}$-measurable.

Proof. Since $\mathcal{B} \subset \mathcal{L}$ and the previous statement has been proven true, the thesis follows trivially. □

Question 3.5

Characterize Lebesgue measurability of functions and prove it.

Solution

Characterization of Lebesgue measurability

All we must do is apply the Characterization of Measurability (3.2.1) taking $(X, \mathcal{A} = \mathcal{L})$, $(X', \mathcal{A}' = \mathcal{B}')$ and $\mathcal{C}'$ the class of open sets of X', since $\mathcal{B}' = \sigma_0(\mathcal{C}')$. We then can write:

$$f : X \to X' \text{ Lebesgue measurable} \iff f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{C}'$$

Proof. Let us prove both sides of the implication:

- **($\implies$):** Suppose f be Lebesgue measurable $\implies \mathcal{C}' \subseteq \mathcal{B}'$ and so we get the thesis.
- **($\impliedby$):** Let us define the following:

  $$\Sigma := \{E \subseteq X' : \ f^{-1}(E) \in \mathcal{L}\}$$

  We can easily see that Σ is a $\sigma -$ algebra so $\mathcal{C}' \subseteq \Sigma$ and thus:

  $$\mathcal{B}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$$

  and we get the thesis.

$\square$

Question 3.6

Establish and show all equivalent statements to the fact that $f : X \to \overline{\mathbb{R}}$ is measurable.

Solution

Equivalent statements of measurability

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a function, the following are equal:

i) f is measurable;

ii) $\{f > \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

iii) $\{f \geq \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

iv) $\{f < \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$;

v) $\{f \leq \alpha\} \in \mathcal{A} \ \forall \alpha \in \mathbb{R}$.

Proof. Let us prove all the coimplications:
(i) $\iff$ (iii):

$$\mathcal{A}' = \mathcal{B}\left(\overline{\mathbb{R}}\right) = \sigma_0(\overbrace{\{(\alpha, +\infty] : \ \alpha \in \mathbb{R}\}}^{\mathcal{C}'})$$
$$f \text{ is measurable} \iff f^{-1}(\underbrace{(\alpha, +\infty]}_{E}) \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$$

(ii) $\implies$ (iii):

$$\{f \geq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f > \alpha - \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(iii) $\implies$ (iv):

$$\{f < \alpha\} = \{f \geq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

(iv) $\implies$ (v):

$$\{f \leq \alpha\} = \cap_{n=1}^{\infty} \overbrace{\{f < \alpha + \frac{1}{n}\}}^{\in \mathcal{A}} \in \mathcal{A}$$

(v) $\implies$ (ii):

$$\{f > \alpha\} = \{f \leq \alpha\}^{\mathsf{c}} \in \mathcal{A}$$

$\square$

Question 3.7

Let $f, g \in \mathcal{M}(X, \mathcal{A})$. What can we say about measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$? Justify the answer.

Solution

Measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$

Let $f, g \in \mathcal{M}(X, \mathcal{A})$, we have:

i) $\{f < g\} \in \mathcal{A}$

ii) $\{f \leq g\} \in \mathcal{A}$

iii) $\{f = g\} \in \mathcal{A}$

Proof.

i) $\{f < g\} = \bigcap_{r \in \mathbb{Q}} \left[\underbrace{\overbrace{\{f < r\}}^{\in \mathcal{A}} \cap \overbrace{\{r < g\}}^{\in \mathcal{A}}}_{\in \mathcal{A}} \right]$

ii) $\{f \leq g\} = \{f > g\}^{\mathsf{c}} \in \mathcal{A}$ by the previous point.

iii) $\{f = g\} = \{f \underset{\in \mathcal{A}}{\leq} g\} \cap \{f \underset{\in \mathcal{A}}{\geq} g\} \in \mathcal{A}$

$\square$

Question 3.8

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$. Show that $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$. Can there exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$? Why?

Solution

Measurability of $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$

Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$, we have:

i) $\sup_n f_n \in \mathcal{M}(X, \mathcal{A})$

ii) $\inf_n f_n \in \mathcal{M}(X, \mathcal{A})$

iii) $\limsup_n f_n \in \mathcal{M}(X, \mathcal{A})$

iv) $\liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$

Proof.

i) $\forall \alpha \in \mathbb{R} \quad \{\sup_{n\in\mathbb{N}} f_n > \alpha\} = \bigcup_{n=1}^{\infty}\{f_n > \alpha\} \in \mathcal{A} \implies \sup_{n\in\mathbb{N}} f_n \in \mathcal{M}$

ii) $\inf_n f_n = -\sup_{n\in\mathbb{N}}(-f_n) \in \mathcal{M}(X,\mathcal{A})$

iii) $\limsup_n f_n = \inf_{k\geq 1}\sup_{n\geq k} f_n \in \mathcal{M}(X,\mathcal{A})$

iv) $\liminf_n f_n = \sup_{k\geq 1}\inf_{n\geq k} f_n \in \mathcal{M}(X,\mathcal{A})$

$\square$

Question 3.9

Let $f, g \in \mathcal{M}(X,\mathcal{A})$. Show that $f+g, f\cdot g \in \mathcal{M}(X,\mathcal{A})$.

Solution

Measurability of $f+g, f\cdot g$

Let $f, g : X \to \mathbb{R}$ and $f, g \in \mathcal{M}(X,\mathcal{A})$, we have that $f+g, f\cdot g \in \mathcal{M}(X,\mathcal{A})$.

Proof. Let us define a few new functions φ, ψ and χ:

$$\begin{cases} \varphi(x) = X \to \mathbb{R}^2 & \varphi(x) := (f(x), g(x)) \\ \psi(x) = \mathbb{R}^2 \to \mathbb{R} & \psi(s,t) := s+t \\ \chi(x) = \mathbb{R}^2 \to \mathbb{R} & \chi(s,t) := s\cdot t \end{cases} \implies \begin{cases} \psi\circ\varphi = f+g \\ \chi\circ\varphi = f\cdot g \end{cases}$$

Now, clearly $\psi, \chi \in C^0(\mathbb{R}^2)$ (hence measurable), let us prove that φ is also measurable. We use the Characterization of Measurability (3.2.1):

$$\varphi : (X,\mathcal{A}) \to (\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right)) \text{ is measurable } \iff \forall E \subseteq \mathbb{R}^2 \text{ open } \varphi^{-1}(E) \in \mathcal{A}$$

We take:

$$\begin{aligned} E = R &:= (a,b)\times(c,d) \\ \varphi^{-1}(R) &= \{x \in X : \ (f(x), g(x)) \in R\} \\ &= \{x \in X : \ f(x) \in (a,b)\} \cap \{x \in X : \ g(x) \in (c,d)\} \\ &= f^{-1}(a,b) \cap g^{-1}(c,d) \in \mathcal{A} \end{aligned}$$

Thus $\forall E \subseteq \mathbb{R}^2$ open, we may write:

$$E = \bigcup_{k=1}^{\infty} R_k, \quad R_k = (a_k, b_k)\times(c_k, d_k)$$
$$\varphi^{-1} = \bigcup_{k=1}^{\infty} \varphi^{-1}(R_k) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k) \cap g^{-1}(c_k, d_k) \in \mathcal{A}$$

Hence $\varphi \in \mathcal{M}(X,\mathcal{A})$, and we have:

$$\psi\circ\varphi,\ \chi\circ\varphi \in \mathcal{M}(X,\mathcal{A})$$

$\square$

Question 3.10

Prove that A is measurable if and only if χ_A is a measurable function.

Solution

A is measurable if and only if χ_A is a measurable function

Let $A \subseteq X$ and χ_A be the indicator function of A. We have:

$$\chi_A \in \mathcal{M}(X, \mathcal{A}) \iff \quad A \in \mathcal{A}$$

Proof.

$$\{\chi_A > \alpha\} = \begin{cases} X & \alpha < 0 \\ A & 1 > \alpha \geq 0 \\ \emptyset & \alpha \geq 1 \end{cases}$$

Now, $X, \emptyset \in \mathcal{A}$ by definition, so:

$$A \in \mathcal{A} \iff \chi_A \in \mathcal{M}$$

□

Question 3.11

Prove or disprove the following statements:

a) $f \in \mathcal{M}(X, \mathcal{A}) \iff f_{\pm} \in \mathcal{M}_+(X, \mathcal{A})$;

b) $f \in \mathcal{M}(X, \mathcal{A}) \iff |f| \in \mathcal{M}(X, \mathcal{A})$.

Solution

Measurability of $f_{\pm}$ and $|f|$

Let $f : X \to \mathbb{R}$, we have:

i) $f \in \mathcal{M}(X, \mathcal{A}) \iff f_{\pm} \in \mathcal{M}_+(X, \mathcal{A})$

ii) $f \in \mathcal{M}(X, \mathcal{A}) \iff |f| \in \mathcal{M}(X, \mathcal{A})$

Proof.

i)
- **($\implies$):** if $f \in \mathcal{M}(X, \mathcal{A})$, then we define f_+ as:

  $$f_+(x) = \max\{f(x), 0\} \geq 0 \quad \forall x \in X$$

  and since $f, 0 \in \mathcal{M}(X, \mathcal{A})$ and max is a measurable function we have that $f_+ = \max \circ (f, 0) \in \mathcal{M}_+(X, \mathcal{A})$ by (3.1.2). We may analogously prove the same for f_-.
- **($\impliedby$):** if $f_+ \in \mathcal{M}_+(X, \mathcal{A})$, then we define $f = f_+ - f_-$, and since $f_+, f_-, f \in \mathcal{M}(X, \mathcal{A})$ we have that $f \in \mathcal{M}(X, \mathcal{A})$ by (3.9.1).

ii) $f \in \mathcal{M} \implies f_+, f_- \in \mathcal{M}$ by the previous point $\implies |f| = f_+ + f_- \in \mathcal{M}$ by (3.9.1).

□

Question 3.12

Write the definition of simple function. What is its canonical form? How can we characterize measurability of a simple function? Write the definition of step function.

Solution

Definition of simple function

Let X be a set and $s : X \to \mathbb{R}$ a function. We say that s is a simple function if $s(X)$ is a finite set.
Furthermore we define the two sets:

$$\mathcal{S}(X, \mathcal{A}) := \{\text{ measurable simple functions}\}$$
$$\mathcal{S}_+(X, \mathcal{A}) := \{\text{ measurable simple functions with non-negative values}\}$$

Canonical form of simple function

The canonical form of a simple function is:

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$

where:

$$\begin{aligned}
&c_i \in \mathbb{R}\ \forall i = 1, \dots, n \\
&E_i = \{x \in X :\ s(x) = c_i\}\ \forall i = 1, \dots, n \\
&X = \bigcup_{i=1}^{n} E_i,\ E_k \cap E_l = \emptyset\ \forall k \neq l
\end{aligned}$$

i.e. E_i is a partition of X.

Measurability of simple function

A simple function is measurable if and only if we have the following:

$$E_i \in \mathcal{A}\ \forall i = 1, \dots, n$$

i.e. :

$$s(x) \in \mathcal{M}(X, \mathcal{A}) \iff E_i \in \mathcal{A}\ \forall i = 1, \dots, n$$

this is because $s(x)$ is a linear combination of indicator functions.

Step Functions

Let $I = [a_0, a_1)$ be an interval and $P = \{a_o \equiv x_0 < x_1 < \cdots < x_n \equiv a_1\}$ a partition of I. A function $f : I \to \mathbb{R}$ is a step function if:

$$f := \sum_{k=0}^{n-1} c_k \chi_{[x_k, x_{k+1})}(x)$$

Question 3.13

State and give a sketch of the proof of the Simple Approximation Theorem.

Solution

Simple Approximation Theorem

Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence of simple functions $\{s_n\}$ such that:

$$s_n \xrightarrow{n\to\infty} f \text{ in } X \text{ (pointwise)}$$

Furthermore:

i) if $f \in \mathcal{M}(X, \mathcal{A})$, then $\{s_n\} \subseteq \mathcal{S}(X, \mathcal{A})$;

ii) if $f \geq 0 \implies \{s_n\} \uparrow$, $0 \leq s_n \leq f$;

iii) f bounded $\implies s_n \xrightarrow{n\to\infty} f$ uniformly in X.

Sketch of proof

Let $f \le 0$, bounded and $0 \le f \le 1 \ \forall x \in X$.

$$f : X \to [0,1]$$

Let us divide $[0,1]$ in 2^n intervals of equal length $\forall n \in \mathbb{N}$, then we define:

$$E_k^{(n)} \coloneqq \left\{ x \in X : \ \frac{k}{2^n} \le f(x) \le \frac{k+1}{2^n} \right\} \quad k = 0, \dots, 2^n - 1$$

$$s_n \coloneqq \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_k^{(n)}}(x) \quad \forall n \in \mathbb{N}$$

Clearly $\{s_n\}$ has the desired properties.

Question 3.14

Write the definitions of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$. State their properties and prove some of them.

Solution

Definition of $\operatorname{ess\,sup}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and f a function on X. We define:

$$\operatorname*{ess\,sup}_X f(x) \coloneqq \inf \left\{ \sup_{x \in N^c} f(x) : \ N \in \mathcal{N}_\mu \right\}$$

Definition of $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measurable space and f a function on X. We define:

$$\operatorname*{ess\,inf}_X f(x) \coloneqq \sup \left\{ \inf_{x \in N^c} f(x) : \ N \in \mathcal{N}_\mu \right\}$$

Properties of $\operatorname{ess\,sup}_X f$ and $\operatorname{ess\,inf}_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g \in \mathcal{M}(x, \mathcal{A})$ two functions on X. We have that:

i) $\exists N \in \mathcal{N}_\mu$ such that $\operatorname{ess\,sup}_X f = \sup_{x \in N^c} f$ and $f \le \operatorname{ess\,sup}_X f$ almost surely $x \in X$;

ii) $\operatorname{ess\,sup}_X f = -\operatorname{ess\,inf}_X -f$;

iii) $\operatorname{ess\,sup}_X k \cdot f = k \cdot \operatorname{ess\,sup}_X f$;

iv) $f \le g \implies \operatorname{ess\,sup}_X f \le \operatorname{ess\,sup}_X g$;

v) $\operatorname{ess\,sup}_X (f+g) \le \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$;

vi) $f = g$ almost everywhere in $X \implies \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$;

vii) $g \ge 0$ almost everywhere in $X \implies f \cdot g \le (\operatorname{ess\,sup}_X f) \cdot g$ almost everywhere in X.

Proof. Let us give a partial proof:

i) Suppose $\operatorname{ess\,sup}_X f < +\infty$, $\forall k \in \mathbb{N} \ \exists N_k \in \mathcal{N}_\mu$ such that:

$$\sup_{x \in N_k} f < \operatorname*{ess\,sup}_X f + \frac{1}{k}$$

We define $N := \bigcup_{k=1}^{\infty} N_k$. Then $N \in \mathcal{N}_\mu$ and:

$$N^{\mathsf{c}} = \bigcap_{k=1}^{\infty} N_k^{\mathsf{c}} \subseteq N_k^{\mathsf{c}} \quad \forall k \in \mathbb{N}$$
$$\implies \operatorname*{ess\,sup}_X f \le \sup_{N^{\mathsf{c}}} f \le \sup_{N_k^{\mathsf{c}}} f < \operatorname*{ess\,sup}_X f + \frac{1}{k} \quad \forall k \in \mathbb{N}$$

Now we pass apply a limit $k \to +\infty$ and we get:

$$\sup_{N^{\mathsf{c}}} f = \operatorname*{ess\,sup}_X f$$
$$N \supseteq \bar{N} := \{x \in X : \ f(x) > \operatorname*{ess\,sup}_X f(x)\} \in \mathcal{A}$$
$$\implies \bar{N} \in \mathcal{N}_\mu \implies f \le \operatorname*{ess\,sup}_X f \text{ almost everywhere in } X$$

□

Question 3.15

What is $\mathcal{L}^\infty$? Which is the relation between functions finite a.e. and essentially bounded functions? Justify the answer.

Solution

Definition of $\mathcal{L}^\infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f \in \mathcal{M}(X, \mathcal{A})$ is said to be essentially bounded if:

$$\operatorname*{ess\,sup}_X f < +\infty$$

and we define the set of essentially bounded functions as:

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : \ \text{ f is essentially bounded } \}$$

Relation between functions finite a.e. and essentially bounded functions

We have that:

1. $f \in \mathcal{L}^\infty \implies f$ is finite a.e. in X;
2. in general if f is finite a.e. in $X \not\Longrightarrow f \in \mathcal{L}^\infty$.

Proof. 1. We can easily see that:

$$|f| \le \operatorname{ess\,sup} |f| < +\infty \text{ almost everywhere in } X$$

thus f is finite almost everywhere in X;

2. Let us assume that:

$$f \text{ is finite a.e. in } X \implies f \in \mathcal{L}^\infty$$

and let us see a clear counterexample of this, take:

$$f(x) : \mathbb{R} \to \overline{\mathbb{R}} := \begin{cases} \frac{1}{|x|} & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

Clearly f is finite in $E = \mathbb{R} \setminus \{0\}$, i.e. f is finite a.e. in $\mathbb{R}$. Let us note that $\lambda(\{0\}) = 0$. Thus:

$$\operatorname*{ess\,sup}_X |f| = +\infty \implies f \notin \mathcal{L}^\infty$$

□

Sheet n. 4

Question 4.1

Define the Cantor set. State its main properties and prove some of them.

Solution

Definition of the Cantor set

The Cantor set is defined iteratively, let us illustrate the first two steps:

Step 1: We start with the interval $[0, 1]$ and remove from it the open interval $(1/3, 2/3)$. We define the following sets:

$$I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right) \quad J_{1,1} = \left[0, \frac{1}{3}\right] \quad J_{1,2} = \left[\frac{2}{3}, 1\right]$$

and:

$$C_1 = \bigcup_{k=1}^{2} J_{1,k} \quad \lambda(C_1) = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

Step 2: We now remove the open set $(1/9,\ 2/9)$ from $J_{1,1}$ and the open set $(7/9,\ 8/9)$ from $J_{1,2}$. We define the following sets:

$$\begin{aligned} I_{2,1} &= \left(\frac{1}{9}, \frac{2}{9}\right) \quad J_{2,1} = \left[0, \frac{1}{9}\right] \quad J_{2,2} = \left[\frac{2}{9}, \frac{1}{3}\right] \\ I_{2,2} &= \left(\frac{7}{9}, \frac{8}{9}\right) \quad J_{2,3} = \left[\frac{2}{3}, \frac{7}{9}\right] \quad J_{2,4} = \left[\frac{8}{9}, 1\right] \end{aligned}$$

and:

$$C_2 = \bigcup_{k=1}^{4} J_{2,k} \quad \lambda(C_2) = 4 \cdot \frac{1}{9} = \frac{4}{9}$$

So at the n-th step we will have:

$$C_n = \bigcup_{k=1}^{2^n} J_{n,k} \quad \lambda(C_n) = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$$

Thus we can finally define the Cantor set $\mathcal{C}$ as:

$$\mathcal{C} = \bigcup_{n=1}^{\infty} C_n$$

let us note that since the endpoints of all the closed intervals are always preserved at each step we have that $C_n \supseteq C_{n+1}$ and thus $C_n \downarrow \mathcal{C}$.

Properties of the Cantor set

i) $\mathcal{C}$ is closed since it is the countable intersection of closed sets (C_n closed $\forall n \in \mathbb{N}$);

ii) $\mathcal{C} \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$ by virtue of its closedness;

iii) $\lambda(\mathcal{C}) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty}(2/3)^n = 0$ since $\lambda(C_1) = 1/3 < +\infty$ and λ is continuos from above (v).

iv) $int(\mathcal{C}) = \emptyset$

Proof.

$$int(\mathcal{C}) \subseteq \mathcal{C} \quad \lambda(\mathcal{C}) = 0 \implies \lambda(int(\mathcal{C})) = 0$$

by the monotonicity of λ (ii). Now, since $int(\mathcal{C})$ is open ($\mathcal{C}$ is closed) it must contain an interval, but intervals have positive measure (this holds true only in $\mathcal{L}(\mathbb{R})$) and thus $int(\mathcal{C}) = \emptyset$. □

Alternatively:

Proof. Let us assume that $int(\mathcal{C}) \neq \emptyset$, then:

$$\exists J \text{ open } \subseteq int(\mathcal{C})$$

now, since $\lambda(J) = l > 0$ we may write that:

$$\lambda(J) = l > \left(\frac{2}{3}\right)^n = \lambda(C_n) \quad \exists n \in \mathbb{N}$$

in other words $\exists n \in \mathbb{N}$ such that $J \supseteq C_n \implies J \not\subseteq C_n$ which is absurd since we assumed that $J \subseteq int(\mathcal{C}) \implies J \subseteq C_n \; \forall n \in \mathbb{N}$. Thus $int(\mathcal{C}) = \emptyset$. □

v) $\mathcal{C}$ is uncountable, indeed each of its elements can be written as an alternating series of 0s and 2s divided by 3^n. This would be equal to approximanting each element by going right or left through the sets $J_{n,k}$ where 0 represents a choice to go right and 2 a choice to go left. We can write this as follows:

$$\mathcal{C} = \left\{ x \in [0,1] | \, x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}, \; x_n \in \{0,2\} \right\}$$

and thus $\mathcal{C}$ can be put into a bijection with $\{0,2\}^{\mathbb{N}}$ which is uncountable.

Question 4.2

Define the Vitali-Lebesgue function. State its main properties and prove some of them.

Solution

Vitali-Lebesgue Function

As for the cantor set we shall define Vitali's function iteratively as a sequence of functions $\{f_n\}$. This sequence is defined as follows:

$$\begin{aligned}
&f_0(x) = 0 \quad x \in [0,1] \\
&f_1(x) = \begin{cases} \frac{3}{2}t & t \in [0,1/3] \\ \frac{1}{2} & t \in (1/3,2/3) \\ \frac{3}{2}t - \frac{1}{2} & t \in [2/3,1] \end{cases} \\
&\vdots \\
&f_n(x) = \begin{cases} \frac{1}{2}f_{n-1}(3t) & t \in [0,1/3] \\ f_{n-1}(t) & t \in (1/3,2/3) \\ \frac{1}{2}f_{n-1}(3t-2) & t \in [2/3,1] \end{cases}
\end{aligned}$$

and we define Vitali's function V as:

$$f_n \to V \in C([0,1])$$

let us prove that such a function exists and is unique.

Proof. Let us prove that f_n is a Cauchy sequence in $C([0,1])$, we may prove that:

$$||f||_\infty = \max_{t\in[0,1]} |f(t)| \to ||f_n - f_{n-1}||_\infty < \frac{1}{2^n}$$

let us assume this to be true, for now, then to prove that $\{f_n\}$ is Cauchy we have to prove that:

$$||f_m - f_n||_\infty < \varepsilon \quad \forall m > n \in \mathbb{N},\ \exists \varepsilon > 0$$

indeed we may write:

$$\begin{aligned}
||f_m - f_n||_\infty &= ||f_m - f_{n+1} + f_{n+1} - f_n||_\infty \\
&\leq ||f_m - f_{n+1}||_\infty + ||f_{n+1} - f_n||_\infty \text{ by the triangular inequality} \\
&\leq \sum_{k=n}^{m} ||f_{k+1} - f_k||_\infty \text{by repeating the previous step} \\
&\leq \sum_{k=n}^{m-1} \frac{1}{2^{k+1}} < \varepsilon \text{ since the series is convergent}
\end{aligned}$$

thus the limit exists and is unique and we have a function:

$$V : [0,1] \to [0,1]$$

□

Properties of Vitali's function

Vitali's function has the following properties:

i) $V(0) = 0$, $V(1) = 1$ and V is continuous since it is the uniform limit of continuous functions.

ii) V is non-decreasing in $[0,1]$ since f_n is non-decreasing for all $n \in \mathbb{N}$.

Proof. Let $0 \leq x < y \leq 1$ then:

$$V(x) = \lim_{n\to\infty} f_n(x) \leq \lim_{n\to\infty} f_n(y) = V(y)$$

□

iii) $V([0,1]) = [0,1]$ since $V \in C([0,1])$ and $V(0) = 0$ and $V(1) = 1$. Thus by Intermediate value theorem V must cross all the values in between.

iv) $V' = 0$ almost everywhere since V is constant on $[0,1] \setminus \mathcal{C}$ and $\lambda(\mathcal{C}) = 0$.

Question 4.3

Write the definitions of the Lebesgue integral of a nonnegative measurable simple function over X and over a measurable subset $E \subseteq X$. Write the main properties of the integral and prove some of them.

Solution

Lebesgue integral of nonnegative simple functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function with canonical form as in (3.12.2). We define the Lebesgue integral of s over X as:

$$\int_X s \, d\mu := \sum_{k=1}^{n} c_k \mu(E_k)$$

And its integral over a measurable subset $E \in \mathcal{A}$ as:

$$\int_E s \, d\mu := \int_X s \cdot \chi_E \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap E)$$

Properties of the Lebesgue integral

i)

$$\int_X \chi_E \, d\mu = \mu(E) \quad \forall E \in \mathcal{A}$$

ii)

$$\int_N s \, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$$

iii) Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $c \geq 0$, then:

$$\int_X c \cdot s \, d\mu = c \cdot \int_X s \, d\mu$$

iv) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, then:

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

v) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, such that $s \leq t$ then:

$$\int_X s \, d\mu \leq \int_X t \, d\mu$$

vi) $s \in \mathcal{S}_+(X, \mathcal{A})$, $E \subseteq F \in \mathcal{A}$ then:

$$\int_E s \, d\mu \leq \int_F s \, d\mu$$

Proof.

i) We may write:

$$\chi_E = \sum_{k=1}^{2} c_k \chi_{E_k} = \begin{cases} c_1 = 1 & E_1 = E \\ c_2 = 0 & E_2 = E^{\mathsf{c}} \end{cases}$$

thus by applying the definition of the Lebesgue integral we get:

$$\int_X \chi_E \, d\mu = \sum_{k=1}^{2} c_k \mu(E_k) = 1 \cdot \mu(E) + 0 \cdot \mu(E^{\mathsf{c}}) = \mu(E)$$

ii) Let us apply the definition:

$$\int_N s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap N)$$

but by the monotonicity of μ (ii) we have:

$$E_k \cap N \subseteq N \implies \mu(E_k \cap N) \leq \mu(N) = 0$$

so the previous sum is equal to 0.

□

Question 4.4

Let $s \in \mathcal{S}_+(X, \mathcal{A})$. For any $E \in \mathcal{A}$, let $\varphi(E) \coloneqq \int_E s d\mu$. Prove that φ is a measure.

Solution

Measure induced by a function

Let $(X, \mathcal{A}, \mu)$ be a measure space and $s \in \mathcal{S}_+(X, \mathcal{A})$ a nonnegative simple function. We define the measure φ induced by s as:

$$\varphi(E) \coloneqq \int_E s\, d\mu \quad \forall E \in \mathcal{A}$$

Proof. Let us see that φ meets the definition of a measure:
Clearly:

$$\varphi : \mathcal{A} \to \overline{\mathbb{R}}_+$$

Furthermore:

i) $\varphi(\emptyset) = 0$ by property 2 of the Lebesgue integral (ii).

ii) Let $\{E_n\} \subseteq \mathcal{A}$ disjoint and $E = \bigcup_{k=1}^{\infty} E_k$, let us write:

$$s \coloneqq \sum_{l=1}^{m} d_l \chi_{F_l} \quad F_l \in \mathcal{A}$$

thus:

$$\begin{aligned}
\varphi(E) &= \int_E s\, d\mu = \sum_{l=1}^{m} d_l \mu(F_l \cap E) \\
&= \sum_{l=1}^{m} d_l \sum_{k=1}^{\infty} \mu(F_l \cap E_k) \text{ by the } \sigma-\text{additivity of } \mu \text{ (ii)} \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{m} d_l \mu(F_l \cap E_k) \\
&= \sum_{k=1}^{\infty} \int_{E_k} s\, d\mu = \sum_{k=1}^{\infty} \varphi(E_k)
\end{aligned}$$

□

Question 4.5

Write the two possible equivalent definitions of Lebesgue integral of a measurable nonnegative function.

Solution

Let $f : X \to \overline{\mathbb{R}}_+$ be a measurable nonnegative function ($f \in \mathcal{M}_+(X, \mathcal{A})$). Let us define the set $\mathcal{S}_f$:

$$\mathcal{S}_f = \{s \in \mathcal{S}_+ : \ s \leq f \text{ in } X\}$$

We then have two possible and equivalent definitions of the Lebesgue integral of f.

Definition by sup

We define the integral of f as:

$$\int_X f\, d\mu = \sup_{s \in \mathcal{S}_f} \int_X s\, d\mu$$

Definition by lim

Thanks to the Simple Approximation Theorem (3.13.1) we know:

$$\exists \{s_n\} \subseteq \mathcal{S}_f \quad s_n \leq s_{n+1} \; n \in \mathbb{N}, \; s_n \xrightarrow{n\to\infty} s \text{ in } X$$

So we can define the integral as:

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X s_n \, d\mu$$

Let us note that the integral must be independent of the choice of the sequence $\{s_n\}$.

Question 4.6

State and prove the Chebychev inequality.

Solution

Chebychev inequality

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ then $\forall c > 0$ we have:

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f\geq c\}} f \; d\mu \leq \frac{1}{c} \int_X f \; d\mu$$

Proof. Clearly

$$E_C \coloneqq \{f \geq c\} \in \mathcal{A} \text{ since } f \in \mathcal{M}_+(X, \mathcal{A}) \text{ (see (iii))}$$

and we have that:

$$c \cdot \chi_{E_C} \leq f \cdot \chi_{E_C}$$

thus by the monotonicity of the integral for functions (v) and for sets (vi) we have:

$$c \cdot \mu(E_C) = \int_X c \cdot \chi_{E_C} \, d\mu \leq \int_X f \cdot \chi_{E_C} \, d\mu = \int_{E_C} f \, d\mu \leq \int_X f \, d\mu$$

so we have the Chebychev inequality. □

Question 4.7

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$. Show that f is finite a.e. in X.

Solution

f is finite a.e. in X if $\int_X f d\mu < +\infty$

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu < +\infty$, then f is finite a.e. in X.

Proof. Let us note that the thesis is equivalent to $\mu(\{f = +\infty\}) = 0$. Let us define:

$$\{f = +\infty\} = \bigcap_{n=1}^{\infty} \{f > n\}$$

Clearly we have that:

a) $\{f > n\} \downarrow \{f = +\infty\}$

b) $\mu(\{f > n\}) \leq \frac{1}{n} \cdot \int_X f \, d\mu \; \forall n \in \mathbb{N}$ by the Chebychev inequality (4.6.1).

So since $\mu(\{f > 1\}) \leq \frac{1}{1} \cdot \int_X f \, d\mu < +\infty$ we may apply the continuity from above of μ (v):

$$\mu(\{f = +\infty\}) = \mu\left(\bigcap_{n=1}^{\infty} \{f > n\}\right) = \lim_{n\to\infty} \mu(\{f > n\}) = \lim_{n\to\infty} \frac{1}{n} \cdot \underbrace{\int_X f \, d\mu}_{<+\infty} \longrightarrow 0$$

□

Question 4.8

State and prove the vanishing lemma for functions $f \in \mathcal{M}_+(X, \mathcal{A})$.

Solution

Vanishing lemma for $f \in \mathcal{M}_+(X, \mathcal{A})$

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f \, d\mu = 0$, then we have that $f = 0$ a.e. in X.

Proof. Let us note that the thesis is equivalent to $\mu(\{f > 0\}) = 0$. Let us define:

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\}$$

Clearly we have that:

a) $\left\{ f > \frac{1}{n} \right\} \uparrow \{f > 0\}$

b) $\frac{1}{n} \cdot \chi_{\{f > \frac{1}{n}\}} \leq f \cdot \chi_{\{f > \frac{1}{n}\}}$

by Chebychev inequality (4.6.1) we have:

$$\mu\left(\left\{ f > \frac{1}{n} \right\} \right) \leq \frac{1}{1/n} \cdot \int_X f \, d\mu = 0 \quad \forall n \in \mathbb{N}$$

thus by the continuity from below of μ (iv) we have:

$$\mu(\{f > 0\}) = \mu\left(\bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\} \right) = \lim_{n \to \infty} \mu\left(\left\{ f > \frac{1}{n} \right\} \right) == 0$$

□

Question 4.9

State and prove the Monotone Convergence Theorem (or Beppo Levi Theorem).

Solution

Monotone Convergence Theorem

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ and $f : X \to \overline{\mathbb{R}}_+$ be such that:

i) $f_n \leq f_{n+1}$ in X $\forall n \in \mathbb{N}$

ii) $f_n \xrightarrow{n \to \infty} f$ pointwise in X

then:

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = \int_X f \, d\mu$$

Proof. $f \in \mathcal{M}_+(X, \mathcal{A})$
by monotonicity of the integral for functions (v) we have:

$$\alpha := \int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu \longrightarrow \alpha \leq \int_X f \, d\mu$$

now, we have to prove that $\alpha \geq \int_X f \, d\mu$. Indeed $\forall \varepsilon \in (0, 1)$, $\forall s \in \mathcal{S}_f$ let:

$$E_n := \{(1 - \varepsilon)s \leq f_n\} \quad n \in \mathbb{N}$$

Let us note that:

a) $\{E_n\} \subseteq \mathcal{A}$;

b) $\{E_n\} \uparrow$, since $\{f_n\} \uparrow$;

c) $X = \bigcup_{n=1}^{\infty} E_n$.

Clearly $\bigcup_{n=1}^{\infty} E_n \subseteq X$, we have to show that $X \subseteq \bigcup_{n=1}^{\infty} E_n$. Now, let us fix $x \in X$, we have two possibilities:

- **$f(x) = +\infty$:** then $\exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$(1-\varepsilon)s(x) < f_n(x) \implies x \in E_n \ \forall n > \bar{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

- **$f(x) < +\infty$:** then $\exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$(1-\varepsilon)s(x) \leq (1-\varepsilon)f(x) < f_n(x) \implies x \in E_n \ \forall n > \bar{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

Thus we have that $X \subseteq \bigcup_{n=1}^{\infty} E_n$ and $\bigcup_{n=1}^{\infty} E_n \subseteq X \implies X = \bigcup_{n=1}^{\infty} E_n$.
It clearly follows that:

$$(1-\varepsilon) \cdot \int_{E_n} s \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f \, d\mu$$

now let $n \to \infty$ ($E_n \xrightarrow{n\to\infty} X$):

$$(1-\varepsilon) \cdot \int_X s \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu = \alpha$$

but since $\varepsilon \in (0,1)$ can be arbitrarily small we have:

$$\int_X s \, d\mu \leq \alpha \implies \sup_{s \in \mathcal{S}_f} \int_X s \, d\mu = \int_X f \, d\mu \leq \alpha$$

thus we have proved that $\int_X f \, d\mu = \alpha$. □

Question 4.10

State and prove Fatou's Lemma.

Solution

Fatou's lemma

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, then:

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \geq \int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu$$

Proof. We already know that $\liminf_{n\to\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$ by (3.8.1).
Let us a define a new sequence $\{g_n\}$ such that:

$$g_k : X \to \overline{\mathbb{R}}_+ \quad g_k := \inf_{n \geq k} f_n$$

We can clearly see that:

a) $\{g_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, $\{g_n\} \uparrow$;

b) $g_k \leq f_k$ for all $k \in \mathbb{N}$;

c) $\lim_{k\to\infty} g_k = \sup_{k\geq 1} g_k = \sup_{k\geq 1} \inf_{n\geq k} f_n = \liminf_{n\to\infty} f_n$.

thus by monotonicty of the integral for functions (v) and (b) we have:

$$\int_X g_k \, d\mu \leq \int_X f_k \, d\mu \quad \forall k \in \mathbb{N}$$

Now, since $\{g_n\}$ is an increasing sequence so is $\int_X g_k \, d\mu$ and thus it admits a limit (which coincides with its $\liminf$), thus, if we apply the $\liminf$ to both sides, we have:

$$\liminf_{k\to\infty} \int_X g_k \, d\mu = \lim_{k\to\infty} \int_X g_k \, d\mu \leq \liminf_{k\to\infty} \int_X f_k \, d\mu$$

Now let us apply the Monotone Convergence Theorem (4.9.1) to the right hand side:

$$\int_X \lim_{k\to\infty} g_k \, d\mu \overset{(c)}{=} \int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu \leq \liminf_{k\to\infty} \int_X f_k \, d\mu$$

and so we have obtained our thesis. □

Sheet n. 5

Question 5.1

State and prove the theorem concerning integration of series with general terms $f_n \in \mathcal{M}_+(X, \mathcal{A})$.

Solution

Integral of a series with positive terms

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, $f_n : X \to \overline{\mathbb{R}}_+$ $\forall n \in \mathbb{N}$ then:

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu \right)$$

Proof. Let us provide a proof of our own making. [1]
Cleary $\sum_{n=1}^{\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$, since each addendum is a non-negative measurable function. Let us now note that:

$$\sum_{k=1}^{n} f_k \underset{n\to\infty}{\uparrow} \sum_{k=1}^{\infty} f_k$$

indeed:

$$\sum_{k=1}^{n} f_k \xrightarrow{n\to\infty} \sum_{k=1}^{\infty} f_k \text{ pointwise in } X$$

$$\sum_{k=1}^{n} f_k \leq \sum_{k=1}^{n} f_k + f_{n+1} = \sum_{k=1}^{n+1} f_k \text{ in } X \ \forall n \in \mathbb{N}$$

so we may apply the Monotone Convergence Theorem (4.9.1) to conclude:

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \int_X \left(\lim_{n\to\infty} \sum_{k=1}^{n} f_k \right) d\mu \overset{MCT}{=} \lim_{n\to\infty} \int_X \left(\sum_{k=1}^{n} f_k \right) d\mu$$

Now, for our last step, let us apply the linearity of the integral:

$$\lim_{n\to\infty} \int_X \left(\sum_{k=1}^{n} f_k \right) d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_X f_k \right) = \sum_{k=1}^{\infty} \left(\int_X f_k \, d\mu \right)$$

□

Question 5.2

Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Show that $\nu(E) := \int_E f \, d\mu$ is a measure; state and prove its properties.

[1] This proof has been reviewed by professor Punzo and stated to be correct.

Solution

Measure induced by a function

Let $f \in \mathcal{M}_+(X, \mathcal{A})$, then $\nu(E) := \int_E f \, d\mu$, $\nu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a measure.

Proof. Let us show that ν meets the definition of a measure:

i) $\nu(\emptyset) = 0$ thanks to the properties of the integral (see ii), since $\mu(\emptyset) = 0$.

ii) Let $\{E_n\} \subseteq \mathcal{A}$ disjoint such that $\bigcup_{k=1}^{\infty} E_k = X$, then:

$$\begin{aligned}\nu(E) &= \int_X f \cdot \chi_E \, d\mu = \int_X \left(f \cdot \sum_{k=1}^{\infty} \chi_{E_k} \right) d\mu \text{ thanks to the disjointedness of } \{E_n\} \\ &= \sum_{k=1}^{\infty} \left(\int_X f \cdot \chi_{E_k} \, d\mu \right) = \sum_{k=1}^{\infty} \nu(E_k)\end{aligned}$$

The penultimate passage was achieved thanks to (5.1.1)

□

Properties of the induced measure

i) Let $g \in \mathcal{M}_+(X, \mathcal{A})$, then:

$$\int_X g \, d\nu = \int_X g \cdot f \, d\mu$$

ii) $\forall E \in \mathcal{A} \ \mu(E) = 0 \implies \nu(E) = 0$

iii) $\forall f \in \mathcal{M}_+ \ \nu(E) = 0 \implies \mu(E) = 0$

Proof.

i) Let us show this equality with $g \equiv s \in \mathcal{S}_+(X, \mathcal{A})$, with canonical form:

$$s = \sum_{k=1}^{n} c_k \cdot \chi_{F_k} \ , \quad \{F_k\} \subseteq \mathcal{A} \ , \quad X = \bigcup_{k=1}^{n} F_k$$

then:

$$\begin{aligned}\int_X s \, d\nu &= \sum_{k=1}^{n} c_k \cdot \nu(F_k) = \sum_{k=1}^{n} c_k \cdot \left(\int_X f \, d\mu \right) \\ &= \int_X \left(\sum_{k=1}^{n} c_k \cdot f \cdot \chi_{F_k} \right)\end{aligned}$$

If $g \in \mathcal{M}_+(X, \mathcal{A})$ then we can get the thesis by approximation (3.13.1).

ii) $\nu(E) = 0$ thanks to the properties of the integral (ii) since $\mu(E) = 0$.

iii) Let us take the function $\chi_E \in \mathcal{M}_+(X, \mathcal{A})$ (see 3.10.1). Then, since the hypothesis is true $\forall f \in \mathcal{M}_+$, we may write:

$$\nu(E) = \int_E \chi_E \, d\mu = 1 \cdot \mu(E) = 0 \implies \mu(E) = 0$$

□

Question 5.3

Let $f, g \in \mathcal{M}_+(X, \mathcal{A})$. Show that if $f = g$ a.e. in X then $\int_X f \, d\mu = \int_X g \, d\mu$.

Solution

$f = g$ a.e. $\implies \int_X f\,d\mu = \int_X g\,d\mu$

Let $f, g \in \mathcal{M}_+(X, \mathcal{A})$ such that $f = g$ a.e. in X, then:

$$\int_X f\,d\mu = \int_X g\,d\mu$$

Proof. Let us define the following set:

$$N := \{x \in X :\ f(x) \neq g(x)\} \in \mathcal{A}$$

Clearly we have that:

$$\mu(N) = 0$$
$$\int_{N^c} f\,d\mu = \int_{N^c} g\,d\mu$$

Both results are a consequence of the definition of almost everywhere. Thus we may write:

$$\begin{aligned}
\int_X f\,d\mu &= \overset{0}{\cancel{\int_N f\,d\mu}} + \int_{N^c} f\,d\mu = \int_{N^c} g\,d\mu \\
&= \underbrace{\int_N g\,d\mu}_{=0} + \int_{N^c} g\,d\mu = \int_X g\,d\mu
\end{aligned}$$

let us note that we have partitioned X with N and N^c. ☐

Question 5.4

Write the definition of: integrable functions; Lebesgue integral; $\mathcal{L}^1(X, \mathcal{A}, \mu)$.

Solution

Integrable function

Let $f : X \to \overline{\mathbb{R}}_+$ be a function, we say that f is integrable on X if $f \in \mathcal{M}(X, A)$ and:

$$\int_X f_+\,d\mu < \infty \qquad \int_X f_-\,d\mu < \infty$$

let us note that both f_+ and f_- are non-negative measurable functions ($f_\pm \in \mathcal{M}_+(X, \mathcal{A})$).

$\mathcal{L}^1(X, \mathcal{A}, \mu)$

We define $\mathcal{L}^1(X, \mathcal{A}, \mu)$ as the set of all integrable functions $f : X \to \overline{\mathbb{R}}$:

$$\mathcal{L}^1(X, \mathcal{A}, \mu) := \left\{ f : X \to \overline{\mathbb{R}} :\ f \in \mathcal{M}(X, \mathcal{A}),\ \int_X f_\pm\,dx < +\infty \right\}$$

Lebesgue integral

Let $f \in \mathcal{L}^1$, then we define its integral as:

$$\int_X f\,d\mu := \int_X f_+\,d\mu - \int_X f_-\,d\mu$$

this is called the Lebesgue integral of f. Moreover we define:

$$\int_E f\,d\mu := \int_X f \cdot \chi_E\,d\mu = \int_E f_+ \cdot \chi_E\,d\mu - \int_E f_- \cdot \chi_E\,d\mu$$

Question 5.5

Let $f : X \to \overline{\mathbb{R}}$. How is the integrability of f related to that of $f_\pm$ and of $|f|$? Justify the answer. Show that if $f \in \mathcal{L}^1$, then $|\int_X f\,d\mu| \le \int_X |f|\,d\mu$. Give an alternative definition of $\mathcal{L}^1(X, \mathcal{A}, \mu)$.

Solution

Properties of the Lebesgue integral

Let $f : X \to \overline{\mathbb{R}}$ be a function, then:

i) $f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$

ii) $f \in \mathcal{L}^1 \iff f \in \mathcal{M}$ and $|f| \in \mathcal{L}^1$

iii) $f \in \mathcal{L}^1 \implies |\int_X f\,d\mu| \le \int_X |f|\,d\mu$

Proof.

i) if $f \in \mathcal{L}^1$ then by definition ($\iff$) we have that:

$$f \in \mathcal{M} \quad \text{and} \quad \int f_\pm\,d\mu < \infty$$

Now, $f \in \mathcal{M} \iff f_\pm \in \mathcal{M}_+$ by (a) thus we have that:

$$f_\pm \in \mathcal{M}, \int f_\pm\,d\mu < \infty \iff f_\pm \in \mathcal{L}^1$$

and the two conditions are equivalent.

ii) [2] as above, if $f \in \mathcal{L}^1$ then by definition ($\iff$) we have that:

$$f \in \mathcal{M} \quad \text{and} \quad \int f_\pm\,d\mu < \infty$$

Now, $f \in \mathcal{M} \iff |f| \in \mathcal{M}_+$ by (b) thus we have that:

$$|f| \in \mathcal{M}, \int |f|\,d\mu < \infty \iff |f| \in \mathcal{L}^1$$

this is true by virtue of the previous point and the fact that $|f| = f_+ + f_-$, indeed:

$$\int |f|\,d\mu = \int f_+\,d\mu + \int f_-\,d\mu < \infty$$

since each addendum is finite. Thus the two conditions are equivalent.

iii) Let $f \in \mathcal{L}^1$, then thanks to the triangular inequality:

$$\begin{aligned}\left|\int_X f\,d\mu\right| &= \left|\int_X f_+\,d\mu - \int_X f_-\,d\mu\right| \\ &\le \left|\int_X f_+\,d\mu\right| + \left|\int_X f_-\,d\mu\right| \\ &= \int_X f_+\,d\mu + \int_X f_-\,d\mu \\ &= \int_X |f|\,d\mu\end{aligned}$$

□

[2] We may also use the previous point and the fact that $\mathcal{L}^1$ is a vector space, but we'll see this later.

ALternative definition of $\mathcal{L}^1$

[3] We can also more compactely define $\mathcal{L}^1(X, \mathcal{A}, \mu)$ through the absolute value:

$$\mathcal{L}^1(X, \mathcal{A}, \mu) \coloneqq \left\{ f : X \to \overline{\mathbb{R}} : \; f \in \mathcal{M}(X, \mathcal{A}), \; \int_X |f| \, dx < +\infty \right\}$$

Question 5.6

Prove that $\mathcal{L}^1$ is a vector space.

Solution

$\mathcal{L}^1$ is a vector space

$\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space.

Proof. Let $f, g \in \mathcal{L}^1$ and $\lambda \in \mathbb{R}$, then:

$$\begin{aligned} &\implies f_\pm, g_\pm \text{ finite a.e. in } X \text{ by (4.7.1)} \\ &\implies f, g \text{ finite a.e. in } X \end{aligned}$$

so we can define:

$$h \coloneqq f + \lambda g \text{ defined a.e. in } X$$

clearly $h \in \mathcal{M}$ by the properties of measurable functions (3.9.1) and:

$$\int_X |h| \, d\mu = \int_X |f| + |\lambda| \int_X |g| \, d\mu < \infty$$

since both addenda are finite. Thus $h \in \mathcal{L}^1$ and $\mathcal{L}^1$ is a vector space. □

Question 5.7

State and prove the vanishing lemma for $\mathcal{L}^1$ functions.

Solution

Vanishing lemma for $f \in \mathcal{L}^1$

Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ be such that:

$$\int_E f \, d\mu = 0 \quad \forall E \in \mathcal{A}$$

then $f = 0$ a.e. in X.

Proof. Let us define two sets:

$$\begin{aligned} E_+ &\coloneqq \{x \in X : f(x) \geq 0\} \\ E_- &\coloneqq \{x \in X : f(x) \leq 0\} \end{aligned}$$

they are both in $\mathcal{A}$ since $f \in \mathcal{M}$ (see iii), so we have that:

$$\int_{E_+} f \, d\mu = 0 \implies f = 0 \text{ a.e. in } E_+$$

$$\int_{E_-} f \, d\mu = 0 \implies f = 0 \text{ a.e. in } E_-$$

so we have that $f = 0$ a.e. in $X = E_+ \cup E_-$. □

[3] This definition has not been directly provided by prof. Punzo. I found no reference to this definition neither in my personal notes nor his, rather I found it in a student's notes from last year's course. Nevertheless, it is clearly attested in the literature and it is equivalent and alternative to the previous definition.

Question 5.8

Let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$, $f = g$ a.e. in X. Show that $g \in \mathcal{L}^1$ and $\int_X g\,d\mu = \int_X f\,d\mu$.

Solution

f = g a.e. $\implies g \in \mathcal{L}^1$ **and** $\int_X g\,d\mu = \int_X f\,d\mu$

Let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$ and $f = g$ a.e. in X. Then:

$$g \in \mathcal{L}^1 \text{ and } \int_X f\,d\mu = \int_X g\,d\mu$$

Proof. We have that:

$$\begin{gathered} f_+ = g_+ \text{ a.e. in } X \text{ and } f_- = g_- \text{ a.e. in } X \\ \implies \int_X f_+\,d\mu = \int_X g_+\,d\mu \text{ and } \int_X f_-\,d\mu = \int_X g_-\,d\mu \end{gathered}$$

thanks to (5.3.1), thus we get:

$$\int_X f\,d\mu = \int_X f_+\,d\mu + \int_X f_-\,d\mu = \int_X g_+\,d\mu + \int_X g_-\,d\mu = \int_X g\,d\mu$$

□

Question 5.9

State and prove the Lebesgue theorem. In which case it is simple to find a dominating function?

Solution

Lebesgue theorem (or Dominated convergence theorem)

Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$ be a sequence of measurable functions and $f \in \mathcal{M}(X, \mathcal{A})$ be a function such that:

$$f_n \xrightarrow{n\to\infty} f \text{ a.e. in } X$$

if $\exists g \in \mathcal{L}^1$ such that:

$$|f_n| \le g \quad \text{ a.e. in } X\ \forall n \in \mathbb{N}$$

then:

$$f_n, f \in \mathcal{L}^1 \text{ and } \int_X |f_n - f|\,d\mu \xrightarrow{n\to\infty} 0$$

in particular:

$$\int_X f_n\,d\mu \xrightarrow{n\to\infty} \int_X f\,d\mu$$

Proof. We shall prove this theorem by applying Fatou's lemma (4.10.1).
Now since $|f_n| \le g$ we can pass the limit and get:

$$|f| \le g \text{ a.e. in } X$$

So we have:

$$\begin{gathered} \int_X |f_n|\,d\mu \le \int_X g\,d\mu \quad \forall n \in \mathbb{N} \\ \int_X |f|\,d\mu \le \int_X g\,d\mu \end{gathered}$$

so since $f_n, f \in \mathcal{M}$ and $g \in \mathcal{L}^1$ we can deduce that:

$$\implies f_n, f \in \mathcal{L}^1$$

So they are also finite a.e., let us now define a new sequence $\{g_n\}$:

$$g_n \coloneqq 2g - |f_n - f| \quad \forall n \in \mathbb{N}$$

by the previous inequalities we get:

$$|f_n - f| \leq |f_n| + |f| \leq 2g \text{ a.e. in } X \; \forall n \in \mathbb{N}$$

therefore:

$$g_n \geq 0 \text{ a.e. in } X \; \forall n \in \mathbb{N} \implies g_n \in Mes_+$$

thus we can write:

$$
\begin{aligned}
2\int_X g\, d\mu &= \int_X \left(\lim_{n\to\infty} g_n\right)\, d\mu \\
&\le \liminf_{n\to\infty} \int_X g_n\, d\mu \text{ by Fatou's lemma 4.10.1} \\
&= \liminf_{n\to\infty} \int_X \left[2g - |f_n - f|\,\right]\, d\mu \\
&= 2\int_X g\, d\mu - \limsup_{n\to\infty} \left(\int_X |f_n - f|\, d\mu\right)
\end{aligned}
$$

thus we may simplify $2\int_X g\, d\mu$ on both sides and invert the inequality sign we get:

$$\limsup_{n\to\infty} \left(\int_X |f_n - f|\, d\mu\right) \le 0$$

so since $\int_X |f_n - f|\, d\mu \ge 0$ it admits a limit and we have:

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$$

Moreover:

$$\left|\int_X f_n\, d\mu - \int_X f\, d\mu\right| \le \lim_{n\to\infty} \int_X |f_n - f|\, d\mu \xrightarrow{n\to\infty} 0$$

□

Simple case for the Lebesgue Theorem

If we have the following situation:

1. $\mu(X) < +\infty$
2. $\exists M > 0$: $|f_n| \le M$ a.e. in X $\forall n \in \mathbb{N}$

Then we can choose $g := M$ and we get:

$$\int_X |g|\, d\mu = \int_X M\, d\mu = M \cdot \mu(X) < +\infty \iff g \in \mathcal{L}^1$$

thus we have easily met the thesis of the Lebesgue Theorem (5.9.1).

Question 5.10

Describe the relations between Peano-Jordan and Lebesgue measures, and between the Riemann (also in the generalized sense) and the Lebesgue integral.

Solution

Every Peano-Jordan-measurable set is Lebesgue measurable

Let $E \subseteq \mathbb{R}^n$, if E is Peano-Jordan-measurable then it is also Lebesgue-measurable ($E \in \mathcal{L}(\mathbb{R}^n)$) and the its measures coincide:

$$m_{PJ}(E) = \lambda(E)$$

Thus the set of Peano-Jordan-measurable sets is strictly included in the set of Lebesgue-measurable sets. Indeed the set $[0,1] \cap \mathbb{Q}$ is Lebesgue-measurable (with measure zero) but is not Peano-Jordan-measurable (this is due to the fact that Peano-Jordan-measurable sets do not form a $\sigma -$ algebra).

The Riemann integral and the Lebesgue integral

Proper integrals

Let $I = [a, b]$ be a closed interval and $R(I)$ the set of Riemann-integrable functions over I. For any function $f \in R(I)$ we have $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and:

$$\int_I f \, d\lambda = \int_a^b f(x) \, dx$$

To state it plainly, we can say that the set of R-integrable functions and the set of $\mathcal{L}$-integrable functions coincide on closed intervals and the integrals of such functions also coincide.

Improper integrals

Let $I = (\alpha, \beta)$ and let $R^i(I)$ be the set of functions $f : I \to \mathbb{R}$ integrable in the generalized (improper) sense. Then we have:

i) $f \in R^i(I) \implies f \in \mathcal{M}(I, \mathcal{L}(I))$

ii) $|f| \in R^i(I) \implies f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and moreover:

$$\int_I f \, d\lambda = \int_\alpha^\beta f(x) \, dx$$

Let us note here the crucial fact that this statement does not imply (as is instead the case for sets) that all R^i-integrable functions are also $\mathcal{L}$-integrable. This is due to the requirement that the absolute value of f be in $\mathcal{L}^1$.

Counter-Example

take the function:

$$f(x) := \begin{cases} \frac{sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad f : I = [0, +\infty) \to \mathbb{R}$$

and on one hand we have that:

$$f \in R^i(I) \qquad \int_0^\infty \frac{sin(x)}{x} \, dx = \frac{\pi}{2}$$

while on the other we have:

$$\int_{\mathbb{R}_+} \left| \frac{sin(x)}{x} \right| d\lambda = \int_0^\infty \left| \frac{sin(x)}{x} \right| dx = +\infty \implies f \notin \mathcal{L}^1$$

Therefore we may conclude that not all R^i-integrable functions are also L-Integrable.

Question 5.11

State the theorem for integration of series (without sign restriction on the general term f_n).

Solution

Integration of series with general terms

Let $\{f_n\} \subseteq \mathcal{L}^1$ be such that:

$$\sum_{n=1}^{\infty} \left(\int_X |f_n| \right) d\mu < +\infty$$

then the series $\sum_{n=1}^{\infty} f_n$ converges a.e. in X and we have that:

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu \right)$$

Question 5.12

Write the definitions of L^1 and of L^∞. Show that they are metric spaces. Are $\mathcal{L}^1$ and $\mathcal{L}^\infty$ metric spaces?

Solution

Definition of L^1

Let $(X, \mathcal{A}, \mu)$ be ameasure space and let R be the equivalence relation (see 1.3.1) such that:

$$fRg \iff f = g \text{ a.e. in } X$$

then we define L^1 as the quotient set of $\mathcal{L}^1$ with respect to this relation:

$$L^1(X, \mathcal{A}, \mu) \coloneqq \mathcal{L}^1(X, \mathcal{A}, \mu)/R$$

and we denote the classes of equivalence inside of it as:

$$[f] \coloneqq \{g \in \mathcal{L}^1 : \ fRg\}$$

Definition of L^∞

As done above we define L^∞ as:

$$L^\infty(X, \mathcal{A}, \mu) \coloneqq \mathcal{L}^\infty(X, \mathcal{A}, \mu)/R$$

L^1 and L^∞ are metric spaces

Both L^1 and L^∞ are metric spaces with the following distance functions:

$$d_1(f, g) \coloneqq \int_X |f - g| \, d\mu \qquad d_\infty(f, g) \coloneqq \operatorname*{ess\,sup}_X |f - g|$$

Proof. Let us prove this for L^1 only, as the proof for L^∞ is analogous.
Now, let us show that d_1 meets the definition of a distance, $d_1 : L^1 \times L^1 \to \mathbb{R}$. Indeed, since $f, g \in L^1$ ($\int_X f \, d\mu, \int_X g \, d\mu < +\infty$), we have:

$$\int_X |f - g| \, d\mu \leq \int_X |f| \, d\mu + \int_X |g| \, d\mu < +\infty$$

moreover:

i) $d(f, g) \geq 0 \ \forall f, g \in L^1$;

ii) $d(f, f) = 0 \ \forall f \in L^1$;

iii) $d(f, g) = 0 \iff \int_X |f - g| \, d\mu = 0$ by the vanishing lemma (4.8.1) we get $|f - g| = 0$ a.e. in X, thus $f = g$ a.e. in X;

iv) $d(f, g) = d(g, f) \ \forall f, g \in L^1$;

v) $d(f, g) \leq d(f, h) + d(h, g) \ \forall f, g, h \in L^1$ by the triangular inequality and monotonicity of the integral.

Let us note that the equality almost everywhere is an exact match under the equivalence relation R. In other words $f, g \in [f]$ and they are the same element with respect to L^1 and we can say that $d_1(f, g) = 0 \implies f \overset{L^1}{=} g$.
Therefore we can say that L^1 is a metric space equipped with the distance d_1.
Let us also note that this isn't true for $\mathcal{L}^1$ since it isn't quotiented by the equivalence relation R, thus it isn't a metric space. The same argument can be applied to L^∞ and $\mathcal{L}^\infty$. □

Question 5.13

For a sequence of functions $\{f_n\} \subset \mathcal{M}$, write the definitions of: pointwise convergence; uniform convergence; almost everywhere convergence; convergence in L^1; convergence in L^∞; convergence in measure.

Solution

Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$, $f_n : X \to \mathbb{R}$, $f : X \to \overline{\mathbb{R}}$. We can define the following:

Pointwise convergence

We say that $f_n \xrightarrow{n\to\infty} f$ pointwise if:

$$f_n(x) \xrightarrow{n\to\infty} f(x) \quad \forall x \in X$$

in the sense of a sequence of real numbers ($f_n(x) \in \mathbb{R}$).

Uniform convergence

We say that $f_n \xrightarrow{n\to\infty} f$ uniformly if:

$$\sup_{x\in X} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$$

Almost everywhere convergence

We say that $f_n \xrightarrow{n\to\infty} f$ almost everywhere if:

$$\{x \in X : \ f_n(x) \xrightarrow{n\,to\infty} f(x)\}^{\mathsf{c}} \in \mathcal{N}_\mu$$

that is to say the set where f_n doesn't converge to f is measurable and has measure zero.

Convergence in L^1

Let $\{f_n\} \subseteq L^1$ and assume (for now) $f \in L^1$. We say that $f_n \xrightarrow{n\to\infty} f$ in L^1 if:

$$d_1(f_n, f) = \int_X |f_n - f| \, d\mu \xrightarrow{n\to\infty} 0$$

Convergence in L^∞

Let $\{f_n\} \subseteq L^\infty$ and assume (for now) $f \in L^\infty$. We say that $f_n \xrightarrow{n\to\infty} f$ in L^∞ if:

$$d_\infty(f_n, f) = \operatorname*{ess\,sup}_X |f_n - f| \xrightarrow{n\to\infty} 0$$

Convergence in measure

We say that $f_n \xrightarrow{n\to\infty} f$ in measure if:

$$\mu(\{|f_n - f| \geq \varepsilon\}) \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon > 0$$

Sheet n. 6

Question 6.1

Is it true that if $f_n \to f$ in measure, then $f_n \to f$ a.e.? Justify the answer.

Solution

Convergence in measure does not imply convergence a.e.

In general, convergence in measure does not imply convergence a.e.. This can be clearly shown by way of Rademacher's sequence (a.k.a. the typewriter sequence):

Rademacher sequence

Let us define the Rademacher sequence iteratively:

$$
\begin{aligned}
&f_1(x) = \mathbb{I}_{[0,1]}(x) \\
&f_2(x) = \mathbb{I}_{[0,1/2]}(x) \\
&f_3(x) = \mathbb{I}_{[1/2,1]}(x) \\
&\quad\vdots \\
&f_n(x) = \mathbb{I}_{\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right]}(x) \quad 2^k \leq n \leq 2^{k+1}\ k \in \mathbb{N}
\end{aligned}
$$

In other words for each $k \in \mathbb{N}$ we divide $[0,1]$ into 2^k intervals and "hover" over them. This way we have a function whose L^1-limit (and thus by extension its limit in measure) is 0, indeed we have:

$$
\begin{aligned}
&\int_{[0,1]} f_1 \, d\mu = 1 \\
&\int_{[0,1]} f_2 \, d\mu = \int_{[0,1]} f_3 \, d\mu = \frac{1}{2} \\
&\quad\vdots \\
&\int_{[0,1]} f_n \, d\mu \xrightarrow{n\to\infty} 0
\end{aligned}
$$

but, on the other hand, if we fix $x \in [0,1]$ the sequence $\{f_n\}$ will oscillate between the value 0 and 1 infinitely many times as $n \to \infty$. Thus f_n cannot be said to converge a.e. in $[0,1]$.

Question 6.2

What is the relation between convergence in measure and convergence a.e. up to subsequences?

Solution

Convergence in measure implies convergence a.e. up to subsequences

Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X. If $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that:

$$f_{n_k} \xrightarrow{k\to\infty} f \text{ a.e. in } X.$$

Question 6.3

Under which hypothesis on X, does convergence a.e. imply convergence in measure? What happens if one omits the key assumption on X?

Solution

Convergence a.e. implies convergence in measure when $\mu(X) < +\infty$

Let $\mu(X) < +\infty$ and $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X. If $f_n \to f$ a.e. in X, then:

$$f_n \xrightarrow{n\to\infty} f \text{ in measure}$$

The assumption that $\mu(X) < +\infty$ is necessary, as the following counterexample shows:

Counterexample

Let us take $f_n := \chi_{[n,+\infty)}$, clearly $f_n \to 0$ pointwise (and thus a.e.) in $\mathbb{R}$ but we have that $\lambda(\mathbb{R}) = +\infty$ and thus $f_n \not\to 0$ in measure. Indeed we have:

$$\mu\left(\left\{f_n \geq \frac{1}{2}\right\}\right) = +\infty \quad \forall n \in \mathbb{N}$$

Question 6.4

Show that convergence in L^1 implies convergence in measure.

Solution

Convergence in L^1 implies convergence in measure

Let $f_n, f \in L^1(X, \mathcal{A}, \mu)$. If $f_n \xrightarrow[n\to\infty]{L^1} f$, then:

$$f_n \to f \text{ in measure}$$

Proof. Suppose by contradiction that:

$$f_n \not\to f \text{ in measure}$$

then, by definition of convergence in measure (5.13.6), $\exists \varepsilon, \sigma > 0$ such that:

$$\mu\left(\{|f_n - f| \geq \varepsilon\}\right) \geq \sigma$$

for infinitely many $n \in \mathbb{N}$. Thus we may write:

$$\begin{aligned}\int_X |f_n - f| \, d\mu &\geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon \, d\mu \\ &= \varepsilon \cdot \mu\left(\{|f_n - f| \geq \varepsilon\}\right) \geq \varepsilon \cdot \sigma\end{aligned}$$

for infinitely many $n \in \mathbb{N}$, thus:

$$\implies f_n \overset{L^1}{\underset{n\to\infty}{\not\longrightarrow}} f$$

which is absurd. ☐

Question 6.5

Show that convergence in L^1 implies convergence a.e. up to subsequences.

Solution

Convergence in L^1 implies convergence a.e. up to subsequences

If $f_n \xrightarrow[n\to\infty]{L^1} f$, then:

$$\exists \{f_{n_k}\} \text{ such that } f_{n_k} \xrightarrow{k\to\infty} f \text{ a.e. in } X$$

Proof. This can be proven by trivially applying the fact that convergence in L^1 implies convergence in measure (6.4.1) and that, in turn, convergence in measure implies convergence a.e. up to subsequences (6.2.1). □

Question 6.6

Does convergence in measure imply convergence in L^1? Does convergence a.e. imply convergence in L^1? Justify the answer.

Solution

Convergence in measure or convergence a.e. do not imply convergence in L^1

Neither convergence in measure nor convergence a.e. imply convergence in L^1. This can be shown by way of the following counterexample:

Counterexample

Let $(X = [0, 1], \mathcal{L}(X), \lambda|_X)$ and $f_n(x) = n \cdot \chi_{[0,1/n]}(x)$ clearly we have that:

$$f_n \xrightarrow[n\to\infty]{a.e.} 0 \text{ in } [0, 1]$$

and, thus, since $\lambda(X) = 1$, we have that:

$$f_n \xrightarrow[n\to\infty]{\lambda} 0 \text{ in } [0, 1]$$

but on the other hand, we have that:

$$\int_0^1 |f_n - 0| \, d\lambda = \int_0^1 f_n \, d\lambda = \int_0^{\frac{1}{n}} n \, d\lambda = n \cdot \frac{1}{n} = 1 \quad \forall n \in \mathbb{N}$$

so $f_n \xrightarrow[n\to\infty]{L^1} 1$ and it cannot be that $f_n \xrightarrow[n\to\infty]{L^1} 0$. So convergence a.e. and in measure do not imply convergence in L^1.

Question 6.7

Write the definitions of: product measurable space, section of a measurable set. What is the product measure? Why is the definition well-posed?

Solution

Product measurable space

Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ be two measurable spaces. Consider the set $R \subseteq \mathcal{P}(X_1 \times X_2)$ defined as follows:

$$R := \{E_1 \times E_2 : \ E_1 \in \mathcal{A}_1, \, E_2 \in \mathcal{A}_2\}$$

let us defined the **product $\sigma -$ algebra** as:

$$\sigma_0(R) \equiv \mathcal{A}_1 \times \mathcal{A}_2$$

then the measurable space $(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is called the **product measurable space** of $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$.

Section of a measurable set

Let $E \subseteq X_1 \times X_2$, then we define the following two sections:

$$E_{x_1} := \{x_2 \in X_2 : \ (x_1, x_2) \in E\} \quad x_1 \in X_1$$
$$E_{x_2} := \{x_1 \in X_1 : \ (x_1, x_2) \in E\} \quad x_2 \in X_2$$

We have that $E_{x_1} \in \mathcal{A}_2 \ \forall x_1 \in X_1$ and $E_{x_2} \in \mathcal{A}_1 \ \forall x_2 \in X_2$.

Product measure

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two $\sigma -$ finite measure spaces with measure μ_1 and μ_2 respectively and let $E \in \mathcal{A}_1 \times \mathcal{A}_2$, then we define the following:

$$\varphi_1 : X_1 \to \overline{\mathbb{R}}_+ \quad \varphi_1(x_1) := \mu_2(E_{x_1}) \quad \forall x_1 \in X_1$$
$$\varphi_2 : X_2 \to \overline{\mathbb{R}}_+ \quad \varphi_2(x_2) := \mu_1(E_{x_2}) \quad \forall x_2 \in X_2$$

these are well defined thanks to the fact that $E_{x_1} \in \mathcal{A}_2$ and $E_{x_2} \in \mathcal{A}_1$.
Moreover we have that:

i) $\varphi_i \in \mathcal{M}_+(X_i, \mathcal{A}_i) \ i = 1, 2$

ii)

$$\int_{X_1} \varphi_1(x_1) \, d\mu_1 = \int_{X_2} \varphi_2(x_2) \, d\mu_2$$

We thus define the **product measure** as the function:

$$\mu_1 \times \mu_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \overline{\mathbb{R}}_+ \quad (\mu_1 \times \mu_2)(E) := \int_{X_1} \varphi_1(x_1) \, d\mu_1 = \int_{X_2} \varphi_2(x_2) \, d\mu_2$$

let us note that this is a $\sigma -$ finite measure and it is well-posed since for both φ_1 and φ_2 the Lebesgue integral is well defined because $\varphi_1 \in \mathcal{M}_+(X_1, \mathcal{A}_1)$ and $\varphi_2 \in \mathcal{M}_+(X_2, \mathcal{A}_2)$. Lastly, let us note that to have this condition it is essential for μ_1 and μ_2 to be $\sigma -$ finite.

Question 6.8

Is the product measure space complete? Justify the answer. Which is the relation between $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ and $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$?

Solution

The product space is incomplete

In general the product measure space is incomplete. Let us show this trough a counterexample:

Counterexample

Let us consider these two spaces:

$$(\mathbb{R}^m \times \mathbb{R}^n, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n) \text{ and } (\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$$

for simplicity's sake here we take $m = n = 1$. As we already know, the space $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$ is a complete space. Now, let us consider Vitali'set: $V \subseteq [0,1]$, $V \notin \mathcal{L}(\mathbb{R})$ and let us take the set:

$$E \coloneqq \{x_0\} \times V \quad (x_0 \in \mathbb{R})$$

Clearly, if we take the section E_{x_0}, we have that:

$$E_{x_0} = V \notin \mathcal{L}(\mathbb{R}) \implies E \notin \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$$

but we have that:

$$E \subseteq F \coloneqq \{x_0\} \times [0,1]$$

and that $F \in \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$, furthermore, by the definition of product mesaure (6.7.3), we observe that:

$$(\lambda \times \lambda)(F) = \int_{[0,1]} \overset{0}{\cancel{\lambda(\{x_0\})}} \, d\lambda = 0$$

therefore we have proved that there exists a set E that is contained within a set F of zero measure but isn't measurable itself. In other words we have proved that $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}), \lambda \times \lambda)$ is not a complete measure space. Futhermore we can observe quite easily that this means that $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$ is the completion of $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}), \lambda \times \lambda)$. This argument can be extended to all pairs of m and n.

Question 6.9

State the Tonelli theorem.

Solution

Tonelli's theorem

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two $\sigma -$ finite measure spaces and $f \in \mathcal{M}_+(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$. let us define the following:

$$\psi_1 : X_1 \to \overline{\mathbb{R}}_+ \quad \psi_1(x_1) \coloneqq \int_{X_2} f(x_1, x_2) \, d\mu_2 \quad \forall x_1 \in X_1$$
$$\psi_2 : X_2 \to \overline{\mathbb{R}}_+ \quad \psi_2(x_2) \coloneqq \int_{X_1} f(x_1, x_2) \, d\mu_1 \quad \forall x_2 \in X_2$$

i) $\psi_i(x_i) \in \mathcal{M}_+(X_i, \mathcal{A}_i)$ $i = 1, 2$

ii)

$$\int_{X_1 \times X_2} f(x_1, x_2) \, d(\mu_1 \times \mu_2) = \int_{X_1} \underbrace{\left[\int_{X_2} f(x_1, x_2) \, d\mu_2 \right]}_{\psi_1(x_1)} d\mu_1 = \int_{X_2} \underbrace{\left[\int_{X_1} f(x_1, x_2) \, d\mu_1 \right]}_{\psi_2(x_2)} d\mu_2$$

Question 6.10

State the Fubini theorem. By means of a counterexample, show that it is not possible to omit the hypothesis $f \in L^1$.

Solution

Fubini's theorem

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two $\sigma-$finite measure spaces and $f \in L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$, then:

i)

$$
\begin{aligned}
&f(x_1, \cdot) \in L^1(X_1, \mathcal{A}_1, \mu_1) \text{ a.e. for } x_1 \in X_1 \\
&f(\cdot, x_2) \in L^1(X_2, \mathcal{A}_2, \mu_2) \text{ a.e. for } x_2 \in X_2
\end{aligned}
$$

ii)

$$
\begin{aligned}
\psi_1(x_1) &\coloneqq \int_{X_2} f(x_1, x_2)\, d\mu_2, \quad \psi_1 \in L^1(X_1, \mathcal{A}_1, \mu_1) \\
\psi_2(x_2) &\coloneqq \int_{X_1} f(x_1, x_2)\, d\mu_1, \quad \psi_2 \in L^1(X_2, \mathcal{A}_2, \mu_2)
\end{aligned}
$$

iii)

$$
\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2) = \int_{X_1} \underbrace{\left[\int_{X_2} f(x_1, x_2)\, d\mu_2\right]}_{\psi_1(x_1)} d\mu_1 = \int_{X_2} \underbrace{\left[\int_{X_1} f(x_1, x_2)\, d\mu_1\right]}_{\psi_2(x_2)} d\mu_2
$$

Counterexample

The hypothesis that $f \in L^1$ is necessary, let us consider the following example:

$$
\begin{aligned}
(X_i, \mathcal{A}_i, \mu_i) &= ((0,1), \mathcal{L}((0,1)), \lambda) \quad i = 1,2 \\
f(x_1, x_2) &= \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} \quad (x_1, x_2) \in (0,1)^2
\end{aligned}
$$

We have that $f \in C\left((0,1)^2\right) \implies f \in \mathcal{M}$ so let us consider the integral of its positive part:

$$
\int_{X_1 \times X_2} f_+(x_1, x_2)\, d(\lambda \times \lambda)
$$

and apply Tonelli's theorem (6.9.1) since ($f_+ \geq 0$):

$$\int_{X_1 \times X_2} f_+(x_1, x_2)\, d(\lambda \times \lambda) = \int_{X_1} \left[\int_{X_2} f_+(x_1, x_2)\, d\lambda \right] d\lambda$$
$$= \int_0^1 \int_0^{x_1} \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}\, dx_2\, dx_1$$
$$= \frac{1}{2} \int_0^1 \frac{1}{x_1}\, dx_1 = +\infty$$

thus $f \notin L^1$ and indeed the equality in (iii) does not hold:

$$\int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2 \right] d\mu_1 = \cdots = \frac{\pi}{4}$$
$$\int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1 \right] d\mu_2 = \cdots = -\frac{\pi}{4}$$

Question 6.11

Write the definition of Lebesgue point. What is about the measure of the set of points that are not Lebesgue points for a function $f \in L^1$?

Solution

Lebesgue point

A point $x_0 \in [a, b]$ is a **Lebesgue point** of a function f if:

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0 + h} |f(t) - f(x_0)|\, dt = 0$$

Integrable functions and Lebesgue points

If $f \in L^1((a, b))$ then almost every $x_0 \in X$ is a Lebesgue point of f. Therefore the set of points that are not Lebesgue points for f has measure zero.

Question 6.12

State and prove the First Fundamental Theorem of Calculus for $f \in L^1$.

Solution

First Fundamental Theorem of Calculus for L^1

Let $X = [a, b]$ and $f \in L^1([a, b])$, we define the integral function F of f as follows:

$$F(x) := \int_a^x f(t)\, dt \quad x \in [a, b]$$

then F is differentiable almost everywhere in (a, b) and:

$$F'(x) = f(x)$$

Proof. Let $x_0 \in [a, b]$ be a Lebesgue point for f and $h \neq 0$ be such that $x_0 + h \in [a, b]$. Let us write the incremental ratio for F:

$$\frac{F(x_0 + h) - F(x_0)}{h} - f(x) = \frac{1}{h} \int_{x_0}^{x_0 + h} [f(t) - f(x)]\, dt$$

we can do this since $f(x)$ is independent of t, we may thus write:

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x)\right| \leq \frac{1}{|h|}\int_{x_0}^{x_0+h}|f(t)-f(x)|\,dt \xrightarrow{h\to 0} 0$$

thanks to the definition of Lebesgue point, hence:

$$F'(x) = f(x)$$

but in view of the previous point (6.11.2) we write:

$$F'(x) = f(x) \text{ a.e. in } (a,b)$$

since almost every x is a Lebesgue point for f. $\square$

Question 6.13

Let $f:[a,b]\to\mathbb{R}$. Write the definitions of: variation of f relative to a partition of $[a,b]$; total variation of f over $[a,b]$; function of bounded variation.

Solution

Variation of f relative to a partition of $[a,b]$

Let $f:[a,b]\to\mathbb{R}$ and P be a partition of $[a,b]$:

$$P \coloneqq \{a\equiv x_0 < x_1 < \cdots < x_n \equiv b\}$$

we define the variation of f with respect to the partition P as:

$$v_a^b(f,P) \coloneqq \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|$$

Total variation of f over $[a,b]$

Let $\mathcal{P}$ be the collection of all partitions P of $[a,b]$. We define the total variation of f over $[a,b]$ as:

$$V_a^b(f) \coloneqq \sup_{P\in\mathcal{P}} v_a^b(f,P)$$

Function of bounded variation

A function $f:[a,b]\to\mathbb{R}$ is said to be of **bounded variation** if $V_a^b(f)<+\infty$. We define the set of functions of bounded variation as:

$$BV([a,b]) \coloneqq \{f:[a,b]\to\mathbb{R} : V_a^b(f)<+\infty\}$$

Question 6.14

Let $f:[a,b]\to\mathbb{R}$ be monotone. Why $f\in BV([a,b])$? Show that if $f\in BV([a,b])$, then f is bounded.

Solution

Monotone functions are of bounded variation

If $f:[a,b]\to\mathbb{R}$ is a monotone function (either decreasing or increasing), then we have that:

$$V_a^b(F) = |f(b)-f(a)| < +\infty \implies f\in BV([a,b])$$

All functions of bounded variation are bounded

If $f \in BV([a,b]) \implies f$ is bounded, in fact we have that:

$$\sup_{x \in [a,b]} |f(x)| \leq |f(a)| + V_a^b(f)$$

thus f must be bounded if $f \in BV$.

Question 6.15

What is the Jordan decomposition of a BV function?

Solution

Jordan decomposition of a BV function

Let $f : [a,b] \to \mathbb{R}$, then the following are equal:

i) $f \in BV([a,b])$

ii) $\exists\, \varphi, \psi : [a,b] \to \mathbb{R}$ both increasing such that:

$$f = \varphi - \psi$$

this is called the **Jordan decomposition** of f.

Question 6.16

Why a function of bounded variation is differentiable a.e.?

Solution

Monotonicity implies a.e. differentiability

Let $f : I \to \mathbb{R}$ be a monotone function, then f is differentiable a.e. in I.

All BV functions are differentiable a.e.

For all functions $f \in BV([a,b])$ we can write its Jordan decomposition as $f = \varphi - \psi$. Both φ and ψ are increasing and thus, by the previous point, are a.e. differentiable in I and so is f since it's the difference of the two and the derivative is a linear operator.

Question 6.17

Let $f : [a,b] \to \mathbb{R}$ be an increasing function. What can we cay about f' and $\int_{[a,b]} f' \, d\lambda$? Justify the answer.

Solution

Derivative and integral of the derivative of an increasing function

If $f : I \to \mathbb{R}$ is increasing, then:

$$f' \text{ exists a.e. in } I$$
$$\int_I f' \, d\lambda \leq f(b) - f(a)$$

Question 6.18

Can there exists a function $f \in BV([a,b])$ with $f' \notin L^1([a,b])$? Justify the answer.

Solution

All BV functions have a Lebesgue-integrable derivative

There cannot exist a BV function with an unintegrable derivative. In other words:

$$f \in BV([a,b]) \implies f' \in L^1([a,b])$$

indeed for any function $f \in BV([a,b])$ we may write it through its Jordan decomposition $f = \varphi - \psi$. Now, since both φ and ψ are increasing we may apply the previous point and say that both φ' and ψ' are in L^1. Thus since L^1 is a vector space $\varphi, \psi \in L^1 \implies f \in L^1$.

Sheet n. 7

Question 7.1

Write the definition of absolutely continuous function. Show that an absolutely continuous function is also uniformly continuous, but the viceversa is not true; furthermore, a Lipschitz function is absolutely continuous, but the viceversa is not true.

Solution

Absolutely continuous function

Let $J = [a, b]$, $f : J \to \mathbb{R}$, we denote by $\mathcal{F}(J)$ the set of finite collections of closed sub-intervals of J without interior points in common. We say that the function f is absolutely continuous in J, if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$\forall \{[a_k, b_k]\} \in \mathcal{F}(J) \quad (k = 1, \dots, n)$$

for which:

$$\sum_{k=1}^{n} (b_k - a_k) < \delta$$

one has:

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

and we denote by $AC([a, b])$ the set of all absolutely continuous functions in $J = [a, b]$.

AC functions are also uniformly continuous

Let $J = [a, b]$, $f : J \to \mathbb{R}$, $f \in AC([a, b])$. If we take:

$$\{[a_k, b_k]\} = \begin{cases} \{[x, y]\} & y \geq x \\ \{[y, x]\} & x > y \end{cases}$$

by the definition of absolute continuity we have that:

$$|x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

which is the definition of uniform continuity. The converse isn't true, let us see a poignant counterexample:

Counterexample

Let us take the following f:

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \in [-1, 1] \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

this function is not absolutely continuous in $[-1, 1]$ but is is uniformly continuous in $[-1, 1]$.

Lipschitz functions are absolutely continuous

Let f be a Lipschitz function in $J = [a, b]$, we have that $f \in AC$.

Proof. Indeed we have that:

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq L \cdot \sum_{k=1}^{n} (b_k - a_k) = L \cdot \delta = \varepsilon$$

thus we can choose $\delta = \varepsilon / L$. □

The converse is not true, let us show this trough a counterexample:

Counterexample

Let $J = [0, 1]$ and $f(x) = \sqrt{x}$. Let us write f as:

$$f(x) = \int_0^x \frac{1}{2\sqrt{x}}\, dt \quad x \in [0, 1]$$

and we have that $\frac{1}{2\sqrt{x}} \in \mathcal{L}^1(J) \implies f \in AC(J)$ but f is clearly not a Lipschitz function.

Question 7.2

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < +\infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{A}$ with $\mu(E) < \delta$ there holds $\int_E f d\mu < \varepsilon$.

Solution

If $f \in \mathcal{M}_+$ and integrable, the integral is continuous in the measure

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < +\infty$, then $\varepsilon > 0$, $\exists \delta > 0$ such that:

$$\forall E \in \mathcal{A} \text{ with } \mu(E) < \delta \quad \int_E f\, d\mu < \varepsilon$$

Proof. Let $F_n := \{f < n\}$ with $n \in \mathbb{N}$. Clealry we have that:

$$F_n \in \mathcal{A} \quad F_n \uparrow X$$
$$X = \{f = +\infty\} \cup \left(\bigcup_{n=1}^{\infty} F_n \right)$$

So we have that:

$$\int_X f\, d\mu < +\infty \implies f \text{ is finite a.e.} \quad \implies \mu\left(\{f = +\infty\}\right) = 0$$

and thus we can write:

$$\int_X f\, d\mu = \lim_{n \to \infty} \int_{F_n} f\, d\mu$$

and so, by the definition of limit, $\forall \varepsilon > 0$ $\exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$\left| \int_X f\, d\mu - \int_{F_n} f\, d\mu \right| = \left| \int_{F_n^c} f\, d\mu \right| < \frac{\varepsilon}{2}$$

therefore, for a fixed $n > \bar{n}$, we get:

$$\begin{aligned}
\int_E f\, d\mu &= \int_{E \cap F_n} f\, d\mu + \int_{E \cap F_n^c} f\, d\mu \\
&< n \cdot \mu(E) + \frac{\varepsilon}{2} \quad \text{by the above and the fact that } E \supset E \cap F_n \\
&< n \cdot \delta + \frac{\varepsilon}{2} = \varepsilon \quad \text{if we choose } \delta = \frac{\varepsilon}{2n}
\end{aligned}$$

In short, we have used the fact that f is finite a.e. to get the limit and, in turn, from this we derived the above inequality which yields us the thesis. □

Question 7.3

Show that if $f \in L^1([a,b])$, then $F(x) := \int_{[a,x]} f \, d\lambda$ is absolutely continuous in $[a,b]$.

Solution

The integral function if AC

Let $I = [a,b]$ and $f \in L^1([a,b])$, then:

$$F(x) := \int_{[a,x]} f \, d\lambda \in AC(I)$$

Proof. Consider the following set E:

$$E := \bigcup_{k=1}^{n} [a_k, b_k] \text{ with } \{[a_k, b_k]\} \in \mathscr{F}(I)$$

then, since such intervals are disjoint, we have that:

$$\lambda(E) = \sum_{k=1}^{n} \lambda([a_k, b_k]) = \sum_{k=1}^{n} (b_k - a_k)$$

thus we may write:

$$\begin{aligned} \sum_{k=1}^{n} |F(b_k) - F(a_k)| &= \sum_{k=1}^{n} \left| \int_{[a_k, b_k]} f \, d\lambda \right| \\ &\leq \sum_{k=1}^{n} \int_{[a_k, b_k]} |f| \, d\lambda \\ &= \int_E |f| \, d\lambda \quad \text{by the definition of E} \end{aligned}$$

and so by the previous point (7.2.1) we get the thesis. □

Question 7.4

Which is the relation between the spaces $BV([a,b])$ and $AC([a,b])$?

Solution

All AC functions are BV

Let $f \in AC([a,b])$, then $f \in BV([a,b])$.

Question 7.5

State and prove the Second Fundamental Theorem of Calculus.

Solution

The Second Fundamental Theorem of Calculus

Let $F : [a,b] \to \mathbb{R}$, the following are equal:

i) $F \in AC([a,b])$

ii) F is differentiable a.e. in $[a,b]$ with $F' \in L^1([a,b])$ and we have:

$$F(x) = \int_a^x F' \, d\lambda + F(a) \quad \forall x \in [a,b]$$

Proof.

- $(i) \implies (ii)$**:** By the previous point we have $F \in AC([a,b]) \implies F \in BV([a,b]) \implies F$ is differentiable a.e. in $[a,b]$ and $F' \in L^1([a,b])$. Let us also suppose that F is increasing.
  We define the following:

  $$G(x) := \int_a^x F' \, d\lambda \quad x \in [a,b]$$

  hence G is differentiabe a.e. in $[a,b]$ by the First Fundamental Theorem of Calculus (6.12.1) and we have:

  $$(F-G)'(x) = F'(x) - G'(x) = F'(x) - F'(x) = 0 \qquad \text{a.e. in } [a,b]$$

  furthermore $G \in AC$ by (7.3.1) which implies $F - G \in AC$ and so $\forall a \leq x_1 \leq x_2 \leq b$ we have:

  $$[F(x_2) - G(x_2)] - [F(x_1) - G(x_1)] = F(x_2) - F(x_1) - \int_{[x_1,x_2]} F' \, d\lambda \geq 0$$

  thanks to the definition of G and the trivial fact that $\int_{[x_1,x_2]} F' \, d\lambda \leq F(x_2) - F(x_1)$. Since this is true for any pair of x_1, x_2 such that $x_1 \leq x_2$, it makes $(F-G)$ increasing. However, we have also proved that $(F-G)' = 0$ a.e. in $[a,b]$. We must thus conclude that:

  $$\exists c \in \mathbb{R} \quad F - G \equiv c \text{ in } [a,b]$$
  $$\implies F(x) - G(x) = F(a) - \cancelto{0}{G(a)}$$

  thus, if we substitute G for its definition and bring to left hand side, we, at last, attain the thesis:

  $$F(x) = \int_a^x F' \, d\lambda + F(a) \quad \forall x \in [a,b]$$

- $(ii) \implies (i)$**:** we already know that the integral function is AC (see 7.3.1), thus its translation by $F(a)$ is also AC.

□

Question 7.6

Write the definitions of: dense set, separable metric space, nowhere dense set, set of first category, set of second category. Provide an example of a nowhere dense and one of a set of first category.

Solution

Let X be a metric space equipped with a metric d.

Dense set

A set $A \subset X$ is dense in X if $\bar{A} = X$, where:

$$\bar{A} = \{y \in X : \exists \{x_n\} \subset A, \, x_n \to y\}$$

Separable metric space

X is a separable metric space if there exists a subset A which is countable and dense in X.

Nowhere dense set

A set $E \subseteq X$ is said to be nowhere dense if:
$$\text{int}(\bar{E}) = \emptyset$$

Example

We take $E = \mathbb{Z} \subset X = \mathbb{R}$, since $\mathbb{Z}$ is the countable union of all integers and thus its interior is empty. We have:

$$E = \bar{E} \implies \text{int}(\bar{E}) = \text{int}(E) = \emptyset$$

Set of first category

A set $E \subseteq X$ is said to be of first category (or meagre) in X if E is the union of countably many nowhere dense sets.

Example

We take $E = \mathbb{Q}$ and $X = \mathbb{R}$, since $\mathbb{Q}$ is the countable union of all rational numbers. Thus $\mathbb{Q}$ is a set of first category in $\mathbb{R}$.

Set of second category

A set $E \subseteq X$ which is not of first category, is said to be of second category in X.

Question 7.7

State the Baire category theorem and its corollary.

Solution

Baire's theorem

Let (X, d) be a complete metric space, then X is of second category in itself.

Corollary to Baire's theorem

The intersection of a countable family of opens sets dense in X is a set dense in X.

Question 7.8

Write the definitions of: compact metric space; sequentially compact metric space, totally bounded metric space. Explain how these properties are related.

Solution

Let (X, d) be a metric space.

Compact metric space

X is said to be compact if from any open cover of X we can extract a finite open subcover.

Sequentially compact metric space

X is said to be sequentially compact if from any sequence $\{x_n\} \subset X$ we can extract a subsequence which converges to some $x_0 \in X$.

Totally bounded metric space

X is said to be totally bounded if $\forall \varepsilon > 0$ $\exists A \subset X$ finite such that:

$$\operatorname{dist}(x, A) < \varepsilon \quad \forall x \in X$$

where:

$$\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y)$$

Relation between compactness, sequential compactness and total boundedness

the following are equal:

i) X is compact

ii) X is sequentially compact

iii) X is totally bounded and complete.

Question 7.9

Write the $\varepsilon - \delta$ definition of equicontinuous subset F of $C^0(X)$, where X is a compact metric space. Explain from which parameters δ depends. In particular, write the definition when $F = \{f_n\}_{n \in \mathbb{N}}$.

Solution

Equicontinuous set

A subset $A \subset C^0(X)$ is said to be equicontinuous if $\forall \varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that:

$$\forall f \in A,\ x, y \in X,\ d(x, y) < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon$$

Let us note that here δ_ε depends on only ε and nothing else. Moreover if we take a set $F = \{f_n\}_{n \in \mathbb{N}}$, then asking that F is equicontinuous is equivalent to asking that every f_n is uniformly continuous with respect to a shared δ_ε.

Question 7.10

State the Ascoli-Arzelà theorem.

Solution

Ascoli-Arzelà theorem

$F \subset C^0(X)$ is bounded and equicontinuous if and only if it is relatively compact (i.e. its closure is compact). More succintly:

$$F \text{ bounded and equicontinuous} \iff \bar{F} \text{ compact}$$

Question 7.11

Write the statement of the Ascoli-Arzelà theorem when the subset of $C^0(X)$ is a sequence $\{f_n\}$.

Solution

Ascoli-Arzelà theorem for sequences

Let $F = \{f_n\}_{n\in\mathbb{N}} \subset C^0(X)$ be a sequence of functions, then F is bounded and equicontinuous:

- $d(x,y) < \delta_\varepsilon \implies |f_n(x) - f_n(y)| < \varepsilon \ \forall n \in \mathbb{N}$
- $\exists k > 0$ such that $\forall x \in X \ f_n(x) < k \ \forall n \in \mathbb{N}$

if and only if its closure is compact (or alternatively, by 7.8.4, sequentially compact).

Sheet n. 8

Question 8.1

Show that $C^0([a,b])$ is separable.

Solution

Stone-Weierstrass theorem

The set of polynomials is dense in $C^0([a,b])$

$C^0([a,b])$ is separable

$C^0([a,b])$ is separable.

Proof. By the Stone-Weierstrass theorem we have that, for any $f \in C^0([a,b])$, given any $\varepsilon > 0$, there exists a polynomial p such that:

$$\operatorname{dist}(f,p) = \sup_{x\in[a,b]} |f-p| < \frac{\varepsilon}{2}$$

So we can find a polynomial r with **rational** coefficients such that:

$$\operatorname{dist}(p,r) < \frac{\varepsilon}{2}$$

hence by the triangular inequality:

$$\operatorname{dist}(f,r) \leq \operatorname{dist}(f,p) + \operatorname{dist}(p,r) = \varepsilon$$

therefore the set of polynomials with rational coefficients is dense in $C^0([a,b])$. So, since such a set is countable, $C^0([a,b])$ is separable and we have the thesis. □

Question 8.2

Write the definition of normed space and provide examples. What is the metric space induced by a given normed space?

Solution

Normed space

Let X be a vector space, a norm on X is a function such that:

$$\|x\| : X \to [0,\infty)$$

and:

i) $\|x\| = 0 \iff x = 0$

ii) $\forall \alpha \in \mathbb{R}, x \in X$: $\|\alpha x\| = |\alpha| \cdot \|x\|$

iii) $\forall x, y \in X$: $\|x + y\| \leq \|x\| + \|y\|$

and we say that the pair $(X, \|\cdot\|)$ is a normed space.

Examples of normed spaces

i) $\mathbb{R}^n$ with a norm of the family:

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad p \in [1, \infty)$$
$$\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$$

ii) $C^0([a,b])$ with the norm:

$$\|f\|_{C^0} := \sup_{x \in [a,b]} |f(x)| = \max_{x \in [a,b]} |f(x)|$$

iii) $L^1(X, \mathcal{A}, \mu)$ with the norm:

$$\|f\|_1 := \int_a^b |f(x)|\, dx$$

iv) $L^\infty(X, \mathcal{A}, \mu)$ with the norm:

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)|$$

v) $C^k([a,b])$ with the norm:

$$\|f\|_{C^k} := \sum_{i=0}^{k} \|f^{(i)}\|_\infty$$

vi) $BV([a,b])$, with two possible norms:

$$\|f\|_{BV} := \begin{cases} |f(a)| + V_a^b(f) \\ \|f\|_1 + V_a^b(f) \end{cases}$$

vii) $AC([a,b])$ with two possible norms:

$$\|f\|_{AC} := \begin{cases} |f(a)| + \|f'\|_1 \\ \|f\|_1 + \|f'\|_1 \end{cases}$$

viii) ℓ^p, ℓ^∞, we take a sequence of real numbers of the form:

$$x = \{x^{(k)}\}_{k \in \mathbb{N}} = (x^{(1)}, x^{(2)}, \dots)$$

and we define the norms:

$$\|x\|_p := \left(\sum_{k=1}^{\infty} |x^{(k)}|^p \right)^{1/p} \qquad p \in [1, \infty)$$
$$\|x\|_\infty := \sup_{k=1,\dots,\infty} |x^{(k)}|$$

we can define two normed spaces as follows:

$$\ell^p := \{x \text{ sequence of real numbers } : \|x\|_p < \infty\}$$
$$\ell^\infty := \{x \text{ sequence of real numbers } : \|x\|_\infty < \infty\}$$

Metric space induced by a normed space

Let $(X, \|\cdot\|)$ be a normed space. The metric space induced by $(X, \|\cdot\|)$ is the pair (X, d) where d is the distance function defined by:

$$\operatorname{dist}(x, y) := \|x - y\|$$

Question 8.3

In a normed space, write the definitions of: convergent sequence; Cauchy sequence; bounded sequence. Which are the relations among these notions? Show that if $x_n \to x$, then $\|x_n\| \to \|x\|$ as $n \to +\infty$.

Solution

Convergent sequence

Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to be convergent to $x \in X$ if:

$$x_n \xrightarrow{n\to+\infty} x \iff d(x_n, x) \xrightarrow{n\to+\infty} 0 \iff \|x_n - x\| \xrightarrow{n\to+\infty} 0$$

Furthermore:

$$x_n \xrightarrow{n\to+\infty} x \implies \|x_n\| \xrightarrow{n\to+\infty} \|x\|$$

Since:

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \quad \forall n \in \mathbb{N}$$

Cauchy sequence

Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{x_n\} \subset X$ is said to be Cauchy if:

$$\forall \varepsilon > 0 \, \forall \bar{n} \in \mathbb{N} \quad \|x_m - x_n\| < \varepsilon \quad \forall m, n \geq \bar{n}$$

Bounded sequence

A sequence $\{x_n\} \subset X$ is said to be bounded if:

$$\exists M > 0 \quad \|x_n\| < M \quad \forall n \in \mathbb{N}$$

Relations among convergent, Cauchy and bounded sequences

i) $\{x_n\}$ is convergent $\implies$ $\{x_n\}$ is Cauchy.

ii) $\{x_n\}$ is Cauchy $\implies$ $\{x_n\}$ is bounded.

Question 8.4

Write the definition of series in a normed space. Is it true that if $\sum_{n=0}^{+\infty} \|x_n\|$ is convergent, then $\sum_{n=0}^{+\infty} x_n$ is convergent.

Solution

Series in a normed space

Let $(X, \|\cdot\|)$ be a normed space and $\{x_n\} \subset X$ be a sequence. Let us define the sequence of partial sums (series) as the following:

$$s_n \coloneqq x_0 + \cdots + x_n + \sum_{k=0}^{n} x_k$$

It is said to be convergent if:

$$\exists x \in X : \; s_n \xrightarrow{n\to+\infty} x \iff \|s_n - x\| \xrightarrow{n\to+\infty} 0$$

and we say that:

$$\sum_{n=0}^{+\infty} x_n \text{ is the sum of the series}$$

Moreover, we have that:

$$\sum_{n=0}^{+\infty} \|x_n\| \text{ is convergent} \quad \not\Longrightarrow \quad \sum_{n=0}^{+\infty} x_n \text{ is convergent}$$

Question 8.5

What is a complete normed space? Write the definition of Banach space, provide examples.

Solution

Complete normed space

Let $(X, \|\cdot\|)$ be a normed space. The space $(X, \|\cdot\|)$ is said to be complete if the metric space induced by $(X, \|\cdot\|)$ is complete.

$$(X, \|\cdot\|) \text{ is complete} \iff (X, d) \text{ is complete} \iff \text{every Cauchy sequence in } X \text{ is convergent}$$

Banach space

A complete normed **vector** space is called a Banach space. Examples of Banach spaces are the same as those given above for normed spaces.

Question 8.6

State the criterion, involving convergence of series, for completeness of a normed space.

Solution

Criterion for completeness of a normed space

i) Let X be a Banach space and $\{x_n\} \subset X$. If $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent, then $\sum_{n=1}^{+\infty} x_n$ is convergent.

ii) Let X be a normed space. If for any $\{x_n\} \subset X$ such that the series $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent, we also have that $\sum_{n=1}^{+\infty} x_n$ is convergent, then X is a Banach space.

Question 8.7

State and prove the Riesz's Lemma.

Solution

Riesz's Lemma

Let X be a normed space, $E \subsetneq X$ a closed subspace, then $\forall \varepsilon > 0\ \exists x \in X$ such that:

$$\|x\| = 1 \text{ and } \operatorname{dist}(x, E) \geq 1 - \varepsilon$$

where $\operatorname{dist}(x, E) := \inf_{\xi \in E} \|x - \xi\|$ and x is called the "almost orthogonal element".

Proof.
Let $y \in X \setminus E$, then:

$$d := \operatorname{dist}(y, E) > 0 \text{ since } E \text{ is closed}$$

Now, let $\varepsilon \in (0,1)$, then, in view of the definition of $\operatorname{dist}(x, E)$, we have:

$$d = \operatorname{dist}(y, E) = \inf_{\xi \in E} \|y - \xi\|$$

and thus we can find $\zeta \in E$ such that:

$$d \leq \|y - \zeta\| \leq \frac{d}{1-\varepsilon}$$

Now, let us define the following:

$$x := \frac{y - \zeta}{\|y - \zeta\|}$$

So by definition x has $\|x\| = 1$ and now, thanks to the homogeneity of the norm and the closedness of E, $\forall \xi \in E$ we also have that:

$$\begin{aligned} \|x - \xi\| &= \left\| \frac{y-\zeta}{\|y-\zeta\|} - \xi \right\| = \frac{1}{\|y-\zeta\|} \|y - \zeta - \xi \cdot \|y - \zeta\|\| \\ &= \frac{1}{\|y-\zeta\|} \left\| y - \underbrace{(\zeta + \xi \cdot \|y-\zeta\|)}_{\in E} \right\| \geq \frac{d}{\|y-\zeta\|} \geq 1 - \varepsilon \end{aligned}$$

therefore we have that $\operatorname{dist}(x, E) \geq 1 - \varepsilon$ and we have the thesis. $\square$

Question 8.8

State and prove the Riesz's Theorem.

Solution

Riesz's Theorem

Let X be a normed space, if the closed ball $\bar{B}_1(0)$ is compact, then $\dim(X) < \infty$.

Proof.
Let $x_1 \in \bar{B}_1(0)$ and $Y_1 := \operatorname{span}\{x_1\}$. Clearly Y_1 is a vector subspace of X and $\dim(Y_1) = 1 < +\infty \iff Y_1$ is closed.

- If $X = Y_1$, then $\dim(X) < \infty$ and we have the thesis.
- If $X \neq Y$, then we can use Riesz's Lemma with $\varepsilon = 1/2$ to find $x - 2 \in \bar{B}_1(0)$ such that:

  $$\|x_1 - x_2\| \geq 1/2$$

  and we define the following set:

  $$Y_2 := \operatorname{span}\{x_1, x_2\}$$

  and we repeat the argument above:

  - If $X = Y_2$, then $\dim(X) < \infty$ and we have the thesis.
  - If $X \neq Y_2$, then we can use again Riesz's Lemma $x_3 \in \bar{B}_1(0)$ such that:

    $$\|x_3 - x_i\| \geq 1/2 \text{ for } i = 1, 2$$

    and we define the following set:

    $$Y_3 := \operatorname{span}\{x_1, x_2, x_3\}$$

    ...

If X is not finite dimensional this argument can be iterated to construct a sequence:

$$\{x_n\} \subseteq \bar{B}_1(0) \text{ such that } \|x_i - x_j\| \geq 1/2 \; i \neq j, \; \forall i, j \in \mathbb{N}$$

hence $\{x_n\}$ is a bounded sequence ($\|x_n\| \leq 1 \; \forall n \in \mathbb{N}$) but $\{x_n\}$ has no convergent subsequence. Thus $\bar{B}_1(0)$ is not sequentially compact and so $\bar{B}_1(0)$ is not compact. $\square$

Question 8.9

Write the definition of equivalent norms. In which type of vector spaces all norms are equivalent? Exhibit an example of a vector space that can be endowed with two norms that are not equivalent.

Solution

Equivalent norms

Let $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ be two normed spaces. We say that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there exist two constants $m, M > 0$ such that:

$$m \cdot \|x\| \leq \|x\|' \leq M \cdot \|x\| \text{ for all } x \in X$$

All norms are equivalent in finite dimensional normed spaces

If X is a normed space and $\dim(X) < \infty$, then all norms are equivalent.

Example of two non equivalent norms

Both $(C^0([a,b]), \|\cdot\|_\infty)$ and $((C^0([a,b]), \|\cdot\|_1)$ are normed spaces, but since $\dim(C^0([a,b])) = \infty$ we have that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent.

Question 8.10

Is it true in general that any vector subspace of a given normed space is closed?

Solution

Closedness of vector subspaces

Let X be a normed space and Y a vector subspace of X. We have the following:

- $\dim(Y) < \infty \implies Y$ is closed.
- $\dim(Y) = \infty \not\Longrightarrow Y$ is closed.

Therefore we can say that in general not all vector subspaces of a normed space are closed.

Question 8.11

Write the definitions of $\mathcal{L}^p$ and L^p . Show that L^p is a vector space (and its preliminary lemma).

Solution

Definition of $\mathcal{L}^p$

Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in [1, +\infty]$. We define the space $\mathcal{L}^p$ as:

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \left\{ f : X \to \overline{\mathbb{R}} \text{ measurable}, \int_X f \, d\mu < +\infty \right\}$$

Definition of L^p

On the space $\mathcal{L}^p(X, \mathcal{A}, \mu)$ we define the following equivalence relation R:

$$f, g \in \mathcal{L}^p \quad fRg \iff f = g \text{ a.e. in } X$$

We define the space L^p as the quotient space $\mathcal{L}^p/R$:

$$L^p(X, \mathcal{A}, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu)/R$$

Question 8.12

Write the definition of conjugate numbers. Show Young's inequality.

Solution

Definition of conjugate numbers

Let $p, q \in [1, +\infty]$. We say that p and q are conjugate if:

- $p, q \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.
- $p = 1$ and $q = +\infty$ or viceversa.

Young's inequality

Let $p, q \in (1, +\infty)$ be conjugate numbers and $a, b > 0$, then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.
Let us define the following convex function:

$$\varphi(x) := e^x \quad \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) \ \forall x, y \in \mathbb{R},\ t \in [0, 1]$$

Now, we choose $t = 1/p$, $1 - t = 1/q$, $x = \log(a^p)$ and $y = \log(b^q)$, thus we have:

$$\begin{aligned} ab &= e^{\log(a)} \cdot e^{\log(b)} = e^{\frac{1}{p}\log(a^p)} \cdot e^{\frac{1}{q}\log(b^q)} \\ &\leq \frac{1}{p}e^{\log(a^p)} \cdot \frac{1}{q}e^{\log(b^q)} \\ &= \frac{1}{p}a^p + \frac{1}{q}b^q \end{aligned}$$

□

Question 8.13

Show Hölder's inequality.

Solution

Hölder's inequality

Let $f, g \in \mathcal{M}(X, \mathcal{A})$ and $p, q \in [0, +\infty]$ be two conjugate numbers, then:

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$$

where we have:

$$\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p} \qquad p \in [1, +\infty)$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_X |f|$$

Proof.
Let us divide the proof into two possible cases:

i) $p, q \in (1, +\infty)$:

- If $\|f\|_p \cdot \|g\|_q = +\infty$, the inequality is trivial.
- If $\|f\|_p \cdot \|g\|_q = 0$, we have that $f = 0$ a.e. $\vee$ $g = 0$ a.e. $\implies$ $f \cdot g = 0$ a.e. $\implies$ $\|f \cdot g\|_1 = 0$ and the inequality holds.
- If$\|f\|_p$ and $\|g\|_q$ exist finite and non-zero, we fix $x \in X$ and define the two following quantities:

$$a := \frac{|f|^p}{\|f\|_p^p} \quad b := \frac{|g|^q}{\|g\|_q^q}$$

Now we apply Young's inequality to these two quantities:

$$\begin{aligned} a^{1/p}b^{1/q} &= \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} \\ &\leq \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q} \end{aligned}$$

Let us integrate both sides of the inequality:

$$\begin{aligned} \frac{1}{\|f\|_p\|g\|_q} \cdot \int_X |f \cdot g| \, d\mu &= \frac{1}{p}\overset{1}{\cancel{\frac{\int_X |f|^p \, d\mu}{\|f\|_p^p}}} + \frac{1}{q}\overset{1}{\cancel{\frac{\int_X |g|^q \, d\mu}{\|g\|_q^q}}} \\ &\leq \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

ii) $p = 1$, $q = +\infty$ (or viceversa):
Let us recall:

$$|g| \leq \|g\|_\infty = \operatorname*{ess\,sup}_X |g| \text{ a.e. in X} \implies |fg| \leq |f|\|g\|_\infty$$

let us integrate both sides of the inequality:

$$\int_X |fg| \, d\mu = \|fg\|_1 \leq \|g\|_\infty \int_X |f| \, d\mu = \|f\|_1\|g\|_\infty$$

and so the inequality holds.

$\square$

Question 8.14

Show Minkowski's inequality.

Solution

Minkowski's inequality

Let $f, g \in \mathcal{M}(X, \mathcal{A})$ and $p \in 1, +\infty$, then:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof.
Let us divide the proof into three possible cases:

- $p \in (1, +\infty)$:

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu = \int_X \underbrace{|f+g|}_{\leq |f|+|g|} |f+g|^{p-1} \, d\mu$$
$$\leq \int_X |f||f+g|^{p-1} \, d\mu + \int_X |g||f+g|^{p-1} \, d\mu$$

We now apply Hölder's inequality to the two integrals, recall $q = p/(p-1)$:

$$\int_X |f||f+g|^{p-1} \, d\mu \leq \|f\|_p \left\| |f+g|^{p-1} \right\|_q$$
$$\int_X |g||f+g|^{p-1} \, d\mu \leq \|g\|_p \left\| |f+g|^{p-1} \right\|_q$$
$$\left\| |f+g|^{p-1} \right\|_q = \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q}$$
$$= \left(\int_X |f+g|^p \, d\mu \right)^{1/q} = \|f+g\|_p^{p/q}$$

It thus follows that:

$$\|f+g\|_p^p \leq (\|f\|_p\| + \|g\|_p) \cdot \||f+g|\|_p^{p/q}$$
$$\implies \|f+g\|_p^{p-p/q=1} \leq \|f\|_p + \|g\|_p$$
$$\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

- $p = 1$, thanks to the triangular inequality we have:

$$\|f+g\|_1 = \int_X |f+g| \, d\mu \leq \int_X |f| \, d\mu + \int_X |g| \, d\mu$$

- $p = +\infty$, thanks to the triangular inequality we have:

$$\|f+g\|_\infty = \operatorname*{ess\,sup}_X |f+g| \leq \operatorname*{ess\,sup}_X (|f|+|g|) \leq \operatorname*{ess\,sup}_X |f| + \operatorname*{ess\,sup}_X |g|$$

□

Question 8.15

Show that L^p is a normed space.

L^p is a normed space

L^p is a normed space with norm:

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} \qquad p \in [1, +\infty)$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_X |f|$$

Proof. Clearly, we have that:

- $\|\cdot\| : L^p \to [0, +\infty)$
- $\|f\|_p = 0 \iff f = 0$ a.e. in X $\iff f = 0$ in L^p thank to its quotientiation with respect to equality a.e.
- $\forall \alpha \in \mathbb{R}$ we have:
$$\|\alpha f\|_p = |\alpha| \|f\|_p$$
- We have the triangular inequality thanks to Minkowski's inequality:
$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

□