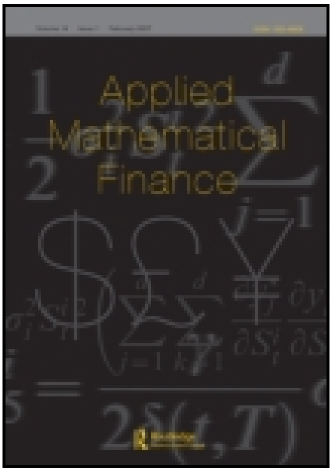

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A Note on Dual-Curve Construction: Mr. Crab's Bootstrap

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ABSTRACT *Observe crabs in the sand of our beaches: they move forward, backward and then forward again. Before the crisis, the standard bootstrap of interest rate curves was a 'Forward'-looking iterative algorithm where only information from previous knots was used to find discounts at subsequent dates.*

In this note we describe a new bootstrapping technique that involves various 'Backward' steps, which are reminiscent of a crab's steps: this new methodology coherently considers now standard dual-curve *framework. Two other major results emerge from the bootstrap methodology described: (i) discounts are independent from the chosen interpolation rule for all practical purposes; and (ii) convexity adjustments to Short-Term Interest Rate futures can be dealt with using a methodology in line with market practice.*

KEY WORDS: Bootstrap, dual-curve construction, STIR futures, convexity adjustments

1. Introduction

When dealing with interest rates derivatives one always starts with a procedure known as *curve construction* (hereinafter *CC*) – the algorithm used to obtain discount factors starting from quoted instruments in the interbank market: cash deposits (depos), Forward Rate Agreements (FRAs), Interest Rate (IR) swaps and Short-Term Interest Rate (STIR) futures. Prior to the 2007 interbank market crisis (hereinafter referred as the *crisis*) a *CC* technique, called *bootstrap*, was market standard (see e.g. Ron 2000); differences among major investment houses arose mainly from the discount factor interpolation rules. Standard bootstrap is a self-sustaining iterative technique that proceeds with time, where in order to compute discounts at the next curve's knot[1] only information from previous knots is used: in this sense, it is a 'Forward'-looking iterative algorithm. Following the *crisis*, the interbank curve is no longer unique since large basis spreads have appeared between different Euribor tenors and consequently the standard *bootstrap* can longer be implemented.

The views expressed here are those of the author and not necessarily those of the Bank.

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Each interbank market has its specific conventions and rules; in the following, for simplicity, we consider (except where stated differently) the EURO curve; however, *mutatis mutandis*, the methodology described can be applied also to other very liquid interbank IR markets such as USD.

A coherent *dual-curve* framework for interest rates following the *crisis* is presented in the article of Henrard (2010); two curves are involved:

- a *discounting* curve: usually chosen as the Effective OverNight Index Average (hereinafter EONIA) curve, since interbank market participants are generally (cash) collateralized, with collateral paying the EONIA rate;
- a *pseudo-discounting* curve[2]: a different curve for each Euribor tenor.

A possible explanation for using a different *pseudo-discounting* curve for each tenor is due to the credit risk related to borrowing/lending at various time horizons in the interbank market; moreover a recent scandal has shown a certain degree of arbitrariness in the fixing procedure for Euribor rates (Choudhury and Vaughan 2012). It is now commonly accepted that each Euribor curve should involve interest rate derivatives with the same tenor underlying rate.

Unfortunately, to date there is no commonly accepted description of a coherent bootstrap in the *dual-curve* framework; the most important info-provider in the interest rates market (i.e. Bloomberg) and very well-known financial software vendors (e.g. Numerix) use, in their standard *dual-curve* bootstrap for the *pseudo-discounting* curve of a given tenor (e.g. 3 months), depos with a shorter tenor (e.g. 1-month and 2-month depos). Starting with 'wrong' rates can be dangerous in a bootstrapping technique: since discounts on future knots depend on discounts on previous knots, an initial error propagates to all maturities.

In this note we propose and describe in detail a technique for post-*crisis* interest rates bootstrap within the theoretical frame proposed by Henrard (2010). This new technique modifies, in an elementary way, the standard bootstrap. As discussed in Section 3, this new bootstrap involves not only a 'Forward'-looking algorithm but also some 'Backward' steps. One example can clarify this point which is crucial in the proposed rates construction. Let us consider the curve relative to the Euribor 3-month: it is not possible to obtain discounts for maturities shorter than 3 months with a standard 'Forward' bootstrap and using only underlyings with the same tenor (3 months); for this reason we consider FRA rates implications on maturities shorter than 3 months. In this sense we shall have to move also 'Backwards'.

This note is organized as follows. In Section 2, we briefly recall key ingredients in the *discounting* curve bootstrap; in Section 3, we introduce and describe the *dual-curve* model under consideration and its consequences for the *pseudo-discounting* curve. Section 4 is devoted to the new 'Forward–Backward' bootstrap technique in the *dual-curve* setting and Section 5 to numerical results. In Section 6, we summarize some impacts of interpolation choice and show some consequences for discrete forward rates. Section 7 includes our concluding remarks. In the appendices, we provide proof of the key results and implications on model calibrations.

2. Discounting Curve

As commonly accepted in today's market practice (see e.g. Bianchetti 2010; Ametrano and Bianchetti 2009; Henrard 2010; Mercurio 2009), in this note we choose the EONIA curve as the *discounting* curve. In this section we recall known results on the EONIA curve (see e.g. Ametrano and Bianchetti 2009 and references therein): the bootstrap technique is standard and uses market quotes of Overnight Indexed Swap (hereinafter OIS) rates with increasing maturity. In the following we define $R^{OIS}(t_0, t_e)$ as the OIS rate starting on settlement date t_0 and lasting till t_e.[3] Swaps with a maturity up to one year and swaps with longer maturities are managed in different ways within the bootstrap.

If OIS lasts less than one year, with a start date $t_s = t_0$ and an end date t_e (i.e. $t_e - t_0 \leq 1y$), the two counterparties involved in the derivative contract exchange at a unique date (t_e) the difference between a rate $R^{OIS}(t_0, t_e)$ fixed in t_0 (two business days before) and a floating rate established in t_e as follows (see Banking Federation of the European Union (2008)):

$$R(t_s, t_e) \equiv \frac{\Pi_{k=t_s}^{t_e-1}\left[1+\delta\left(t_k^E, t_{k+1}^E\right)E\left(t_k^E\right)\right]-1}{\delta(t_s, t_e)}, \tag{1}$$

where $\delta(t_s, t_e)$ is the year fraction corresponding to the lag (t_s, t_e) and $E\left(t_k^E\right)$ is the EONIA rate on the kth day.[4]

Choosing EONIA as the *discounting* curve is equivalent to stating that $P^D(t_0, t_e)$, the value in t_0 of an instrument paying 1 in $t_e > t_0$, is the expected value of the stochastic discount factor $D(t_0, t_e)$:

$$P^D(t_0, t_e) \equiv E[D(t_0, t_e)],$$

where, in the following, we denote with $E[\cdot]$ the expectation $E[\cdot|t_0]$ given the information in t_0.

Using the above definitions, the *discounting* curve up to 1 year is (see Appendix A for details)

$$P^D(t_0, t_e) = \frac{1}{1+\delta(t_0, t_e)R^{OIS}(t_0, t_e)}. \tag{2}$$

If the OIS has a maturity t_i longer than one year, the two counterparties in the swap contract exchange at the end of each period the difference between the OIS rate $R^{OIS}(t_0, t_i)$ (i.e. the fixed rate) and the OIS floating rate $R(t_{k-1}, t_k)$ (with $k = 1, \ldots, i$) defined in (1). In these OIS contracts each period lasts one year with a short stub in advance, and in this case we get (see Appendix A for details)

$$P^D(t_0, t_i) = \frac{1-R^{OIS}(t_0, t_i)\Sigma_{k=1}^{i-1}\delta_k P^D(t_0, t_k)}{1+\delta_i R^{OIS}(t_0, t_i)}, \tag{3}$$

where δ_k is the year fraction for the calculation period (t_{k-1}, t_k).

Starting from quoted OIS rates, bootstrapping the *discounting* curve requires only Equations (2) and (3).

A natural question arises: which swap rates should be considered in the bootstrap of liquid curves? After the recent increase in electronic trading, all main market-makers quote bid/ask rates on a platform for EURO swaps (OIS, swap *vs* Euribor 3*m*, swap *vs* Euribor 6*m*) and USD swaps (OIS, swap *vs* USD Libor 3*m*) that allows end users to trade for market sizes at these prices on Bloomberg^{TM} terminals. Let us stress that these swap rates represent not just indications but tradable prices and they include all yearly rates up to 20*y* and also 25*y* and 30*y*. Adding information coming from other electronic platforms (e.g. Tradeweb) not only on swap outrights but also on swap spreads, since the beginning of 2006 a complete information set is available (i.e. every year up to 30*y* is contributed to by live prices for market size).

When bootstrapping liquid curves such as EURO and USD, it is possible to consider the live prices of one market-maker: however, even if each market-maker quotes all swap rates in these liquid markets with tight bid/ask spreads (often within one bp[5]), its prices can be influenced by its book positions. We prefer a different solution that is common practice in the market: we consider the rates published by one main broker (ICAP) on main yearly knots (2*y* up to 10*y*, then 12*y*, 15*y*, 20*y*, 25*y* and 30*y*) and we use a *natural cubic spline* (de Boor (1978) 2001) on mid-swap rates in order to obtain yearly rates (Garofalo 2006; Hagan and West 2006). Even if these rates are just indications and not tradable prices, this information is public and available to every player on the market for the whole trading day: in our experience, these yearly rates always fall within market bid/ask spreads.

Remark 2.1. The main property of OIS derivative contracts is that they allow curve bootstrap avoiding to consider EONIA fixings (even if these daily fixings appear in the definition (1) of floating rate) or the expected values of these rates. Only OIS rates are needed for bootstrapping, as shown in Equations (2) and (3). We note that the *discounting* curve on OIS curve knots (1*w*, 2*w*, 3*w*, 1*m*, 2*m*, …, 11*m*, 12*m*, 15*m*, 18*m*, 21*m*, 2*y*, 3*y*, …, 30*y*) does not depend on the chosen interpolation method in liquid markets (i.e. when a complete information set is available).

Remark 2.2. The particularity of EONIA FRA is that underlying rate is the corresponding swap rate $R^{OIS}(t_s, t_e)$ and not the floating rate $R(t_s, t_e)$ as in standard FRAs. In the EONIA curve, FRA rate $R^{FRA}(t_0, t_s, t_e)$ and EONIA forward rate are equivalent (see Appendix A for details).

3. Euribor Curve: Model and Theoretical Results

In this section, we introduce a new model which is *dual-curve* and multi-factor; as we will show, it allows an accurate description of various derivative instruments (i.e. STIR futures) involved in the bootstrap technique described in the next section. We obtain simple closed formulas needed in the bootstrap and we also generalize results obtained in the single-curve/single-factor case in Henrard (2005). This is a model within the theoretical framework described in Henrard (2010): we briefly recall model main hypothesis and some theoretical results that relate the *pseudo-discounting* curve $P(t_0, t)$ to the financial quantities of interest (e.g. Euribor rate and Euribor forward rate).

A j – Euribor floating coupon instrument is an asset that pays in t_e the Euribor fixing $L(t_s, t_e)$ for the tenor j on the period $[t_s, t_e]$[6] multiplied by the conventional year fraction $\delta(t_s, t_e)$. Its value in t_0 is

$$P^D(t_0, t_e)F(t_0; t_s, t_e) \equiv E[D(t_0, t_e)L(t_s, t_e)],$$

which is the standard definition of forward rate $F(t_0; t_s, t_e)$.

Each *pseudo-discounting* curve involves contracts defined only on the same tenor. As already noted in the introduction, there is a high degree of arbitrariness in Euribor fixing: for this reason one has to consider only contracts with the same underlying rate when bootstrapping each *pseudo-discounting* curve. In the following we consider in detail the 3-month Euribor curve.[7] Results can be extended to different tenors: in Section 4 we discuss similarities and differences with the 6-month Euribor case. Furthermore, given $F(t_0; t_s, t_e)$ the 3-month Euribor forward rate at time t_0 over the period $[t_s, t_e]$, with $t_0 \le t_s < t_e$, the forward value in t_0 of the *pseudo-discount* between t_s and t_e is defined as

$$P(t_0; t_s, t_e) \equiv \frac{1}{1 + \delta(t_s, t_e)F(t_0; t_s, t_e)}, \tag{4}$$

where the *pseudo-discounting* is

$$P(t_0; t_s, t_e) \equiv \frac{P(t_0, t_e)}{P(t_0, t_s)}; \tag{5}$$

in particular, when t_s is equal to t_0 in definition (4), the value of the *pseudo-discount* 3-months after the settlement date is equal to

$$P(t_0, t_e) \equiv \frac{1}{1 + \delta(t_0, t_e)L(t_0, t_e)}. \tag{6}$$

Definition (Henrard 2010).

The multiplicative spread (hereinafter spread) between the two curves (*discounting* curve and *pseudo-discounting* curve) is defined as

$$\beta(t; t_s, t_e) \equiv \frac{P^D(t; t_s, t_e)}{P(t; t_s, t_e)} \Diamond. \tag{7}$$

The following hypothesis is the main one in the theoretical framework proposed by Henrard (2010).

Hypothesis SI (Independence hypothesis).

The spread $\beta(t; t_s, t_e)$ is independent of the (forward) *discounting* value $P^D(t; t_s, t_e)$ on the same calculation period $\Diamond$.

Lemma 1. The spread $\beta(t; t_s, t_e)$ is martingale under the t_s-martingale measure. If hypothesis SI also holds, $\beta(t; t_s, t_e)$ is also martingale under the spot measure.

Proof. See Appendix A.

Let us introduce the model we consider in this note.

Hypothesis M (Model).

Under hypothesis SI, the dynamics for *discounting* curve is modelled by a Gaussian Heath–Jarrow–Morton (hereafter GHJM) model (Heath, Jarrow, and Morton 1992):

$$P^D(t;t_s,t_e) = P^D(t_0;t_s,t_e)$$
$$\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left[v^D(u,t_e)^2 - v^D(u,t_s)^2\right]du - \int_{t_0}^{t}\left[v^D(u,t_e) - v^D(u,t_s)\right]\cdot dW_u\right\},$$

while the spread $\beta(t;t_s,t_e)$ is

$$\beta(t;t_s,t_e) = \beta(t_0;t_s,t_e)\exp\left\{-\frac{1}{2}\int_{t_0}^{t_s}[\eta(u,t_e)-\eta(u,t_s)]^2 du + \int_{t_0}^{t_s}[\eta(u,t_e)-\eta(u,t_s)]\cdot dW_u\right\}$$

and (orthogonality property)

$$\eta(u,t)\cdot\rho v^D(u,s) = 0 \quad \forall t,s>u.$$

We indicate by $v^D(u,\ t)$ and $\eta(u,\ t)$ the two M-dimensional vectors of deterministic functions with $v^D(t,\ t) = \eta(t,\ t) = 0$, and W is a M-dimensional Brownian motion with instantaneous covariance matrix $\rho = (\rho_{ij=1,\ldots,M})$:

$$dW_i(u)\ dW_j(u) = \rho_{ij}\ du,$$

where $y \cdot z$ is the scalar product in $\mathbb{R}^M$ between two vectors y and z, while z^2 is a short notation for $z \cdot \rho z$ ◊.

Finally the *pseudo-discounting* volatility is defined as

$$v(u,t) \equiv v^D(u,t) + \eta(u,t) \in \Re^M. \tag{8}$$

Hypothesis SI is the main one in the framework described in Henrard (2010) and, together with hypothesis M, defines the *dual-curve model* we consider in this note.[8] In the remaining part of this section we address model consequences for each class of IR derivatives involved in the bootstrapping technique.

3.1 FRA

A FRA is a derivative contract linked to a forward fixing of a 3-month Euribor rate. Contractual payment in t_s of a receiver FRA is

$$\delta(t_s, t_e)\frac{F^{FRA}(t_0; t_s, t_e) - L(t_s, t_e)}{1 + \delta(t_s, t_e)L(t_s, t_e)}.$$

The following theorem relates a FRA rate to the corresponding forward rate.

Theorem 1. Under hypothesis SI,

$$F(t_0; t_s, t_e) = F^{FRA}(t_0; t_s, t_e). \tag{9}$$

Proof. See Theorem 1 in Henrard (2010).

3.2 STIR Futures

The most liquid STIR Euribor future contracts have as underlying the 3*m* Euribor rate and they are traded in the Euronext-LIFFE exchange. Future fixing dates are standard IMM dates.[9]

At time t_0, future price $\Phi(t_0)$ has as underlying the 3*m* Euribor rate $L(t_s, t_e)$. On the fixing date the relation between the Euribor fixing rate and future's price is

$$L(t_s, t_e) = 1 - \Phi(t_s).$$

At a generic settlement date $t_0 \leq t_s$ the corresponding future rate is defined as

$$F^{fut}(t_0; t_s, t_e) \equiv 1 - \Phi(t_0).$$

Theorem 2. In the *dual-curve model* Euribor, forward rate and future rate are related by

$$F(t_0; t_s, t_e) = \frac{1}{\gamma_1}F^{fut}(t_0; t_s, t_e) - \frac{1}{\delta(t_s, t_e)}\frac{\gamma_1 - 1}{\gamma_1} \simeq F^{fut}(t_0; t_s, t_e) - \frac{\tilde{\gamma}_1}{\delta(t_s, t_e)} \tag{10}$$

where (future's) convexity adjustment is

$$\gamma_1 = \exp\left\{\int_{t_0}^{t_s} v^D(u, t_e) \cdot \rho(v^D(u, t_e) - v^D(u, t_s))du\right\} \equiv 1 + \tilde{\gamma}_1.$$

Proof. It is a straightforward generalization to the multidimensional case of Theorem 2 in Henrard (2010).

Remark 3.1. Observing that $\tilde{\gamma}_1 \geq 0$, forward rates in Equation (10) are always lower than the corresponding future rates; convexity adjustment correction of future rates can even be of some basis points. Let us also stress that γ_1 depends only on *discounting curve* volatility v^D.

Remark 3.2. Let us note that, nowadays, both STIR futures and FRAs are collateralized contracts[10]: interbank market participants in the first case exchange margins with the clearing house, in the second case with the OTC derivative counterpart. However in order to compute margins, future price is used as reference price in the first case while FRA NPV is used in the second derivative contract: this justifies the main difference in valuation and the presence of the convexity adjustment in the future case (see Jarrow and Oldfield (1981) for details).

Volatilities that appear in the convexity adjustment γ_1 can be calibrated on futures calls/puts. In the Euribor case futures options are physical delivery and American: their price can be determined as in the European case. The most liquid options are the quarterly ones with expiry on underlying fixing date (i.e. on the standard IMM dates); in the following we consider only these options.

Theorem 3. In the *dual-curve model* a call and a put on a STIR future with expiry t_s, year fraction δ and strike K are

$$\begin{cases} \mathcal{C}_0 = \frac{P^D(t_0,t_s)}{\delta}\left[\hat{X}N(-d_2) - \hat{P}_0^{-1}N(-d_1)\right] \\ \mathcal{P}_0 = \frac{P^D(t_0,t_s)}{\delta}\left[\hat{P}_0^{-1}N(d_1) - \hat{X}N(d_2)\right] \end{cases}, \tag{11}$$

where

$$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{t_s - t_0}}\ln\frac{\hat{P}_0^{-1}}{\hat{X}} + \frac{1}{2}\sigma\sqrt{t_s - t_0} \\ d_2 = d_1 - \sigma\sqrt{t_s - t_0} \end{cases}$$

and

$$\begin{cases} \hat{P}_0^{-1} \equiv \gamma_2(P(t_0;t_s,t_e))^{-1} = \gamma_2(1 + \delta F(t_0;t_s,t_e)), \\ \hat{X} \equiv 1 + \delta(1 - K) \end{cases}$$

with

$$\sigma^2 = \frac{1}{t_s - t_0}\int_{t_0}^{t_s}(\nu(u,t_e) - \nu(u,t_s))^2 du$$

and (option's) convexity adjustment is

$$\gamma_2 = \exp\left\{\int_{t_0}^{t_s}\left(\nu^D(u,t_e) - \nu^D(u,t_s)\right)^2 du\right\} \equiv 1 + \tilde{\gamma}_2.$$

Proof. See Appendix A.

Remark 3.3. Note that STIR future option prices (11) depend on both *discounting curve* volatility v^D and *pseudo-discounting* volatility v. Unfortunately nowadays no liquid market exists for options on EONIA curve derivatives, so in order to calibrate both classes of volatilities we have just one market information (STIR option prices). For this reason, as we discuss in detail in Section 5, we need a further hypothesis in order to calibrate volatilities.

Remark 3.4. Another new feature of this note is to consider STIR future options in order to compute STIR future convexity adjustments. Since STIR future options have become, since the *crisis*, the most liquid options on this part of the curve, it has become more natural (and more accurate) to use them instead of OTC caps/floors (see details in Appendix B).

3.3 IR Swaps

Swap contracts in the $3m$ Euribor case are used to build the curve starting from the second year.[11] For a shorter notation we denote with $\tilde{t}_k$ swap's floating leg reset dates with Quarterly/Quarterly frequency and with $\tilde{\delta}_k$ the floating leg year fractions for the calculation period $(\tilde{t}_{k-1}, \tilde{t}_k)$. Directly from the definition of swap rate and having imposed equal to zero swaps' NPVs, one gets the following relation for forward Euribor rates $F_k(t_0) \equiv F(t_0; t_{k-1}, t_k)$:

$$\sum_{k=1}^{f \times i} w_k F_k(t_0) = \mathcal{I}(i) \qquad i \geq 2, \tag{12}$$

where

$$w_k \equiv \tilde{\delta}_k P^D(t_0, \tilde{t}_k),$$

with the frequency $f = 4$ in $3m$ case ($f = 2$ in the $6m$ case). The quantity $\mathcal{I}(i)$ is defined as follows

$$\mathcal{I}(i) \equiv S(t_0, t_i) \sum_{k=1}^{i} \delta_k P^D(t_0, t_k) \qquad i \geq 2.$$

Remark 3.5. The quantity $\mathcal{I}(i)$ is not dependent on interpolation methods in liquid markets. This property has important consequences for the *pseudo-discounts* bootstrap: as we discuss in detail in Section 6, *pseudo-discounts* depend weakly on the chosen interpolation rule.

4. Euribor Curve: Mr. Crab's Bootstrap

The most liquid IR contracts in the EURO area are Euribor-related derivatives. In this section we show how to bootstrap *pseudo-discounting* (or Euribor) curves in the *dual-curve* model described in the previous section.

In the following we first consider in detail the 3-month Euribor curve; at the end of this section we describe the main differences with the 6-month case. For the 3-month Euribor curve, reference contracts are the 3-month depo, 3-month FRAs (the 1 × 4, 2 × 5 and 3 × 6 contracts), STIR futures and swaps *vs* the 3-month Euribor.

In a standard 'Forward' bootstrap, Equations (4) and (5) are used to obtain the value of $P(t_0, t_e)$ given the known value of $P(t_0, t_s)$. For expiries lower than 3 months, depos would underestimate the credit spread of the 3*m* Euribor curve: for this reason, in order to obtain the *pseudo-discounts* for time horizons shorter than 3 months, we implement a 'Forward–Backward' technique (i.e. moving as a crab) based on FRA contracts. This technique is an alternative to the use of 'synthetic deposits' recently proposed by Ametrano (2011), having the advantage of being based only on tradeable contracts with the same underlying tenor.

We define $\hat{t}_k$ as the bootstrap knot *k* months after the settlement date with mod-foll convention.

Before entering into the details of the bootstrapping methodology, a few words should be said on the interpolation methodology. We have used a *linear-on-zero-rates* interpolation, where the zero-rate at time t_i is equal to $-\ln P(t_0, t_i)/(t_i - t_0)$ with the lag $(t_i - t_0)$ computed with an Act/365 day-count convention.[12] Let us mention that the interpolation can be much more general: in Section 6 we discuss in detail the consequences of a more generic choice.

Having bootstrapped the *discounting* curve using Equations (2) and (3), the proposed methodology for *pseudo-discount* construction in the 3-month case is the following:

(1) Consider the 3-month Euribor rate for the 3-month depo[13] and compute $P(t_0, \hat{t}_3)$ using (6).

(2) Consider the 3 × 6 FRA and calculate the 6-month discount $P(t_0, \hat{t}_6)$ moving 'Forward' from $\hat{t}_3$ using definitions (4) and (5):

$$P(t_0, \hat{t}_6) = P(t_0, \hat{t}_3) P(t_0; \hat{t}_3, \hat{t}_6).$$

(3) Consider the 1 × 4 and 2 × 5 FRAs in order to obtain the 1- and 2-month discounts. To this end, obtain the 4- and 5-month discounts $P(t_0, \hat{t}_{i+3})$with $i = 1, 2$ via interpolation; then move 'Backwards' with the formula

$$P(t_0, \hat{t}_i) = \frac{P(t_0, \hat{t}_{i+3})}{P(t_0; \hat{t}_i, \hat{t}_{i+3})},$$

with $i = 1, 2$ and where $P(t_0; \hat{t}_i, \hat{t}_{i+3})$ is obtained via (4).

(4) For expiries up to two years we consider the first $m = 7$ STIR future contracts. STIR futures have a 3-month Euribor rate underlying with start date $t_{s,i}$ and end date $t_{e,i}$ after 3 months mod-foll (with $i = 1, \ldots, m$); forward rates could be calculated from the future prices $\Phi_i(t_0)$ using Equation (10):

$$P(t_0, t_{e,i}) = P(t_0, t_{s,i}) P(t_0; t_{s,i}, t_{e,i}),$$

where $P(t_0, t_{s,i})$ is obtained via interpolation.[14] The convexity adjustment γ_1 is computed using futures option prices (11) for volatility calibration as described in Appendix B.

(5) Compute the last forward rate of the second year by using the 2-year swap rate in Equation (12):

$$F_8(t_0) = \frac{1}{w_8}\left[\mathcal{I}(2) - \sum_{k=1}^{2\times f-1} w_k F_k(t_0)\right],$$

where $f = 4$ and the first seven values of $F_k(t_0)$ forward rates are obtained by interpolating the *pseudo-discounts* on curve knots. In the 6-month case the technique is slightly different and it is described at the end of this section.

(6) Consider the quoted swap rates starting from the 3-year one: once we have chosen an interpolation rule, from Equation (12) we obtain the interpolated forward rates. The *pseudo-discounts* are computed by

$$P(t_0; t_{i-1}, t_i) \equiv \prod_{k\in Y(i)} \frac{1}{1+\tilde{\delta}_k F_k(t_0)} \quad i \geq 3, \tag{13}$$

where $k \in Y(i)$ indicates the 3-month forward rates in the ith year and the value of $P(t_0, t_{i-1})$ has already been obtained in previous bootstrap steps.[15]

In Figure 1 we show the key ingredients in the proposed 'Forward–Backward' bootstrap: after a 'Forward' movement using $3m$-depo and 3 × 6 FRA, via a 'Backward' movement one gets $1m$ and $2m$ curves using 1 × 4 and 2 × 5 FRAs. This construction is crucial in the bootstrap, in order to obtain curve values coherent with a $3m$ curve since the first STIR future starts always before $\hat{t}_3$.

In the 6-month Euribor case the bootstrap technique is analogous to the 3-month case.[16] *Mutatis mutandis*, the first three steps in the bootstrap described above are the same: we get the 6-month curve value from the corresponding $6m$ deposit and the one-year value with the 6 × 12 FRA; then, moving 'Backwards' with 1 × 7, 2 × 8, …, 5 × 11 FRAs, we calculate monthly discounts up to 5 months. The main difference is that future contracts do not exist with a 6-month Euribor rate as underlying; for this

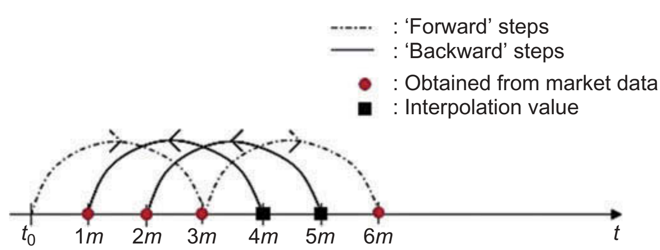

Figure 1. 'Forward–Backward' bootstrap: how to obtain the $3m$ Euribor curve $1m$ and $2m$ after the settlement date using 1 × 4 and 2 × 5 FRAs.

reason, steps 4 and 5 in the bootstrap are modified and the forward rates of the second year are computed by using the 18-month swap rate:

$$F_3(t_0) = \frac{1}{w_3}\left[\mathcal{I}(1) - \sum_{k=1}^{2\times f-1} w_k F_k(t_0)\right],$$

with $f = 2$, where the first two forward rates $\{F_k(t_0)\}_{k=0,1}$ have been computed in previous steps and

$$\mathcal{I}(1) \equiv S(t_0, \hat{t}_{18})[\delta(t_0, \hat{t}_6)P^D(t_0, \hat{t}_6) + \delta(\hat{t}_6, \hat{t}_{18})P^D(t_0, \hat{t}_{18})],$$

uses the quoted 18*m* swap rate $S(t_0, \hat{t}_{18})$. Furthermore, we have

$$F_4(t_0) = \frac{1}{w_4}[\mathcal{I}(2) - \mathcal{I}(1)].$$

Step 6 in the bootstrap is the same as in the 3-month case and it involves swaps (*vs* 6 months) with maturities longer than or equal to 3 years.

5. Numerical Results

In this section we show some numerical results of the described bootstrap methodology. We deal with the EURO interbank market on 13 September 2012 at 16:18 CET.

The *discounting* curve bootstrap described in Section 2 needs no further comment. We stress once more that results are, by construction, independent of the chosen interpolation rule on bootstrap knots.

When bootstrapping the 3*m* Euribor curve we need one more hypothesis that relates the volatilities of the *discounting* and *pseudo-discounting* curves; in fact, as underlined above (see Remark 3.3), in the *dual-curve* model we have two sets of volatilities but one market value, which is the STIR future option price.

The two simplest alternative assumptions (both more restrictive than SI) are

- the so-called hypothesis S0 (Henrard 2010): volatilities for *discounting* and *pseudo-discounting* curves are the same (i.e. we choose $v(s, t) = v^D(s, t)$);
- hypothesis S1: *discounting* volatility is equal to the volatility of the spread (i.e. $v(s, t) = \eta(s, t)$).

These hypotheses are the two limit cases of the *dual-curve* model since, by definition (8), *pseudo-discounting* volatility v is the sum of *discounting* curve volatility v^D and of volatility η.

Under hypothesis S0, one considers a deterministic spread between *discounting* and *pseudo-discounting* curves when calibrating IR models to options. In this case, since Equation (11) has σ as the only free parameter (under this additional hypothesis we have $\gamma_2 = \exp[\sigma^2(t_s - t_0)]$), obtaining its implied value from quoted options is as simple as in a standard Black formula. Under hypothesis S1 instead, convexity adjustments are equal to zero and Equations (11) are standard Black formulas with σ the only free parameter.

Table 1. STIR options implied volatilities and convexity adjustment corrections. As discussed in the text, we consider both the hypothesis S1 (with no adjustment) and the hypothesis S0 (with a deterministic spread β): we report the volatilities in the Euribor case σ_i^{mkt} (with $i = 1, \ldots, 7$) and the values obtained after the calibration with a HW model σ_i^{HW}. The market value for the first STIR option σ_i^{mkt} has not been computed since in the market no bid prices were present. The HW fit for market volatilities appears to be excellent with an order of error of one-hundredth per cent. We observe that the calibrated volatilities are very similar under the two hypotheses. In the last column we indicate the convexity adjustment corrections (in bps) under the hypothesis S0 (they are equal to zero under hypothesis S1).

	No adjustment (hypothesis S1)		*Deterministic spread* (hypothesis S0)		
Expiry date	$\sigma_i^{mkt}(\%)$	$\sigma_i^{HW}(\%)$	$\sigma_i^{mkt}(\%)$	$\sigma_i^{HW}(\%)$	$\tilde{\gamma}_1$ (bps)
19 December 2012	–	0.23	–	0.24	0.00
19 March 2013	0.23	0.25	0.23	0.25	0.10
20 June 2013	0.28	0.28	0.27	0.28	0.29
19 September 2013	0.32	0.30	0.32	0.30	0.63
18 December 2013	0.35	0.33	0.34	0.32	1.15
18 March 2014	0.34	0.36	0.33	0.34	1.94
19 June 2014	0.40	0.40	0.38	0.38	3.14

Table 2. HW parameters under the two hypotheses.

	$\hat{\sigma}$	a
Hypothesis S1	0.855	−0.613
Hypothesis S0	0.868	−0.560

In Table 1 we indicate STIR implied volatilities and the Hull–White (HW hereinafter) implied volatility (Hull and White 1990) obtained following the methodology described in Appendix B. We observe that Euribor volatilities are very similar in the two cases; in Table 2 we report HW calibration parameters obtained under the two hypotheses.

At this point, it may be useful to stop and comment. The *dual-curve* model is quite general and allows different volatility choices for the *discounting* and *pseudo-discounting* curves. Unfortunately, options on OIS derivatives are not liquid in the market and we can use only options on Euribor rates in order to calibrate the model. In particular, in the part of the curve relevant for convexity adjustments (less than 2 years), the most liquid ones are the STIR futures options. Hypothesis S0 could seem reasonable when computing convexity adjustments and is often considered a standard approach in the literature (see e.g. Henrard 2010): in this case we have only one implied volatility in Equation (11) that can easily be obtained inverting a Black-type formula. A calibration with a one-factor HW model already appears very accurate. Unfortunately this does not seem to be in line with market practice: when computing convexity adjustments, market participants seem to neglect *discounting* curve volatility (in line with hypothesis S1).

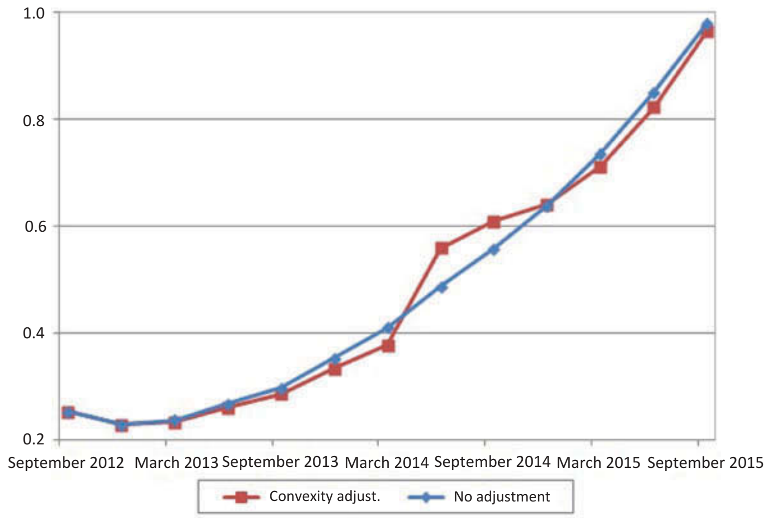

Figure 2. $3m$ Euribor curve forward rates in percentage points with convexity adjustments (hypothesis S0) and with no adjustment (hypothesis S1).

In Figure 2 we have plotted $3m$ forward rates; we observe that, including convexity adjustments (computed according to S0), a sharp and unjustified discontinuity appears while values are much more regular when not considering any adjustment (hypothesis S1). Smoothness is the first criterion that allows us to prefer hypothesis S1. A further indication that currently market practice neglects convexity adjustments can be obtained computing $18m$ swap rate from the bootstrapped curves and comparing this value with the quoted swap rate: the unadjusted value (27.5 bps) is practically equal to the mid-market value (27.4 bps), whereas when we consider S0 convexity adjustments we get a significantly lower value (26.8 bps).

In Figure 3 we consider the 3-month Euribor curve and compare it to the Bloomberg curve. We observe that results differ significantly from the Bloomberg

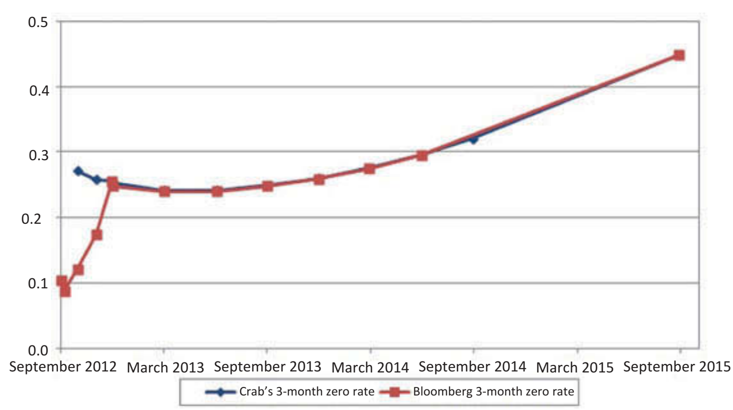

Figure 3. 3-month Euribor zero-rates bootstrapped with the linear interpolation rule compared to the Bloomberg curve.

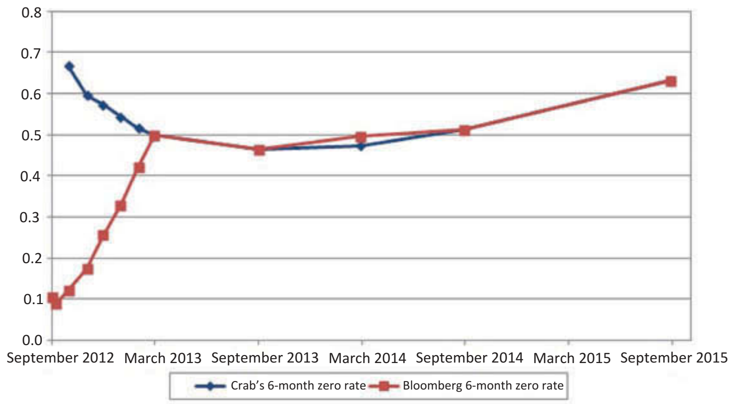

Figure 4. 6-month Euribor zero-rates bootstrapped with the linear interpolation rule compared to the Bloomberg curve.

curve. Crab's bootstrap considers only contracts with the same tenor as underlying: in this way curves do not present the cusp that characterizes curves built with classic bootstrapping methods based on shorter tenor deposits. In Figure 4 we compare the curve obtained with Crab's bootstrap to the Bloomberg curve in the 6*m* Euribor case: in this case the cusp in the Bloomberg curve is even more pronounced than in the 3*m* case.

In Figure 5 we show the zero-rates obtained for the EURO market (OIS, 3-month Euribor and 6-month Euribor) on 13 September 2012 at 16:18 CET. Similar results hold true in the USD case: in Figure 6 we show the zero-rates for the USD market (OIS and 3-month Libor) for the same date and time.[17]

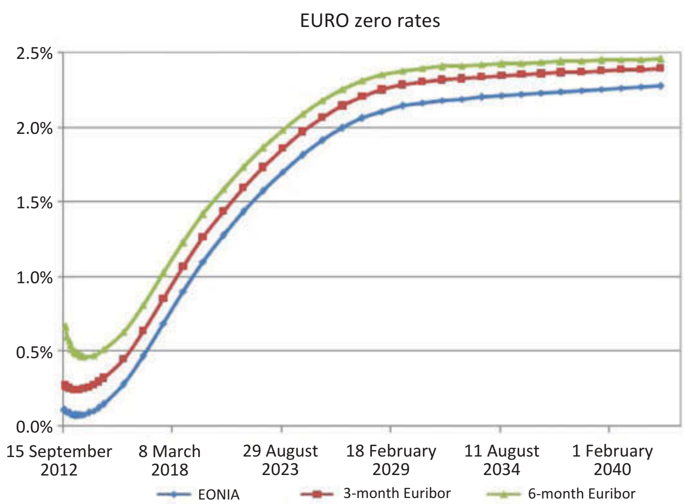

Figure 5. EURO zero-rate curves on 13 September 2012 at 16:18 CET. We plot the *discounting* curve and the *pseudo-discounting* curves in the 3*m* Euribor and 6*m* Euribor cases.

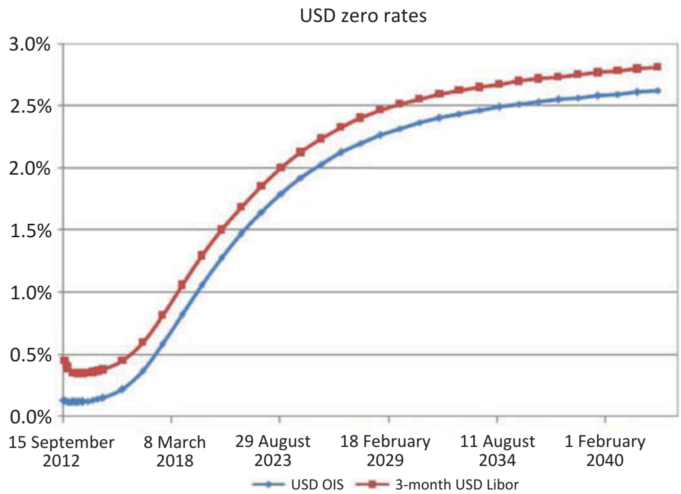

Figure 6. USD zero-rate curves on 13 September 2012 at 16:18 CET. We plot the *discounting* curve (OIS) and the *pseudo-discounting* curve in the 3*m* USD Libor case.

6. Interpolations, Stability and Forward Rates

The choice of the interpolation rule generally plays a central role in *CC*, and the set of interpolation rules that can be used in financial markets can be extremely wide. In this note, curve bootstrap was realized using a *linear-on-zero-rates* interpolation; however, for liquid IR markets where a complete information set is available, discounts do not depend on interpolation rule while *pseudo-discount* values on curve knots depend negligibly on interpolation. The former feature was mentioned in Section 2 and the latter result is discussed in this section. We also mention some possible consequences for algorithm choice and forward calculations.

In order to show this stability result we apply to several quantities (zero-rates, discounts, log-discounts, spreads) certain different interpolation rules: one global (spline) and two local (piecewise constant and linear). Local interpolation functions in each lag between two knots depend (at most) on the two knots' values; we obtain a constant value between two subsequent knots that includes the value at the ending knot for the piecewise constant interpolation and a straight line that connects the two values for the linear interpolation. With spline we mean a *natural cubic spline* if the number of knots is greater than two and a linear interpolation when only two knots are involved. It is well known that spline is a global interpolator, since it depends on all values used in the interpolation. The use of such a global interpolation is unorthodox in a bootstrap. Since we add one knot after the other, the number of knots changes during this *CC*: it might appear unsound to use an interpolation function that depends at each step on all knots; the interpolated values are clearly different at the beginning of the bootstrap (e.g. when we include just four knots) and near the end of *CC* (e.g. when we include 30 knots). However it is exactly what we need in order to show the stability of the proposed bootstrap: for (almost) every interpolation rule we obtain practically the same value on curve knots.

Besides *linear-on-zero-rates* interpolation we also considered *log-linear on discounts, spline on discounts, spline on zero-rates, piecewise constant spread* and *linear spread.*

When considering the *piecewise constant spread* and *linear spread* interpolations, step 6 in the bootstrap technique described in Section 4 is slightly modified. Rather than Equation (12) we use the following linear relation for the spread:

$$\sum_{k \in Y(i)} \left[P^D(t_0,\ \tilde{t}_{k-1})\ \beta(t_0; \tilde{t}_{k-1}, \tilde{t}_k) - P^D(t_0, \tilde{t}_k) \right] = \mathcal{I}(i) - \mathcal{I}(i-1) \qquad i \geq 3.$$

This equation can be derived by Equation (12) and it can be solved in the *piecewise constant spread* and *linear spread* cases.[18]

In Table 3 we show discount differences on curve knots between *linear-on-zero-rates* and the other interpolation techniques in the 3-month case (similar results, not reported in this note, hold true in the 6-month case). Interpolation errors are of the order of some hundredths of bp: negligible for all practical purposes.

This result is important: the following lemma states that for *pseudo-discounts* on curve knots obtained from swap rates the impact of the interpolation method is negligible when a complete information set is available.

Lemma 2. Using an interpolation rule *I*, one gets an interpolated value for the Euribor forward rate $L_k^{(I)}(t_0)$ that differs from the 'true' Euribor forward rate $F_k(t_0)$ for an error

$$\epsilon_k^{(I)} \equiv L_k^{(I)}(t_0) - F_k(t_0).$$

When a complete information set is available, the error on the forward *pseudo-discount* value between two following years' t_{i-1} and t_i (with $i > 2$) is equal to

$$\begin{aligned} &P^{(I)}(t_0; t_{i-1}, t_i) - P(t_0; t_{i-1},\ t_i) = P(t_0; t_{i-1}, t_i) \\ &\quad \sum_{k \in Y(i)} \tilde{\delta}_k\, \epsilon_k^{(I)} \left[\tilde{\delta}_k F_k(t_0) + \delta(\tilde{t}_k, t_i) R^{FRA}(t_0, \tilde{t}_k, t_i) \right] + O(\epsilon^2) + O(\epsilon F^2) \quad i \geq 3 \end{aligned}$$

where $O(\cdot)$ is the standard Landau symbol.

Proof. See Appendix A.

Let us comment on this simple result that has particularly relevant consequences in practice. Interpolation error ϵ on forward rates can be of several bps depending on the chosen interpolation rule; instead, *pseudo-discount* error is of a few hundredths of bp on bootstrap knots starting from the second year. This is a consequence of Lemma 2, since interest rates (forward and OIS FRA rates) are of some percentage points. This result is in line with the values shown in Table 3 in the 3*m* Euribor case: *pseudo-discount* values on curve knots can be considered independent of the interpolation rule for all practical purposes in liquid markets.

This stability result has (at least) two important practical consequences.

Table 3. 3-month Euribor *pseudo-discounts*: in the first column we report bootstrap knots (for the $3m$ curve); in the second the *pseudo-discounts* $P^{3m}(t_0, t_e)$ computed with the *linear-on-zero-rates* interpolation rule; and in the other five columns the differences (Δ in 10^{-1} bps) between the curves computed with the linear-on-zero-rates interpolation rule and the other five interpolation rules considered (*log-linear-on-discounts, spline-on-zero-rates, spline on discounts, PWC spline and linear spline*).

Date	*Pseudo-discounts*	*Log-linear discount* Δ (10^{-1}bps)	*Spline zero-rates* Δ (10^{-1}bps)	*Spline discount* Δ (10^{-1}bps)	*PWC spread* Δ (10^{-1}bps)	*Linear spread* Δ (10^{-1}bps)
17 September 2012	1.00000	0.00	0.00	0.00	0.00	0.00
17 October 2012	0.99978	−0.74	0.00	−0.74	0.00	0.00
19 November 2012	0.99956	−0.68	0.00	−0.69	0.00	0.00
17 December 2012	0.99936	0.00	0.00	0.00	0.00	0.00
19 December 2012	0.99936	1.14	0.05	2.14	0.00	0.00
19 March 2013	0.99879	1.14	0.05	2.14	0.00	0.00
20 June 2013	0.99818	1.15	0.05	2.15	0.00	0.00
19 September 2013	0.99750	1.15	1.30	1.46	0.00	0.00
18 December 2013	0.99676	1.16	0.77	1.61	0.00	0.00
18 March 2014	0.99588	1.16	0.77	1.61	0.00	0.00
19 June 2014	0.99483	1.16	0.77	1.61	0.00	0.00
17 September 2014	0.99361	0.00	0.00	0.00	0.00	0.00
17 September 2015	0.98663	0.03	−0.05	−0.04	0.00	−0.01
19 September 2016	0.97480	0.08	0.01	0.00	0.00	0.01
18 September 2017	0.95817	0.14	−0.05	0.00	0.00	−0.02
17 September 2018	0.93796	0.19	−0.08	−0.05	0.00	0.01
17 September 2019	0.91530	0.23	−0.03	−0.02	0.00	−0.04
17 September 2020	0.89117	0.27	−0.03	−0.03	0.00	0.00

Table 3. Continued.

Date	*Pseudo-discounts*	*Log-linear discount* Δ (10^{-1}bps)	*Spline zero-rates* Δ (10^{-1}bps)	*Spline discount* Δ (10^{-1}bps)	*PWC spread* Δ (10^{-1}bps)	*Linear spread* Δ (10^{-1}bps)
17 September 2021	0.86638	0.30	−0.02	−0.02	0.01	−0.09
19 September 2022	0.84089	0.32	0.03	0.02	0.00	0.02
18 September 2023	0.81516	0.34	0.08	0.07	0.00	−0.10
17 September 2024	0.78948	0.35	0.06	0.06	0.00	0.02
17 September 2025	0.76450	0.36	0.03	0.03	0.00	−0.10
17 September 2026	0.74068	0.38	0.01	0.01	0.01	−0.10
17 September 2027	0.71830	0.38	0.04	0.04	0.02	−0.11
18 September 2028	0.69750	0.39	0.04	0.04	0.03	−0.09
17 September 2029	0.67840	0.39	0.11	0.11	0.03	−0.06
17 September 2030	0.66063	0.38	0.11	0.11	0.03	−0.05
17 September 2031	0.64394	0.38	0.13	0.12	0.03	−0.04
17 September 2032	0.62799	0.38	0.11	0.10	0.03	−0.06
19 September 2033	0.61235	0.37	0.14	0.14	0.03	−0.05
18 September 2034	0.59716	0.37	0.10	0.10	0.03	−0.07
17 September 2035	0.58230	0.36	0.10	0.10	0.03	−0.06
17 September 2036	0.56773	0.36	0.08	0.08	0.04	−0.08
17 September 2037	0.55350	0.35	0.11	0.11	0.04	−0.06
17 September 2038	0.53962	0.35	0.09	0.09	0.04	−0.07
19 September 2039	0.52602	0.35	0.10	0.10	0.04	−0.05
17 September 2040	0.51291	0.34	0.10	0.10	0.04	−0.06
17 September 2041	0.50009	0.33	0.13	0.13	0.04	−0.03
17 September 2042	0.48762	0.33	0.13	0.13	0.04	−0.03

On the one hand, we can decide to use a different interpolation rule when bootstrapping *pseudo-discounts* from IR swaps and from other IR contracts. For example, the *piecewise constant* or *linear spread* interpolations fall within this class and they allow one to obtain a completely linear bootstrapping technique. The algorithm can be extremely efficient since the number of operations (and thus the computational time) is linear in the number of curve knots.

On the other hand, since we obtain almost the same *pseudo-discounts* on curve knots whatever interpolation rule we consider, when computing forward rates starting from pseudo-discount values on curve knots we can use an interpolation rule different from that used in the bootstrap (and even a global one).

We need to discuss briefly how to obtain discrete forward rates from pseudo-discount curves. From Equation (4) we obtain

$$F(t_0; t_s, t_e) = \frac{1}{\delta(t_s, t_e)}\left(\frac{P(t_0, t_s)}{P(t_0, t_e)} - 1\right) \simeq -\frac{\ln P(t_0, t_e) - \ln P(t_0, t_s)}{\delta(t_s, t_e)}$$

(i.e. discrete forward rates are related to the (discrete) slope of the logarithms of *pseudo-discounts*). In particular, when dealing with forward rates after the second year, as already discussed by Hagan and West (2006) in the continuous case, we obtain three results on forwards from the above formula: (i) forwards are (approximately) piecewise constant if we use a *log-linear on discounts* interpolation; (ii) they are (approximately) linear in time and discontinuous at knots using a *linear-on-zero-rates* interpolation; and (iii) they are continuous when *pseudo-discounts* are differentiable.

If we aim to obtain continuous forward rates we can use, for example, a *natural cubic spline* on *pseudo-discounts* (e.g. in Figure 7 we plot forward rates in the EURO market obtained in this way).

7. Conclusions

After the 2007 financial crisis, it has become clear that interest rate bootstraps should involve a *discounting* curve and a *pseudo-discounting* curve for each underlying tenor.

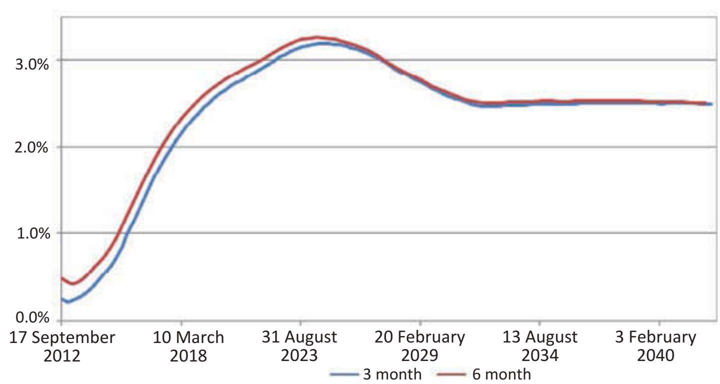

Figure 7. EURO forward rates in percentage points on 13 September 2012 at 16:18 CET for the 3-month and 6-month cases.

Unfortunately the standard bootstrap, which is only 'Forward' looking, involves contracts of shorter tenors in 3- and 6-month *CC*. We propose a new bootstrap where some 'Backward' steps are involved, a technique that considers only the same tenor contracts. We have considered in detail the EURO market, focusing on the 3- and 6-month Euribor cases; however, the results can be generalized for liquid interest rate markets.

Within the coherent theoretical framework widely accepted among practitioners (Henrard 2010), we have introduced a new *dual-curve* model that can be calibrated on STIR future options in a simple, fast and accurate way: the approach described solves the well-known chicken and egg problem in convexity adjustment, since it allows estimating first the volatility parameters and then the adjusted futures rates. This fact allows us to consider very precisely the impact of convexity adjustments on interest rates bootstrap and to verify that, currently, market participants neglect *discounting* curve volatility when computing convexity adjustments: we have shown that the proposed *dual-curve* model has this as a possible outcome.

Finally we observed that the choice of the interpolation rule has a negligible impact on *CC*. *Discounting* curve is independent of the interpolation rule by construction; we have also shown that the *pseudo-discounting* curve is, for all practical purposes, independent of the chosen interpolation rule on curve knots that involve swap rates.

Let us mention here that our approach is not the only possible one in the *CC* problem. In order to avoid standard bootstrap problems on first knots, a general multidimensional root finding is proposed that simultaneously obtains all knots in the discounting curve and the *pseudo-discounting* curve (Andersen and Piterbarg 2010). Even if solving a set of non-linear equations is technically feasible and it has been implemented by a software vendor (OpenGamma), from a practitioner point of view a multidimensional root finder is a 'black box' where, in general, the user is not guaranteed to obtain a unique curve.

Clearly, as already stated in several reference papers, 'there is no single correct way to complete the term structure of a yield curve from a set of rates' (Hagan and West 2006, 90). We consider that, when judging a *CC*, a practitioner adopts the main criteria as those proposed in Hagan and West (2006):

(1) Precision. Is the *CC* able to reproduce all liquid derivatives prices used in the construction? And how it works with some liquid derivatives prices not included in the construction?
(2) Speed. Is the algorithm fast?
(3) Local interpolation. How local is the interpolation method?
(4) Stability. Is the algorithm also stable?
(5) Continuity. How do the discrete forward rates look? Are they sufficiently smooth?

We have shown that Crab's bootstrap generalizes, in a coherent way, the *standard* bootstrap to the multi-curve framework for very liquid interest rates markets such as EURO and USD.

In this note we have shown that the proposed methodology can be a suitable candidate for the five desirable features above. It allows exact reproduction not only of derivatives prices included in the construction but also some very liquid contracts not included (e.g. the 18*m* swap rate *vs* 3*m*). It is extremely fast, being linear in the number of knots. The *linear-on-zero-rates* interpolation rule allows the proposed methodology to be local. Furthermore we have also shown that results are incredibly stable for knots that correspond to swap maturities, showing differences for *pseudo-discounts* of the order of the hundredth of a basis point using different interpolation rules. A consequence of this 'uniqueness' of curve values on bootstrap knots is that we can use a different interpolation rule when computing forward rates and, in this way, we can obtain smooth forward values.

Finally the 'uniqueness' of curve values on knots opens up a way forward for obtaining a reference set for interest rate discounting curves given a set of reference rates for the underlying instruments: this could have significant consequences to risk management policies and central clearing for OTC derivatives.

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Notes

[1]Discount curves are generally provided as the discount values on a discrete set of relevant dates called curve knots, or simply knots.

[2]In the literature also termed *forward* curve, *estimation* curve and *forecast* curve.

[3]Recall that OIS swap rate is the fixed rate in the swap. In the interbank market, quoted swap rates correspond to swaps with Net Present Value (hereinafter NPV) equal to zero.

[4]A 3-month OIS with trade date 16 January 2012 has t_0 equal to 18 January 2012 and t_e to 18 April 2012. The first EONIA fixing in on 18 January 2012 (at approximately 13:00 CET) and on 17 April 2012 (at 13:00 CET). In this note we do not stress the difference between trade date and settlement date t_0: time schedule starts always from the settlement date, while trade and fixing dates fall always, in the EURO market, two business days before the relevant reset dates.

[5]1 bp (basis point) is equal to 0.01%.

[6]Fixing takes place two business days before t_s at 11:00 CET.

[7]$3m$ curve knots are the settlement date t_0, $1m$, $2m$ and $3m$, the payment date of the underlying $3m$ depo of each of the $m = 7$ STIR futures, and annual knots from $2y$ up to $30y$.

[8]The model described is not the only stochastic description of spread; Henrard (2013) and Mercurio and Xie (2012) recently introduced other models where the spread is dependent on OIS forward rates. The approach in these studies differs significantly from ours: hypothesis SI (and thus Theorem 1) does not hold. Bootstrap is more complex due to the presence of convexity adjustments, even for simple FRAs.

[9]Two business days before the third Wednesday in March, June, September and December.

[10]OTC derivatives considered in this note are fully collateralized contracts between interbank market counterparties that have signed an ISDA Master agreement with a Collateral Support Annex.

[11]We recall that in the EURO swap market *vs* $3m$, the fixed leg is paid annually with a modified-following adjustment rule (mod-foll hereinafter), while the floating leg is paid quarterly mod-foll.

[12]The same rule is used when one needs to extrapolate, generally for a few days outside the already bootstrapped ending-knots.

[13]Due to the relative illiquidity of the 3-month depo, for bootstrapping often the best estimation is Euribor fixing after 11:00 CET and, before that hour, previous business day fixing. In any case after 11:00 CET Euribor fixing determines the first term of the swap floating leg.

[14]We recall that $t_{s,i+1}$ does not coincide in general with $t_{e,i}$ where $i = 1, \ldots, m - 1$.

[15]In the *linear-on-zero-rates* interpolation this is the only step where a one-dimensional root finder is involved. Since Equation (12) states that a weighted sum of four positive exponentials is equal to a

positive number, due to monotonicity there is a unique solution and due to convexity convergence this is extremely fast. We thank the referee for having underlined this point.

[16]6*m* curve knots are the settlement date t_0, each month up to 6*m*, each semester up to 2*y* and then annually up to 30*y*.

[17]Also in the 3-month USD Libor case, liquid swaps have a fixed leg that is paid annually; the only difference is that it has an Act/360 day-count.

[18]We use *piecewise constant spread* and *linear spread* when bootstrapping curve knots after the second year, while the *linear-on-zero-rates* interpolation is used in previous knots. The underlying research market data for EURO and USD interbank markets on 13 September 2012 at 16:18 CET can be accessed at http://www.mate.polimi.it/qfinlab/baviera/data/MarketData_Crab.xls.

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Appendix A

Proof of Equation (2). Stochastic discount, for the EONIA discounting curve, is equal to

$$D(t_0,t) = \frac{1}{\prod_{k=t_0}^{t-1}\left[1+\delta(t_k^E,t_{k+1}^E)E(t_k^E)\right]} = \frac{1}{1+\delta(t_0,t)R(t_0,t)},$$

where the last equality derives from the definition of the floating rate $R(t_0, t)$ (1).

For OIS with a maturity of less than 1 year, imposing OIS NPV equal to zero is equivalent to

$$E\left[D(t_0,t)R^{OIS}(t_0,t)\right] = E[D(t_0,t)R(t_0,t)], \tag{14}$$

or, equivalently

$$\delta(t_0,t)P^D(t_0,t)R^{OIS}(t_0,t) = \delta(t_0,t)E[D(t_0,t)R(t_0,t)] = 1 - E[D(t_0,t)],$$

we prove Equation (2) observing that, by definition,

$$P^D(t_0,t) \equiv E[D(t_0,t)]\clubsuit$$

Proof of Remark 2.2. We prove the equivalence between $R^{FRA}(t_0,\ t_s,\ t_e)$ and EONIA forward rate. A receiver EONIA FRA is characterized by a settlement payment at expiry date t_s (see definition (2) in Banking Federation of the European Union (2008) for details):

$$\delta(t_s,t_e)\frac{R^{FRA}(t_0;t_s,t_e) - R^{OIS}(t_s,t_e)}{1+\delta(t_s,t_e)R^{OIS}(t_s,t_e)} \quad t_0 \le t_s \le t_e$$

where EONIA FRA rate $R^{FRA}(t_0;\ t_s,\ t_e)$ is established in t_0 (two business days before) and it has a calculation period start in t_s and a calculation period end in t_e; and R^{OIS} $(t_s,\ t_e)$ is the OIS rate for the same calculation period that will fix at a future date t_s (less two business days) at 11:00 CET. Quoted EONIA FRA rate has zero FRA NPV; after some algebra, one obtains

$$P^D(t_0,t_e)R^{FRA}(t_0;t_s,t_e) = E[D(t_0,t_e)R^{OIS}(t_s,t_e)]$$

and

$$R^{FRA}(t_0;t_s,t_e) = \frac{1}{\delta(t_s,t_e)}\left(\frac{P^D(t_0,t_s)}{P^D(t_0,t_e)} - 1\right). \tag{15}$$

Using (14) with $E[\cdot|t_0] = E[E[\cdot|t_s]|t_0]$ we get

$$E[D(t_0,t_e)R^{OIS}(t_s,t_e)] = E[D(t_0,t_e)R(t_s,t_e)]$$

or

$$P^D(t_0,t_e)R^{FRA}(t_0;t_s,t_e)] = E[D(t_0,t_e)R(t_s,t_e)],$$

where the previous equation states that EONIA FRA rate $R^{FRA}(t_0,t_s,t_e)$ is equal to EONIA forward rate. As is well known, this result is true only in the EONIA case since it is also the *discounting* curve; it is not true in general for the *pseudo-discounting* curve (see Theorem 1) ♣

Proof of Equation (3). OIS rate sets to zero the swap NPV at t_0.
OIS receiver NPV for a longer than 1-year swap is

$$\begin{aligned} NPV &= E\left[R^{OIS}(t_0,t_i)\sum_{k=1}^{i}\delta_k D(t_0,t_k) - \sum_{k=1}^{i}\delta_k R(t_{k-1},t_k)D(t_0,t_k)\right] \\ &= R^{OIS}(t_0,t_i)\sum_{k=1}^{i}\delta_k P^D(t_0,t_k) - \sum_{k=1}^{i}\delta_k R^{FRA}(t_0;t_{k-1},t_k)P^D(t_0,t_k) \end{aligned},$$

where the last equality is obtained by using the definition of the *discounting* curve and the above result. Let us stress that Equation (16), instead of involving rates with daily fixings, has a floating leg characterized by the discounted sum of EONIA FRA rates $R^{FRA}(t_0;t_k,t_{k+1})$, as in Ametrano and Bianchetti (2009). From Equation (15) we see that a telescopic sum holds:

$$\sum_{k=1}^{i}\delta_k R^{FRA}(t_0;t_{k-1},t_k)P^D(t_0,t_k) = 1 - P^D(t_0,t_i).$$

By substituting the above equation into (16) and setting NPV to zero, we get Equation (3) ♣

Proof of Lemma 1. Forward Euribor definition is

$$P^D(t_0,t_e)F(t_0;t_s,t_e) = E[D(t_0,t_e)L(t_s,t_e)].$$

This is equivalent to

$$P^D(t_0,t_e)\frac{\beta(t_0;t_s,t_e)}{P^D(t_0;t_s,t_e)} = E\left[D(t_0,t_e)\frac{\beta(t_s,t_e)}{P^D(t_s,t_e)}\right] = E[D(t_0,t_s)\beta(t_s,t_e)] \tag{16}$$

or, equivalently, under the t_s-forward measure $\beta(t;\ t_s,\ t_e)$ is martingale

$$P^D(t_0,t_s)\beta(t_0;t_s,t_e) = P^D(t_0,t_s)E^{(s)}[\beta(t_s,t_e)].$$

If also hypothesis SI holds, Equation (16) becomes

$$P^D(t_0,t_s)\beta(t_0;t_s,t_e) = E[D(t_0,t_s)]E[\beta(t_s,t_e)],$$

i.e. $\beta(t;t_s,t_e)$ is martingale even under the spot measure ♣

The following technical Lemma is needed in the proof of Theorem 3.

Lemma A1. Under hypothesis M, the following relation for the *pseudo-discounts* holds:

$$P^{-1}(t_s,t_e) = \hat{P}_0^{-1}\exp\left\{-\frac{1}{2}\int_{t_0}^{t_s}[v(u,t_e)-v(u,t_s)]^2 du + \int_{t_o}^{t_s}[v(u,t_e)-v(u,t_s)]\cdot dW_u^{(s)}\right\},$$

where $\hat{P}_0^{-1}$ is defined in Theorem 3.

Proof. Hypothesis M for the *discounting* curve can be rewritten as

$$\begin{aligned}(P^D(t_s,t_e))^{-1} = (P^D(t_0;t_s,t_e))^{-1}\exp\Bigg\{&\int_{t_0}^{t_s} v^D(u,t_e)\cdot\rho\left[v^D(u,t_e)-v^D(u,t_s)\right]du \\ &-\frac{1}{2}\int_{t_0}^{t_s}\left[v^D(u,t_e)-v^D(u,t_s)\right]^2 du + \int_{t_0}^{t_s}\left[v^D(u,t_e)-v^D(u,t_s)\right]\cdot dW_u\Bigg\},\end{aligned}$$

or equivalently in the t_s-forward measure:

$$\begin{aligned}(P^D(t_s,t_e))^{-1} = \gamma^2(P^D(t_0;t_s,t_e))^{-1}\exp\Bigg\{&-\frac{1}{2}\int_{t_0}^{t_s}\left[v^D(u,t_e)-v^D(u,t_s)\right]^2 du \\ &+\int_{t_0}^{t_s}\left[v^D(u,t_e)-v^D(u,t_s)\right]\cdot dW_u^{(s)}\Bigg\},\end{aligned}$$

where

$$dW^{(s)}(u) = dW(u) + \rho v^D(u,t_s)du$$

and $\gamma_2$is defined in Theorem 3.

Using the definition of spread $\beta(t;t_s,t_e)$

$$(P(t;t_s,t_e))^{-1} = (P^D(t;t_s,t_e))^{-1}\beta(t;t_s,t_e),$$

the dynamics for $\beta(t;t_s,t_e)$ (see hypothesis M) and the ortogonality property we prove Lemma ♣

Proof of Theorem 3. We consider the case put. *Mutatis mutandis*, the proof is the same in the call case.

STIR future price is

$$\Phi(t_0) = 1 - E[L(t_s, t_e)]$$

while STIR put option is equal to

$$\begin{aligned}\mathcal{P}_0 &= E[D(t_0, t_s)[K - \Phi(t_s)]^+] = E[D(t_0, t_s)[L(t_s, t_e) - (1 - K)]^+] \\ &= \frac{1}{\delta} E[D(t_0, t_s)[P^{-1}(t_s, t_e) - \hat{X}]^+].\end{aligned} \tag{17}$$

Using Lemma A1 we get

$$P^{-1}(t_s, t_e) = \hat{P}_0^{-1} \exp\left\{ -\frac{\sigma^2}{2}(t_s - t_0) + \sigma\sqrt{t_s - t_0}g \right\},$$

where g is a standard normal variable. The theorem is proven once we rewrite (17) in the t_s-forward measure:

$$\mathcal{P}_0 = \frac{P^D(t_0, t_s)}{\delta} E^{(s)}[P^{-1}(t_s, t_e) - \hat{X}]^+ = \frac{P^D(t_0, t_s)}{\delta} E[\hat{P}_0^{-1} e^{-\frac{\sigma^2}{2}(t_s - t_0) + \sigma\sqrt{t_s - t_0}\, g} - \hat{X}]^+ \clubsuit$$

Proof of Lemma 2. First we show that the following relation holds for $i > 2$ when a complete information set is available:

$$\sum_{k \in Y(i)} \tilde{\delta}_k \epsilon_k^{(I)} = -\sum_{k \in Y(i)} \tilde{\delta}_k \epsilon_k^{(I)} \delta(\tilde{t}_k, t_i) R^{OIS}(t_0; \tilde{t}_k, t_i). \tag{18}$$

It is straightforward from Equation (12) after having observed that

$$0 = \sum_{k \in Y(i)} \tilde{\delta}_k P^D(t_0, \tilde{t}_k) \epsilon_k^{(I)} = P^D(t_0, t_i) \sum_{k \in Y(i)} \tilde{\delta}_k \epsilon_k^{(I)} [1 + \delta(\tilde{t}_k, t_i) R^{OIS}(t_0; \tilde{t}_k, t_i)].$$

Thus we have the fact that between two following years for $i > 2$:

$$\begin{aligned} P^{(I)}(t_0; t_{i-1}, t_i)^{-1} &\equiv \prod_{k \in Y(i)} \left[1 + \tilde{\delta}_k L_k^{(I)}(t_0)\right] \\ &= P(t_0; t_{i-1}, t_i)^{-1} \left[1 + \sum_{k \in Y(i)} \frac{\tilde{\delta}_k \epsilon_k^{(I)}}{1 + \tilde{\delta}_k F_k}\right] + O(\epsilon^2) \\ &= P(t_0; t_{i-1}, t_i)^{-1} \left[1 + \sum_{k \in Y(i)} \tilde{\delta}_k \epsilon_k^{(I)} \left(1 - \tilde{\delta}_k F_k\right)\right] + O(\varepsilon^2) + O(\epsilon F^2). \end{aligned}$$

After substituting (18) and some algebra, the lemma is proven ♣

Appendix B: Volatility Calibration

In order to calibrate model volatilities STIR future options are, nowadays, the most liquid options on the part of the IR curves where convexity adjustments can be relevant in the bootstrap.

As already stressed in Section 3.2, we can use only options on Euribor rates since no liquid market exists for options on EONIA derivatives; for this reason a further hypothesis is needed in order to calibrate the model since we have two sets of volatility curves and one of market data. The two most frequently adopted are either to consider a deterministic spread β (S0 hypothesis) or to neglect *discounting* curve volatility when computing convexity adjustments (S1 hypothesis). These are the two limiting hypotheses ($v^D = v$ and $v^D = 0$) in the *dual-curve* model under consideration.

We use ATM straddle prices since these are very liquid. Implied volatilities $\{\sigma_i^{mkt}\}_{i=1,\dots,m}$ can be obtained from market prices inverting Black formulas (11) in the S1 hypothesis. In a similar way, in the S0 hypothesis, since ATM straddle delta is almost equal to zero, it allows one to ignore the difference between forward and future rates as a first step in an iterative process: for this reason, implied volatility values for the *pseudo-discounting* curve (i.e. $\{\sigma_i^{mkt}\}_{i=1,\dots,m}$) are very similar under the two hypotheses.

Once $\{\sigma_i^{mkt}\}_{i=1,\dots,m}$ values are obtained, it is straightforward to calibrate HW parameters, with a standard Least Square Minimization:

$$d = \sum_{i=1}^{m} \left(\sigma_i^{HW} - \sigma_i^{mkt}\right)^2,$$

where m is the number of future contracts utilized in the bootstrap ($m = 7$ in this note).

We recall that in HW:

$$v(s,t) = \frac{\hat{\sigma}}{a}\left[1 - e^{-a(t-s)}\right] \text{ with } s \le t$$

and for options' on the ith future implied volatilities can be written as

$$\sigma_i^{HW} = \hat{\sigma}\,\frac{1 - e^{-a(t_{i+1}-t_i)}}{a}\sqrt{\frac{1 - e^{-2a(t_i-t_0)}}{2a(t_i - t_0)}}.$$

In a similar way it is also simple to calibrate a more generic GHJM to market prices (see e.g. Baviera 2006).

While there are no convexity adjustments in the S1 hypothesis, in the S0 hypothesis case, when considering the HW model above, convexity adjustments are equal to

$$\begin{cases} \ln \gamma_1(i) & = \hat{\sigma}^2\,\frac{1-e^{-a(t_{i+1}-t_i)}}{a} \qquad \left[\frac{1-e^{-a(t_i-t_0)}}{a^2} - e^{-a(t_{i+1}-t_i)}\,\frac{1-e^{-2a(t_i-t_0)}}{2a^2}\right] \\ \ln \gamma_2(i) & = \left(\sigma_i^{HW}\right)^2 (t_i - t_0). \end{cases}$$